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Asymptotic expansions and Stokes multipliers of the confluent hypergeometric function Φ_2 II. Behaviour near (∞,∞) in $P^1(C)\times P^1(C)$

By Shun SHIMOMURA

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1. Introduction.

The confluent hypergeometric function

$$\Phi_2(\beta,\beta',\gamma,x,y)=\sum_{m,n\geqq 0}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}(1)_m(1)_n}x^m y^n \tag{1.1}$$

with $(\beta)_m=\Gamma(\beta+m)/\Gamma(\beta)$ satisfies a system of partial differential equations

$$\begin{aligned} &xz_{xx}+yz_{xy}+(\gamma-x)z_x-\beta z=0,\\ &yz_{yy}+xz_{xy}+(\gamma-y)z_y-\beta' z=0 \end{aligned} \tag{1.2}$$

(see [**4**, §5.9]) for $(x,y)\in P^1(C)\times P^1(C)$. Observing that $(x-y)z_{xy}-\beta' z_x+\beta z_y=0$, we can verify that (z,xz_x,yz_y) satisfies a Pfaffian system which possesses the singular loci $x=0$, $y=0$, $x=y$ of regular type and $x=\infty$, $y=\infty$ of irregular type, and that the solutions of (1.2) constitute a three-dimensional vector space over C. In the previous paper [**6**], we defined linearly independent solutions z_+, z_0, z_- admitting integral representations. Modifying the paths of integration, we obtained monodromy matrices with respect to them. The main theorems in [**6**] give the asymptotic properties of them near the singular loci $x=\infty$ (y is bounded) and $y=\infty$ (x is bounded), that is to say, asymptotic expansions in powers of $1/x$ and $1/y$, respectively, and Stokes multipliers. By a connection formula, the asymptotic behaviour of $\Phi_2(\beta,\beta',\gamma,x,y)$ itself is also clarified near these singular loci.

The present paper gives asymptotic expansions and Stokes multipliers of linearly independent solutions as (x,y) tends to (∞,∞). Consequently, we know the asymptotic behaviour of the general solutions in the whole tubular neighbourhood around the singular loci of irregular type. As in [**6**], an integral of the form

$$\int_C t^{\beta+\beta'-\gamma}(t-x)^{-\beta}(t-y)^{-\beta'}e^t\,dt \tag{1.3}$$

(by Erdélyi [**2**], [**3**]) satisfying (1.2) plays an important role. (For the integral see also [**1**], [**5**].) The difficulty of our problem is caused by the fact that the three singular loci $x=\infty, y=\infty, x=y$ meet at one point $(x,y)=(\infty,\infty)$. In Section 2, we define four solutions of (1.2) expressible in the form (1.3) and five domains of which the union covers the full neighbourhood of $(x, y) = (\infty, \infty)$. In each domain, we examine the asymptotic behaviour of a suitably chosen triplet of linearly independent solutions. In Sections 3 and 5, we state main theorems which give asymptotic expansions and Stokes multipliers, respectively. They are proved in Sections 4 and 6. In the final section, we explain the asymptotic behaviour of Φ_2.

Throughout this paper, we assume that none of the complex numbers β, β', $\gamma - \beta - \beta'$, $\beta - \gamma, \beta' - \gamma$ and $\beta + \beta'$ is an integer. For a complex number λ, we use the notation

$$e^{(\lambda)} = \exp(2\pi i \lambda). \tag{1.4}$$

2. Preliminaries.

2.1. Integrals.

Let $\mathscr{R}$ be the universal covering space of the domain $\{(x, y) \in \boldsymbol{C}^2 \mid |x| > M, |y| > M, y - x \neq 0\}$, where M is a sufficiently large positive constant. Consider the domain

$$\Delta = \{(x, y) \in \mathscr{R} | 0 < \arg x < \pi < \arg y < 2\pi,\ \pi < \arg(y - x) < 2\pi\}, \tag{2.1}$$

which is simply connected. For $(x, y) \in \Delta$, we put

$$z_{+}(\beta, \beta', \gamma, x, y) = (1 - e^{(\beta)})^{-1} \int_{C(x)} f(x, y, t) dt, \tag{2.2}$$

$$z_{0}(\beta, \beta', \gamma, x, y) = (1 - e^{(\gamma-\beta-\beta')})^{-1} \int_{C(0)} f(x, y, t) dt, \tag{2.3}$$

$$z_{-}(\beta, \beta', \gamma, x, y) = (1 - e^{(\beta')})^{-1} \int_{C(y)} f(x, y, t) dt, \tag{2.4}$$

$$z_{*}(\beta, \beta', \gamma, x, y) = (1 - e^{(\gamma-\beta-\beta')})^{-1} \int_{C_*(0)} f(x, y, t) dt \tag{2.5}$$

with

$$f(x, y, t) = t^{\beta+\beta'-\gamma} (t - x)^{-\beta} (t - y)^{-\beta'} e^{t}. \tag{2.6}$$

Here the paths of integration and the branch of each integrand are taken in such a way that they have the following properties:

(i) $C(0)$, $C(x)$, $C(y)$ and $C_*(0)$ are loops which start from $t = -\infty$, encircle the points 0, x, y and 0, respectively, in the positive sense, and end at $t = -\infty$.

(ii) These paths are located as described in Fig.1.

(iii) The branch of $f(x, y, t)$ in each integral is taken such that

$$\arg t = \pi, \quad \arg(t - x) = \pi, \quad \arg(t - y) = \pi \tag{2.7}$$

at the endpoint $t = -\infty$ of the corresponding path of integration.

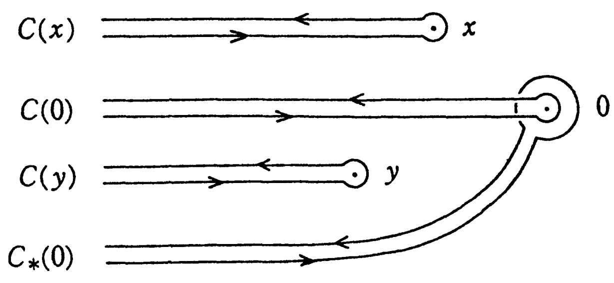

Fig. 1.

By modifying the paths of integration, we obtain the analytic continuations of these integrals to the whole domain $\mathscr{R}$, which are also denoted by the same notation. Furthermore they are often expressed by appropriate abbreviations. For example the function $z_+(\beta,\beta',\gamma,x,y)$ is written as z_+ or $z_+(x,y)$, when it is not necessary to indicate the variables or the parameters. As was shown in [6], the integrals z_+, z_0, z_- are linearly independent solutions of (1.2). Note that $(1-e^{(\gamma-\beta-\beta')})z_* = I_1 + I_2 + I_3$. Here I_1, I_2, I_3 are integrals along the contours $C(y), C(0), -C(y)$, respectively, of which integrands are determined in such a way that, at the endpoints of the paths of integration, $(\arg t$, $\arg(t-x)$, $\arg(t-y)) = (-\pi,\pi,3\pi)$, $(\pi,\pi,3\pi)$, (π,π,π), respectively (see Fig.2). Since $I_1 = e^{(\gamma-\beta-\beta')}(e^{(-\beta')}-1)z_-$, $I_2 = e^{(-\beta')}(1-e^{(\gamma-\beta-\beta')})z_0$, $I_3 = (1-e^{(-\beta')})z_-$, the new integral z_* is written in the form

$$z_* = e^{(-\beta')}z_0 + (1-e^{(-\beta')})z_-; \tag{2.8}$$

hence z_+, z_*, z_- are also linearly independent solutions.

Note that integral representation (2.3), with unmodified path $C(0)$, gives the analytic continuation of z_0 to the domain

$$-\pi/2 < \arg x < 0, \quad \pi < \arg(y-x) < \arg y < 3\pi/2. \tag{2.9}$$

Assume that $(x,y) = (e^{-\pi i}x', y'-x')$ satisfies (2.9) and that $\arg((y'-x') - e^{-\pi i}x') = \arg y'$. Then we have $\pi/2 < \arg x' < \pi < \arg y' < \arg(y'-x') < 3\pi/2$. In (2.3), we replace $(\beta,\beta',\gamma,x,y)$ by $(\gamma-\beta-\beta'$, β', γ, $e^{-\pi i}x'$, $y'-x')$, and put $t = v - x'$. Then the resulting expression is $e^{-x'}z_+(\beta,\beta',\gamma,x',y')$. Applying the same replacement to (2.4) in

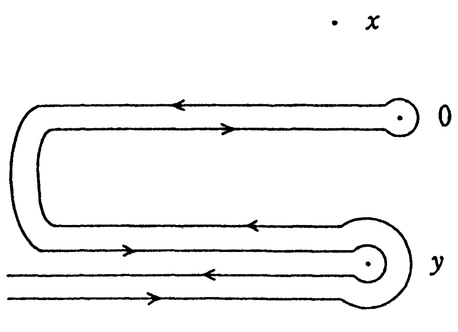

Fig. 2.

domain (2.9), we obtain the integral $e^{-x'}z_-(\beta,\beta',\gamma,x',y')$. Using the uniqueness of analytic continuation, we have the following proposition.

Proposition 2.1. *We have*

$$z_+(\beta,\beta',\gamma,x,y) = e^x z_0(\gamma-\beta-\beta',\beta',\gamma,e^{-\pi i}x,y-x), \tag{2.10}$$

$$z_-(\beta,\beta',\gamma,x,y) = e^x z_-(\gamma-\beta-\beta',\beta',\gamma,e^{-\pi i}x,y-x) \tag{2.11}$$

on $\mathscr{R}$, *where* $\arg((y-x)-e^{-\pi i}x)$ *is to be* $\arg y$.

Starting from integral representations (2.2) and (2.4) in the domain $\pi < \arg x < 3\pi/2 < \arg y < 2\pi < \arg(y-x) < 5\pi/2$, we arrive at the following proposition:

Proposition 2.2. *We have*

$$z_-(\beta,\beta',\gamma,x,y) = e^x z_+(\beta',\gamma-\beta-\beta',\gamma,y-x,e^{\pi i}x), \tag{2.12}$$

$$z_*(\beta,\beta',\gamma,x,y) = e^x z_-(\beta',\gamma-\beta-\beta',\gamma,y-x,e^{\pi i}x) \tag{2.13}$$

on $\mathscr{R}$, *where* $\arg(e^{\pi i}x-(y-x))$ *is to be* $\pi+\arg y$.

