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Normal homogeneous spaces admitting totally geodesic hypersurfaces

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1. Introduction.

In [3], Chen and Verstraelen treated the classification of (locally) symmetric spaces admitting a hypersurface with a particular property. Also, in [6], Koiso treated similar problem among the class of Einstein manifolds.

In this paper we shall treat a similar problem to the above among normal homogeneous spaces which are known as a generalization of Riemannian symmetric spaces of compact type. More precisely, our main purpose is to prove Theorem A which is a generalization of the following fact (see Chen and Nagano [2]).

FACT 1.1. Spheres and hyperbolic spaces are the only simply connected, irreducible symmetric spaces admitting a totally geodesic hypersurface.

THEOREM A. *Let G be a compact simple Lie group and K a closed subgroup of G. If a normal homogeneous space G/K admits a totally geodesic hypersurface, then G/K has* (*positive*) *constant sectional curvature.*

We shall prove some properties of root systems of simple Lie algebras in Section 3. In Section 4 we shall give a description of $\mathfrak{k}$ (the Lie algebra of K) in terms of a root system of $\mathfrak{g}$ (the Lie algebra of G). Particularly, in the case $\mathrm{rk}(G)=\mathrm{rk}(K)+1$, the forms of root spaces with respect to a maximal abelian subalgebra of $\mathfrak{k}$ is determined. Using some results in [7] and in Sections 3, 4 we shall prove Theorem A in Sections 7, 8.

Particularly, in the case $\mathrm{rk}(G)=\mathrm{rk}(K)$, the same result as in Theorem A is derived from a weaker hypothesis.

THEOREM B. *Let G be a compact simple Lie group and K a closed subgroup of G with $\mathrm{rk}(G)=\mathrm{rk}(K)$. If there exists a curvature invariant hyperplane V in a tangent space of the normal space G/K, then G/K has constant sectional curvature.*

The above theorem will be proved in section 5. In section 9, we shall give

Recently, Tsukuda [8] has improved Theorem A.

a counterexample of Theorem B in the case $\mathrm{rk}(G)>\mathrm{rk}(K)$.

2. Preliminaries.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Put $l=\dim_C\mathfrak{h}$. Let Δ be the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Φ the Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, Φ is nondegenerate. Then we can define $H_\alpha\in\mathfrak{h}(\alpha\in\Delta)$ by $\Phi(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{h}$.

As is well-known, there exist the root vectors $\{E_\alpha : \alpha\in\Delta\}$ such that for all α, $\beta\in\Delta$ and for all $H\in\mathfrak{h}$,

$$\Phi(E_\alpha, E_{-\alpha})=1, \qquad [E_\alpha, E_{-\alpha}]=H_\alpha, \tag{2.1}$$
$$[H, E_\alpha]=\alpha(H)E_\alpha, \qquad [E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta},$$

where $N_{\alpha,\beta}$ are real numbers satisfying the relation

$$N_{\alpha,\beta}=-N_{-\alpha,-\beta}. \tag{2.2}$$

Moreover the following holds.

$$N_{\alpha,\beta}=N_{\beta,\gamma}=N_{\gamma,\alpha} \quad \text{if} \quad \alpha+\beta+\gamma=0 \tag{2.3}$$

$$(N_{\alpha,\beta})^2=\frac{q(1-p)}{2}\alpha(H_\alpha) \tag{2.4}$$

where $\{\beta+n\alpha : p\leqq n\leqq q\}$ is the α-series containing β. Set $a_{\beta,\alpha}=2\alpha(H_\beta)/\alpha(H_\alpha)$, then

$$a_{\beta,\alpha}=-(p+q) \tag{2.5}$$

$$0\leqq a_{\alpha,\beta}a_{\beta,\alpha}\leqq 3 \quad \text{if} \quad H_\alpha \text{ is not parallel to } H_\beta \tag{2.6}$$

(see Helgason [4] for details).

Next, we recall some properties of naturally reductive homogeneous spaces.

Let $(M, g)=G/K$ be a naturally reductive homogeneous space with a Lie group G and a compact subgroup K of G. Then there exists an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G ($\mathfrak{k}$ is the Lie algebra of K) such that for $X, Y, Z\in\mathfrak{p}$ the following holds.

$$\langle[X, Y]_\mathfrak{p}, Z\rangle+\langle Y, [X, Z]_\mathfrak{p}\rangle=0 \tag{2.7}$$

Here $\langle\ ,\ \rangle$ denotes the inner product on $\mathfrak{p}$ induced by g under the canonical identification of $\mathfrak{p}$ with the tangent space $T_oM(o=\{K\})$. Then $\langle\ ,\ \rangle$ is $\mathrm{Ad}(K)$-invariant and $[\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p}$. For $W\in\mathfrak{g}$ we write $W=W_\mathfrak{k}+W_\mathfrak{p}$ to identify components under the decomposition.

As is well-known, $\gamma_x(t)=\tau(\exp tx)(o)$ $(x\in\mathfrak{p})$ is the geodesic of (M, g) such that $\gamma_x(0)=o$ and $\gamma_x'(0)=x$ where $\tau(h)$ denotes the left transformation of G/K induced by $h\in G$. Furthermore, the curvature tensor R at o is given by

$$R(X, Y)Z = [[X, Y]_{\mathfrak{k}}, Z]+\frac{1}{2}[[X, Y]_{\mathfrak{p}}, Z]_{\mathfrak{p}} -\frac{1}{4}[X, [Y, Z]_{\mathfrak{p}}]_{\mathfrak{p}}+\frac{1}{4}[Y, [X, Z]_{\mathfrak{p}}]_{\mathfrak{p}} \tag{2.8}$$

(cf. Kobayashi and Nomizu [**5**]).

In particular, let G/K be a normal homogeneous space, that is, G/K is a reductive homogeneous space equipped with a G-invariant metric induced by a biinvariant metric $\langle\ ,\ \rangle$ of compact Lie group G. More precisely, let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the orthogonal decomposition with respect to $\langle\ ,\ \rangle$. Then, since $\langle\ ,\ \rangle$ is biinvariant, we have for $X, Y, Z\in\mathfrak{p}$

$$\langle[X, Y]_{\mathfrak{p}}, Z\rangle+\langle Y, [X, Z]_{\mathfrak{p}}\rangle = 0.$$

Hence normal homogeneous spaces are naturally reductive. From now on, we use the same symbol $\langle\ ,\ \rangle$ for the biinvariant metric $\langle\ ,\ \rangle$ restricted to $\mathfrak{p}$. Then for $X, Y\in\mathfrak{p}$, we have from (2.8)

$$\langle R(X, Y)X, Y\rangle = \|[X, Y]_{\mathfrak{k}}\|^2+\frac{1}{4}\|[X, Y]_{\mathfrak{p}}\|^2.$$

Therefore we obtain

$$\langle R(X, Y)X, Y\rangle = 0 \Longleftrightarrow [X, Y] = 0. \tag{2.9}$$

In the remaining part of this section we give some lemmas which are concerned with the totally geodesic hypersurfaces of naturally reductive homogeneous spaces (see [**7**]).

