Characterization of the group association scheme of
A_5 by its intersection numbers

By Masato TOMIYAMA

1. Introduction

Let X be a finite set and let $R_i(i = 0, 1, \ldots, d)$ be relations on X, i.e., subsets of $X \times X$. $\mathscr{X} = (X, \{R_i\}_{0 \le i \le d})$ is a *commutative association scheme of d classes* if the following conditions hold.

(1) $R_0 = \{(x, x) \,|\, x \in X\}$,

(2) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ and $R_i \cap R_j = \phi$ if $i \ne j$,

(3) ${}^t R_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^t R_i = \{(x, y) \,|\, (y, x) \in R_i\}$,

(4) for $i, j, k \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k whenever $(x, y) \in R_k$,

(5) $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in \{0, 1, \ldots, d\}$.

The non-negative integers p_{ij}^k are called the *intersection numbers* of $\mathscr{X}$.

An association scheme $\mathscr{X}$ is called *imprimitive* if some union of relations is an equivalence relation distinct from R_0 and $X \times X$, and *primitive* otherwise.

For an imprimitive association scheme $\mathscr{X}$, by rearranging the indices of the relations of $\mathscr{X}$, let $\bigcup_{i=0}^s R_i$ is an equivalence relation. For each equivalence class X' of $\bigcup_{i=0}^s R_i$, we find an association scheme $\mathscr{X}' = (X', \{R_i'\}_{0 \le i \le s})$, where $R_i' = R_i \cap (X' \times X')$. We write $\mathscr{X} \supseteq \mathscr{X}'$. Let $\sim$ be the relation on $\{0, 1, \ldots, d\}$ defined by $i \sim j$ if and only if $p_{j,\alpha}^i \ne 0$ for some $0 \le \alpha \le s$. Then $\sim$ is an equivalence relation. Let $T_0 = \{0, 1, \ldots, s\}$, $T_1, \ldots, T_r$ be the equivalence classes. Then $\mathscr{X}/\mathscr{X}' = (\tilde{X}, \{\tilde{R}_i\}_{0 \le i \le r})$, where $\tilde{X}$ is the set of the equivalence classes of $\bigcup_{i=0}^s R_i$ on X and $\tilde{R}_i = \{(\tilde{x}, \tilde{y}) \,|\, \text{for } x \in \tilde{x} \text{ and } y \in \tilde{y} \text{ we have } (x, y) \in R_\alpha \text{ with } \alpha \in T_i\}$, is a primitive association scheme.

The reader is refered to [2] and [3] for the general theory of association schemes, and other terminology.

Let G be a finite group. Let $C_0 = \{1\}, C_1, \ldots, C_d$ be the conjugacy classes of G. Define relations $R_i(i = 0, 1, \ldots, d)$ on G by $R_i = \{(x, y) \,|\, yx^{-1} \in C_i\}$. Then $\mathscr{X}(G) = (G, \{R_i\}_{0 \le i \le d})$ is a commutative association scheme of d classes, called the *group association scheme* of G. (See [2].)

It is well known that G is simple if and only if the group association scheme $\mathscr{X}(G)$ is primitive.

In the study of association schemes, primitive association schemes play an important role, similar to the role simple groups play in finite groups. Namely, they are building blocks of general association schemes in the following sense. For any commutative

association scheme $\mathscr{X}$, there exists a composition series, i.e., a sequence $\mathscr{X} = \mathscr{X}_0 \supseteq \mathscr{X}_1 \supseteq \cdots \supseteq \mathscr{X}_r = (\{x\}, R_0)$ such that the composition factors $\mathscr{X}_i/\mathscr{X}_{i+1}$ are primitive for $0 \leq i \leq r - 1$. (If we fix $x \in X$, then the schemes $\mathscr{X}_i/\mathscr{X}_{i+1}$ are uniquely determined up to isomorphism as association schemes. See [5].)

The classification of primitive commutative association schemes is a hard problem, too hard to be expected to be solved completely. It would be interesting to classify some special classes of primitive commutative association schemes. In [1] E. Bannai proposed that in order to study finite simple groups from the viewpoint of algebraic combinatorics, it would be interesting and necessary to determine whether the association scheme $\mathscr{X}(G)$ is the only association scheme having the same intersection numbers as those of $\mathscr{X}(G)$ for a given simple group G. He posed this question, in particular, for the alternating group of degree 5 A_5, the smallest non-abelian finite simple group. In this paper, we solve this question; namely we prove the following:

THEOREM 1.1. *Let $\mathscr{Y}$ be an association scheme having the same intersection numbers as those of the group association scheme $\mathscr{X}(A_5)$. Then $\mathscr{Y}$ is isomorphic to $\mathscr{X}(A_5)$.*

REMARKS. (1) Most known characterizations of association schemes by intersection numbers concern $P-$ (and $Q-$) polynomial association schemes. But $\mathscr{X}(A_5)$ is not P-polynomial.

(2) In Section 5, we consider the case $G = SL(2,5)$, which is isomorphic to the nonsplit central extension of A_5. We also show that $\mathscr{X}(SL(2,5))$ is the only association scheme having the same intersection numbers as those of $\mathscr{X}(SL(2,5))$. (See Theorem 5.1.)