Consider the analytic continuations of (2.4) and (2.5) to the domain $0 < \arg x < \pi/2 < \arg y < \arg(y-x) < \pi$, and those of (2.3) and (2.4) to the domain $\pi/2 < \arg x < \pi < \arg(y-x) < 2\pi < \arg y < 5\pi/2$. Then, putting $t = v - y$, in these integrals, we have the following results:

Proposition 2.3. *We have*

$$z_+(\beta,\beta',\gamma,x,y) = e^y z_-(\gamma-\beta-\beta',\beta,\gamma,e^{-\pi i}y,e^{-\pi i}(y-x)), \tag{2.14}$$

$$z_-(\beta,\beta',\gamma,x,y) = e^y z_*(\gamma-\beta-\beta',\beta,\gamma,e^{-\pi i}y,e^{-\pi i}(y-x)) \tag{2.15}$$

on $\mathscr{R}$, *where* $\arg(e^{-\pi i}(y-x)-e^{-\pi i}y)$ *is to be* $\arg x$.

Proposition 2.4. *We have*

$$z_-(\beta,\beta',\gamma,x,y) = e^y z_0(\beta,\gamma-\beta-\beta',\gamma,e^{-\pi i}(y-x),e^{\pi i}y), \tag{2.16}$$

$$z_*(\beta,\beta',\gamma,x,y) = e^y z_-(\beta,\gamma-\beta-\beta',\gamma,e^{-\pi i}(y-x),e^{\pi i}y) \tag{2.17}$$

on $\mathscr{R}$, *where* $\arg(e^{\pi i}y-e^{-\pi i}(y-x))$ *is to be* $\pi+\arg x$.

2.2. Domains.

Let r, r', R and ε be arbitrary positive constants satisfying $1 < r < r' \leqq 5/4$, $R > 1$, $0 < \varepsilon \leqq 1/24$. Consider five domains defined by

$$D_1(r) = \{(x,y) \in \mathscr{R} \mid |x| > r|y|\},$$

$$D_2(r) = \{(x,y) \in \mathscr{R} \mid |y| > r|x|\},$$

$$D_0(R) = \{(x,y) \in \mathscr{R} \mid 0 < |y-x| < R\},$$

$$D_0'(R) = \{(x,y) \in \mathscr{R} \mid R/2 < |y-x| < |x|/3\},$$

$$D_*(r',\varepsilon) = \{(x,y) \in \mathscr{R} \mid 1/r' < |y/x| < r', |y-x| > (1/3-\varepsilon)|x|\}.$$

The union of them covers $\mathscr{R}$ completely. By E we denote an arbitrary one among them. Then $E \cap \Delta$ is connected and not empty. Let $c(E)$ denote the connected component of E including $E \cap \Delta$. In the subsequent sections, we are concerned with the asymptotic behaviour of the solutions z_+, z_0, z_-, z_* in $c(E)$. In the other connected components of E, we can also derive the asymptotic representations from those in $c(E)$, using the monodromy matrices M_0, M_1 and M_2 (cf. Section 6 and [**6**, Proposition 2.1]).

3. Asymptotic expansions.

Before the statement of our results, we give some definitions of asymptotic expansions in two variables. Let $K(U, V)$ be a formal power series defined by $\sum a_{mn} U^m V^n$ $(0 \leqq m < +\infty, 0 \leqq n < +\infty)$. Denote by $K_N(U, V)$ $(N \in \boldsymbol{N})$ the partial sum of $K(U, V)$ over (m, n) such that $m + n \leqq N$. We write $\eta = y/x$. Let Y be a variable y or $y - x$. Let $f(x, y), u(\eta)$ and $v(\eta)$ be functions holomorphic in a sector of the form

$$S_0 = \{(x, y) \in E \,|\, |\arg x - \theta_1| < \Theta_1, |\arg Y - \theta_2| < \Theta_2\}$$

$(\theta_j \in \boldsymbol{R}, \Theta_j > 0$ $(j = 1, 2))$, where E denotes one of the domains defined in the preceding section except $D_0(R)$. If, for every positive integer N, there exists a positive constant M_N such that

$$|f(x, y) - K_N(u(\eta)/x, v(\eta)/Y)| \leqq M_N(|x|^{-N-1} + |Y|^{-N-1})$$

in S_0, then we say that $f(x, y)$ *admits the asymptotic expansion* $K(u(\eta)/x, v(\eta)/Y)$ *as* (x, Y) *tends to* (∞, ∞) *through the sector* S_0. In E, at least either η or η^{-1} is bounded. Suppose that, for every $(x, y) \in E$, the variable η belongs to a bounded set E' $(\subset \boldsymbol{C})$. Let $t(\eta)$ and $w(\eta)$ be functions holomorphic in E'. If, for every positive integer N, there exists a positive constant M_N such that

$$|f(x, y) - K_N(t(\eta)/x, w(\eta)/x)| < M_N|x|^{-N-1}$$

in a sector of the form $\{(x, y) \in E \,|\, |\arg x - \theta_1'| < \Theta_1', \eta \in E'\}$ $(\theta_1' \in \boldsymbol{R}, \Theta_1' > 0)$, then we say that $f(x, y)$ *admits the asymptotic expansion* $K(t(\eta)/x, w(\eta)/x)$ *uniformly for* $\eta \in E'$ *as* x *tends to* ∞ *through the sector* $|\arg x - \theta_1'| < \Theta_1'$. For each fixed $\eta \in E'$, we can analogously define an asymptotic expansion in powers of $(t(\eta)/x, w(\eta)/x)$ as x tends to ∞ through the sector $|\arg x - \theta_1'| < \Theta_1'$. In what follows, to indicate the asymptotic relations defined above, we use the notation

$$f(x, y) \sim K(u(\eta)/x, v(\eta)/Y) \quad ((x, Y) \longrightarrow (\infty, \infty) \text{ through } S_0),$$

$$f(x, y) \sim K(t(\eta)/x, w(\eta)/x) \quad (x \longrightarrow \infty \text{ through the sector } |\arg x - \theta_1'| < \Theta_1' \text{ uniformly for } \eta \in E'),$$

respectively. If the quotient $f(x, y)/g(x, y)$, where $g(x, y)$ is a given function, admits an asymptotic expansion $K(u(\eta)/x, v(\eta)/Y)$ (or $K(t(\eta)/x, w(\eta)/x)$), then we write in the form $f(x, y) \sim g(x, y)K(u(\eta)/x, v(\eta)/Y)$ (or $f(x, y) \sim g(x, y)K(t(\eta)/x, w(\eta)/x)$).

Let $H(\beta,\beta',\gamma;t,u)$ be a formal power series defined by

$$H(\beta,\beta',\gamma;t,u)=\sum_{\substack{m\geqq 0\\ n\geqq 0}}\frac{(\beta)_m(\beta')_n(\beta+\beta'-\gamma+1)_{m+n}}{(1)_m(1)_n}t^m u^n.$$

Recall the power series expressions (cf. [**6**, §3])

$$T(\beta,\beta',\gamma,s;u)=\sum_{n\geqq 0}\frac{(\beta)_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta-\gamma+2)_n}$$
$$\times s^n{}_1F_1(\beta+\beta'-\gamma+n+1,\beta-\gamma+n+2,s)u^n,$$

$$U(\beta,\beta',\gamma,s;u)=\sum_{n\geqq 0}\frac{(\beta)_n(\beta-\gamma)_{n+1}}{(1)_n}\,{}_1F_1(\beta',\gamma-\beta-n,s)u^n,$$

and

$$V(\beta,\beta',\gamma,s;u)=\sum_{n\geqq 0}(1-\beta)_n P_n(\beta,\beta',\gamma,s)u^n$$

with

$$P_n(\beta,\beta',\gamma,s)=\sum_{m=0}^{n}\frac{(\beta')_m(\gamma-\beta-\beta')_{n-m}}{(\beta-n)_m(1)_{n-m}}L_m^{(\beta-n-1)}(s).$$

Here ${}_1F_1(a,c,s)$ is the confluent hypergeometric function and $L_m^{(a)}(s)$ is the Laguerre polynomial

$$L_m^{(a)}(s)=\sum_{j=0}^{m}\binom{m+a}{m-j}\frac{(-s)^j}{(1)_j}.$$

In what follows, we write $\eta=y/x,\xi=\eta^{-1}$; δ denotes an arbitrary small positive constant and δ_r an arbitrary constant satisfying $\sin^{-1}(1/r)<\delta_r<\pi/2$.

3.1. Asymptotic expansions in $c(D_1(r))$.
Note that $|\eta|<1/r$ in $c(D_1(r))$.

THEOREM 3.1. *In the domain $c(D_1(r))$ we have the following asymptotic expansions*:

(i) $$z_+\sim -e^{-\beta\pi i}\Gamma(1-\beta)x^{\beta-\gamma}(1-\eta)^{-\beta'}e^x H(\gamma-\beta-\beta',\beta',\gamma;1/x,(1-\eta)^{-1}/x) \tag{3.1}$$

uniformly for $|\eta|<1/r$ as x tends to ∞ through the sector $|\arg x-\pi|<3\pi/2-\delta_r$;

(ii) $$z_0\sim -e^{(2\beta'-\gamma)\pi i}\Gamma(\beta+\beta'-\gamma+1)x^{-\beta}y^{-\beta'}H(\beta,\beta',\gamma;-1/x,-1/y) \tag{3.2}$$

as (x,y) tends to (∞,∞) through the sector $|\arg x|<3\pi/2-\delta_r, |\arg y-2\pi|<3\pi/2-\delta$;

(iii) $$z_-\sim -e^{-(\beta+\beta')\pi i}\Gamma(1-\beta')x^{-\beta}y^{\beta+\beta'-\gamma}(1-\eta)^{-\beta}e^y$$
$$\times H(\gamma-\beta-\beta',\beta,\gamma;1/y,-(1-\eta)^{-1}/x) \tag{3.3}$$

as (x,y) tends to (∞,∞) through the sector $|\arg x|<3\pi/2-\delta_r, |\arg y-\pi|<3\pi/2-\delta$.