As before, let G/K be a naturally reductive homogeneous space. Define an isometry $e^{-\varphi_X}(X\in\mathfrak{p})$ of $(\mathfrak{p}, \langle\ ,\ \rangle)$ as follows:

$$e^{-\varphi_X}(Y) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l}\varphi_{X^l}(Y)$$

where $\varphi_X(Y)=(1/2)[X, Y]_{\mathfrak{p}}$. Then $\tau(\exp tX)_*(e^{-\varphi_{tX}}(Y))$ is a parallel vector field along the geodesic $\tau(\exp tX)(o)$. (see [**7**].)

For $\nu\in\mathfrak{p}$, we say that ν satisfies condition (T-G) if there exists a totally geodesic hypersurface tangent to the hyperplane $\nu^\perp$ in $\mathfrak{p}$ at o. Then the following lemmas hold.

LEMMA 2.1. *A vector ν satisfies condition (T-G) if and only if for any $X\in V (V=\nu^\perp)$ the following is satisfied*:

$$R(X, e^{-\varphi_X}(V))e^{-\varphi_X}(\nu) = \{0\}$$

LEMMA 2.2. *If ν satisfies condition (T-G), then* $\mathrm{Ad}(k)(\nu)(k\in K)$ *and* $e^{-\varphi_X}(\nu)$

$(X \in \nu^{\perp})$ *satisfy condition* (*T-G*).

In particular, if ν satisfies condition (T-G), then the following holds.

$$R(V, V)V \subset V \tag{2.10}$$

3. Simple root systems.

In this section we shall give some properties of the root systems of complex simple Lie algebras for later use (cf. [4]).

Let $\mathfrak{g}$ be a simple Lie algebra over $\boldsymbol{C}$ and $\mathfrak{h}$, Δ the same as in section 2. We call $\mathfrak{g}$ is of type I (resp. of type II) if it is a Lie algebra associated with one of the following types:

$$A_l,\ D_l,\ E_6,\ E_7,\ E_8 \text{ (resp. } B_l,\ C_l,\ F_4)$$

Then the following holds.

LEMMA 3.1. *Let* $\{\beta+n\alpha : p \leqq n \leqq q\}$ *be the* α*-series containing* β $(\alpha, \beta \in \Delta)$. *Then*:

(1) $0 \leqq q-p \leqq 1$ *if* $\mathfrak{g}$ *is of type I.*
(2) $0 \leqq q-p \leqq 2$ *if* $\mathfrak{g}$ *is of type II.*
(3) $0 \leqq q-p \leqq 3$ *if* $\mathfrak{g}$ *is of type* G_2.

PROOF. By using descriptions of the simple root systems in [4, p. 462-p. 474], we prove the lemma.

(1) In this case, for any $\alpha, \beta \in \Delta (\alpha \neq \pm \beta)$ we can easily check $\alpha(H_\alpha) = \beta(H_\beta)$. Hence we get $a_{\beta,\alpha} = a_{\alpha,\beta}$. Then by (2.6), we get $|a_{\beta,\alpha}| = 0$ or 1. Therefore we obtain $q-p \leqq 1$. (If $q-p \geqq 2$, then $a_{\beta+p\alpha,\alpha} = p-q$. This contradicts the above.) We have thus proved (1).

(2) In this case, for any $\alpha, \beta \in \Delta (\alpha \neq \pm \beta)$ we can see

$$\frac{\beta(H_\beta)}{\alpha(H_\alpha)} = 1,\ 2 \quad \text{or} \quad \frac{1}{2}. \tag{3.1}$$

From (2.5), if $q-p \geqq 3$, then we get

$$a_{\beta+p\alpha,\alpha} \leqq -3.$$

Then it follows from (2.6) that

$$a_{\beta+p\beta,\alpha} = -3, \qquad a_{\alpha,\beta+p\alpha} = -1.$$

Therefore we have

$$\frac{\beta(H_\alpha)}{\alpha(H_\alpha)} = 3.$$

This contradicts (3.1) and finishes the proof of (2).

Finally (3) is obvious from (2.6). We have thus proved the lemma. □

Let α_1, α_2 and α_3 in Δ such that $\alpha_1 \neq \pm\alpha_2$. We consider the case where $(H_{\alpha_1}-H_{\alpha_2})$ is parallel to $(H_{\alpha_1}-H_{\alpha_3})$. Set

$$H_{\alpha_1}-H_{\alpha_3} = c(H_{\alpha_1}-H_{\alpha_2}) \ (c \in \boldsymbol{R} \ c \neq 0),$$

that is,

$$H_{\alpha_3} = (1-c)H_{\alpha_1}+cH_{\alpha_2}. \tag{3.2}$$

Hence

$$\alpha_3(H_{\alpha_3}) = (1-c)^2\alpha_1(H_{\alpha_1})+c^2\alpha_2(H_{\alpha_2})+2c(1-c)\alpha_1(H_{\alpha_2}). \tag{3.3}$$

If $\mathfrak{g}$ is of type I, then it follows from (3.3) that

$$c(1-c)(2-a_{\alpha_1,\alpha_2}) = 0.$$

From (1) of Lemma 3.1, we have $|a_{\alpha_1,\alpha_2}| \leqq 1$. Therefore we have $c=1$ and $\alpha_3=\alpha_2$ from (3.2). We have thus the following lemma.

LEMMA 3.2. *Let $\mathfrak{g}$ be of type I and α, $\beta \in \Delta$. If $\gamma \neq \pm\alpha$, $\pm\beta$ $(\gamma \in \Delta)$, then $(H_\gamma - H_\alpha)$ is not parallel to $(H_\alpha - H_\beta)$.*

Next, we shall check the root systems of type II and of type G_2.

TYPE II.

Put $a_i=\alpha_i(H_{\alpha_i})$ $(i=1, 2, 3)$. By (3.1), for all $\alpha \in \Delta$, there is a positive number e such that $\alpha(H_\alpha)=e$ or $2e$. Then the following eight cases are possible:

$$(a_1, a_2, a_3) = (1): (e, e, e) \quad (2): (2e, e, e) \quad (3): (e, 2e, e)$$
$$(4): (e, e, 2e) \quad (5): (2e, 2e, e) \quad (6): (2e, e, 2e)$$
$$(7): (e, 2e, 2e) \quad (8): (2e, 2e, 2e)$$

(1) From (3.3), we get

$$2c(1-c)\{e-\alpha_1(H_{\alpha_2})\} = 0.$$

Therefore we obtain $c=1$ or $\alpha_1(H_{\alpha_2})=e$. If $\alpha_1(H_{\alpha_2})=e$, then

$$a_{\alpha_1,\alpha_2}a_{\alpha_2,\alpha_1} = \frac{4e^2}{e^2} = 4.$$

Hence we conclude $c=1$, i.e. $\alpha_3=\alpha_2$.

