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# Sato's conjecture on recurrence conditions for multidimensional processes of Ornstein-Uhlenbeck type

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## 1. Introduction

A stochastic process of Ornstein-Uhlenbeck type (OU type process) $\{X_t\}$ was introduced in one dimension by Wolfe [7] and in multidimension by Sato and Yamazato [4]. It is a Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, P^x, X_t)$ on the $d$-dimensional Euclidean space $\boldsymbol{R}^d$ obtained from a spatially homogeneous Markov process undergoing a linear drift force determined by a matrix $-Q$. The purpose of this paper is to give an integral condition of recurrence and transience for OU type processes. Let $\{Z_t\}$ be a Lévy process on $\boldsymbol{R}^d$, that is, a stochastically continuous process with stationary independent increments, starting at the origin. Let $Q$ be a real $d \times d$ matrix of which all eigenvalues have positive real parts. An OU type process $\{X_t\}$ on $\boldsymbol{R}^d$ is, under the measure $P^x$, equivalent to the process $\{\overline{X}_t\}$ defined by

$$\overline{X}_t = e^{-tQ}x + \int_0^t e^{-(t-u)Q}\,dZ_u, \tag{1.1}$$

where the stochastic integral with respect to the Lévy process $\{Z_t\}$ is defined by stochastic convergence from integrals of simple functions. It is the unique solution of the equation

$$\overline{X}_t = x + Z_t - \int_0^t Q\overline{X}_u\,du. \tag{1.2}$$

An OU type process is determined by the Lévy process $\{Z_t\}$ and the matrix $Q$. When $\{Z_t\}$ is a Brownian motion and $Q$ is a positive constant multiple of the unit matrix, it is a classical Ornstein-Uhlenbeck process. Precise definition of an OU type process by its infinitesimal generator is given in [2] and [4]. The process $\{X_t\}$ is called recurrent if there is $y \in \boldsymbol{R}^d$ such that

$$P^x(\liminf_{t\to\infty} |X_t - y| = 0) = 1 \quad \text{for every } x \in \boldsymbol{R}^d.$$

The process $\{X_t\}$ is called transient if

$$P^x(\lim_{t\to\infty} |X_t| = \infty) = 1 \quad \text{for every } x \in \boldsymbol{R}^d.$$

OU type processes are necessarily recurrent if they have limit distributions. Sato and Yamazato [3, 4] obtain a necessary and sufficient condition for OU type processes to have limit distributions. Moreover they show in [4], by giving a concrete example, that there is a recurrent OU type process which does not have a limit distribution. Later Shiga [6] gives a recurrence criterion for OU type processes in one dimension and discusses several symmetric multidimensional cases. He also shows that any OU type process is either recurrent or transient. However his proof of the recurrence criterion includes probabilistic argument which is peculiar to one dimension. Sato, Watanabe and Yamazato [2] discover a purely Fourier analytic method to overcome difficulty in nonsymmetric multidimensional case and give a recurrence criterion when $Q$ is diagonalizable and all eigenvalues of $Q$ are real and positive. Some related remarks are discussed in Sato and Yamazato [5]. After that, Sato, Watanabe, Yamamuro and Yamazato [1] give a criterion when $Q$ is a Jordan cell matrix with a positive eigenvalue and also obtain a criterion which unifies the results when $Q$ is diagonalizable or a Jordan cell. Concerning recurrence and transience of two-dimensional OU type processes, they make comparison of the case where $Q$ is diagonal and the case where $Q$ is a Jordan cell matrix. Through these studies, it is conjectured by K. Sato that the unified criterion in [1] should be a general recurrence criterion for the OU type process. In this paper we shall answer Sato's conjecture in the affermative. A lemma (Lemma 1) on boundedness of some integrals involving exponential functions and trigonometric functions is crucial for the proof. Let $|x|$ and $\langle x, y\rangle$ denote the norm and the inner product in $\boldsymbol{R}^d$ in the usual sense. The characteristic function of $Z_t$ is represented as

$$Ee^{i\langle z, Z_t\rangle} = e^{t\psi(z)}$$

$$\psi(z) = i\langle \gamma, z\rangle - 2^{-1}\langle z, Bz\rangle + \int_{\boldsymbol{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle(1+|x|^2)^{-1})\rho(dx), \tag{1.3}$$

where $\gamma$ is a constant vector in $\boldsymbol{R}^d$, $B$ is a symmetric and nonnegative definite real $d \times d$ matrix, and $\rho$ is a measure on $\boldsymbol{R}^d$ satisfying $\rho(\{0\}) = 0$ and $\int_{\boldsymbol{R}^d}(|x|^2 \wedge 1)\rho(dx) < \infty$. The measure $\rho$ is called the Lévy measure of the Lévy process $\{Z_t\}$. Our main result is the following theorem.

THEOREM 1 (Sato's conjecture). *Fix $c > 0$. Then the OU type process $\{X_t\}$ associated with the Lévy process $\{Z_t\}$ and the matrix $Q$ is recurrent if and only if*

$$\int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x| \geq c} (\exp(-|u^Q x|) - 1)\rho(dx)\right] = \infty, \tag{1.4}$$

*where $u^Q = e^{(\log u)Q}$.*

We state the following corollaries, which are pointed out by M. Yamazato. Let $\|x\|$ be an arbitrary norm in $\boldsymbol{R}^d$.