3.2. Asymptotic expansions in $c(D_2(r))$.

Note that $|\xi| < 1/r$ in $c(D_2(r))$.

THEOREM 3.2. *In the domain $c(D_2(r))$ we have the following asymptotic expansions:*

(i)
$$z_+ \sim -e^{(\beta'-\beta)\pi i}\Gamma(1-\beta)x^{\beta+\beta'-\gamma}y^{-\beta'}(1-\xi)^{-\beta'}e^x \times H(\gamma-\beta-\beta',\beta',\gamma;1/x,-(1-\xi)^{-1}/y) \tag{3.4}$$

as (x,y) tends to (∞,∞) through the sector $|\arg x-\pi| < 3\pi/2-\delta, |\arg y-2\pi| < 3\pi/2-\delta_r$;

(ii)
$$z_0 \sim -e^{(2\beta'-\gamma)\pi i}\Gamma(\beta+\beta'-\gamma+1)x^{-\beta}y^{-\beta'}H(\beta,\beta',\gamma;-1/x,-1/y) \tag{3.5}$$

as (x,y) tends to (∞,∞) through the sector $|\arg x| < 3\pi/2-\delta, |\arg y-2\pi| < 3\pi/2-\delta_r$;

(iii)
$$z_- \sim -e^{-\beta'\pi i}\Gamma(1-\beta')y^{\beta'-\gamma}(1-\xi)^{-\beta}e^y \times H(\gamma-\beta-\beta',\beta,\gamma;1/y,(1-\xi)^{-1}/y) \tag{3.6}$$

uniformly for $|\xi| < 1/r$ as y tends to ∞ through the sector $|\arg y-\pi| < 3\pi/2-\delta_r$.

3.3. Asymptotic expansions in $c(D_0(R))$.

It is easy to see that, $|\arg y-\arg x| < \pi/2$ for $(x,y)\in c(D_0(R))\cup c(D_0'(R))$. In $c(D_0(R))$, when $x\to\infty$ through each sector, y also tends to ∞ and satisfies $\arg y-\arg x\to 0$.

THEOREM 3.3. *In the domain $c(D_0(R))$ we have a convergent series expansion*

$$z_+-z_- = e^{\beta'\pi i}\frac{\Gamma(1-\beta)\Gamma(1-\beta')}{\Gamma(2-\beta-\beta')}x^{\beta+\beta'-\gamma}(y-x)^{1-\beta-\beta'}e^x \times T(\gamma-\beta-\beta',\beta',\gamma,y-x;-1/x) \tag{3.7}$$

and the following asymptotic expansions:

(i)
$$(1-e^{(\beta)})z_+ + e^{(\beta)}(1-e^{(\beta')})z_- \sim \frac{2\pi i}{\Gamma(\beta+\beta')}x^{\beta+\beta'-\gamma}e^xU(\gamma-\beta-\beta',\beta',\gamma,y-x;1/x) \tag{3.8}$$

uniformly for $|y-x| < R$ as x tends to ∞ through the sector $|\arg x-\pi| < 3\pi/2-\delta$;

(ii)
$$z_* \sim -e^{-\gamma\pi i}\Gamma(\beta+\beta'-\gamma+1)x^{-\beta-\beta'}V(\gamma-\beta-\beta',\beta',\gamma,y-x;-1/x) \tag{3.9}$$

uniformly for $|y-x| < R$ as x tends to ∞ through the sector $|\arg x| < 3\pi/2-\delta$.

3.4. Asymptotic expansions in $c(D_0'(R))$.

Note that $|\eta-1| < 1/3$ in $c(D_0'(R))$.

THEOREM 3.4. *In the domain $c(D_0'(R))$ we have the following asymptotic expansions*:

(i)
$$z_+ \sim -e^{(\beta'-\beta)\pi i}\Gamma(1-\beta)x^{\beta+\beta'-\gamma}(y-x)^{-\beta'}e^x \times H(\gamma-\beta-\beta',\beta',\gamma;1/x,-1/(y-x)) \tag{3.10}$$

as $(x,y-x)$ tends to (∞,∞) through the sector $|\arg x-\pi|<4\pi/3$, $|\arg(y-x)-2\pi|<3\pi/2-\delta$;

(ii)
$$z_* \sim -e^{-\gamma\pi i}\Gamma(\beta+\beta'-\gamma+1)x^{-\beta-\beta'}\eta^{-\beta'}H(\beta,\beta',\gamma;-1/x,-\eta^{-1}/x) \tag{3.11}$$

uniformly for $|\eta-1|<1/3$ as x tends to ∞ through the sector $|\arg x|<4\pi/3$;

(iii)
$$z_- \sim -e^{-\beta'\pi i}\Gamma(1-\beta')(x\eta)^{\beta+\beta'-\gamma}(y-x)^{-\beta}e^{\eta x} \times H(\gamma-\beta-\beta',\beta,\gamma;\eta^{-1}/x,1/(y-x)) \tag{3.12}$$

as $(x,y-x)$ tends to (∞,∞) through the sector $|\arg x-\pi|<4\pi/3$, $|\arg(y-x)-\pi|<3\pi/2-\delta$.

3.5. Asymptotic expansions in $c(D_*(r',\varepsilon))$.

Note that $D_*(r',\varepsilon)$ is expressible in the form

$$D_*(r',\varepsilon)=\{(x,x\eta)\in\mathscr{R}|1/r'<|\eta|<r',|1-\eta|>1/3-\varepsilon\}.$$

Since $0<\arg y-\arg x<2\pi$ for $(x,y)\in c(D_*(r',\varepsilon))$, there exists a constant θ_0 $(\delta<\theta_0<\pi/6)$ such that $|\arg\eta-\pi|=|\arg y-\arg x-\pi|<\pi-\theta_0$ in this domain. In fact, if θ_0 satisfies $|r'e^{i\theta_0}-1|<1/3-\varepsilon$ and $|(1/r')e^{i\theta_0}-1|<1/3-\varepsilon$, then this inequality is valid. For instance a numerical computation shows that we can take $\theta_0=\pi/16$, if $r'=5/4$, $\varepsilon\leqq 1/1600$, and $\theta_0=\pi/12$, if $r'=9/8$, $\varepsilon\leqq 1/35$. Consider the subdomains

$$c_+(r',\varepsilon)=\{(x,x\eta)\in c(D_*(r',\varepsilon))|\theta_0<\arg\eta<\pi-\theta_0\},$$
$$c_0(r',\varepsilon)=\{(x,x\eta)\in c(D_*(r',\varepsilon))|5\pi/6<\arg\eta<7\pi/6\},$$
$$c_-(r',\varepsilon)=\{(x,x\eta)\in c(D_*(r',\varepsilon))|\pi+\theta_0<\arg\eta<2\pi-\theta_0\},$$

which satisfy

$$c(D_*(r',\varepsilon))=c_+(r',\varepsilon)\cup c_0(r',\varepsilon)\cup c_-(r',\varepsilon).$$

We can regard z_+,z_0,z_-,z_* as functions of (x,η).

THEOREM 3.5. *Put $K_+=\{s\in\boldsymbol{C}|1/r'<|s|<r',\ |1-s|>1/3-\varepsilon,\ \theta_0<\arg s<\pi-\theta_0\}$ and $\theta=\arg\eta$. Then, in the domain $c_+(r',\varepsilon)$, we have the following asymptotic expansions*:

(i) *for each $\eta\in K_+$,*

$$z_+ \sim -e^{-\beta\pi i}\Gamma(1-\beta)(1-\eta)^{-\beta'}x^{\beta-\gamma}e^xH(\gamma-\beta-\beta',\beta',\gamma;1/x,(1-\eta)^{-1}/x) \tag{3.13}$$

as x tends to ∞ through the sector $\pi/2-\theta<\arg x<5\pi/2-\delta$;

(ii) *for each* $\eta \in K_+$,

$$z_* \sim -e^{-\gamma\pi i}\Gamma(\beta+\beta'-\gamma+1)\eta^{-\beta'}x^{-\beta-\beta'}H(\beta,\beta',\gamma;-1/x,-\eta^{-1}/x) \tag{3.14}$$

as x *tends to* ∞ *through the sector* $-3\pi/2+\delta < \arg x < 3\pi/2-\theta-\delta$;

(iii) *for each* $\eta \in K_+$,

$$\begin{aligned} z_- \sim & -e^{-(\beta+\beta')\pi i}\Gamma(1-\beta')\eta^{\beta+\beta'-\gamma}(1-\eta)^{-\beta}x^{\beta'-\gamma}e^{\eta x} \\ & \times H(\gamma-\beta-\beta',\beta,\gamma;\eta^{-1}/x,-(1-\eta)^{-1}/x) \end{aligned} \tag{3.15}$$

as x *tends to* ∞ *through the sector* $-\pi/2-\theta+\delta < \arg x < 3\pi/2$.

The asymptotic representations of z_+, of z_* and of z_- given above are uniformly valid for $\eta \in K_+$ as x tends to ∞ through the sectors $\pi/2-\theta_0 < \arg x < 5\pi/2-\delta$, $-3\pi/2+\delta < \arg x < \pi/2+\theta_0-\delta$, and $-\pi/2-\theta_0+\delta < \arg x < 3\pi/2$, respectively.