(3) For a finite group G, the character table of G determines the intersection numbers of $\mathscr{X}(G)$, and vice versa. (See [2].) But the classification of the groups having the same character table as that of G and the classification of the association schemes having the same intersection numbers as those of $\mathscr{X}(G)$ are different problems. For example, the dihedral and quaternion groups of order 8, D_8 and Q_8, are non-isomorphic but the association schemes $\mathscr{X}(D_8)$ and $\mathscr{X}(Q_8)$ are isomorphic. On the other hand, there are exactly three non-isomorphic association schemes having the same intersection numbers as those of $\mathscr{X}(S_4)$, where S_4 is the symmetric group of degree 4. (See Theorem 6.1.)

2. Intersection numbers

Let $\mathscr{Y} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme and let $A_0, \ldots, A_d$ be the adjacency matrices of $\mathscr{Y}$.

Let

$$R_i(x) = \{y \in X \mid (x, y) \in R_i\},$$

$$R_{\leq 3}(x) = \{y \in X \mid (x, y) \in R_0 \cup R_1 \cup R_2 \cup R_3\}.$$

We define the graph $(R_i(x), R_j)$ as the graph with vertex set $R_i(x)$ and edge set $(R_i(x) \times R_i(x)) \cap R_j$.

We denote by $\boldsymbol{B}_i$ the ith *intersection matrix* of $\mathscr{Y}$, defined by

$$(\boldsymbol{B}_i)_{j,k} = p_{ij}^k.$$

We order the conjugacy classes of A_5 as follows:

	C_0	C_1	C_2	C_3	C_4
representative :	( )	(12345)	(123)	(12354)	(12)(34)
$|C_i|$:	1	12	20	12	15

Then the intersection matrices of $\mathscr{X}(A_5)$ are as follows:

$$\boldsymbol{B}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{B}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 12 & 5 & 3 & 1 & 0 \\ 0 & 5 & 3 & 5 & 4 \\ 0 & 1 & 3 & 1 & 4 \\ 0 & 0 & 3 & 5 & 4 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 3 & 5 & 4 \\ 20 & 5 & 7 & 5 & 8 \\ 0 & 5 & 3 & 5 & 4 \\ 0 & 5 & 6 & 5 & 4 \end{pmatrix},$$

$$\boldsymbol{B}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 4 \\ 0 & 5 & 3 & 5 & 4 \\ 12 & 1 & 3 & 5 & 0 \\ 0 & 5 & 3 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{B}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 5 & 4 \\ 0 & 5 & 6 & 5 & 4 \\ 0 & 5 & 3 & 0 & 4 \\ 15 & 5 & 3 & 5 & 2 \end{pmatrix}.$$

We assume $\mathscr{Y} = (X, \{R_i\}_{0 \leq i \leq 4})$ is an association scheme having the same intersection matrices as $\mathscr{X}(A_5)$.

Let $\Gamma = (X, R_1)$, then the adjacency matrix of Γ is the 1st adjacency matrix A_1 of $\mathscr{Y}$. If we know the matrix A_1, then we have the other adjacency matrices of $\mathscr{Y}$ by using the equation

$$A_1{}^2 = \sum_{k=0}^{4} p_{11}^k A_k{}^2 = 12A_0 + 5A_1 + 3A_2 + 1A_3 + 0A_4.$$

So to prove Theorem 1.1, we determine the graph Γ uniquely up to isomorphism.

3. Local structure of Γ

In this section we show Γ is locally an icosahedron. We take any $x \in X$ and let $\Delta = (R_1(x), R_1)$.

For vertices $u, v \in R_1(x)$, let $\partial(u, v)$ denote the distance between u and v in Δ. Set

$$\Delta_i(u) = \{v \in R_1(x) \mid \partial(u, v) = i\},$$

$$S_i = (R_1(x) \times R_1(x)) \cap R_i,$$

$$S_i(u) = \{v \in R_1(x) \mid (u, v) \in S_i\}.$$

By definition, $|S_i(u)| = p_{1i}^1$ for every $u \in R_1(x)$. Since $p_{14}^1 = 0, S_4 = \phi$. From $p_{11}^1 = p_{12}^1 = 5$ and $p_{13}^1 = 1$, we have $|S_1(u)| = |S_2(u)| = 5$ and $|S_3(u)| = 1$ for every $u \in R_1(x)$.

LEMMA 3.1. *For* $0 \leq i \leq 3$, S_i *is the distance relation in* Δ, *i.e.*, $S_i = \{(u, v)| \partial(u, v) = i\}$.

PROOF. Take any $(u, v) \in S_2$. Suppose $\partial(u, v) \geq 3$. Since Δ is regular of valency 5 on 12 vertices, $R_1(x) = \{u\} \cup S_1(u) \cup \{v\} \cup S_1(v)$ (disjoint union). So we get

$$S_1(v) = S_3(u) \cup (S_2(u) \backslash \{v\})$$

and

$$4 = |S_2(u) \backslash \{v\}| = |S_2(u) \cap S_1(v)| \leq |R_2(u) \cap R_1(v)|.$$

This contradicts $p_{21}^2 = 3$. Hence we have $\partial(u, v) = 2$.