COROLLARY 1. *Fix $c > 0$. The equation (1.4) is equivalent to*

$$\int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x| \geq c} (\exp(-\|u^Q x\|) - 1)\rho(dx)\right] = \infty, \tag{1.5}$$

*and hence the OU type process $\{X_t\}$ is recurrent if and only if (1.5) holds.*

Let $S = RQR^{-1}$ be the real Jordan canonical form of $Q$ with $R$ being a real invertible $d \times d$ matrix. Let $\gamma_j = \alpha_j + \sqrt{-1}\beta_j$ $(1 \le j \le n)$ be distinct eigenvalues of $Q$. We define a matrix $\tilde{S}$ as a matrix substituting 0 for all $\beta_j$ in $S$ and define a matrix $\tilde{Q}$ as $\tilde{Q} = R^{-1}\tilde{S}R$.

Corollary 2. *Let $\{X_t\}$ and $\{\tilde{X}_t\}$ be OU type processes associated with a common Lévy process $\{Z_t\}$ and matrices $Q$ and $\tilde{Q}$, respectively. Then $\{X_t\}$ is recurrent if and only if $\{\tilde{X}_t\}$ is recurrent.*

After we show a lemma on boundedness of some integrals in Section 2, we prove the results above in Section 3.

## 2. Boundedness of some integrals

In this section we prove a key lemma which plays an essential role in the proof of Theorem 1. Let $\boldsymbol{R}$ be the set of all real numbers. Let $m$ and $n$ be positive integers. Fix $n$ distinct complex numbers $\gamma_j$ $(1 \le j \le n)$ such that $\gamma_j = \alpha_j + \sqrt{-1}\beta_j$ with $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_j \in \boldsymbol{R}$. Let $P_j(s)$ and $Q_j(s)$ $(1 \le j \le n)$ be polynomials with complex coefficients and with degrees being at most $m$. We assume that if $\gamma_j$ is real, then polynomials $P_j(s)$ and $Q_j(s)$ have real coefficients and that if $\gamma_j$ is not real, then there exists $k$ $(1 \le k \le n)$ such that $\gamma_k = \overline{\gamma_j}$, $P_k(s) = \overline{P_j(s)}$, and $Q_k(s) = \overline{Q_j(s)}$. Here $\bar{z}$ stands for the complex conjugate of a complex number $z$. Define functions $f(s)$ and $g(s)$ on $\boldsymbol{R}$ as

$$f(s) = \sum_{j=1}^{n} e^{\gamma_j s} P_j(s) \quad \text{and} \quad g(s) = \sum_{j=1}^{n} e^{\gamma_j s} Q_j(s). \tag{2.1}$$

Note that $f(s)$ and $g(s)$ are real valued by virtue of our assumption on $P_j(s)$ and $Q_j(s)$. Let $I(x)$ be a real bounded measurable function on $\boldsymbol{R}$ continuous at $x = 0$ and let $J(x) = \int_0^x I(u)\,du$. From now on, denote by $K_l$ $(l = 1, 2, \dots)$ positive constants depending only on $m, n$, and $\{\gamma_j\}_{j=1}^n$. We state the following lemma in a version improved by M. Yamazato.

Lemma 1. *Suppose that*

$$\sup_{x \in \boldsymbol{R}} |I(x)| \le 1 \quad \textit{and} \quad \sup_{x \in \boldsymbol{R}} |J(x)| \le 1. \tag{2.2}$$

(i) *We have, for every $N > 0$,*

$$\left| \int_0^N I(f(s))\,ds \right| \le K_1 + K_2 \log\left( \frac{1}{|f(0)|} \vee 1 \right). \tag{2.3}$$

(ii) *In addition to* (2.2), *suppose that*

$$\sup_{x \neq 0} \frac{|I(x)|}{|x|} \le 1. \tag{2.4}$$

*Then we have, for every* $M$ *and* $N$ *with* $M < N$,

$$\left|\int_M^N I(f(s))\,ds\right| \le K_3. \tag{2.5}$$

REMARK 1. *We change the variable as* $s = \log u$. *Setting* $I(x) = e^{-|x|}$ *and letting* $N \to \infty$ *in* (i) *of Lemma* 1, *we get*

$$\int_1^\infty e^{-|f(\log u)|}\frac{du}{u} \le K_1 + K_2 \log\left(\frac{1}{|f(0)|} \vee 1\right). \tag{2.6}$$

*On the other hand, setting* $I(x) = \sin x$ *and* $2^{-1}(\cos x - e^{-|x|})$, *respectively in* (ii) *of Lemma* 1, *we obtain that, for every* $M$ *and* $N$ *with* $0 < M < N$,

$$\left|\int_M^N \sin(f(\log u))\frac{du}{u}\right| \le K_3 \tag{2.7}$$

*and*

$$\left|\int_M^N (\cos(f(\log u)) - e^{-|f(\log u)|})\frac{du}{u}\right| \le K_4. \tag{2.8}$$

We need several lemmas for the proof of Lemma 1. Denote by $\{\theta_l\}_{l=1}^{L_0}$ the set of all zeros of the polynomials $\{Q_j(s)\}_{j=1}^n$. Define a set $T_\eta$ for $\eta > 0$ as

$$T_\eta = \bigcap_{l=1}^{L_0}\{s \in \boldsymbol{R} : |s - Re\,\theta_l| \ge \eta\}, \tag{2.9}$$

where $Re\,z$ stands for the real part of a complex number $z$. Here we define $T_\eta = \boldsymbol{R}$ if all polynomials $\{Q_j(s)\}_{j=1}^n$ are constants. For every $\delta_0$ with $0 < \delta_0 < \alpha_1/4$, there exists a sufficiently large $\eta_0 > n$ depending only on $\delta_0, m, n$, and $\{\gamma_j\}_{j=1}^n$ such that, for any $s, t \in T_{\eta_0}$ satisfying $|s - t| \le n$ and for $1 \le j \le n$ and $1 \le k \le 2n$,

$$|Q_j(s) - Q_j(t)| \le \delta_0 |Q_j(s)| \tag{2.10}$$

and

$$|Q_j^{(k)}(t)| \le \delta_0 |Q_j(s)|, \tag{2.11}$$

where $Q_j^{(k)}(t)$ stands for the $k$-th derivative of $Q_j(t)$.