THEOREM 3.6. *Put* $K_- = \{s \in \boldsymbol{C} \mid 1/r' < |s| < r',\ |1-s| > 1/3-\varepsilon,\ \pi+\theta_0 < \arg s < 2\pi-\theta_0\}$ *and* $\theta' = 2\pi - \arg\eta$. *Then, in the domain* $c_-(r',\varepsilon)$, *for each* $\eta \in K_-$, *the integrals* $z_+, e^{(-\beta')}z_0$ *and* z_- *admit the asymptotic expansions in the right-hand members of* (3.13), (3.14) *and* (3.15), *respectively, as* x *tends to* ∞ *through the sectors* $-\pi/2+\delta < \arg x < 3\pi/2+\theta'$, $-3\pi/2+\theta'+\delta < \arg x < 3\pi/2-\delta$, *and* $-3\pi/2 < \arg x < \pi/2+\theta'-\delta$, *respectively.*

THEOREM 3.7. *Put* $K_0 = \{s \in \boldsymbol{C} \mid 1/r' < |s| < r',\ 5\pi/6 < \arg s < 7\pi/6\}$ *and* $\theta'' = |\pi - \arg\eta|$. *Then, in the domain* $c_0(r',\varepsilon)$, *for each* $\eta \in K_0$, *the integrals* $z_+, z_*, e^{(-\beta')}z_0$ *and* z_- *admit the asymptotic expansions in the right-hand members of* (3.13), (3.14), (3.14) *and* (3.15), *respectively, as* x *tends to* ∞ *through the sectors* $-\pi/2+\theta''+\delta < \arg x < 5\pi/2-\theta''-\delta$, $-3\pi/2+\delta < \arg x < \pi/2-\theta''-\delta$, $-\pi/2+\theta''+\delta < \arg x < 3\pi/2-\delta$, *and* $-3\pi/2+\theta''+\delta < \arg x < 3\pi/2-\theta''-\delta$, *respectively.*

4. Proofs of the theorems in Section 3.

4.1. Proofs of Theorems 3.1 and 3.2.

We prove Theorem 3.1 only. Theorem 3.2 is proved in a similar way. First we consider the function z_0. Let δ be an arbitrary small positive constant. In order to calculate an asymptotic expansion in $c(D_1(r))$, we have to modify continuously the path $C(0)$, which is originally defined for $(x,y) \in \Delta$, so that it has the following properties:

(a) $C(0)$ lies outside the circles $|t-x| = \delta'|x|, |t-y| = \delta'|x|$, where δ' is some positive constant;

(b) $C(0)$ consists of $C_{x,y}$, the circle $|t| = 1$, and $-C_{x,y}$, where $C_{x,y}$ denotes a curve defined by $t = \tau + ig(\tau)$ $(-\infty < \tau < 0, |t| \geqq 1)$, $g(\tau)$ being a real-valued piecewise smooth function such that $g'(\tau) = O(1)$ for $-\infty < \tau < 0$.

To verify this modifiability, assume that $(x,y) \in c(D_1(r))$ belongs to the sector

$$|\arg x| < 3\pi/2 - \delta_r, \quad |\arg y - 2\pi| < 3\pi/2 - \delta, \tag{4.1}$$

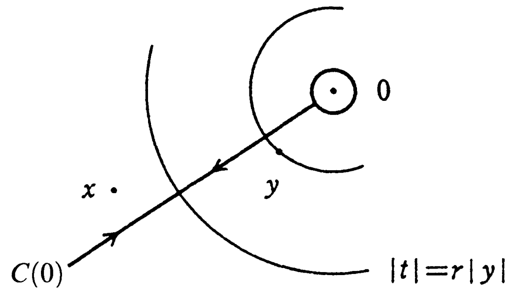

Fig. 3.

where δ_r is a constant satisfying $\sin^{-1}(1/r) < \delta_r < \pi/2$. Then, as is shown below, such a modification of $C(0)$ is possible. Note that $t = x$ moves outside the circle $|t| = r|y|$. If $|\arg x - (\arg y - 2\pi)| < 2\pi - \delta$, then we can take $C_{x,y}$ to be the half line $t = \tau + ig(\tau)$ with $g(\tau) = \tau \tan \phi_0$, where $\phi_0 = \phi_0(x, y)$ satisfies $|\phi_0 - \pi| < \pi/2 - \delta/2$ (cf. Fig.3); hence conditions (a) and (b) are satisfied. In the remaining case where $|\arg x - (\arg y - 2\pi)| \geqq 2\pi - \delta$, we can also modify $C(0)$ preserving the properties above. For example, when $\pi/2 < \arg y - \delta \leqq \arg x < 3\pi/2 - \delta_r$, we can take $C_{x,y}$ to be the broken line $t = \tau + ig(\tau)$ with

$$g(\tau) = \begin{cases} \tau \tan(\arg y - \delta''), & \text{if } \operatorname{Re} y \leqq \tau < 0, \\ (\tau - \operatorname{Re} y)\tan \phi_1 + g(\operatorname{Re} y), & \text{if } -\infty < \tau < \operatorname{Re} y, \end{cases}$$

such that $C(0)$ satisfies (a) and (b), where $\delta'' = \delta''(y, \delta')$ $(< \delta)$ is a small positive constant and $\phi_1 = \phi_1(x, y, \delta')$ is a constant satisfying $|\phi_1 - \pi| \leqq \pi/2 - (\delta_r - \sin^{-1}(1/r))/2$ (cf. Fig.4). Thus, under (4.1), there exists a desired modification of $C(0)$. If $\delta_r \leqq \sin^{-1}(1/r)$, we cannot take $C_{x,y}$ any longer. This implies that the inequality on δ_r is essential.

In view of condition (2.7), we can write $t - x = e^{\pi i}x(1 - t/x)$ and $t - y = e^{-\pi i}y(1 - t/y)$ for $t \in C(0)$, in which $\arg(1 - t/x) \to 0$ (as $t/x \to 0$) and

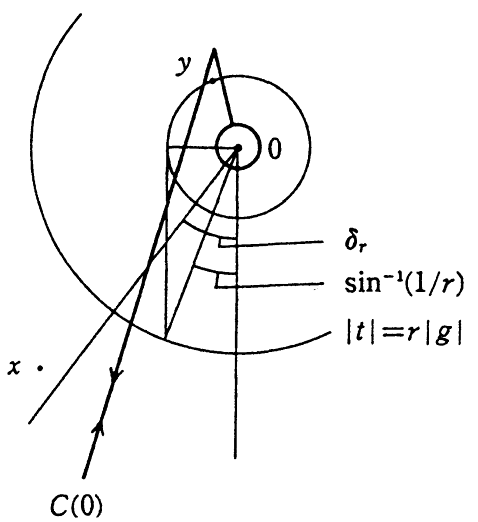

Fig. 4.

$\arg(1-t/y) \to 0$ (as $t/y \to 0$). Applying [**6**, Lemma 4.1] to the factors $(1-t/x)^{-\beta}$ and $(1-t/y)^{-\beta'}$, we write the integrand in the form

$$e^{(\beta'-\beta)\pi i}x^{-\beta}y^{-\beta'}\left(\sum_{m+n\leqq N}\frac{(\beta)_m(\beta')_n}{(1)_m(1)_n}t^{m+n}x^{-m}y^{-n}+O(t^{N+1}(|x|^{-N-1}+|y|^{-N-1}))\right)t^{\beta+\beta'-\gamma}e^t$$

for $t \in C(0)$, where N is an arbitrary large positive integer. For each $\lambda \in \boldsymbol{C}-\boldsymbol{Z}$ and $\nu \in \boldsymbol{N}$ $(\nu > \operatorname{Re}\lambda)$, using property (b), we have

$$(1-e^{(-\lambda)})^{-1}\int_{C(0)} t^{\lambda+\nu}e^t\,dt = (-1)^{\nu+1}e^{\lambda\pi i}(\lambda+1)_\nu \Gamma(\lambda+1),$$

and

$$\int_{C(0)} |t^{\lambda+\nu}e^t|\,|dt| = O\left(\int_{-\infty}^{0}|\sigma|^{\operatorname{Re}\lambda+\nu}e^{\sigma}d\sigma\right) = O(1).$$

Using these estimates, we arrive at asymptotic expansion (3.2) as $(x,y)\to(\infty,\infty)$ through the sector given above.

Next consider the function z_-. As long as (x,y) belongs to the sector $|\arg x| < 3\pi/2-\delta_r, |\arg y-\pi| < 3\pi/2-\delta$, we can modify the path $C(y)$ continuously in such a way that it has the following properties:

(a′) $C(y)$ lies outside the circles $|t-x| = \delta'|x|, |t| = \delta'|x|$;

(b′) $C(y)-y = \{t-y \mid t \in C(y)\}$ consists of the same circle and the same curves as those of (b).

The verification of this fact is similar to that of the modifiability of $C(0)$. Putting $t = v+y$, we have

$$z_- = (1-e^{(\beta')})^{-1}e^y\int_{C(y)-y} v^{-\beta'}(v+y)^{\beta+\beta'-\gamma}(v+y-x)^{-\beta}e^v dv.$$

In view of condition (2.7), the integrand is written as

$$e^{-\beta\pi i}x^{-\beta}y^{\beta+\beta'-\gamma}(1-\eta)^{-\beta}v^{-\beta'}(1+v/y)^{\beta+\beta'-\gamma}(1-v/(x-y))^{-\beta}e^v$$

with $\eta = y/x$, where $\arg(1-\eta)\to 0$ (as $\eta\to 0$), $\arg(1+v/y)\to 0$ (as $v/y\to 0$) and $\arg(1-v/(x-y))\to 0$ (as $v/(x-y)\to 0$). Applying [**6**, Lemma 4.1] and using properties (a′) and (b′), we obtain asymptotic expansion (3.3) uniformly for $|\eta| < 1/r$.

Since $|x| > r|y|$, as long as $|\arg x-\pi| < 3\pi/2-\delta_r$, we can modify the path $C(x)$ of z_+ continuously in such a way that it has the following properties:

(a″) $C(x)$ lies outside the circles $|t| = \delta'|x|, |t-y| = \delta'|x|$;

(b″) $C(x)$ consists of the circle $t-x = e^{\sigma i}$ $(-\pi+\rho \leqq \sigma \leqq \pi+\rho)$ and the two half lines $t-x = \tau e^{-(\pi+\rho)i}, t-x = \tau e^{(\pi+\rho)i}$ $(\tau \geqq 1)$, where ρ is some real constant depending on x and satisfying $|\rho| < \pi/2$.