(2) Since

$$a_{\alpha_1,\alpha_2}a_{\alpha_2,\alpha_1} = \frac{2\alpha_1(H_{\alpha_2})^2}{e^2} \quad \text{and} \quad a_{\alpha_i,\alpha_j} \in \boldsymbol{Z},$$

the possible values of $\alpha_1(H_{\alpha_2})$ are 0 and $\pm e$. If $\alpha_1(H_{\alpha_2})=0$, then from (3.3) we get $c=1$ or $1/3$. Therefore

$$\alpha_3=\alpha_2 \quad \text{or} \quad \frac{2\alpha_1+\alpha_2}{3}.$$

However, by making use of the description of the root systems in [4, p. 462–p. 473], we can see that $(2\alpha_1+\alpha_2)/3$ is not a root.

If $\alpha_1(H_{\alpha_2})=e$, then $\alpha_3=\alpha_2$.

If $\alpha_1(H_{\alpha_2})=-e$, then

$$\alpha_3=\alpha_2 \quad \text{or} \quad \frac{4\alpha_1+\alpha_2}{5}.$$

However, we can see that $(4\alpha_1+\alpha_2)/5$ is not a root. Consequently, in this case we have $\alpha_3=\alpha_2$.

(3) As in (2), we can check

$$\alpha_1(H_{\alpha_2})=0 \quad \text{or} \quad \pm e.$$

If $\alpha_1(H_{\alpha_2})=0$, then from (3.3), we have $3c^2-2c=0$. hence $c=2/3(c\neq 0)$. Then $\alpha_3=(\alpha_1+2\alpha_2)/3$. But it is not a root.

If $\alpha_1(H_{\alpha_2})=e$, then $c=0$. This contradicts the assumption.

If $\alpha_1(H_{\alpha_2})=-e$, then $c=4/5$, that is, $\alpha_3=(\alpha_1+4\alpha_2)/5$. But it is not a root.

(4) In this case we have $\alpha_1(H_{\alpha_2})=0$ or $\pm(e/2)$. However, for each $\alpha_1(H_{\alpha_2})$, we see that $(1-c)\alpha_1+c\alpha_2$ is not a root.

(5) In this case we see $\alpha_1(H_{\alpha_2})=0$ or $\pm e$.

If $\alpha_1(H_{\alpha_2})=\pm e$, then $(1-c)\alpha_1+c\alpha_2$ is not a root.

If $\alpha_1(H_{\alpha_2})=0$, then we have $c=1/2$, i.e. $\alpha_3=(\alpha_1+\alpha_2)/2$. Hence we get

$$\alpha_3=\frac{\alpha_1+\alpha_2}{2} \quad \text{with} \quad a_{\alpha_1,\alpha_2}=0,\ \ \frac{a_1}{a_2}=1. \tag{3.4}$$

(6) Also in this case, we get $\alpha_1(H_{\alpha_2})=0$ or $\pm e$.

If $\alpha_1(H_{\alpha_2})=0$ or $-e$, then we can see that $(1-c)\alpha_1+c\alpha_2$ is not a root. If $\alpha_1(H_{\alpha_2})=e$, then we obtain $\alpha_3=2\alpha_2-\alpha_1$. Thus we have

$$\alpha_3=2\alpha_2-\alpha_1 \quad \text{with} \quad a_{\alpha_1,\alpha_2}=2,\ \ a_{\alpha_2,\alpha_1}=1. \tag{3.5}$$

(7) In this case we get $\alpha_1(H_{\alpha_2})=0$ or $\pm e$.

If $\alpha_1(H_{\alpha_2})=0$ or $-e$, then $(1-c)\alpha_1+c\alpha_2$ is not a root.

If $\alpha_1(H_{\alpha_2})=e$, then we find that $\alpha_3=2\alpha_1-\alpha_2$. Consequently we get

$$\alpha_1=2\alpha_1-\alpha_2 \quad \text{with} \quad a_{\alpha_1,\alpha_2}=1,\ \ a_{\alpha_2,\alpha_1}=2. \tag{3.6}$$

(8) In this case we see that $\alpha_1(H_{\alpha_2})=0$ or $\pm e$. Then for any $\alpha_1(H_{\alpha_2})$, we have $c(1-c)=0$. Hence $\alpha_3=\alpha_2$.

Remark. Put $\hat{\alpha}_1=\alpha_3$, $\hat{\alpha}_2=\alpha_2$ and $\hat{\alpha}_3=\alpha_1$ in (3.4). Then we obtain

$$\hat{\alpha}_3=2\hat{\alpha}_1-\hat{\alpha}_2 \quad \text{with} \quad a_{\hat{\alpha}_1,\hat{\alpha}_2}=1,\ \ a_{\hat{\alpha}_2,\hat{\alpha}_1}=2.$$

Thus, from (3.4), (3.5) and (3.6), we have the following.

LEMMA 3.3. *Let* $\mathfrak{g}$ *be of type II. If* $\alpha_1, \alpha_2, \alpha_3 \in \Delta(\alpha_1 \neq \pm\alpha_2, \alpha_2 \neq \pm\alpha_3, \alpha_3 \neq \pm\alpha_1)$ *satisfy* (3.2), *then*

$$\{\alpha_1, \alpha_2, \alpha_3\} = \{\alpha, \beta, \gamma\} \quad \text{with} \quad a_{\alpha,\beta} = 1, \quad a_{\beta,\alpha} = 2$$

where $\gamma = 2\alpha - \beta$.

TYPE G_2.

Under the same notation as in type II, we can assume that $a_i = e$ or $3e$ $(i=1, 2, 3)$. Then we examine the following eight cases:

$$(a_1, a_2, a_3) = (e, e, e), \quad (3e, e, e), \quad (e, 3e, e),$$
$$(e, e, 3e), \quad (3e, 3e, e), \quad (3e, e, 3e),$$
$$(e, 3e, 3e), \quad (3e, 3e, 3e).$$

By the same computation as in type II, we have the following lemma.

LEMMA 3.4. *Let* $\mathfrak{g}$ *be of type* G_2. *If* $\alpha_1, \alpha_2, \alpha_3 \in \Delta(\alpha_1 \neq \pm\alpha_2, \alpha_2 \neq \pm\alpha_3, \alpha_3 \neq \pm\alpha_1)$ *satisfy* (3.2), *then* $\{\alpha_1, \alpha_2, \alpha_3\}$ *is one of the following two cases.*

(1) $\{\alpha, \beta, 2\alpha - \beta\}$ *with* $a_{\alpha,\beta} = 1$, $a_{\beta,\alpha} = 1$.