LEMMA 2. *There exists* $\rho_0 \in (0,1)$ *and* $\eta_0 > n$ *depending only on* $m$, $n$, *and* $\{\gamma_j\}_{j=1}^n$ *such that if* $\rho \in (0, \rho_0]$ *and* $s_0 \in T_{\eta_0+n}$ *satisfy*

$$|g(s_0)| < \rho^n \sum_{j=1}^n e^{\alpha_j s_0}|Q_j(s_0)|, \tag{2.12}$$

*then, for some* $l$ *with* $0 \le l \le n-1$, *there is a real zero* $\zeta_l$ *of* $g^{(l)}(s)$ *satisfying* $|s_0 - \zeta_l| < n\rho$.

**Proof.** Fix $\rho_0 \in (0,1)$ and $\eta_0 > n$ temporarily and let $\rho \in (0, \rho_0]$ and $s_0 \in T_{\eta_0+n}$ satisfy (2.12). Denote

$$I_0 = \sum_{j=1}^{n} e^{\alpha_j s_0} |Q_j(s_0)|. \tag{2.13}$$

Suppose that, for any $l$ with $0 \le l \le n-1$, $g^{(l)}(s)$ is non-zero for any $s$ satisfying $|s_0 - s| < n\rho$. We show the following assertion $(a)$.

$(a)$ For every $l$ with $0 \le l \le n-1$, there exists $s_l$ satisfying

$$|s_l - s_0| \le l\rho \quad \text{and} \quad |g^{(l)}(s_l)| \le \rho^{n-l} I_0. \tag{2.14}$$

If we let $l = 0$ in (2.14), it is obviously true. Suppose that, for some $k$ with $0 \le k \le n-2$, and for some $s_k$,

$$|s_k - s_0| \le k\rho \quad \text{and} \quad |g^{(k)}(s_k)| \le \rho^{n-k} I_0, \tag{2.15}$$

and, for every $s$ satisfying $|s - s_0| \le (k+1)\rho$,

$$|g^{(k+1)}(s)| > \rho^{n-k-1} I_0. \tag{2.16}$$

By virtue of the mean value theorem, we find from (2.15) and (2.16) that there are two numbers $c_\pm$ satisfying $|c_\pm - s_k| \le \rho$ and

$$|g^{(k)}(s_k \pm \rho) - g^{(k)}(s_k)| = \rho |g^{(k+1)}(c_\pm)| > \rho^{n-k} I_0. \tag{2.17}$$

Since $g^{(k+1)}(s)$ cannot change its sign on $(s_k - \rho, s_k + \rho) \subset (s_0 - (k+1)\rho, s_0 + (k+1)\rho)$, it follows from (2.15) that either $g^{(k)}(s_k + \rho)$ or $g^{(k)}(s_k - \rho)$ has the opposite sign with $g^{(k)}(s_k)$. Hence there exists a real zero $\zeta_k$ of $g^{(k)}(s)$ satisfying $|\zeta_k - s_k| \le \rho$, which contradicts our assumption. This proves the assertion $(a)$. Now define an $n \times n$ matrix $\Gamma$ as

$$\Gamma = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_1^{n-1} & \gamma_2^{n-1} & \cdots & \gamma_n^{n-1} \end{pmatrix} \tag{2.18}$$

Note that $\det \Gamma \ne 0$, since it is Vandermonde's determinant. Let $\Gamma^{-1} = (\gamma_{ij})$ $(1 \le i, j \le n)$ and let $\max_{1 \le i,j \le n} |\gamma_{ij}| = K_5$. If we choose $\rho_0$ small enough and $\eta_0$ large enough beforehand, then, by virtue of (2.10) and (2.11), for some $\delta_1 > 0$ and for each $l$ with $0 \le l \le n-1$,

$$\begin{gathered} g^{(l)}(s_l) = \sum_{j=1}^{n} (\gamma_j)^l e^{\gamma_j s_0} Q_j(s_0) + R_l, \\ |R_l| \le \delta_1 I_0, \quad \text{and} \quad n^2(\delta_1 + \rho_0) K_5 < 1. \end{gathered} \tag{2.19}$$

The smallness of $\rho_0$ and the largeness of $\eta_0$ depend only on $m, n$, and $\{\gamma_j\}_{j=1}^{n}$. We define complex numbers $x_j$ $(0 \le j \le n-1)$ and $y_l$ $(0 \le l \le n-1)$ as

$$x_j = e^{\gamma_{j+1} s_0} Q_{j+1}(s_0) \tag{2.20}$$

and

$$y_l = g^{(l)}(s_l) - R_l. \tag{2.21}$$

Denote $\boldsymbol{x} = {}^t(x_0, \ldots, x_{n-1})$ and $\boldsymbol{y} = {}^t(y_0, \ldots, y_{n-1})$. Then we see from (2.19) that

$$\boldsymbol{x} = \Gamma^{-1}\boldsymbol{y}. \tag{2.22}$$

Hence we obtain from the assertion $(a)$ and (2.19) that

$$I_0 = \sum_{j=0}^{n-1} |x_j| \le n^2(\delta_1 + \rho_0)K_5 I_0 < I_0, \tag{2.23}$$

which is a contradiction. This proves Lemma 2.