Next we take any $(u, u') \in S_3$. Suppose $\partial(u, u') = 2$, then there is some vertex $w \in S_1(u) \cap S_1(u')$. So $w, x \in R_1(u) \cap R_1(u')$. This contradicts $p_{11}^3 = 1$. So $S_1(u') = S_2(u)$. Hence we have $\partial(u, u') = 3$.   □

COROLLARY 3.2. *For every* $u, u' \in R_1(x)$ *with* $\partial(u, u') = 3$, *we have* $\Delta_1(u) = \Delta_2(u')$.

For any $u, v \in R_1(x)$ with $\partial(u, v) = 1$, let

$$\lambda(u, v) = |\{w \in R_1(x)| u \sim w, v \sim w\}|,$$

and for any $u, v \in R_1(x)$ with $\partial(u, v) = 2$, let

$$\mu(u, v) = |\{w \in R_1(x)| u \sim w, v \sim w\}|.$$

LEMMA 3.3. *We have the following.*
(1) $\mu(u, v) = 2$ *for every* $u, v \in R_1(x)$ *with* $\partial(u, v) = 2$.
(2) $\lambda(u, v) = 2$ *for every* $u, v \in R_1(x)$ *with* $\partial(u, v) = 1$.

PROOF. For every $v, w \in R_1(x)$ with $\partial(v, w) = 1$, there is $v' \in R_1(x)$ such that $\partial(v, v') = 3$. Then by Corollary 3.2, $\Delta_1(v) = \Delta_2(v')$. So $\lambda(v, w) + \mu(w, v') = 4$. Hence it is enough to show (1).

CLAIM. $1 \leq \mu(u, v) \leq 2$ for every $u, v \in R_1(x)$ with $\partial(u, v) = 2$.

Suppose $\mu(u, v) \geq 3$ for some $u, v \in R_1(x)$ with $\partial(u, v) = 2$. Then at least 3 vertices of $R_1(x)$ and x are in $R_1(u) \cap R_1(v)$. But this contradicts $p_{11}^2 = 3$. Therefore we get the claim.

By Corollary 3.2 and the claim, we also get $2 \leq \lambda(v, w) \leq 3$ for every $v, w \in R_1(x)$ with $\partial(v, w) = 1$. Suppose $\mu(u, v) = 1$ for some $u, v \in R_1(x)$ with $\partial(u, v) = 2$. Let $\{w\} = \Delta_1(u) \cap \Delta_1(v)$. Take $u' \in \Delta_3(u)$. Then by Corollary 3.2 and $\{w\} = \Delta_1(u) \cap \Delta_1(v)$,

$$\{v\} \cup (\Delta_1(w) \cap \Delta_1(v)) \subseteq \Delta_1(w) \cap \Delta_1(u').$$

So $3 \leq 1 + \lambda(v, w) \leq \mu(w, u')$, which is impossible. Hence we have (1).   □

LEMMA 3.4. Δ *is isomorphic to the icosahedron.*

PROOF. By Lemma 3.3, Δ is distance-regular with intersection array

$$\left\{\begin{matrix} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{matrix}\right\}.$$

This array easily determines that Δ is isomorphic to the icosahedron.   □

By Lemma 3.4, we have Γ is locally an icosahedron. The locally icosahedral graphs are completely classified in [4]. There are precisely three locally icosahedral graphs, namely the point graph of the 600-cell on 120 vertices, and quotients of this graph on 60 vertices and 40 vertices, respectively. But we will give the complete proof which shows Γ is uniquely determined in our situation, because it is elementary and self-contained.

4. Structure of Γ

We set

$$E\Delta = \{e \subseteq R_1(x) | \, e \text{ is an edge in } \Delta\},$$

$$F\Delta = \{F \subseteq R_1(x) | \, F \text{ is a triangle in } \Delta\}.$$

For $u \in R_1(x)$, let

$$\{u'\} = \Delta_3(u).$$

Similarly for $e = \{u, v\} \in E\Delta$ and $F = \{u, v, w\} \in F\Delta$, let

$$e' = \{u', v'\} \quad \text{and} \quad F' = \{u', v', w'\}.$$

Since $p^1_{31} = p^3_{11} = 1$, for every $u \in R_1(x)$ and $\alpha \in R_3(x)$, we have $|R_3(x) \cap R_1(u)| = |R_1(x) \cap R_1(\alpha)| = 1$. So let

$$\{\hat{u}\} = R_3(x) \cap R_1(u),$$

then

$$R_3(x) = \{\hat{u} \, | \, u \in R_1(x)\}.$$

So we get the following lemma.

LEMMA 4.1. *For $u \in R_1(x)$ and $\hat{v} \in R_3(x)$, $u \sim \hat{v}$ if and only if $u = v$.*

We show the graph $(R_2(x), R_1)$ is isomorphic to the dual graph Δ^* of Δ, where Δ^* is the graph whose vertex set is $F\Delta$ and two distinct vertices F and G are adjacent if and only if $|F \cap G| = 2$.

For any $u \in R_1(x)$, let

$$P(u) = R_2(x) = \cap R_1(u).$$

Then $|P(u)| = 5$ because $p^1_{21} = 5$.