Put $t = v+x$ and observe that $x+v = x(1+v/x)$, $x-y+v = x(1-\eta)(1+v/(x-y))$, where $\arg(1+v/x)\to 0$ (as $v/x\to 0$), $\arg(1-\eta)\to 0$ (as $\eta\to 0$), and

$\arg(1+v/(x-y)) \to 0$ (as $v/(x-y) \to 0$). Using these facts and [**6**, Lemma 4.1], we obtain asymptotic expansion of (3.1). This completes the proof of Theorem 3.1.

4.2. Proof of Theorem 3.3.

By virtue of [**6**, Propositions 5.1 and 5.2] combined with formulae (2.10) and (2.11), we can easily obtain the convergent series expansion of $z_+ - z_-$ and the asymptotic representation of $(1-e^{(\beta)})z_+ + e^{(\beta)}(1-e^{(\beta')})z_-$. Using formula (2.13), we immediately derive the asymptotic representation of z_* from [**6**, Theorem 3.4].

4.3. Proof of Theorem 3.4.

Note that Theorem 3.1 is valid for $r > 1$. From formula (2.10) and Theorem 3.1, (ii) (with $r = 3$), we derive the desired expansion of z_+ as $(x, y-x)$ tends to (∞, ∞) through the sector $|\arg x - \pi| < 3\pi/2 - \delta_3$, $|\arg(y-x) - 2\pi| < 3\pi/2 - \delta$ with $\pi/2 - \cos^{-1}(1/3) < \delta_3 < \pi/2$. Taking $\delta_3 = \pi/6$, we arrive at assertion (i). Using formula (2.11) and Theorem 3.1, (iii), we can prove assertion (iii) in a similar way. Observe that $|\arg x - \arg y| < \sin^{-1}(1/3) < \pi/6$ in $c(D_0'(R))$. Assertion (ii) immediately follows from this fact and the proposition below:

Proposition 4.1. *Write $\Delta_\delta = \{(x,y) \in \mathscr{R} \mid |\arg y - \arg x| < 2\pi - \delta\}$. Denote by $c(\Delta_\delta)$ the connected component of Δ_δ including $\Delta_\delta \cap \Delta$. In $c(\Delta_\delta)$ we have*

$$z_* \sim -e^{-\gamma\pi i}\Gamma(\beta+\beta'-\gamma+1)x^{-\beta}y^{-\beta'}H(\beta,\beta',\gamma;-1/x,-1/y)$$

as (x,y) tends to (∞,∞) through the sector $|\arg x| < 3\pi/2 - \delta$, $|\arg y| < 3\pi/2 - \delta$.

Proof. Under the assumption $|\arg x| < 3\pi/2 - \delta$, $|\arg y| < 3\pi/2 - \delta$, $|\arg y - \arg x| < 2\pi - \delta$, we can modify the path $C_*(0)$ continuously in such a way that it has the following properties:

(a_*) $C_*(0)$ lies outside the circles $|t-x| = \delta'|x|, |t-y| = \delta'|x|$, where δ' is some positive constant;

(b_*) $C_*(0)$ consists of the circle $t = e^{\sigma i}$ $(-\pi+\rho \leqq \sigma \leqq \pi+\rho)$ and the two half lines $t = \tau e^{(-\pi+\rho)i}, t = \tau e^{(\pi+\rho)i}$ $(\tau \geqq 1)$, where ρ is some real constant depending on x and satisfying $|\rho| < \pi/2$.

Considering condition (2.7), we put $t - x = e^{\pi i}x(1-t/x), t-y = e^{\pi i}y(1-t/y)$, where $\arg(1-t/x) \to 0$ (as $t/x \to 0$), $\arg(1-t/y) \to 0$ (as $t/y \to 0$). By the same calculation as that of z_0 (cf. the proof of Theorem 3.1), we arrive at the desired asymptotic representation of z_*.

4.4. Proof of Theorem 3.5.

Since assertion (ii) is an immediate consequence of Proposition 4.1, we show the remaining ones. By the definition of z_* and [**6**, Corollary 2.3, (1)], we have $z_*(x,y) = z_0(x, ye^{2\pi i})$. Then we arrive at the following corollary of Proposition 4.1.

Corollary 4.2. *Write $\Delta_\delta' = \{(x,y) \in \mathscr{R} \mid |\arg y - \arg x - 2\pi| < 2\pi - \delta\}$. Denote by $c(\Delta_\delta')$ the connected component of Δ_δ' including $\Delta_\delta' \cap \Delta$. In $c(\Delta_\delta')$ we have*

$$z_0 \sim -e^{(2\beta'-\gamma)\pi i}\Gamma(\beta+\beta'-\gamma+1)x^{-\beta}y^{-\beta'}H(\beta,\beta',\gamma;-1/x,-1/y)$$

as (x,y) *tends to* (∞,∞) *through the sector* $|\arg x| < 3\pi/2-\delta$, $|\arg y - 2\pi| < 3\pi/2-\delta$.

Assume that $(x,y)\in c_+(r',\varepsilon)\subset c(\Delta_\delta)\cap c(\Delta'_\delta)$. Note that $\theta_0<\theta<\pi-\theta_0$ $(\theta=\arg y-\arg x)$. Considering the triangle with vertices $x,0,y$, we see that

$$\theta+\delta_0<\arg(y-x)-\arg x<\pi-\delta_0, \tag{4.2}$$

where δ_0 is some positive constant satisfying $0<\delta_0<\theta_0$. Consider the function z_+. The inequality $\pi/2-\theta<\arg x<5\pi/2-\delta$ combined with (4.1) yields $-3\pi/2+\delta<-\pi/2-\theta<\arg(e^{-\pi i}x)<3\pi/2-\delta$, $\pi/2+\delta_0<\pi/2-\theta<\arg(y-x)<7\pi/2-\delta$, and $\pi<\pi+\theta+\delta_0<\arg(y-x)-\arg(e^{-\pi i}x)<2\pi$. Hence, using formula (2.10) and Corollary 4.2, and observing that $y-x=e^{\pi i}x(1-\eta)$, we have the desired asymptotic expansion of z_+ uniformly for $1/r'<|\eta|<r'$ as $x\to\infty$ through the sector $\pi/2-\theta<\arg x<5\pi/2-\delta$.

Finally consider the function z_-. If $-\pi/2-\theta+\delta<\arg x<3\pi/2$, then, by (4.2) and the definition of θ, we have $-3\pi/2+\delta<\arg(e^{-\pi i}y)<\arg x-\theta_0<3\pi/2-\theta_0$, $-3\pi/2+\delta<\arg(e^{-\pi i}(y-x))<3\pi/2-\delta_0$, and $\delta_0<\arg(e^{-\pi i}(y-x))-\arg(e^{-\pi i}y)<\pi-\theta-\delta_0<\pi$. Therefore, by Proposition 4.1 combined with (2.15), we obtain the asymptotic expansion in assertion (iii) uniformly for $1/r'<|\eta|<r'$ as $x\to\infty$ through the sector $-\pi/2-\theta+\delta<\arg x<3\pi/2$, which completes the proof of the theorem.

4.5. Proofs of Theorems 3.6 and 3.7.

Note that, under the assumption of Theorem 3.6, $\pi+\delta_0<\arg(y-x)-\arg x<2\pi-\theta'-\delta_0$, and that, under that of Theorem 3.7, $\pi-\theta''+\delta'_0<\arg(y-x)-\arg x<\pi+\theta''-\delta'_0$, where δ_0 and δ'_0 are some positive constants. Using these inequalities, we have the asymptotic expressions of z_+, $e^{(-\beta')}z_0$, z_-, z_* in the same way as in the proof of Theorem 3.5.

5. Stokes multipliers.

5.1. Stokes multipliers in $c(D_1(r))$.

To indicate a sector in $c(D_1(r))$ we use the notation

$$S_1(\theta_1,\theta_2)=\{(x,y)\in c(D_1(r))\,|\,|\arg x-\theta_1|<\pi-\delta_r, |\arg y-\theta_2|<\pi-\delta\}.$$

Consider the linearly independent solutions z_+,z_0 and z_- on $c(D_1(r))$. By Theorem 3.1 we have

$$z_+\sim Z_+(x,y),\quad z_0\sim Z_0(x,y),\quad z_-\sim Z_-(x,y), \tag{5.1}$$

as $(x,y)\to(\infty,\infty)$ through the sector $S_1(\pi/2,3\pi/2)$, where $Z_+(x,y)$, $Z_0(x,y)$ and $Z_-(x,y)$ denote the asymptotic expansions in the right-hand members of (3.1), (3.2) and (3.3), respectively. For a given sector S $(\subset c(D_1(r)))$, assume that linearly independent solutions z_+^S, z_0^S, z_-^S satisfy

$$z_+^S\sim Z_+(x,y),\quad z_0^S\sim Z_0(x,y),\quad z_-^S\sim Z_-(x,y), \tag{5.2}$$

as $(x, y) \to (\infty, \infty)$ through the sector S. There exists a matrix $C_S \in GL(3, \boldsymbol{C})$ such that

$$ {}^t(z_+, z_0, z_-) = C_S {}^t(z_+^S, z_0^S, z_-^S). \tag{5.3} $$

(The notation ${}^t\boldsymbol{v}$ denotes the transposed vector of $\boldsymbol{v}$.) Then we call C_S *the Stokes multiplier* (*for the sector* S) *with respect to* (z_+, z_0, z_-). From (5.2) and (5.3) we obtain the asymptotic representation

$$ {}^t(z_+, z_0, z_-) \sim C_S {}^t(Z_+(x, y), Z_0(x, y), Z_-(x, y)) $$

as $(x, y) \to (\infty, \infty)$ through the sector S. In the other domains treated afterward, the Stokes multipliers are similarly defined.

THEOREM 5.1. *In the domain $c(D_1(r))$, we have the Stokes multipliers with respect to (z_+, z_0, z_-) listed below*:

(a) $S_1(\pi/2, \pi/2)$

$$ C_{11}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 - e^{(\beta')} \\ 0 & 0 & 1 \end{pmatrix}, $$

(b) $S_1(\pi/2, 3\pi/2)$

$$ C_{12}^{(1)} = I, $$

(c) $S_1(3\pi/2, \pi/2)$

$$ C_{21}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(-\beta)} & 1 & 1 - e^{(\beta')} \\ 1 - e^{(-\beta)} & 0 & 1 \end{pmatrix}, $$

(d) $S_1(3\pi/2, 3\pi/2)$

$$ C_{22}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(-\beta)} & 1 & 0 \\ 1 - e^{(-\beta)} & 0 & 1 \end{pmatrix}, $$

where I denote the identity matrix.