LEMMA 3. *There exists sufficiently large $\eta_0 > n$ and small $\delta_2 \in (0,1)$ each depending only on $m, n$, and $\{\gamma_j\}_{j=1}^n$ such that, for every $t \in \boldsymbol{R}$ and every $l$ with $0 \le l \le n-1$, $g^{(l)}(s)$ has at most $n-1$ real zeros on $[t, t+\delta_2] \cap T_{\eta_0}$.*

PROOF. We use again $I_0$ defined in (2.13). We see from (2.10) and (2.11) that, for every $\varepsilon > 0$, there exist sufficiently large $\eta_0 > n$ and small $\delta_2 > 0$ depending only on $\varepsilon, m, n$, and $\{\gamma_j\}_{j=1}^n$ such that, for any $s_0, s \in T_{\eta_0}$ satisfying $|s_0 - s| \le \delta_2$ and for $0 \le j \le 2n$,

$$|g^{(j)}(s) - g^{(j)}(s_0)| \le \varepsilon I_0. \tag{2.24}$$

Suppose that, for some $l$ with $0 \le l \le n-1$ and for some $t$, there are at least $n$ distinct real zeros of $g^{(l)}(s)$ on $[t, t+\delta_2] \cap T_{\eta_0}$. Then we find from Rolle's theorem that, for $0 \le j \le n-1$, $g^{(l+j)}(s)$ has at least $n-j$ distinct real zeros on $[t, t+\delta_2] \cap T_{\eta_0}$. Choosing $s_0 \in [t, t+\delta_2] \cap T_{\eta_0}$, we see that, for $0 \le j \le n-1$, there exists a real zero $\zeta_{l+j}$ of $g^{(l+j)}(s)$ satisfying $|\zeta_{l+j} - s_0| \le \delta_2$. Hence, from (2.24),

$$|g^{(l+j)}(s_0)| \le \varepsilon I_0 \quad \text{for } 0 \le j \le n-1, \tag{2.25}$$

which leads to a contradiction by argument similar to the proof of Lemma 2. The proof of Lemma 3 is complete.

Let $\alpha > 0$ and let $P(s)$ be a polynomial with complex coefficients and with the degree being at most $m$. Let $\lambda_j$ $(1 \le j \le l)$ be all zeros of $P(s)$. Define a set $T$ as

$$T = \bigcap_{j=1}^{l} \{s \in \boldsymbol{R} : |s - Re\,\lambda_j| \ge 1\}. \tag{2.26}$$

Here we define $T = \boldsymbol{R}$ if $P(s)$ is a constant.

LEMMA 4. *There exists a positive constant $K$ depending only on $\alpha$ and $m$ such that*

$$e^{\alpha s}|P(s)| \ge K|P(0)| \quad \textit{for all } s \ge 0 \quad \textit{with } s \in T. \tag{2.27}$$

PROOF. If $P(0)=0$ or $P(s)$ is a constant, then the lemma is obviously true. Hence we can and do assume that $P(0)\neq 0$ and $l\geq 1$. Define a function $R(s)$ on $\boldsymbol{R}$ as

$$R(s)=\frac{e^{\alpha s}|P(s)|}{|P(0)|}. \tag{2.28}$$

We may assume the zeros are repeated according to their multiplicity. Then $R(s)$ is expressed as

$$R(s)=e^{\alpha s}\prod_{j=1}^{l}\frac{|s-\lambda_j|}{|\lambda_j|}. \tag{2.29}$$

Without loss of generality, we can and do assume that $0<|\lambda_1|\leq|\lambda_2|\leq\cdots\leq|\lambda_l|$. We shall prove (2.27) considering the following three possible cases:

Case (i). If $2^{-1}|\lambda_l|\leq s$, then we have

$$R(s)\geq\frac{e^{\alpha|\lambda_l|/2}}{|\lambda_l|^l}. \tag{2.30}$$

Case (ii). If $2^{-1}|\lambda_j|\leq s<2^{-1}|\lambda_{j+1}|$ for some $j$ with $1\leq j\leq l-1$, then we get

$$R(s)\geq\frac{2^{j-l}e^{\alpha|\lambda_j|/2}}{|\lambda_j|^j}. \tag{2.31}$$

Case (iii). If $0<s<2^{-1}|\lambda_1|$, then we find that

$$R(s)\geq 2^{-l}. \tag{2.32}$$

Let $C=\inf_{1\leq l\leq m}\inf_{x>0}x^{-l}e^{\alpha x/2}$ and let $K=2^{-m}(C\wedge 1)$. Then (2.27) is evident from (2.30), (2.31), and (2.32).