LEMMA 4.2. *We have the following.*
(1) $P(u) = R_2(x) \cap R_1(\hat{u})$ *for every* $u \in R_1(x)$.
(2) $P(u)$ *is a pentagon for every* $u \in R_1(x)$.
(3) $P(u) \cap P(v)$ *is an edge if* $\{u, v\} \in E\Delta$.
(4) $|P(u) \cap P(v) \cap P(w)| = 1$ *if* $\{u, v, w\} \in F\Delta$.
(5) $P(u) \cap P(v) = \phi$ *if* $\{u, v\} \notin E\Delta$.

PROOF. (1)–(4) Take $u \in R_1(x)$, then we have

$$R_1(u) = \{x\} \cup (R_1(x) \cap R_1(u)) \cup P(u) \cup \{\hat{u}\}.$$

$(R_1(u), R_1)$ is isomorphic to the icosahedron by Lemma 3.4. By Lemma 3.1 and $|R_2(x) \cap R_1(\hat{u})| = p_{21}^3 = 5$, we get (1)–(4).

(5) Take $\{u, v\} \notin E\Delta$, then $(u, v) \in R_2$ or R_3. If $(u, v) \in R_3$, then $R_1(u) \cap R_1(v) = \{x\}$ because $p_{11}^3 = 1$. If $(u, v) \in R_2$, then by Lemma 3.3 and $p_{11}^2 = 3$, $R_1(u) \cap R_1(v) = \{x, \alpha, \beta\}$ where $\{\alpha, \beta\} = R_1(x) \cap (R_1(u) \cap R_1(v))$. Hence we get (5). $\square$

By Lemma 4.2, there is a subgraph of $(R_2(x), R_1)$ which is isomorphic to Δ^*. Since both $(R_2(x), R_1)$ and Δ^* are regular of valency 3 on 20 vertices, they are isomorphic. So there is a one-to-one correspondence between $R_2(x)$ and $F\Delta$. Hence we set

$$R_2(x) = \{\tilde{F} \mid F \in F\Delta\}.$$

COROLLARY 4.3. *For* $u \in R_1(x)$, $\tilde{F}, \tilde{G} \in R_2(x)$, *we have the following.*
(1) $\tilde{F} \sim \tilde{G}$ *if and only if* $|F \cap G| = 2$.
(2) $u \sim \tilde{F}$ *if and only if* $\hat{u} \sim \tilde{F}$ *if and only if* $u \in F$.

Next consider $(R_3(x), R_1)$.

LEMMA 4.4. *For* $\hat{u}, \hat{v} \in R_3(x), \hat{u} \sim \hat{v}$ *if and only if* $v = u'$.

PROOF. Since

$$(p_{30}^3, p_{33}^3, p_{32}^3, p_{31}^3, p_{34}^3) = (p_{10}^1, p_{11}^1, p_{12}^1, p_{13}^1, p_{14}^1) = (1, 5, 5, 1, 0),$$

we have $(R_3(x), R_3)$ is isomorphic to the icosahedron by changing the role R_1 and R_3. We note the distance-2 graph of the icosahedron is isomorphic to the icosahedron. So $(R_3(x), R_2)$ is isomorphic to the icosahedron. It is enough to show $(\hat{u}, \hat{v}) \in R_2$ if $(u, v) \in R_1$. Because the mapping $\hat{\ } : (R_1(x), R_1) \to (R_3(x), R_2)(u \mapsto \hat{u})$ becomes a graph isomorphism, the distance-3 graph of $(R_1(x), R_1)$ is $(R_1(x), R_3)$ and the distance-3 graph of $(R_3(x), R_2)$ is $(R_3(x), R_1)$.

For $u, v \in R_1(x)$ with $\partial(u, v) = 1, |P(u) \cap P(v)| = |R_2(x) \cap R_1(\hat{u}) \cap R_1(\hat{v})| = 2$ by Lemma 4.2(1)(3). So $(\hat{u}, \hat{v}) \notin R_3$ because $p_{11}^3 = 1$. Since $v, \hat{u} \in R_1(u)$ and they are at distance 2 in $(R_1(u), R_1)$, we get $(v, \hat{u}) \in R_2$ by Lemma 3.1. Since $p_{11}^2 = 3$,

$$\{u\} \cup (P(u) \cap P(v)) = R_1(v) \cap R_1(\hat{u}).$$

So $(\hat{u}, \hat{v}) \notin R_1$. Hence we get $(\hat{u}, \hat{v}) \in R_2$. $\square$

For any $\hat{u} \in R_3(x)$, let

$$Q(\hat{u}) = R_4(x) \cap R_1(\hat{u}).$$

Then $|Q(\hat{u})| = 5$ because $p_{41}^3 = 5$.

LEMMA 4.5. *We have the following.*
(1) $Q(\hat{u}) = Q(\hat{u}')$ *for* $\hat{u} \in R_3(x)$,
(2) $|Q(\hat{u}) \cap Q(\hat{v})| = 1$ *if* $\hat{u} \not\sim \hat{v}$,
(3)   *let* $\{\alpha\} = Q(\hat{u}) \cap Q(\hat{v})$ *for* $\hat{u} \not\sim \hat{v}$, *then* $R_3(x) \cap R_1(\alpha) = \{\hat{u}, \hat{u}', \hat{v}, \hat{v}'\}$.