(2) $\{\alpha, \beta, 3\alpha - 2\beta\}$ *with* $a_{\alpha,\beta} = 1$, $a_{\beta,\alpha} = 3$.

4. Description of $(\mathfrak{g}, \mathfrak{k})$.

Let G/K be a normal homogeneous space with a compact simple Lie group G and a closed subgroup K of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then $\mathfrak{g}$ is isomorphic to a compact real form

$$\mathfrak{g}_u = \sqrt{-1}\mathfrak{h}_R + \sum_{\alpha \in \Delta} \boldsymbol{R} A_\alpha + \sum_{\alpha \in \Delta} \boldsymbol{R} B_\alpha$$

of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$, where $A_\alpha = E_\alpha - E_{-\alpha}$, $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$ and $\mathfrak{h}_R = \sum_{\alpha \in \Delta} \boldsymbol{R} H_\alpha$ (see [4]). Therefore we consider $\mathfrak{g}_u$ as the Lie algebra $\mathfrak{g}$ of G. Moreover, since any maximal abelian subalgebra of $\mathfrak{g}$ is conjugate to $\sqrt{-1}\mathfrak{h}_R$, we suppose that a maximal abelian subalgebra of $\mathfrak{k}$ is contained in $\sqrt{-1}\mathfrak{h}_R$.

Generally, the dimension of a maximal abelian subalgebra of $\mathfrak{g}$ is called the *rank* of G (denoted by $\mathrm{rk}(G)$).

Let $\langle\ ,\ \rangle$ be a biinvariant metric of G. Since G is simple, then $\langle\ ,\ \rangle$ is unique up to a scalar factor by Schur's lemma. Thus we put $\langle\ ,\ \rangle = -\Phi(\ ,\ )$ (Φ is the Killing form).

Let $\mathfrak{g}_\alpha (\alpha \in \Delta)$ be a subspace of $\mathfrak{g}$ spanned by A_α and B_α. We note that for $\alpha, \beta \in \Delta$

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] = \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha-\beta}. \tag{4.1}$$

Moreover the following is immediately given by (2.1):

$$\left\{ \begin{array}{l} [A_{\alpha}, A_{\beta}] = N_{\alpha,\beta}A_{\alpha+\beta} - N_{\alpha,-\beta}A_{\alpha-\beta} \\ [A_{\alpha}, B_{\beta}] = N_{\alpha,\beta}B_{\alpha+\beta} + N_{\alpha,-\beta}B_{\alpha-\beta} \\ [B_{\alpha}, B_{\beta}] = -N_{\alpha,\beta}A_{\alpha+\beta} - N_{\alpha,-\beta}A_{\alpha-\beta} \end{array} \right\} \tag{4.2}$$

We define subsets $\Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{p}}$ and Δ' of Δ as follows:

$$\Delta_{\mathfrak{k}} = \{\alpha \in \Delta : \mathfrak{g}_{\alpha} \subset \mathfrak{k}\}$$
$$\Delta_{\mathfrak{p}} = \{\alpha \in \Delta : \mathfrak{g}_{\alpha} \subset \mathfrak{p}\}$$
$$\Delta' = \Delta \setminus (\Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}})$$

Then the following holds.

LEMMA 4.1. *If* $\mathrm{rk}(G) = \mathrm{rk}(K)$, *then*

$$\alpha \in \Delta_{\mathfrak{k}} \quad or \quad \alpha \in \Delta_{\mathfrak{p}}$$

for each $\alpha \in \Delta$.

PROOF. See Berger [1]. □

Next, we consider the case $\mathrm{rk}(G) > \mathrm{rk}(K)$. From the above assumption, we get the orthogonal decomposition $\sqrt{-1}\mathfrak{h}_R = \sqrt{-1}\mathfrak{h}_1 \oplus \sqrt{-1}\mathfrak{h}_2$ with $\sqrt{-1}\mathfrak{h}_1 \subset \mathfrak{k}$ and $\sqrt{-1}\mathfrak{h}_2 \subset \mathfrak{p}$.

Let $\bar{\alpha}(\alpha \in \Delta)$ be the restriction of α to $\sqrt{-1}\mathfrak{h}_1$. For $X, X' \in \mathfrak{g}_{\alpha}$, we call that the couple (X, X') *belongs to* α (denoted by $(X, X') \in \alpha$) if $[\sqrt{-1}H, X] = \alpha(H)X'$ and $[\sqrt{-1}H, X'] = -\alpha(H)X$ for any $H \in \mathfrak{h}_R$. Obviously, if $(X, X') \in \alpha$, then $(X', X) \in -\alpha$. For $(X, X') \in \alpha$ and $\sqrt{-1}H \in \sqrt{-1}\mathfrak{h}_1$, we have

$$[\sqrt{-1}H, X] = [\sqrt{-1}H, X_{\mathfrak{k}} + X_{\mathfrak{p}}]$$
$$\alpha(H)X' = \alpha(H)X'_{\mathfrak{k}} + \alpha(H)X'_{\mathfrak{p}}.$$

Since $[\sqrt{-1}H, X_{\mathfrak{k}}] \in \mathfrak{k}$ and $[\sqrt{-1}H, X_{\mathfrak{p}}] \in \mathfrak{p}$, it follows that

$$[\sqrt{-1}H, X_{\mathfrak{k}}] = \alpha(H)X'_{\mathfrak{k}} \tag{4.3}$$

$$[\sqrt{-1}H, X_{\mathfrak{p}}] = \alpha(H)X'_{\mathfrak{p}}. \tag{4.4}$$

Similarly we have

$$[\sqrt{-1}H, X'_{\mathfrak{k}}] = -\alpha(H)X_{\mathfrak{k}} \tag{4.5}$$

$$[\sqrt{-1}H, X'_{\mathfrak{p}}] = -\alpha(H)X_{\mathfrak{p}}. \tag{4.6}$$

Then the following lemma holds (cf. [1]).

LEMMA 4.2. *If* $(X, X')\in\alpha$, *then* $X_{\mathfrak{t}}$ *has the following form.*

$$X_{\mathfrak{t}}=\sum_{\lambda\in\Delta(\alpha)} p_{\lambda}X_{\lambda} \quad \textit{where}:$$

(1) $\Delta(\alpha)=\{\lambda\in\Delta:\ \bar{\lambda}=\bar{\alpha}\}$; (2) $p_{\lambda}\geqq 0$; (3) *there exist* X'_{λ} *such that* $(X_{\lambda}, X'_{\lambda})\in\lambda$; (4) $\sum_{\lambda\in\Delta(\alpha)}p_{\lambda}^{2}\lambda(H)=0$ *for all* $H\in\mathfrak{h}_{2}$.

From (4.3) and (4.5), we have the following.