PROOF OF LEMMA 1. The assertion is trivial if $f(s)$ identically vanishes. Hence we assume that, for some $j, P_j(s)$ does not vanish identically. We first prove (i). Denote $\{\tilde{\theta}_l\}_{l=1}^{L_1}$ be all zeros of the polynomials $\{P_j(s)\}_{j=1}^n$. Define a set $S_\eta$ for $\eta>0$ as

$$S_\eta=\bigcap_{l=1}^{L_1}\{s\in\boldsymbol{R}:|s-Re\,\tilde{\theta}_l|\geq\eta\}. \tag{2.33}$$

Here we define $S_\eta=\boldsymbol{R}$ if all polynomials $\{P_j(s)\}_{j=1}^n$ are constants. There exists sufficiently large $\eta_1>1$ depending only on $m$ and $\alpha_1$ such that, for any $s\in S_{\eta_1}$ and for $1\leq j\leq n$,

$$|P_j'(s)|\leq 4^{-1}\alpha_1|P_j(s)|. \tag{2.34}$$

Now we choose $Q_j(s)=\gamma_jP_j(s)+P_j'(s)$. Then $g(s)=f'(s)$. Choose sufficiently large $\eta_0>n$ and small $\rho_0,\delta_2\in(0,1)$ as in Lemmas 2 and 3. Denote

$$U=T_{\eta_0+n}\cap S_{\eta_1}=\bigcup_{l=0}^{L_2}[a_l,b_l]\quad\text{and}\quad V=U^c\cap\boldsymbol{R}, \tag{2.35}$$

where $0 \leq L_2 \leq 2mn+1$ and $a_0 = -\infty$ and $b_{L_2} = +\infty$. Then we have

$$\int_V |I(f(s))|\,ds \leq 2(\eta_0 + n)mn + 2\eta_1 mn = K_6. \tag{2.36}$$

Define functions $f_0(s)$, $g_0(s)$, and $g_1(s)$ on $\boldsymbol{R}$ as

$$f_0(s) = \sum_{j=1}^{n} e^{\alpha_j s}|P_j(s)|, \tag{2.37}$$

$$g_0(s) = \sum_{j=1}^{n} e^{\alpha_j s}|Q_j(s)|, \tag{2.38}$$

and

$$g_1(s) = e^{-\alpha_1 s/2} g_0(s). \tag{2.39}$$

Hereafter we fix an arbitrary integer $l$ in $0 \leq l \leq L_2$. We see from (2.11) that

$$g_1'(s) \geq 4^{-1}\alpha_1 g_1(s) > 0 \quad \text{on } [a_l, b_l]. \tag{2.40}$$

Since $Q_j(s) = \gamma_j P_j(s) + P_j'(s)$, it follows from (2.34) that

$$f_0(s) \leq K_7 g_0(s) \quad \text{on } [a_l, b_l]. \tag{2.41}$$

Note from (2.40) that $g_0'(s) > 0$ on $[a_l, b_l]$. We define $c_l \in [a_l, b_l]$ considering the following three possible cases:

(i) If there exists $s_l \in [a_l, b_l]$ such that $g_0(s_l) = 1$, then we set $c_l = s_l$.

(ii) If $g_0(a_l) > 1$, then we set $c_l = a_l$.

(iii) If $g_0(b_l) < 1$, then we set $c_l = b_l$.

We shall show that if $c_l < b_l$, then

$$\left| \int_{c_l}^{N} I(f(s))\,ds \right| \leq K_8 \quad \text{for every } N \in [c_l, b_l]. \tag{2.42}$$

Let $\varepsilon$ be a positive number. Denote $\rho_k = \rho_0 \varepsilon^k$ for integers $k \geq 0$ and $\lambda = \varepsilon^{2n} e^{\alpha_1 \delta_2/2}$. We choose $\varepsilon$ depending only on $m, n$, and $\{\gamma_j\}_{j=1}^{n}$ so that $0 < \varepsilon < 1$ and $\lambda > 1$. Let $\{\zeta_j\}_{j=1}^{L_3}$ be all real zeros of the functions $\{g^{(k)}(s)\}_{k=0}^{n-1}$ ($L_3$ may be infinity). For $k \geq 0$ define sets $W_k, U_k$, and $V_k$ as

$$W_k = \bigcap_{j=1}^{L_3} \{s \in \boldsymbol{R} : |s - \zeta_j| \geq n\rho_k\}, \tag{2.43}$$

$$U_k = [c_l, N] \cap [c_l + k\delta_2, c_l + (k+1)\delta_2] \cap W_k, \tag{2.44}$$

and

$$V_k = [c_l, N] \cap [c_l + k\delta_2, c_l + (k+1)\delta_2] \cap W_k^c. \tag{2.45}$$

Here we define $W_k = \boldsymbol{R}$ if there are no real zeros of the functions $\{g^{(k)}(s)\}_{k=0}^{n-1}$. We represent $U_k$ as

$$U_k = \bigcup_{j=1}^{L_4} [d_{2j}, d_{2j+1}]. \tag{2.46}$$

Note from Lemma 3 that $L_4 \le n^2 + 1$ and that

$$\int_{V_k} |I(f(s))|\, ds \le n^2 2n\rho_k = 2n^3 \rho_0 \varepsilon^k. \tag{2.47}$$