PROOF. Take $\hat{u}, \hat{v} \in R_3(x)$ such that $\hat{u} \neq \hat{v}$. By Lemma 4.2(1),

$$R_{\leq 3}(x) \cap (R_1(\hat{u}) \cap R_1(\hat{v})) = R_2(x) \cap (R_1(\hat{u}) \cap R_1(\hat{v})) = P(u) \cap P(v).$$

By the proof of Lemma 4.4, $(u, v) \in R_1$ if and only if $(\hat{u}, \hat{v}) \in R_2$. So from Lemma 4.2(3)(5), we get

$$|R_{\leq 3}(x) \cap (R_1(\hat{u}) \cap R_1(\hat{v}))| = \begin{cases} 0 & \text{if } (\hat{u}, \hat{v}) \in R_1, \\ 2 & \text{if } (\hat{u}, \hat{v}) \in R_2, \\ 0 & \text{if } (\hat{u}, \hat{v}) \in R_3. \end{cases}$$

(1)   Since $(\hat{u}, \hat{u}') \in R_1$ and $p_{11}^1 = 5$, we get (1).
(2) $p_{11}^2 = 3$ and $p_{11}^3 = 1$ imply (2).
(3)   It is easy from (1)(2) and $p_{31}^4 = 4$. $\square$

Take any $e = \{u, v\} \in E\Delta$. As $\hat{u} \not\sim \hat{v}, e \cup e'$ determines one vertex in $R_4(x)$. Since $|E\Delta|/2 = |R_4(x)|$, we set

$$R_4(x) = \{\bar{e} \mid \bar{e} = \bar{e}' = e \cup e', e \in E\Delta\}.$$

COROLLARY 4.6. *For* $\hat{u} \in R_3(x)$, $\bar{e} \in R_4(x)$, $\hat{u} \sim \bar{e}$ *if and only if* $u \in e \cup e'$.

For any $\tilde{F} \in R_2(x)$, let

$$T(\tilde{F}) = R_4(x) \cap R_1(\tilde{F}).$$

Then $|T(\tilde{F})| = 3$ because $p_{41}^2 = 3$.

LEMMA 4.7. *We have the following.*
(1) $T(\tilde{F})$ *is a triangle for every* $\tilde{F} \in R_2(x)$.
(2) $|T(\tilde{F}) \cap T(\tilde{G})| = 1$ *if* $|F \cap G| = 2$.
(3)   *For* $\tilde{F} \sim \tilde{G}$, $T(\tilde{F}) \cap T(\tilde{G}) = \{\bar{e}\}$ *where* $e = F \cap G$.
(4)   *For* $\tilde{F} \sim \tilde{G}$, *let* $\{\bar{e}\} = T(\tilde{F}) \cap T(\tilde{G})$. *Then* $R_2(x) \cap R_1(\bar{e}) = \{\tilde{F}, \tilde{G}, \tilde{F}', \tilde{G}'\}$.

PROOF. Let $F = \{u, v, w\}$ and $\{\alpha, \beta, \gamma\} = R_4(x) \cap R_1(\tilde{F})$.
(1)(2)(3) Take $G = \{u, v, w_1\}$, $H = \{u, v_1, w\}$, $I = \{u_1, v, w\} \in F\Delta$. Then by Corollary 4.3,

$$\{u, v, w\} = R_1(x) \cap R_1(\tilde{F}),$$

$$\{\tilde{G}, \tilde{H}, \tilde{I}\} = R_2(x) \cap R_1(\tilde{F}),$$

$$\{\hat{u}, \hat{v}, \hat{w}\} = R_3(x) \cap R_1(\tilde{F}).$$

By Lemma 4.1 and Corollary 4.3, we know the subgraph on these vertices. $(R_1(\tilde{F}), R_1)$ is the icosahedron by Lemma 3.4. So we get

$$\alpha \sim \beta, \beta \sim \gamma, \gamma \sim \alpha,$$

and we may assume

$$\alpha \sim \tilde{G}, \alpha \sim \hat{u}, \alpha \sim \hat{v}.$$

So we get (1)(2). By Corollary 4.6, $\alpha = \{u, v, u', v'\}$. Hence we have (3).

(4) Since $F' \cap G' = e'$ and $\overline{e'} = \overline{e} = e \cup e'$, we get $T(\tilde{F'}) \cap T(\tilde{G'}) = \{\overline{e}\}$ by (3). So $p_{21}^4 = 4$ implies (4).   □

Let Φ be a graph whose vertex set is $R_4(x)$ and two distinct vertices $\overline{e}, \overline{f}$ are adjacent if and only if there is some $F \in F\Delta$ such that $e, f \subseteq F$ or $e, f' \subseteq F$. Then Φ is regular of valency 4 on 15 vertices.

COROLLARY 4.8. *For* $\tilde{F} \in R_2(x)$, $\overline{e}, \overline{f} \in R_4(x)$, *we have the following.*

(1) $\tilde{F} \sim \overline{e}$ *if and only if* $e \subseteq F$ *or* $e' \subseteq F$

(2) $\overline{e} \sim \overline{f}$ *if and only if there is some* $F \in F\Delta$ *such that* $e, f \subseteq F$ *or* $e, f' \subseteq F$.

PROOF. (1) It is clear from Lemma 4.7(3)(4).