LEMMA 4.3. *If* $(X, X')\in\alpha$, *then*

$$[X_{\mathfrak{t}}, X'_{\mathfrak{t}}]\in\sqrt{-1}\mathfrak{h}_{1}.$$

Under the same notation as in Lemma 4.2, we define a subspace $\mathfrak{g}(\alpha)$ of $\mathfrak{g}$ as follows:

$$\mathfrak{g}(\alpha)=\bigoplus_{\substack{\lambda\in\Delta(\alpha)\\ p_{\lambda}\neq 0}}\mathfrak{g}_{\lambda}.$$

Particularly, in the case $\mathfrak{g}(\alpha)=\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta}$ for some $\beta\in\Delta'$, the following holds.

LEMMA 4.4. *If* $\mathfrak{g}(\alpha)=\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta}$, *then*

$$\alpha-\beta\notin\Delta$$

PROOF. Take $X_{\mathfrak{t}}=p_{\alpha}X_{\alpha}+p_{\beta}X_{\beta}$ and $X'_{\mathfrak{t}}=p_{\alpha}X'_{\alpha}+p_{\beta}X'_{\beta}(p_{\alpha}\cdot p_{\beta}\neq 0)$ with $X_{\alpha}=A_{\alpha}$, $X'_{\alpha}=B_{\alpha}$, $X_{\beta}=aA_{\beta}+bB_{\beta}$ and $X'_{\beta}=aB_{\beta}-bA_{\beta}(a^{2}+b^{2}=1)$. Then

$$\begin{aligned}[X_{\mathfrak{t}}, X'_{\mathfrak{t}}] &= 2\sqrt{-1}(p_{\alpha}^{2}H_{\alpha}+p_{\beta}^{2}H_{\beta})\\ &\quad +p_{\alpha}p_{\beta}([A_{\alpha}, aB_{\beta}-bA_{\beta}]+[aA_{\beta}+bB_{\beta}, B_{\alpha}])\\ &= 2\sqrt{-1}(p_{\alpha}^{2}H_{\alpha}+p_{\beta}^{2}H_{\beta})+2p_{\alpha}p_{\beta}N_{\alpha,-\beta}(aB_{\alpha-\beta}+bA_{\alpha-\beta}).\end{aligned}$$

It follows from Lemma 4.3 that $N_{\alpha,-\beta}=0$. This completes the proof of the lemma. □

From Lemmas 3.2, 3.3 and 3.4, we can determine the form of $\mathfrak{g}(\alpha)$ in the case $\mathrm{rk}(G)=\mathrm{rk}(K)+1$.

PROPOSITION 4.5. *Suppose* $\mathrm{rk}(G)=\mathrm{rk}(K)+1$ *and* $\alpha\in\Delta'$. *Then*

(1) *If* $\mathfrak{g}$ *is of type I, then* $\mathfrak{g}(\alpha)=\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta}$ *for some* $\beta\in\Delta'$.

(2) *If* $\mathfrak{g}$ *is of type II, then* $\mathfrak{g}(\alpha)$ *has one of the following two forms.*

(a) $\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta}$ *for some* $\beta\in\Delta'$.

(b) $\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta}\oplus\mathfrak{g}_{\gamma}$, *where* $\{\alpha, \beta, \gamma\}$ *coincides with* $\{\alpha_{1}, \alpha_{2}, \alpha_{3}\}$ *in Lemma* 3.3.

(3) *If* $\mathfrak{g}$ *is type* G_{2}, *then* $\mathfrak{g}(\alpha)$ *has one of the following four forms.*

(a) $\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta}$ *for some* $\beta\in\Delta'$.

(b) $\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\beta_{1}}\oplus\mathfrak{g}_{\beta_{2}}$, *where* $\{\alpha, \beta_{1}, \beta_{2}\}$ *coincides with* (1) *in Lemma* 3.4.

(c) $\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\gamma_{1}}\oplus\mathfrak{g}_{\gamma_{2}}$, *where* $\{\alpha, \gamma_{1}, \gamma_{2}\}$ *coincides with* (2) *in Lemma* 3.4.

(d) $\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{\delta_{1}}\oplus\mathfrak{g}_{\delta_{2}}\oplus\mathfrak{g}_{\delta_{3}}$, *where* $\{\alpha, \delta_{1}, \delta_{2}, \delta_{3}\}=\{\alpha_{1}, \alpha_{2}, 2\alpha_{1}-\alpha_{2}, 2\alpha_{2}-\alpha_{1}\}$ *with*

$a_{\alpha_1,\alpha_2}=a_{\alpha_2,\alpha_1}=1$.

Moreover, if $\mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_2}\subset\mathfrak{g}(\alpha)$, *then* $\sqrt{-1}\mathfrak{h}_2=\boldsymbol{R}\sqrt{-1}(H_{\alpha_1}-H_{\alpha_2})$.

PROOF. If $\mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_2}\subset\mathfrak{g}(\alpha)$, then $(\alpha_1-\alpha_2)(\mathfrak{h}_1)=0$. Therefore we have for any $\sqrt{-1}H\in\sqrt{-1}\mathfrak{h}_1$ that

$$\langle\sqrt{-1}(H_{\alpha_1}-H_{\alpha_2}),\ \sqrt{-1}H\rangle=(\alpha_1-\alpha_2)(H)=0\,. \tag{4.7}$$

Since $\sqrt{-1}\mathfrak{h}_1\perp\sqrt{-1}\mathfrak{h}_2$ and $\dim\sqrt{-1}\mathfrak{h}_2=1$, (4.7) gives

$$\sqrt{-1}\mathfrak{h}_2=\boldsymbol{R}\sqrt{-1}(H_{\alpha_1}-H_{\alpha_2})\,.$$

Next, suppose $\mathfrak{g}(\alpha)=\mathfrak{g}_{\alpha_1}\oplus\cdots\oplus\mathfrak{g}_{\alpha_m}$. From the above argument, $(H_{\alpha_1}-H_{\alpha_i})$ $(i\geqq2)$ must be parallel to $(H_{\alpha_1}-H_{\alpha_2})$. Therefore by Lemma 3.2 and Lemma 3.3, the statements (1) and (2) in the proposition are obvious.

Similarly, if $\mathfrak{g}$ is of type G_2 and $m\leqq3$, the $\mathfrak{g}(\alpha)$ has one of the three cases (a), (b) and (c) in (3).