5.2. Stokes multipliers in $c(D_2(r))$.

In the domain $c(D_2(r))$, we write

$$ S_2(\theta_1, \theta_2) = \{(x, y) \in c(D_2(r)) \mid |\arg x - \theta_1| < \pi - \delta, |\arg y - \theta_2| < \pi - \delta_r\}. $$

THEOREM 5.2. *In the domain $c(D_2(r))$, we have the Stokes multipliers with respect to (z_+, z_0, z_-) listed below*:

(a) $S_2(\pi/2, \pi/2)$

$$ C_{11}^{(2)} = \begin{pmatrix} 1 & 0 & 1 - e^{(\beta')} \\ 0 & 1 & 1 - e^{(\beta')} \\ 0 & 0 & 1 \end{pmatrix}, $$

(b) $S_2(\pi/2, 3\pi/2)$

$$ C_{12}^{(2)} = I, $$

(c) $S_2(3\pi/2, \pi/2)$

$$ C_{21}^{(2)} = \begin{pmatrix} 1 & 0 & 1 - e^{(\beta')} \\ 1 - e^{(-\beta)} & 1 & 1 - e^{(\beta')} \\ 0 & 0 & 1 \end{pmatrix}, $$

(d) $S_2(3\pi/2, 3\pi/2)$

$$ C_{22}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{(-\beta)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

5.3. Stokes multipliers in $c(D_0(R))$.

We put

$$ S_0(\theta) = \{(x, y) \in c(D_0(R)) \mid |\arg x - \theta| < \pi - \delta\}, $$

in which $\arg y$ necessarily satisfies $|\arg y - \arg x| < \delta$, if $|x|$, and hence $|y|$, is sufficiently large.

THEOREM 5.3. *In the domain $c(D_0(R))$, we have the Stokes multipliers with respect to $(z_{-1}, z_1, z_*) = (z_+ - z_-, (1 - e^{(\beta)})z_+ + e^{(\beta)}(1 - e^{(\beta')})z_-, z_*)$ listed below:*

(a) $S_0(-\pi/2)$

$$C_1^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & (1 - e^{(\beta+\beta')})(1 - e^{(\gamma-\beta-\beta')}) \\ 0 & 0 & 1 \end{pmatrix},$$

(b) $S_0(\pi/2)$

$$C_2^{(0)} = I.$$

5.4. Stokes multipliers in $c(D_0'(R))$.

Consider the sector

$$S'(\theta_1, \theta_2) = \{(x, y) \in c(D_0'(R)) \mid |\arg x - \theta_1| < 5\pi/6, |\arg(y - x) - \theta_2| < \pi - \delta\}.$$

THEOREM 5.4. *In the domain $c(D_0'(R))$, we have the Stokes multipliers with respect to (z_+, z_*, z_-) listed below:*

(a) $S'(-\pi/2, \pi/2)$

$$C_{11}' = \begin{pmatrix} 1 & 1 - e^{(\gamma-\beta-\beta')} & 1 - e^{(\beta')} \\ 0 & 1 & 0 \\ 0 & 1 - e^{(\gamma-\beta-\beta')} & 1 \end{pmatrix},$$

(b) $S'(-\pi/2, 3\pi/2)$

$$C_{12}' = \begin{pmatrix} 1 & 1 - e^{(\gamma-\beta-\beta')} & 0 \\ 0 & 1 & 0 \\ 0 & 1 - e^{(\gamma-\beta-\beta')} & 1 \end{pmatrix},$$

(c) $S'(\pi/2, \pi/2)$

$$C_{21}' = \begin{pmatrix} 1 & 0 & 1 - e^{(\beta')} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(d) $S'(\pi/2, 3\pi/2)$

$$C_{22}' = I.$$

5.5. Stokes multipliers in $c(D_*(r', \varepsilon))$.

For each $\eta \in K_+ \cup K_0 \cup K_-$ (see Theorems 3.5, 3.6 and 3.7), we put

$$S_*^+(\eta; f_1(\theta), f_2(\theta)) = \{(x, \eta x) \in c_+(r', \varepsilon) \mid f_1(\theta) < \arg x < f_2(\theta)\} \qquad (\text{if } \eta \in K_+),$$

$$S_*^-(\eta; g_1(\theta'), g_2(\theta')) = \{(x, \eta x) \in c_-(r', \varepsilon) \mid g_1(\theta') < \arg x < g_2(\theta')\} \quad (\text{if } \eta \in K_-),$$

$$S_*^0(\eta; h_1(\theta''), h_2(\theta'')) = \{(x, \eta x) \in c_0(r', \varepsilon) \mid h_1(\theta'') < \arg x < h_2(\theta'')\} \quad (\text{if } \eta \in K_0),$$

where $f_j(\theta), g_j(\theta')$ and $h_j(\theta'')(j = 1, 2)$ are linear functions of $\theta = \arg \eta, \theta' = 2\pi - \arg \eta$ and $\theta'' = |\pi - \arg \eta|$, respectively.

THEOREM 5.5. *In the domain $c_+(r', \varepsilon)$, for each $\eta \in K_+$, we have the Stokes multipliers with respect to (z_+, z_*, z_-) listed below:*

(a) $S_*^+(\eta; -\pi/2 - \theta + \delta, \pi/2 - \delta)$

$$C_{+,-1}^* = \begin{pmatrix} 1 & e^{(\beta')} - e^{(\gamma-\beta)} & 1 - e^{(\beta')} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(b) $S_*^+(\eta; \pi/2 - \theta, 3\pi/2 - \theta - \delta)$
$C_{+,0}^* = I$,

(c) $S_*^+(\eta; \pi/2 + \delta, 3\pi/2)$

$$C_{+,1}^* = \begin{pmatrix} 1 & 0 & 0 \\ e^{(-\beta')} - e^{(-\beta-\beta')} & 1 & 1 - e^{(-\beta')} \\ 0 & 0 & 1 \end{pmatrix}.$$

THEOREM 5.6. *In the domain $c_-(r', \varepsilon)$, for each $\eta \in K_-$, we have the Stokes multipliers with respect to (z_+, z_0, z_-) listed below*:

(a) $S_*^-(\eta; -3\pi/2 + \theta' + \delta, -\pi/2 + \theta')$

$$C_{-,-1}^* = \begin{pmatrix} 1 & 1 - e^{(\gamma-\beta-\beta')} & e^{(\gamma-\beta-\beta')} - e^{(\gamma-\beta)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(b) $S_*^-(\eta; -\pi/2 + \delta, \pi/2 + \theta' - \delta)$
$C_{-,0}^* = I$,

(c) $S_*^-(\eta; \pi/2, 3\pi/2 - \delta)$

$$C_{-,1}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{(\beta+\beta'-\gamma)} - e^{(\beta'-\gamma)} & 1 - e^{(\beta+\beta'-\gamma)} & 1 \end{pmatrix}.$$

THEOREM 5.7. *In the domain $c_0(r', \varepsilon)$, for each $\eta \in K_0$, we have the Stokes multipliers with respect to (z_+, z_0, z_-) (or (z_+, z_*, z_-)) given by* (a), (b) (*or* (a′), (b′)):

(a) $S_*^0(\eta; -3\pi/2 + \theta'' + \delta, \pi/2 - \theta'' - \delta)$

$$C_{0,-1}^* = \begin{pmatrix} 1 & 1 - e^{(\gamma-\beta-\beta')} & 1 - e^{(\beta')} \\ 0 & 1 & 1 - e^{(\beta')} \\ 0 & 0 & 1 \end{pmatrix},$$

(b) $S_*^0(\eta; -\pi/2 + \theta'' + \delta, 3\pi/2 - \theta'' - \delta)$
$C_{0,1}^* = I$,

(a′) $S_*^0(\eta; -3\pi/2 + \theta'' + \delta, \pi/2 - \theta'' - \delta)$

$$C_{*,-1}^* = \begin{pmatrix} 1 & e^{(\beta')} - e^{(\gamma-\beta)} & 1 - e^{(\beta')} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(b′) $S_*^0(\eta; -\pi/2 + \theta'' + \delta, 3\pi/2 - \theta'' - \delta)$

$$C_{*,1}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 - e^{(-\beta')} \\ 0 & 0 & 1 \end{pmatrix}.$$

REMARK 5.1. For each $\eta_0 \in K_+$, consider the projection $p : D(\eta_0) \to V(\eta_0)$, where $D(\eta_0)$ and $V(\eta_0)$ denote domains defined by $y/x = \eta_0$ in $c_+(r', \varepsilon)$ and in $\boldsymbol{C}^2$, respectively. Then the images of the three sectors in Theorem 5.5 under p cover the full neighbourhood of $(x,y)=(\infty,\infty)$ in $V(\eta_0)$. The sectors in Theorems 5.6 and 5.7 possess the same property.

REMARK 5.2. In the three theorems above, we can replace the sectors by ones which are independent of $\theta=\arg\eta$. For example, in Theorem 5.5, the asymptotic formulae in (a), (b) and (c) are valid uniformly for $\eta\in K_+$ as x tends to ∞ through $\cup_\eta S_*^+(\eta;-\pi/2-\theta_0+\delta,\pi/2-\delta)$, $\cup_\eta S_*^+(\eta;\pi/2-\theta_0,\pi/2+\theta_0-\delta)$ and $\cup_\eta S_*^+(\eta;\pi/2+\delta,3\pi/2)$, respectively, where the union is over all $\eta\in K_+$.