(2) If there is $F \in F\Delta$ such that $e, f \subseteq F$, then $\tilde{F} \sim \overline{e}$ and $\tilde{F} \sim \overline{f}$ from (1). By Lemma 4.7(1), $\overline{e} \sim \overline{f}$. So Φ is the subgraph of $(R_4(x), R_1)$. Since $(R_4(x), R_1)$ is also regular of valency 4 on 15 vertices, we get $\Phi = (R_4(x), R_1)$. So we have (2).   □

PROOF OF THEOREM 1.1. From these Lemmas and Corollaries, (X, R_1) is expressed in terms of $\Delta = (R_1(x), R_1)$ as follows:

$R_2(x) = \{\tilde{F} \mid F \in F\Delta\}$,

$R_3(x) = \{\hat{u} \mid u \in R_1(x)\}$,

$R_4(x) = \{\overline{e} \mid \overline{e} = \overline{e'}, e \in E\Delta\}$,

$u \sim \hat{v} \Leftrightarrow u = v$,

$\tilde{F} \sim \tilde{G} \Leftrightarrow |F \cap G| = 2$,

$u \sim \tilde{F} \Leftrightarrow \hat{u} \sim \tilde{F} \Leftrightarrow u \in F$,

$\hat{u} \sim \hat{v} \Leftrightarrow v = u'$,

$\hat{u} \sim \overline{e} \Leftrightarrow u \in e \cup e'$,

$\tilde{F} \sim \overline{e} \Leftrightarrow e \subseteq F$ or $e' \subseteq F$,

$\overline{e} \sim \overline{f} \Leftrightarrow$ there is some $F \in F\Delta$ such that $e, f \subseteq F$ or $e, f' \subseteq F$.

So we know Γ is uniquely determined. Therefore we have completed the proof of Theorem 1.1.   □

5. Group association scheme of $SL(2, 5)$

In this section, we consider $\mathscr{X}(SL(2, 5))$. We order the conjugacy classes of $SL(2, 5)$ as follows:

	C_0	C_1	C_2	C_3	C_4
representative :	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$
$|C_i|$:	1	12	20	12	30

	C_5	C_6	C_7	C_8
representative :	$\begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
$|C_i|$:	12	20	12	1

Then the intersection matrices of $\mathscr{X}(SL(2,5))$ are as follows:

$$B_0 = I \text{(identity matrix)},$$

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 5 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 5 & 4 & 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 5 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 5 & 2 & 0 & 0 & 0 & 0 \\ 20 & 5 & 6 & 0 & 4 & 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 5 & 2 & 0 & 3 & 0 & 0 \\ 0 & 5 & 6 & 5 & 4 & 5 & 6 & 5 & 0 \\ 0 & 0 & 3 & 0 & 2 & 5 & 0 & 5 & 0 \\ 0 & 0 & 1 & 5 & 4 & 0 & 6 & 5 & 20 \\ 0 & 0 & 0 & 0 & 2 & 5 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 5 & 2 & 0 & 3 & 0 & 0 \\ 12 & 0 & 3 & 0 & 0 & 5 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 & 4 & 0 & 3 & 5 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 3 & 0 & 12 \\ 0 & 0 & 3 & 0 & 2 & 5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 5 & 4 & 5 & 3 & 0 & 0 \\ 0 & 5 & 6 & 5 & 4 & 5 & 6 & 5 & 0 \\ 0 & 5 & 3 & 0 & 4 & 0 & 3 & 5 & 0 \\ 30 & 10 & 6 & 10 & 4 & 10 & 6 & 10 & 30 \\ 0 & 5 & 3 & 0 & 4 & 0 & 3 & 5 & 0 \\ 0 & 5 & 6 & 5 & 4 & 5 & 6 & 5 & 0 \\ 0 & 0 & 3 & 5 & 4 & 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 & 2 & 5 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 3 & 0 & 12 \\ 0 & 5 & 3 & 0 & 4 & 0 & 3 & 5 & 0 \\ 12 & 0 & 3 & 0 & 0 & 5 & 0 & 1 & 0 \\ 0 & 5 & 0 & 5 & 2 & 0 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 & 3 & 5 & 0 \\ 0 & 0 & 1 & 5 & 4 & 0 & 6 & 5 & 20 \\ 0 & 0 & 3 & 0 & 2 & 5 & 0 & 5 & 0 \\ 0 & 5 & 6 & 5 & 4 & 5 & 6 & 5 & 0 \\ 0 & 5 & 0 & 5 & 2 & 0 & 3 & 0 & 0 \\ 20 & 5 & 6 & 0 & 4 & 5 & 1 & 0 & 0 \\ 0 & 5 & 3 & 5 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 5 & 12 \\ 0 & 0 & 0 & 0 & 2 & 5 & 3 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 5 & 4 & 5 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 5 & 2 & 0 & 0 & 0 & 0 \\ 12 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We assume $\mathscr{Y} = (X, \{R_i\}_{0 \le i \le 8})$ is an association scheme having the same intersection numbers as those of $\mathscr{X}(SL(2,5))$.