Finally, we check the case $m\geqq4$. As in section 3, we can set $a_i=\alpha_i(H_{\alpha_i})$ $=e$ or $3e$ for some $e>0(1\leqq i\leqq m)$. If $\{\alpha_1, \alpha_2, \alpha_3\}$ is of type (1) in Lemma 3.4 (set $\alpha_3=2\alpha_1-\alpha_2$), then we can easily see that $a_1=a_2=e$ and $a_3=3e$. Similarly, if $\{\alpha_1, \alpha_2, \alpha_3\}$ is of type (2) in Lemma 3.4, then we have $a_1=e$ and $a_2=a_3=3e$. Therefore by Lemma 3.4, we get for $i\geqq4$

$$a_i=\begin{cases}3e, & \text{if } \{\alpha_1, \alpha_2, \alpha_3\} \text{ is of type (1)}\\ e, & \text{if } \{\alpha_1, \alpha_2, \alpha_3\} \text{ is of type (2).}\end{cases}$$

Now, we assume that $\{\alpha_1, \alpha_2, \alpha_3\}$ is of type (1). Since $\{\alpha_1, \alpha_2, \alpha_i\}$ satisfies the relation (3.2), we obtain $\alpha_i=2\alpha_1-\alpha_2$ or $2\alpha_2-\alpha_1$ by Lemma 3.4. Hence we have $\alpha_i=2\alpha_2-\alpha_1(\alpha_i\neq\alpha_3)$.

Next, we assume that $\{\alpha_1, \alpha_2, \alpha_3\}$ is type (2). Then by Lemma 3.4 we can put $\alpha_3=3\alpha_1-2\alpha_2$. Therefore $\alpha_1=(2\alpha_2+\alpha_3)/3$. Since $\{\alpha_i, \alpha_2, \alpha_3\}$ satisfies (3.2), Lemma 3.4 implies that $\alpha_i=(2\alpha_3+\alpha_2)/3$ $(\alpha_i\neq\alpha_1)$. Hence we have

$$\alpha_2=2\alpha_1-\alpha_i,\qquad \alpha_3=2\alpha_i-\alpha_1$$

with $a_{\alpha_1,\alpha_i}=a_{\alpha_i,\alpha_1}=1$. We have thus proved the proposition. □

5. $\mathrm{rk}(G)=\mathrm{rk}(K)$

We devote this section to prove Theorem B.

Let G be a compact simple Lie group and K a closed subgroup of G. Then the following lemma plays an important part to prove the theorem.

LEMMA 5.1. *Let G/K be a normal homogeneous space as in section 2. If there exists a subspace V of $\mathfrak{p}$ satisfying* (1) $[V, \mathfrak{p}]_{\mathfrak{p}}\subset V$ *and* (2) $[\mathfrak{k}, V]\subset V$, *then*

$V=\{0\}$ *or* $V=\mathfrak{p}$.

PROOF. Put $\mathfrak{n}=V\oplus[V, V]_{\mathfrak{k}}$. We shall show that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$.

Let $u, v, w\in V$ and $x, y, z\in V^{\perp}$. Then if follows from (2.7) and assumption (1) that

$$\langle[u, x]_{\mathfrak{p}}, v\rangle=-\langle[u, v]_{\mathfrak{p}}, x\rangle=0.$$

Therefore we get

$$[V, V^{\perp}]_{\mathfrak{p}}=\{0\}. \tag{5.1}$$

Since $\langle\ ,\ \rangle$ is $\mathrm{Ad}(K)$-invariant, we have

$$[\mathfrak{k}, V^{\perp}]\subset V^{\perp} \tag{5.2}$$

from assumption (2). Therefore (5.1) and (5.2) give

$$[x, [y, u]],\qquad [y, [u, x]]\in V^{\perp}.$$

Similarly, (5.1) and assumption (2) give

$$[u, [x, y]]=[u, [x, y]_{\mathfrak{p}}]+[u, [x, y]_{\mathfrak{k}}]\in\mathfrak{k}\oplus V.$$

On the other hand, it follows from (5.1) and (5.2) that

$$[u, [x, y]]=-[x, [y, u]]-[y, [u, x]]\in V^{\perp}.$$

Hence

$$[V, [V^{\perp}, V^{\perp}]]=\{0\}. \tag{5.3}$$

Similarly we have

$$[V^{\perp}, [V, V]]=\{0\}. \tag{5.4}$$

For $A\in\mathfrak{k}$, we get

$$[A, [u, x]]=-[u, [x, A]]-[x, [A, u]]\in[V, V^{\perp}],$$

that is,

$$[\mathfrak{k}, [V, V^{\perp}]]\subset[V, V^{\perp}]. \tag{5.5}$$

From (5.4), we obtain

$$\begin{aligned}\langle[[u, x], v], w\rangle&=-\langle[[u, x], w], v\rangle\\&=\langle[[x, w], u], v\rangle\\&=-\langle[[x, w], v], u\rangle\\&=\langle[[v, x], w], u\rangle\\&=-\langle[[v, x], u], w\rangle\\&=-\langle[[u, x], v], w\rangle.\end{aligned}$$

Therefore we get

$$[[V, V^{\perp}], V] = \{0\}. \tag{5.6}$$

Similarly

$$[[V, V^{\perp}], V^{\perp}] = \{0\}. \tag{5.7}$$

It follows from (5.5), (5.6) and (5.7) that $[V, V^{\perp}]$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$. Since $\mathfrak{g}$ is simple, it follows that $[V, V^{\perp}]=\{0\}$.

For $A\in\mathfrak{k}$, we have

$$\begin{aligned}[A, [u, v]_{\mathfrak{k}}] &= [A, [u, v]]-[A, [u, v]_{\mathfrak{p}}] \\ &= -[u, [v, A]]-[v, [A, u]]-[A, [u, v]_{\mathfrak{p}}] \\ &\in \mathfrak{n}\end{aligned}$$

from the assumptions. For this reason, we conclude that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ and $V=\{0\}$ or $\mathfrak{p}$. □

COROLLARY 5.2. *If a subspace* $V=\{x\in\mathfrak{p}: [x, \mathfrak{p}]_{\mathfrak{p}}=\{0\}\}$ *is not zero, then* $V=\mathfrak{p}$ *and the pair* $(\mathfrak{g}, \mathfrak{k})$ *is symmetric.*

PROOF. For $A\in\mathfrak{k}$, $x\in V$ and $u\in\mathfrak{p}$, we have

$$\begin{aligned}[[A, x], u]_{\mathfrak{p}} &= -[[x, u], A]_{\mathfrak{p}}-[[u, A], x]_{\mathfrak{p}} \\ &= -[[x, u]_{\mathfrak{p}}, A]_{\mathfrak{p}} = 0.\end{aligned}$$

Hence $[\mathfrak{k}, V]\subset V$. Then from Lemma 5.1, we see $V=\mathfrak{p}$. Consequently $[\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{k}$ and $(\mathfrak{g}, \mathfrak{k})$ is symmetric. □

Now, we prove the theorem.

Throughout the remaining part of this section, we suppose that a hyperplane V of $\mathfrak{p}$ satisfies (2.10) and $\nu(\neq 0)$ is a vector of $\mathfrak{p}$ perpendicular to V. Then $R(x, \nu)x$ is parallel to ν for $x\in V$. Note that $\mathrm{Ad}(k)(V)(k\in K)$ satisfies (2.10).