Fix an arbitrary integer $j$ in $1 \le j \le L_4$. We see from Lemma 2 that

$$|g(s)| \ge \rho_k^n g_0(s) \quad \text{on } [d_{2j}, d_{2j+1}]. \tag{2.48}$$

On the other hand, we find from (2.40) that, for $s \in [d_{2j}, d_{2j+1}]$,

$$g_0(s) \ge e^{\alpha_1 s/2} g_1(c_l) \ge e^{k\alpha_1\delta_2/2}. \tag{2.49}$$

Here note from $c_l < b_l$ that $g_0(c_l) \ge 1$. Hence we obtain from (2.48) that

$$|g(d_{2j})| \ge \rho_k^n g_0(d_{2j}) \ge \rho_0^n \lambda^k \tag{2.50}$$

and likewise

$$|g(d_{2j+1})| \ge \rho_0^n \lambda^k. \tag{2.51}$$

We see from (2.11) that

$$|g'(s)| \le K_9 g_0(s) \quad \text{on } [d_{2j}, d_{2j+1}]. \tag{2.52}$$

Hence we get by (2.40), (2.48), and (2.49) that

$$\begin{aligned}\int_{d_{2j}}^{d_{2j+1}} \frac{|g'(s)|}{g(s)^2}\, ds &\le \frac{K_9}{\rho_k^{2n}} \int_{d_{2j}}^{d_{2j+1}} \frac{ds}{g_0(s)} \\ &\le \frac{K_9}{\rho_k^{2n} g_1(d_{2j})} \int_{d_{2j}}^{d_{2j+1}} e^{-\alpha_1 s/2}\, ds \le \frac{2K_9}{\alpha_1 \rho_k^{2n} g_0(d_{2j})} \le K_{10}\lambda^{-k}.\end{aligned} \tag{2.53}$$

By using integration by parts, we obtain from (2.50), (2.51), and (2.53) that

$$\begin{aligned}\left| \int_{d_{2j}}^{d_{2j+1}} I(f(s))\, ds \right| &\le \left| \left[ \frac{J(f(s))}{g(s)} \right]_{d_{2j}}^{d_{2j+1}} \right| + \left| \int_{d_{2j}}^{d_{2j+1}} \frac{J(f(s)) g'(s)}{g(s)^2}\, ds \right| \\ &\le 2\rho_0^{-n}\lambda^{-k} + K_{10}\lambda^{-k} = K_{11}\lambda^{-k}.\end{aligned} \tag{2.54}$$

It follows that, for every $N \in [c_l, b_l]$,

$$\left| \int_{c_l}^{N} I(f(s))\, ds \right| \le \sum_{k=0}^{\infty} \{(n^2+1) K_{11} \lambda^{-k} + 2n^3 \rho_0 \varepsilon^k\} = K_8. \tag{2.55}$$

Thus we have proved (2.42). Next we shall prove that if $a_l < c_l$ and $0 < c_l$, then

$$\int_{a_l \vee 0}^{c_l} |I(f(s))|\, ds \le K_{12} + K_{13} \log\left( \frac{1}{|f(0)|} \vee 1 \right). \tag{2.56}$$

Note from $a_l < c_l$ that $g_0(c_l) \le 1$. Hence we have

$$c_l \le -2\alpha_1^{-1} \log g_1(c_l). \tag{2.57}$$

By using Lemma 4 with $\alpha = \alpha_j - \alpha_1/2$, $s = c_l$, and $P(s) = P_j(s)$ for $1 \le j \le n$, we find from (2.41) that

$$|f(0)| \le K_{14} e^{-\alpha_1 c_l/2} f_0(c_l) \le K_{15} g_1(c_l). \tag{2.58}$$

Hence we get (2.56) by (2.57). It follows that, for every $N > 0$,

$$\begin{aligned} \left| \int_0^N I(f(s))\,ds \right| &\le K_6 + (2mn+1)\left(2K_8 + K_{12} + K_{13} \log\left(\frac{1}{|f(0)|} \vee 1\right)\right) \\ &= K_1 + K_2 \log\left(\frac{1}{|f(0)|} \vee 1\right). \end{aligned} \tag{2.59}$$

Thus we have established (i). Secondly we prove (ii). We see from (2.40) and (2.41) that if $a_l < c_l$, then

$$\begin{aligned} \int_{a_l}^{c_l} |I(f(s))|\,ds &\le K_7 \int_{a_l}^{c_l} g_0(s)\,ds \\ &\le K_7 g_1(c_l) \int_{a_l}^{c_l} e^{s\alpha_1/2}\,ds \le 2K_7 \alpha_1^{-1} g_0(c_l) \le K_{16}, \end{aligned} \tag{2.60}$$

where we use the inequality $g_0(c_l) \le 1$. Recalling (2.42), we conclude that, for every $M$ and $N$ with $M < N$,

$$\left| \int_M^N I(f(s))\,ds \right| \le K_6 + (2mn+1)(2K_8 + K_{16}) = K_3. \tag{2.61}$$

The proof of Lemma 1 is complete.

## 3. Proof of results

In this section we prove the results which are stated in Section 1. Our argument used in the proof of Theorem 1 is similar to the proof of Theorem A of [**1**] and so the same part of the proof is omitted.