For any $x \in X$, let $\varDelta = (R_1(x), R_1)$. Since

$$(p_{10}^1, p_{11}^1, p_{12}^1, p_{13}^1, p_{14}^1, \ldots, p_{18}^1) = (1, 5, 5, 1, 0, \ldots, 0),$$

$$(p_{11}^1, p_{11}^2, p_{11}^3, p_{11}^4, \ldots, p_{11}^8) = (12, 5, 3, 1, 0, \ldots, 0) \text{ and } p_{12}^2 = 3,$$

by the same arguement in section 3 we have $\varDelta$ is isomorphic to the icosahedron. So we have (X, R_1) is the point graph of the 600-cell on 120 vertices from the classification of the locally icosahedral graphs in [4].

So we get 1st adjacency matrix A_1 is unique. By the equation

$$A_1{}^2 = 12A_0 + 5A_1 + 3A_2 + A_3,$$

we have A_2 and A_3. Similarly by two equations

$$A_2{}^2 = 20A_0 + 5A_1 + 6A_2 + 4A_4 + 5A_5 + A_6,$$

$$A_3{}^2 = 12A_0 + 3A_2 + 5A_5 + 7A_7$$

we have all the other adjacency matrices. So we have $\mathscr{Y}$ is unique. Hence we have the following theorem.

THEOREM 5.1. *Let $\mathscr{Y}$ be an association scheme having the same intersection numbers as those of the group association scheme $\mathscr{X}(SL(2,5))$. Then $\mathscr{Y}$ is isomorphic to $\mathscr{X}(SL(2,5))$.*

REMARK. We also show that (X, R_1) is unique by the similar arguement in Section 4. (X, R_1) is expressed in terms of $\varDelta = (R_1(x), R_1)$ as follows:

$R_2(x) = \{\tilde{F} \mid F \in F\varDelta\}$,
$R_3(x) = \{\hat{u} \mid u \in R_1(x)\}$,
$R_4(x) = \{\bar{e} \mid e \in E\varDelta\}$,
$R_5(x) = \{\hat{\underline{u}} \mid u \in R_1(x)\}$,
$R_6(x) = \{\underline{\tilde{F}} \mid F \in F\varDelta\}$,
$R_7(x) = \{\underline{u} \mid u \in R_1(x)\}$,
$R_8(x) = \{\underline{x}\}$,

$u \sim \hat{v} \Leftrightarrow \underline{u} \sim \hat{v} \Leftrightarrow \hat{u} \sim \hat{\underline{v}} \Leftrightarrow u = v,$

$u \sim \tilde{F} \Leftrightarrow \hat{u} \sim \tilde{F} \Leftrightarrow \underline{u} \sim \underline{\tilde{F}} \Leftrightarrow \hat{\underline{u}} \sim \underline{\tilde{F}} \Leftrightarrow u \in F,$

$\tilde{F} \sim \tilde{G} \Leftrightarrow \underline{\tilde{F}} \sim \underline{\tilde{G}} \Leftrightarrow |F \cap G| = 2,$

$\tilde{F} \sim \bar{e} \Leftrightarrow \underline{\tilde{F}} \sim \bar{e} \Leftrightarrow e \subseteq F,$

$\hat{u} \sim \bar{e} \Leftrightarrow \hat{\underline{u}} \sim \bar{e} \Leftrightarrow u \in e,$

$\bar{e} \sim \bar{f} \Leftrightarrow$ there is some $F \in F\Delta$ such that $e, f \subseteq F,$

$\underline{u} \sim \underline{v} \Leftrightarrow u \sim v,$

$\underline{u} \sim \underline{x}.$

6. Group association scheme of S_4

We order the conjugacy classes of S_4 as follows:

	C_0	C_1	C_2	C_3	C_4
representative :	( )	(12)(34)	(123)	(1234)	(12)
$|C_i|$:	1	3	8	6	6

Then the intersection matrices of $\mathscr{X}(S_4)$ are as follows:

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 8 & 8 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 \\ 6 & 2 & 3 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 4 & 3 & 0 & 0 \\ 6 & 2 & 3 & 0 & 0 \end{pmatrix}.$$

We assume $\mathscr{Y} = (X, \{R_i\}_{0 \le i \le 4})$ is an association scheme having the same intersection matrices as $\mathscr{X}(S_4)$. Let $R = \sum_{i=0}^{4} iA_i$ be the relation matrix of $\mathscr{Y}$.

REMARK. If we order the conjugacy classes of S_4 as follows:

	C_0	C_1	C_2	C_3	C_4
representative :	( )	(12)(34)	(123)	(12)	(1234)
$|C_i|$:	1	3	8	6	6

then we have the same intersection matrices. So if we have two sets of adjacency matrices $\{A_0, \ldots, A_4\}$ and $\{A'_0, \ldots, A'_4\}$ such that $A_3 = A'_4$ and $A_4 = A'_3$, then the corresponding association schemes are isomorphic.

By the intersection matrices, $\mathscr{Y}$ is imprimitive and $\bigcup_{i=0}^{2} R_i$ is an equivalence relation.

For each equivalence class $X', \mathcal{Y}' = (X', \{R_i'\}_{i=0,1,2})$ has the intersection matrices

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 8 & 8 & 4 \end{pmatrix}.$$

These matrices determine an association scheme uniquely. So we have $R = \begin{pmatrix} D & E \\ {}^tE & D \end{pmatrix}$, where

$$D = \begin{pmatrix}
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0
\end{pmatrix}.$$

Let $X_1, X_2, \ldots, X_6$ be partition of X such that $X_1 \cup X_2 \cup X_3$ and $X_4 \cup X_5 \cup X_6$ are equivalence classes of $\bigcup_{i=0}^{2} R_i$ and $(x,y) \in R_1$ if and only if $x, y \in X_i$ for some i with $1 \le i \le 6$.