6. Proofs of the theorems in Section 5.

6.1. Proofs of Theorems 5.1 and 5.2.

Recall the monodromy matrices

$$M_0=\begin{pmatrix} e^{(-\beta')} & 0 & 1-e^{(-\beta')} \\ -(1-e^{(-\beta)})(1-e^{(-\beta')}) & 1 & (1-e^{(-\beta)})(1-e^{(-\beta')}) \\ e^{(-\beta')}(1-e^{(-\beta)}) & 0 & 1-e^{(-\beta')}+e^{(-\beta-\beta')} \end{pmatrix},$$

$$M_1=\begin{pmatrix} e^{(\beta+\beta'-\gamma)} & 1-e^{(\beta+\beta'-\gamma)} & 0 \\ e^{(\beta+\beta'-\gamma)}-e^{(\beta'-\gamma)} & 1-e^{(\beta+\beta'-\gamma)}+e^{(\beta'-\gamma)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{(-\beta')} & 1-e^{(-\beta')} \\ 0 & e^{(-\beta')}-e^{(\beta-\gamma)} & 1-e^{(-\beta')}+e^{(\beta-\gamma)} \end{pmatrix}$$

(cf. [**6**, Proposition 2.1]). To prove Theorem 5.1, it is sufficient to show assertions (a), (c) and (d). We prepare the following lemma, which is easily obtained from [**6**, Corollary 2.3, (1)] and the monodromy matrices above.

LEMMA 6.1. *In the domain $c(D_1(r))$, we have*

$$z_0(xe^{-2\pi i},y)=(1-e^{(\beta)})z_+(x,y)+e^{(\beta)}z_0(x,y), \tag{6.1}$$

$$z_-(xe^{-2\pi i},y)=(1-e^{(\beta)})z_+(x,y)+e^{(\beta)}z_-(x,y), \tag{6.2}$$

$$z_0(x,ye^{2\pi i})=e^{(-\beta')}z_0(x,y)+(1-e^{(-\beta')})z_-(x,y). \tag{6.3}$$

In $S_1(\pi/2,\pi/2)$, the asymptotic relations $z_+(x,y)\sim Z_+(x,y)$ and $z_-(x,y)\sim Z_-(x,y)$ immediately follow from Theorem 3.1, (i), (iii). Observing that $|\arg x|<3\pi/2-\delta_r$, $|\arg(ye^{2\pi i})-2\pi|<3\pi/2-\delta$ in $S_1(\pi/2,\pi/2)$, and using Theorem 3.1, (ii), we have $z_0(x,ye^{2\pi i})\sim Z_0(x,ye^{2\pi i})=e^{(-\beta')}Z_0(x,y)$. Substitution of this into (6.3) yields

$$z_0(x,y)+(e^{(\beta')}-1)z_-(x,y)\sim Z_0(x,y).$$

Thus we have obtained a column vector of solutions

$${}^t(z_+^{11}(x,y),z_0^{11}(x,y),z_-^{11}(x,y))=T_{11}\,{}^t(z_+(x,y),z_0(x,y),z_-(x,y))$$

satisfying $z_+^{11}(x,y) \sim Z_+(x,y)$, $z_0^{11}(x,y) \sim Z_0(x,y)$, $z_-^{11}(x,y) \sim Z_-(x,y)$ as $(x,y) \to (\infty,\infty)$ through $S_1(\pi/2,\pi/2)$, where

$$T_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{(\beta')}-1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the Stokes multiplier $C_{11}^{(1)} = T_{11}^{-1}$ immediately follows.

Since $|\arg(xe^{-2\pi i})| < 3\pi/2 - \delta_r$, $|\arg y - 3\pi/2| < \pi - \delta$ in $S_1(3\pi/2, 3\pi/2)$, Theorem 3.1 yields $z_0(xe^{-2\pi i}, y) \sim e^{(\beta)} Z_0(x,y)$ and $z_-(xe^{-2\pi i}, y) \sim e^{(\beta)} Z_-(x,y)$. Using these relations together with (6.1) and (6.2), we derive the Stokes multiplier $C_{22}^{(1)}$ by the same argument as above.

Suppose that $(x,y) \in S_1(3\pi/2, \pi/2)$. Since $|\arg(xe^{-2\pi i})| < 3\pi/2 - \delta_r$, $|\arg(ye^{2\pi i}) - 2\pi| < 3\pi/2 - \delta$, we have $z_0(xe^{-2\pi i}, ye^{2\pi i}) \sim Z_0(xe^{-2\pi i}, ye^{2\pi i}) = e^{(\beta-\beta')} Z_0(x,y)$. On the other hand, from Lemma 6.1, it follows that

$$\begin{aligned} z_0(xe^{-2\pi i}, ye^{2\pi i}) &= e^{(-\beta')} z_0(xe^{-2\pi i}, y) + (1 - e^{(-\beta')}) z_-(xe^{-2\pi i}, y) \\ &= (1 - e^{(\beta)}) z_+(x,y) + e^{(\beta-\beta')} z_0(x,y) + (e^{(\beta)} - e^{(\beta-\beta')}) z_-(x,y). \end{aligned}$$

Hence,

$$(e^{(\beta'-\beta)} - e^{(\beta')}) z_+(x,y) + z_0(x,y) + (e^{(\beta')} - 1) z_-(x,y) \sim Z_0(x,y). \tag{6.4}$$

Observing that $|\arg(xe^{-2\pi i})| < 3\pi/2 - \delta_r$, $|\arg y - \pi| < 3\pi/2 - \delta$, and using (6.2), we have

$$(e^{(-\beta)} - 1) z_+(x,y) + z_-(x,y) = e^{(-\beta)} z_-(xe^{-2\pi i}, y) \sim Z_-(x,y). \tag{6.5}$$

Adding a trivial relation $z_+(x,y) \sim Z_+(x,y)$ to (6.4) and (6.5), we arrive at linearly independent solutions $z_+^{21}(x,y) \sim Z_+(x,y)$, $z_0^{21}(x,y) \sim Z_0(x,y)$, $z_-^{21}(x,y) \sim Z_-(x,y)$ (as $(x,y) \to (\infty,\infty)$ through $S_1(3\pi/2, \pi/2)$) given by

$${}^t(z_+^{21}(x,y), z_0^{21}(x,y), z_-^{21}(x,y)) = T_{21}\, {}^t(z_+(x,y), z_0(x,y), z_-(x,y))$$

with

$$T_{21} = \begin{pmatrix} 1 & 0 & 0 \\ e^{(\beta'-\beta)} - e^{(\beta')} & 1 & e^{(\beta')} - 1 \\ e^{(-\beta)} - 1 & 0 & 1 \end{pmatrix}.$$

This gives the Stokes multiplier $C_{21}^{(1)} = T_{21}^{-1}$, and the proof of Theorem 5.1 is completed.

Theorem 5.2 can be verified in a similar way.

6.2. Proof of Theorem 5.3.

Recall the fundamental group $\pi_1(D)$ generated by the loops c_0, c_1 and c_2, where $D = \boldsymbol{C}^2 - (\{X=0\} \cup \{Y=0\} \cup \{X-Y=0\})$ (see [**6**, §2]). They satisfy the relation $c_0c_1c_2 = c_1c_2c_0 = c_2c_0c_1$. Let E_0 $(\subset \boldsymbol{C}^2)$ be a domain defined by $0 < |Y - X| < (1/3)|x|$, $|X| > M$, $|Y| > M$. Then $\pi_1(E_0)$ is generated by c_0 and c_1c_2. In

$c(D_0(R)) \cup c(D_0'(R))$ $(\ni (x,y))$, which is a covering of E_0, we regard $z = {}^t(z_+, z_0, z_-)$ as a function of $(x, y-x)$, and write it in the form $z = z(x, y-x)$. Then we easily obtain the following proposition:

PROPOSITION 6.2. *In the domain* $c(D_0(R)) \cup c(D_0'(R))$, *we have*

$$[c_0]z(x,y-x) = z(x,(y-x)e^{2\pi i}) = M_0 z(x,y-x), \tag{6.6}$$

$$[c_1c_2]z(x,y-x) = z(xe^{2\pi i},y-x) = M_1M_2 z(x,y-x), \tag{6.7}$$

where $[c]z(x,y-x)$ *denotes the analytic continuation of* $z(x,y-x)$ *along the loop* c.

Consider the convergent series expansion $z_{-1} = T_{-1}(x,y-x)$ and the asymptotic expansions $z_1 \sim T_1(x,y-x)$, $z_* \sim T_*(x,y-x)$ given by Theorem 3.3. It is sufficient to deduce $C_1^{(0)}$. If we put $w = {}^t(z_1, z_*)$, then, by (6.7), we have

$$[c_1c_2]w(x,y-x) = w(xe^{2\pi i},y-x) = M'w(x,y-x)$$

with

$$M' = \begin{pmatrix} e^{(\beta+\beta'-\gamma)} & (1-e^{(\beta+\beta')})(1-e^{(\beta+\beta'-\gamma)}) \\ -e^{(-\gamma)} & 1+e^{(-\gamma)}-e^{(\beta+\beta'-\gamma)} \end{pmatrix}.$$

This implies the relation

$$z_1(xe^{2\pi i},y-x) = e^{(\beta+\beta'-\gamma)}z_1(x,y-x) + (1-e^{(\beta+\beta')})(1-e^{(\beta+\beta'-\gamma)})z_*(x,y-x).$$

Since $|\arg(xe^{2\pi i})-\pi| < 3\pi/2-\delta$ in $S_0(-\pi/2)$, we have $z_1(xe^{2\pi i},y-x) \sim e^{(\beta+\beta'-\gamma)}T_1(x,y-x)$ in $S_0(-\pi/2)$. Hence,

$$z_1(x,y-x) - (1-e^{(\beta+\beta')})(1-e^{(\gamma-\beta-\beta')})z_*(x,y-x) \sim T_1(x,y-x).$$

From this and trivial relations $z_{-1}(x,y-x) = T_{-1}(x,y-x)$, $z_*(x,y-x) \sim T_*(x,y-x)$, we derive the Stokes multiplier $C_1^{(0)}$, which completes the proof of the theorem.

6.3. Proof of Theorem 5.4.

We write the asymptotic expansions of Theorem 3.4 in the form $z_+ \sim U_+(x,y-x)$, $z_* \sim U_*(x,\eta)$, $z_- \sim U_-(x,y-x,\eta)$. When x, and hence y, turns round the point (∞,∞) along the loop c_1c_2, then $\arg\eta$ remains invariant. Using Proposition 6.2, we have the following lemma.