At first, we suppose $\nu\in\mathfrak{g}_{\alpha}$ for some $\alpha\in\Delta_{\mathfrak{p}}$. Considering the isotropy representation $\mathrm{Ad}(\exp\sqrt{-1}\mathfrak{h}_{R})$, we may assume that ν is parallel to B_{α} (then $A_{\alpha}\in V$).

For each $\beta\in\Delta_{\mathfrak{p}}(\beta\neq\pm\alpha)$, a vector $R(A_{\alpha}, B_{\alpha})A_{\beta}$ is parallel to B_{β} from (2.10). It follows from (2.8) and (4.2) that

$$\begin{aligned}R(B_{\alpha}, A_{\alpha})B_{\alpha} &= -[B_{\alpha}, [B_{\alpha}, A_{\alpha}]] \\ &= 2\alpha(H_{\alpha})A_{\alpha}.\end{aligned} \tag{5.8}$$

Then (5.8) gives

$$\langle R(A_{\alpha}, B_{\alpha})A_{\beta}, B_{\alpha}\rangle = \langle R(B_{\alpha}, A_{\alpha})B_{\alpha}, A_{\beta}\rangle = 0.$$

Hence we obtain $R(A_\alpha, B_\alpha)A_\beta=0$, i.e.

$$[[A_\alpha, B_\alpha], A_\beta]-\frac{1}{4}[A_\alpha, [B_\alpha, A_\beta]_{\mathfrak{p}}]_{\mathfrak{p}}+\frac{1}{4}[B_\alpha, [A_\alpha, A_\beta]_{\mathfrak{p}}]_{\mathfrak{p}}=0. \tag{5.9}$$

From Corollary 5.2, we consider the case $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]_{\mathfrak{p}}\neq\{0\}$. For this reason, we check the following three cases:

(1) $\alpha+\beta\in\Delta_{\mathfrak{p}}$, $\alpha-\beta\notin\Delta_{\mathfrak{p}}$
(2) $\alpha-\beta\in\Delta_{\mathfrak{p}}$, $\alpha+\beta\notin\Delta_{\mathfrak{p}}$
(3) $\alpha\pm\beta\in\Delta_{\mathfrak{p}}$

(1) $\alpha+\beta\in\Delta_{\mathfrak{p}}$, $\alpha-\beta\notin\Delta_{\mathfrak{p}}$.
By computing the coefficient of B_β in (5.9), we have

$$2\beta(H_\alpha)=\frac{1}{2}N_{\alpha,\beta}N_{\alpha,-(\alpha+\beta)}.$$

Therefore by (2.3) and (2.4), we get

$$2\beta(H_\alpha)=-\frac{1}{2}(N_{\alpha,\beta})^2=-\frac{q(1-p)}{4}\alpha(H_\alpha)$$

where $\{\beta+n\alpha: p\leqq n\leqq q\}$ is the α-series containing β. Then it follows from (2.5) that

$$p+q=\frac{q(1-p)}{4}. \tag{5.10}$$

Since $\alpha+\beta\in\Delta_{\mathfrak{p}}$, we get $q\geqq 1$. Furthermore, by lemma 3.1, we obtain $1\leqq q\leqq 3$ $(p\leqq 0)$. Hence the only pair (p, q) satisfying the equation (5.10) is $(-1, 2)$.

(2) $\alpha-\beta\in\Delta_{\mathfrak{p}}$, $\alpha+\beta\notin\Delta_{\mathfrak{p}}$.
By the same computation as in (1), we get

$$2\beta(H_\alpha)=\frac{1}{2}(N_{\alpha,-\beta})^2.$$

Therefore

$$p+q=\frac{p(1+q)}{4}. \tag{5.11}$$

In this case we have $-3\leqq p\leqq -1$. Consequently (5.11) implies

$$(p, q)=(-2, 1).$$

(3) $\alpha\pm\beta\in\Delta_{\mathfrak{p}}$.
Once more, by making use of the same method as in the above, we have

$$p+q=\frac{1}{4}\{q(1-p)+p(1+q)\},$$

that is, $p+q=0$. Therefore, by Lemma 3.1, we have $(p, q)=(-1, 1)$. Then the α-series containing $\alpha+\beta$ is

$$\{\beta-\alpha=(\alpha+\beta)-2\alpha,\ \beta=(\alpha+\beta)-\beta,\ \beta+\alpha\}$$

However, from the above argument, this implies $R(A_\alpha, B_\alpha)A_{\alpha+\beta}\neq 0(\alpha+\beta\in\Delta_\mathfrak{p})$. This contradicts (5.9).

We have thus the following lemma.

LEMMA 5.3. *Suppose that* $\nu\in\mathfrak{g}_\alpha(\alpha\in\Delta_\mathfrak{p})$ *and* $\nu^\perp$ *satisfies* (2.10). *Then each* $\beta\in\Delta_\mathfrak{p}(\beta\neq\pm\alpha)$ *satisfies one of the following conditions.*

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\subset\mathfrak{k}$.

(2) *The* α*-series containing* β *is* $\{\beta-\alpha, \beta, \beta+\alpha, \beta+2\alpha\}$ *with* $\alpha+\beta\in\Delta_\mathfrak{p}$ *and* $\beta-\alpha\in\Delta_\mathfrak{k}$.

(3) *The* α*-series containing* β *is* $\{\beta-2\alpha, \beta-\alpha, \beta, \beta+\alpha\}$ *with* $\alpha+\beta\in\Delta_\mathfrak{k}$ *and* $\beta-\alpha\in\Delta_\mathfrak{p}$.

REMARK 5.4. Put $\gamma=-\beta$ in (3) of Lemma 5.3. Then, since $\mathfrak{g}_\beta=\mathfrak{g}_\gamma$, (3) is reduced to (2). Moreover, from Lemma 3.1, the Lie algebra $\mathfrak{g}$ is necessary of type G_2 in case (2).

Next, we examine the case $\nu\in\mathfrak{g}_{\alpha_1}\oplus\cdots\oplus\mathfrak{g}_{\alpha_m}(m\geqq 2)$. Take α, $\beta(\alpha\neq\pm\beta)$ out of $\{\alpha_1, \cdots, \alpha_m\}$ and suppose A_α is perpendicular to ν as in the case $m=1$. Moreover, using the isotropy representation $\mathrm{Ad}(\exp t\sqrt{-1}H)(\sqrt{-1}H\perp\sqrt{-1}H_\alpha$, $H\in\mathfrak{h}_1)$, we assume $B_\beta\perp\nu$. Since $R(A_\alpha, B_\alpha)A_\alpha=2\alpha(H_\alpha)B_\alpha$ (as (5.8)), we have

$$R(A_\alpha, \nu)A_\alpha=2\alpha(H_\alpha)\nu. \tag{5.12}$$

Then the coefficient of A_β in $R(A_\alpha, A_\beta)A_\alpha$ must be equal to $2\alpha(H_\alpha)$. In fact, if not, then

$$2\alpha+\beta \quad\text{or}\quad 2\alpha-\beta\in\{\alpha_1, \cdots, \alpha_m\}$$

because $R(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)\mathfrak{g}_\alpha\subset\mathfrak{g}_\beta\oplus\mathfrak{g}_{2\alpha\pm\beta}$. However, it is obvious that $[\mathfrak{g}_\beta, \mathfrak{g}_{2\alpha\pm\beta}]=\{0\}$. Hence $R(B_\beta, \nu)B_\beta=0$. This contradicts (5.12) (substitute B_β for A_α).