To find A_3, let F be the submatrix of A_3, whose rows and columns are indexed by $X_1 \cup X_2 \cup X_3$ and $X_4 \cup X_5 \cup X_6$, respectively. We decompose F into 9 submatrices $F_{[i,j]}$ whose rows and columns are indexed by X_i and X_j, respectively.

We set F_x is a xth row of F for $x \in X_1 \cup X_2 \cup X_3$ and F^x is a xth column of F for $x \in X_4 \cup X_5 \cup X_6$. Then

$$\langle F_x, F_y \rangle = \begin{cases} 2 & \text{if } x \ne y \text{ and } x, y \in X_i \ (1 \le i \le 3), \\ 3 & \text{if } x \in X_i, y \in X_j, i \ne j \ (1 \le i, j \le 3), \end{cases}$$

and

$$\langle F^x, F^y \rangle = \begin{cases} 2 & \text{if } x \ne y \text{ and } x, y \in X_i \ (4 \le i \le 6), \\ 3 & \text{if } x \in X_i, y \in X_j, i \ne j \ (4 \le i, j \le 6), \end{cases}$$

where $\langle \, , \, \rangle$ is a usual inner product. We know for $1 \le i \le 3$ and $4 \le j \le 6$, there are some permutation matrices P and Q such that

$$PF_{[i,j]}Q = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

By these properties, we determine F.

Assume we have $F = F_1$ or F_2. The corresponding association schemes are isomorphic if and only if there are some permutation matrices P and Q such that $PF_1Q = F_2$ or $J - F_2$, where J is the all one matrix.

Therefore, up to isomorphism, there are exactly three possibilities for F. So we have E is one of the following:

$$
E_1 = \begin{pmatrix}
3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 \\
3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 \\
4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 \\
4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 \\
3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 \\
3 & 4 & 4 & 3 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 \\
4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 \\
4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 & 3 & 4 \\
3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 \\
3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 \\
4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 & 4 & 3
\end{pmatrix},
$$

$$
E_2 = \begin{pmatrix}
3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 \\
3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 4 & 3 & 3 \\
4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 3 \\
4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 \\
3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 4 & 3 & 4 \\
3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 4 & 3 & 4 & 3 \\
4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 4 & 3 \\
4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 3 & 4 & 3 & 4 \\
3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 & 4 & 3 & 3 & 4 \\
3 & 4 & 4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 \\
4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 \\
4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 & 3 & 4
\end{pmatrix},
$$

$$
E_3 = \begin{pmatrix}
3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 \\
3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 4 & 3 & 3 \\
4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 3 \\
4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 \\
3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 4 & 3 & 4 \\
3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 \\
4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 3 & 4 & 3 & 4 \\
3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 \\
3 & 4 & 4 & 3 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 \\
4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 & 3 & 4 \\
4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3
\end{pmatrix}.
$$

THEOREM 6.1.   *There are exactly three non-isomorphic association schemes having the same intersection numbers as those of $\mathfrak{X}(S_4)$.*

REMARKS.   (1) If $E = E_1$, then $\mathcal{Y} \simeq \mathfrak{X}(S_4)$.

(2) The number of $F_{[i,j]}$ such that $PF_{[i,j]}Q = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ for some permu-

tation matrices P, Q is $\begin{cases} 0 & \text{if} & E = E_1, \\ 6 & \text{if} & E = E_2, \\ 4 & \text{if} & E = E_3. \end{cases}$

(3) By using computer, we have the order of $Aut(\mathcal{Y})$, the automorphism group of $\mathcal{Y}$.

$$|Aut(\mathcal{Y})| = \begin{cases} 1152 & \text{if} & E = E_1, \\ 24 & \text{if} & E = E_2, \\ 64 & \text{if} & E = E_3. \end{cases}$$

References

[1]   E. Bannai,   "Algebraic combinatorics: recent topics on association schemes", (in Japanese),   Sugaku (Mathematics, a publication of Math. Soc. of Japan), 45 (1993), 55–75; English version, Sugaku Expositions (American Math. Soc.), 7 (1994), 181–207.

[2]   E. Bannai and T. Ito,   Algebraic Combinatrics I,   Benjamin-Cummings, California, 1984.

[3]   A. E. Brouwer, A. M. Cohen and A. Neumaier,   Distance-Regular Graphs,   Springer, Berline-Heidelberg, 1989.

[4]   A. Blokhuis, A. E. Brouwer, D. Buset and A. M. Cohen,   "The locally icosahedral graphs", pp. 19–22 in: Finite geometries, Proc. Winnipeg 1984,   Lecture Notes in Pure and Applied Math. 103 (C. A. Baker & L. M. Batten, eds.), Marcel Dekker, New York, 1985.

[5]   M. Rassy and P. H. Zieschang,   "Basic structure theory of association schemes", preprint, 1994.

Masato TOMIYAMA

Division of Mathematical Sciences
Osaka Kyoiku University
Kashiwara, Osaka 582,
Japan