LEMMA 6.3. *In the domain* $c(D_0'(R))$, *we have*

$$z_+(xe^{2\pi i},y-x) = e^{(\beta+\beta'-\gamma)}z_+(x,y-x) + (1-e^{(\beta+\beta'-\gamma)})z_*(x,y-x), \tag{6.8}$$

$$z_-(xe^{2\pi i},y-x) = (1-e^{(\beta+\beta'-\gamma)})z_*(x,y-x) + e^{(\beta+\beta'-\gamma)}z_-(x,y-x), \tag{6.9}$$

$$z_+(x,(y-x)e^{2\pi i}) = e^{(-\beta')}z_+(x,y-x) + (1-e^{(-\beta')})z_-(x,y-x). \tag{6.10}$$

From (6.8) and (6.10), it follows that

$$z_+(xe^{2\pi i},(y-x)e^{2\pi i}) = e^{(\beta-\gamma)}z_+(x,y-x) + (1-e^{(\beta+\beta'-\gamma)})z_*(x,y-x)$$
$$+ e^{(\beta-\gamma)}(e^{(\beta')}-1)z_-(x,y-x). \tag{6.11}$$

Assume that $(x,y) \in S'(-\pi/2,\pi/2)$. Since $|\arg(xe^{2\pi i})-\pi| < 4\pi/3$, $|\arg((y-x)e^{2\pi i})-2\pi| < 3\pi/2-\delta$, we obtain $z_+(xe^{2\pi i},(y-x)e^{2\pi i}) \sim e^{(\beta-\gamma)}U_+(x,y-x)$. Substitution of this into (6.11) yields

$$z_+(x,y-x) + (e^{(\gamma-\beta)}-e^{(\beta')})z_*(x,y-x) + (e^{(\beta')}-1)z_-(x,y-x)$$
$$\sim U_+(x,y-x). \tag{6.12}$$

Since $|\arg(xe^{2\pi i})-\pi| < 4\pi/3$, $|\arg(y-x)-\pi| < 3\pi/2-\delta$, from (6.9), we obtain

$$(e^{(\gamma-\beta-\beta')}-1)z_*(x,y-x) + z_-(x,y-x)$$
$$\sim e^{(\gamma-\beta-\beta')}U_-(xe^{2\pi i},y-x,\eta) = U_-(x,y-x,\eta). \tag{6.13}$$

From (6.12), (6.13) and a relation $z_*(x,y-x) \sim U_*(x,\eta)$, we derive the Stokes multiplier C'_{11}. Other assertions are verified by analogous arguments.

6.4. Proofs of Theorems 5.5, 5.6 and 5.7.

Let E_* ($\subset C^2$) be a domain defined by $1/r' < |Y/X| < r'$, $|Y-X| > (1/3-\varepsilon)|X|$. Then $\pi_1(E_*)$ is a free group generated by $c_0c_1c_2$. In $c(D_*(r',\varepsilon))$, we regard $\boldsymbol{z} = {}^t(z_+,z_0,z_-)$ and $\boldsymbol{u} = {}^t(z_+,z_*,z_-)$ as functions of (x,η), and write them in the form $\boldsymbol{z} = \boldsymbol{z}(x,\eta)$ and $\boldsymbol{u} = \boldsymbol{u}(x,\eta)$.

PROPOSITION 6.4. *In the domain $c(D_*(r',\varepsilon))$, we have*

$$[c_0c_1c_2]\boldsymbol{z}(x,\eta) = \boldsymbol{z}(xe^{2\pi i},\eta) = M_0M_1M_2\boldsymbol{z}(x,\eta), \tag{6.14}$$
$$[c_0c_1c_2]\boldsymbol{u}(x,\eta) = \boldsymbol{u}(xe^{2\pi i},\eta) = QM_0M_1M_2Q^{-1}\boldsymbol{u}(x,\eta), \tag{6.15}$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{(-\beta')} & 1-e^{(-\beta')} \\ 0 & 0 & 1 \end{pmatrix}.$$

To prove Theorem 5.5, it is sufficient to show assertions (a) and (c). From (6.15) we deduce that

$$z_+(xe^{2\pi i},\eta) = e^{(\beta-\gamma)}z_+(x,\eta) + (1-e^{(\beta+\beta'-\gamma)})z_*(x,\eta)$$
$$+ (e^{(\beta+\beta'-\gamma)}-e^{(\beta-\gamma)})z_-(x,\eta). \tag{6.16}$$

We write the asymptotic expansions given by Theorem 3.5 as $z_+ \sim W_+(x,\eta)$, $z_* \sim W_*(x,\eta)$, $z_- \sim W_-(x,\eta)$. Assume that $(x,x\eta) \in S_*^+(\eta;-\pi/2-\theta+\delta,\ \pi/2-\delta)$.

Observing that $3\pi/2 - \theta + \delta < \arg(xe^{2\pi i}) < 5\pi/2 - \delta$, and using (6.16), we have

$$\begin{aligned} &z_+(x,\eta) + (e^{(\gamma-\beta)} - e^{(\beta')})z_*(x,\eta) + (e^{(\beta')} - 1)z_-(x,\eta) \\ &\quad \sim e^{(\gamma-\beta)} W_+(xe^{2\pi i},\eta) = W_+(x,\eta). \end{aligned}$$

Combining this with $z_*(x,\eta) \sim W_*(x,\eta)$, $z_-(x,\eta) \sim W_-(x,\eta)$, we derive the Stokes multiplier $C^*_{+,-1}$. Relation (6.15) implies $\boldsymbol{u}(xe^{-2\pi i},\eta) = QM_2^{-1}M_1^{-1}M_0^{-1}Q^{-1}\boldsymbol{u}(x,\eta)$, from which it follows that

$$z_*(xe^{-2\pi i},\eta) = (1 - e^{(\beta)})z_+(x,\eta) + e^{(\beta+\beta')}z_*(x,\eta) + (e^{(\beta)} - e^{(\beta+\beta')})z_-(x,\eta). \tag{6.17}$$

The matrix $C^*_{+,1}$ can also be obtained from (6.17) in a similar way. Thus Theorem 5.5 is proved.

The asymptotic formula of z_0 in Theorem 3.6 is written in the form $z_0 \sim W_0(x,\eta) = e^{(\beta')}W_*(x,\eta)$. From (6.14) we obtain

$$z_+(xe^{2\pi i},\eta) = e^{(\beta-\gamma)}z_+(x,\eta) + (e^{(-\beta')} - e^{(\beta-\gamma)})z_0(x,\eta) + (1 - e^{(-\beta')})z_-(x,\eta), \tag{6.18}$$

and

$$z_-(xe^{-2\pi i},\eta) = (1 - e^{(\beta)})z_+(x,\eta) + (e^{(\beta)} - e^{(\gamma-\beta')})z_0(x,\eta) + e^{(\gamma-\beta')}z_-(x,\eta). \tag{6.19}$$

By virtue of Theorem 3.6 combined with the use of them, we can prove Theorem 5.6 in a similar way.

Using (6.16) and Theorem 3.7, we easily obtain the matrices $C^*_{*,-1}, C^*_{0,1}$ of Theorem 5.7. The matrices $C^*_{0,-1}, C^*_{*,1}$ immediately follow from them by virtue of relation (2.8).

7. Asymptotic behaviour of Φ_2.

By [**6**, Proposition 2.4], the function Φ_2 given by (1.1) is expressible in the form

$$\Phi_2(\beta,\beta',\gamma,x,y) = \boldsymbol{a}^t(z_+,z_0,z_-)$$

with $\boldsymbol{a} = (2\pi i)^{-1}\Gamma(\gamma)(1 - e^{(\beta)}, e^{(\beta)} - e^{(\gamma-\beta')}, e^{(\gamma-\beta')} - e^{(\gamma)})$. By (2.8) it is also written in the form

$$\Phi_2(\beta,\beta',\gamma,x,y) = \boldsymbol{a}_*{}^t(z_+,z_*,z_-) = \boldsymbol{b}^t(z_{-1},z_1,z_*)$$

with $\boldsymbol{a}_* = (2\pi i)^{-1}\Gamma(\gamma)(1 - e^{(\beta)},\ e^{(\beta+\beta')} - e^{(\gamma)},\ e^{(\beta)} - e^{(\beta+\beta')})$, $\boldsymbol{b} = (2\pi i)^{-1}\Gamma(\gamma)$ $(0, 1, e^{(\beta+\beta')} - e^{(\gamma)})$ (for z_{-1}, z_1 see Theorem 5.3). Using these formulae, we can derive an asymptotic expansion of Φ_2 in each sector. Recall the notation $Z_+(x,y)$, $Z_0(x,y)$, $Z_-(x,y)$ and $S_1(\theta_1,\theta_2)$ defined in Section 5.1. For example, in the domain $c(D_1(r))$, by

Theorem 5.1, we have

$$\Phi_2 \sim c_{11}^{(1)}\,{}^t(Z_+(x,y), Z_0(x,y), Z_-(x,y)) \quad \text{in } S_1(\pi/2, \pi/2),$$

$$\Phi_2 \sim c_{12}^{(1)}\,{}^t(Z_+(x,y), Z_0(x,y), Z_-(x,y)) \quad \text{in } S_1(\pi/2, 3\pi/2),$$

$$\Phi_2 \sim c_{21}^{(1)}\,{}^t(Z_+(x,y), Z_0(x,y), Z_-(x,y)) \quad \text{in } S_1(3\pi/2, \pi/2),$$

$$\Phi_2 \sim c_{22}^{(1)}\,{}^t(Z_+(x,y), Z_0(x,y), Z_-(x,y)) \quad \text{in } S_1(3\pi/2, 3\pi/2),$$

where the vector multipliers $c_{pq}^{(1)}$ $(p, q = 1, 2)$ are given by $\boldsymbol{a}C_{pq}^{(1)}$.

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Shun SHIMOMURA
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223-0061
E-mail address: shimomur@math.Keio.ac.jp