Proof of Theorem 1. Denote by $\gamma_j = \alpha_j + \sqrt{-1}\beta_j$ $(1 \le j \le n)$ distinct eigenvalues of $Q$ with $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_j \in \boldsymbol{R}$. Suppose that $\{X_t\}$ is transient. Then, by using (2.7) of Remark 1, we find as in the proof of Theorem A of [**1**] that there is $z \in \boldsymbol{R}^d$ with $0 < |z| \le 1$ such that

$$\int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x| \ge c} (\cos\langle z, u^Q x\rangle - 1)\rho(dx)\right] < \infty. \tag{3.1}$$

Here $c$ is an arbitrary positive constant. Setting $f(\log u) = \langle z, u^Q x\rangle$, we see from (2.8) of Remark 1 that

$$\left|\int_v^1 (\cos\langle z, u^Q x\rangle - e^{-|\langle z, u^Q x\rangle|})\frac{du}{u}\right| \le K_4, \tag{3.2}$$

and hence

$$\begin{aligned}
&\int_v^1 (\cos\langle z, u^Q x\rangle - 1)\frac{du}{u} \\
&\quad = \int_v^1 [(\cos\langle z, u^Q x\rangle - e^{-|\langle z, u^Q x\rangle|}) + (e^{-|\langle z, u^Q x\rangle|} - e^{-|u^Q x|}) + (e^{-|u^Q x|} - 1)]\frac{du}{u} \\
&\quad \ge -K_4 + \int_v^1 (e^{-|u^Q x|} - 1)\frac{du}{u}.
\end{aligned} \tag{3.3}$$

Here we note that $e^{-|\langle z, u^Q x\rangle|} - e^{-|u^Q x|} \ge 0$. Hence we obtain from (3.1) that

$$\int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x|\ge c} (\exp(-|u^Q x|) - 1)\rho(dx)\right] < \infty. \tag{3.4}$$

Conversely, suppose that (3.4) is true for each $c > 0$. We shall prove that $\{X_t\}$ is transient. Again, as in the proof of Theorem A of [1], it is enough to prove that, for some $c > 0$,

$$\int_{|z|\le 1} dz \int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x|\ge 0} (\cos\langle z, u^Q x\rangle - 1)\rho(dx)\right] < \infty. \tag{3.5}$$

In general there are positive constants $C_j$ $(1 \le j \le 4)$ depending only on $Q$ such that, for $x \in \boldsymbol{R}^d$,

$$C_4 u^{C_2}|x| \le |u^Q x| \le C_3 u^{C_1}|x| \quad \text{for } u \in (0,1], \tag{3.6}$$

and

$$C_3^{-1} u^{C_1}|x| \le |u^Q x| \le C_4^{-1} u^{C_2}|x| \quad \text{for } u \in [1,\infty). \tag{3.7}$$

For $c > 0$ denote by $\rho_c$ the ristriction of the Lévy measure $\rho$ to the set $\{x \in \boldsymbol{R}^d : |x| \ge c\}$. Denote by $\boldsymbol{S}^{d-1}$ the $d-1$ dimensional unit sphere. Define a set $S_Q$ as

$$S_Q = \{\xi \in \boldsymbol{S}^{d-1} : |u^Q \xi| > 1 \text{ for } u > 1\}. \tag{3.8}$$

We define a probability measure $\sigma$ on $S_Q$ and measures $\tau_\xi$ $(\xi \in \boldsymbol{S}^{d-1})$ on $(0,\infty)$ such that $\tau_\xi(B)$ is measurable in $\xi$ for any Borel set $B$ in $(0,\infty)$, $\tau_\xi((0,\infty)) = \rho_c(\boldsymbol{R}^d)$, and, for each Borel set $E$ in $\boldsymbol{R}^d$,

$$\rho_c(E) = \int_{S_Q} \sigma(d\xi) \int_0^\infty 1_E(r^Q \xi)\tau_\xi(dr). \tag{3.9}$$

We get by (3.6) that, for $\xi \in S_Q$ and for $z \in \boldsymbol{R}^d$ with $|z| \le 1$,

$$\int_0^1 (e^{-|\langle z, w^Q \xi\rangle|} - e^{-|w^Q \xi|}) \frac{dw}{w} \le \int_0^1 |w^Q \xi| \frac{dw}{w} \le C_1^{-1} C_3. \tag{3.10}$$

Setting $f(\log w) = \langle z, w^Q \xi\rangle$, we find from (2.6) of Remark 1 that

$$\int_1^\infty (e^{-|\langle z, w^Q \xi\rangle|} - e^{-|w^Q \xi|}) \frac{dw}{w} \le \int_1^\infty e^{-|\langle z, w^Q \xi\rangle|} \frac{dw}{w} \le K_1 + K_2 \log \frac{1}{|\langle z, \xi\rangle|}. \tag{3.11}$$

We see from Lemma 2.2 of [2] that, for sufficiently large $c > 0$,

$$\int_{|z| \le 1} dz \exp\left[\rho_c(\boldsymbol{R}^d) K_2 \int_{S_Q} \log \frac{1}{|\langle z, \xi\rangle|} \sigma(d\xi)\right] < \infty. \tag{3.12}$$

Letting $ru = w$, we obtain from (3.10), (3.11), and (3.12) that, for sufficiently large $c > 0$,

$$\begin{aligned} &\int_{|z| \le 1} dz \exp\left[\int_0^1 \frac{du}{u} \int_{|x| \ge c} (e^{-|\langle z, u^Q x\rangle|} - e^{-|u^Q x|}) \rho(dx)\right] \\ &\le \int_{|z| \le 1} dz \exp\left[\rho_c(\boldsymbol{R}^d) \int_{S_Q} \sigma(d\xi) \int_0^\infty (e^{-|\langle z, w^Q \xi\rangle|} - e^{-|w^Q \xi|}) \frac{dw}{w}\right] \\ &\le const \cdot \int_{|z| \le 1} dz \exp\left[\rho_c(\boldsymbol{R}^d) K_2 \int_{S_Q} \log \frac{1}{|\langle z, \xi\rangle|} \sigma(d\xi)\right] < \infty. \end{aligned} \tag{3.13}$$

Recalling (3.2), we get that, for sufficiently large $c > 0$,

$$\begin{aligned} &\int_{|z| \le 1} dz \int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x| \ge c} (\cos\langle z, u^Q x\rangle - 1) \rho(dx)\right] \\ &= \int_{|z| \le 1} dz \int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x| \ge c} \{(\cos\langle z, u^Q x\rangle - e^{-|\langle z, u^Q x\rangle|}) \right. \\ &\qquad \left. + (e^{-|\langle z, u^Q x\rangle|} - e^{-|u^Q x|}) + (e^{-|u^Q x|} - 1)\} \rho(dx)\right] \\ &\le const \cdot \int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x| \ge c} (\exp(-|u^Q x|) - 1) \rho(dx)\right] < \infty. \end{aligned} \tag{3.14}$$

Thus we have established (3.5). The proof of Theorem 1 is complete.

PROOF OF COROLLARY 1. In the proof of Corollaries 1 and 2, we continue to use the notations above. Obviously there are positive constants $C_5$ and $C_6$ such that

$$C_5 |x| \le \|x\| \le C_6 |x| \quad \text{for } x \in \boldsymbol{R}^d. \tag{3.15}$$

Let $a$ and $b$ be constants satisfying $0 < a < b$. We see from (3.6) and (3.7) that, for $\xi \in S_Q$,

$$\int_0^1 (e^{-a|w^Q \xi|} - e^{-b|w^Q \xi|}) \frac{dw}{w} \le (b - a) \int_0^1 |w^Q \xi| \frac{dw}{w} \le (b - a) C_1^{-1} C_3, \tag{3.16}$$

and

$$\int_1^\infty (e^{-a|w^Q\xi|} - e^{-b|w^Q\xi|})\frac{dw}{w} \le \int_1^\infty e^{-a|w^Q\xi|}\frac{dw}{w} \le a^{-1}C_1^{-1}C_3. \tag{3.17}$$

Letting $ur = w$, we obtain from (3.16) and (3.17) that

$$\begin{aligned} &\int_{|x|\ge c}\rho(dx)\int_0^1 (e^{-a|u^Q x|} - e^{-b|u^Q x|})\frac{du}{u} \\ &\le \rho_c(\boldsymbol{R}^d)\int_{S_Q}\sigma(d\xi)\int_0^\infty (e^{-a|w^Q\xi|} - e^{-b|w^Q\xi|})\frac{dw}{w} < \infty. \end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned} &\int_{|x|\ge c}\rho(dx)\int_0^1 |e^{-\|u^Q x\|} - e^{-|u^Q x|}|\frac{du}{u} \\ &\le \int_{|x|\ge c}\rho(dx)\int_0^1 \{(e^{-C_5|u^Q x|} - e^{-C_6|u^Q x|}) + |e^{-C_6|u^Q x|} - e^{-|u^Q x|}|\}\frac{du}{u} < \infty. \end{aligned} \tag{3.19}$$

Hence Corollary 1 is evident from Theorem 1.

Proof of Corollary 2. Let $u > 0$. Denote

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{3.20}$$

with $\alpha, \beta \in \boldsymbol{R}$. Obviously we have

$$|u^{D_2}x| = u^\alpha |x| \quad \text{for } x \in \boldsymbol{R}^2. \tag{3.21}$$

Let $A_j$ $(1 \le j \le l)$ be real $2 \times 2$ matrices. We define real $2l \times 2l$ matrix $M(A_1, A_2, \ldots, A_l)$ as

$$M(A_1, A_2, \ldots, A_l) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_l \\ & A_1 & A_2 & \cdots & A_{l-1} \\ & & \cdots & \cdots & \cdots \\ & & & A_1 & A_2 \\ 0 & & & & A_1 \end{pmatrix} \tag{3.22}$$

Denote $J_l = M(D_2, E_2, 0, \ldots, 0)$ and $\tilde{J}_l = M(\alpha E_2, E_2, 0, \ldots, 0)$. Then we find that

$$\begin{aligned} u^{J_l} &= M\left(u^{D_2}, (\log u)u^{D_2}, \ldots, \frac{(\log u)^{l-1}}{(l-1)!}u^{D_2}\right) \\ &= M(u^{D_2}, 0, \ldots, 0)M\left(E_2, (\log u)E_2, \ldots, \frac{(\log u)^{l-1}}{(l-1)!}E_2\right) \end{aligned} \tag{3.23}$$

and

$$u^{\tilde{J}_l} = M(u^\alpha E_2, 0, \ldots, 0)M\left(E_2, (\log u)E_2, \ldots, \frac{(\log u)^{l-1}}{(l-1)!}E_2\right). \tag{3.24}$$

Hence we obtain from (3.21) that

$$|u^{J_l}x| = |u^{\tilde{J}_l}x| \quad \text{for } x \in \boldsymbol{R}^{2l}. \tag{3.25}$$

It follows that

$$|u^{S}x| = |u^{\tilde{S}}x| \quad \text{for } x \in \boldsymbol{R}^{d}. \tag{3.26}$$

Define the norm $\|x\| = |Rx|$. Note from (3.26) that

$$\|u^{Q}x\| = |u^{S}Rx| = |u^{\tilde{S}}Rx| = \|u^{\tilde{Q}}x\|. \tag{3.27}$$

Therefore, Corollary 2 is evident from Corollary 1.

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