Let $\{\beta+n\alpha\colon p\leqq n\leqq q\}$ be the α-series containing β as before. Then $0\leqq q-p\leqq 3$ $(p\leqq 0, q\geqq 0)$ by Lemma 3.1. We shall compute the coefficient c_β of A_β in $R(A_\alpha, A_\beta)A_\alpha$. Since

$$R(A_\alpha, A_\beta)A_\alpha=-[A_\alpha, [A_\alpha, A_\beta]_\mathfrak{k}]-\frac{1}{4}[A_\alpha, [A_\alpha, A_\beta]_\mathfrak{p}]_\mathfrak{p},$$

it follows from (4.2) that

$$c_\beta=\begin{cases}(N_{\alpha,\beta})^2+(N_{\alpha,-\beta})^2, & (\alpha\pm\beta\notin\Delta_\mathfrak{p})\\ (N_{\alpha,\beta})^2+\frac{1}{4}(N_{\alpha,-\beta})^2, & (\alpha+\beta\notin\Delta_\mathfrak{p},\ \alpha-\beta\notin\Delta_\mathfrak{k})\\ \frac{1}{4}(N_{\alpha,\beta})^2+(N_{\alpha,-\beta})^2, & (\alpha+\beta\notin\Delta_\mathfrak{k},\ \alpha-\beta\notin\Delta_\mathfrak{p})\\ \frac{1}{4}\{(N_{\alpha,\beta})^2+(N_{\alpha,-\beta})^2\}, & (\alpha\pm\beta\notin\Delta_\mathfrak{k}).\end{cases}$$

Hence, we can easily verify that the pair (p, q) satisfying the equation $c_\beta = 2\alpha(H_\alpha)$ is one of the following:

(i) $(p, q)=(-1, 1)$ with $\alpha\pm\beta\in\Delta_{\mathfrak{k}}$.

(ii) $(p, q)=(-2, 1)$ with $\alpha+\beta\in\Delta_{\mathfrak{k}}$, $\alpha-\beta\in\Delta_{\mathfrak{p}}$.

(iii) $(p, q)=(-1, 2)$ with $\alpha+\beta\in\Delta_{\mathfrak{p}}$, $\alpha-\beta\in\Delta_{\mathfrak{k}}$.

REMARK 5.5. We do not distinguish between (ii) and (iii) for the reason mentioned in Remark 5.4. Moreover, the case (2) in Lemma 5.3 is reduced to the case (ii) in the above. In fact, take $\hat{\nu}=\mathrm{Ad}(\exp\mathfrak{g}_{\alpha-\beta})(\nu)$ instead of ν in (2), then

$$\hat{\nu}\not\perp\mathfrak{g}_\alpha \quad\text{and}\quad \hat{\nu}\not\perp\mathfrak{g}_\beta .$$

LEMMA 5.6. *In the case* (ii), *a normal homogeneous space* G/K (*G is necessarily of type* G_2) *has constant sectional curvature.*

PROOF. By a similar argument under (5.12), we see that $\beta-2\alpha\in\Delta_{\mathfrak{k}}$. Using the same method as in the above, we see that the β-series containing α is $\{\alpha-2\beta, \alpha-\beta, \alpha, \alpha+\beta\}$.

Let $\{\gamma_1, \gamma_2\}$ be the set of simple roots of type G_2 with the heighest root $2\gamma_1+3\gamma_2$. Then, by a straightforward computation, it can be checked that $\{\alpha, \beta\}$ is one of the following:

$$\pm\{\gamma_2, -(\gamma_1+\gamma_2)\}, \quad \pm\{\gamma_2, \gamma_1+2\gamma_2\}, \quad \pm\{\gamma_1+\gamma_2, \gamma_1+2\gamma_2\}.$$

Therefore we have

$$\mathfrak{p} = \mathfrak{g}_{\gamma_2}\oplus\mathfrak{g}_{\gamma_1+\gamma_2}\oplus\mathfrak{g}_{\gamma_1+2\gamma_2}$$

for any $\{\alpha, \beta\}$. Then, as is well-known, G/K has constant sectional curvature (cf. [**1**]). □

From Lemma 5.6, we may assume that for any $\alpha, \beta\in\{\alpha_1, \cdots, \alpha_m\}$ $(\nu\in\mathfrak{g}_{\alpha_1}\oplus\cdots\oplus\mathfrak{g}_{\alpha_m})$ the α-series containing β is $\{\beta-\alpha, \beta, \beta+\alpha\}$ with $\alpha\pm\beta\in\Delta_{\mathfrak{k}}$. Then

LEMMA 5.7. *For* $\alpha, \beta, \gamma\in\{\alpha_1, \cdots, \alpha_m\}$ *with* $\alpha\neq\pm\beta$, $\beta\neq\pm\gamma$ *and* $\gamma\neq\pm\alpha$, *the following equations hold.*

$$N_{\gamma,\alpha+\beta} = N_{\gamma,-(\alpha+\beta)} = N_{\gamma,\alpha-\beta} = N_{\gamma,-\alpha+\beta} = 0 \tag{5.13}$$

PROOF. As before, we assume $A_\alpha, B_\beta\perp\nu$. Set $\nu_{|\mathfrak{g}_\gamma} = aA_\gamma+bB_\gamma (a, b\in\boldsymbol{R})$. Since

$$\alpha_i\pm\alpha_j \in \Delta_{\mathfrak{k}} (i\neq j,\ i, j = 1, \cdots, m),$$

it follows that

$$\begin{aligned} R(A_\alpha, B_\beta)(aA_\gamma+bB_\gamma) &= [[A_\alpha, B_\beta], aA_\gamma+bB_\gamma] \\ &= [N_{\alpha,\beta}B_{\alpha+\beta}+N_{\alpha,-\beta}B_{\alpha-\beta}, aA_\gamma+bB_\gamma] \\ &= N_{\alpha,\beta}N_{\alpha+\beta,\gamma}(-bA_{\alpha+\beta+\gamma}+aB_{\alpha+\beta+\gamma}) \end{aligned} \tag{5.14}$$

$$-N_{\alpha,\beta}N_{\alpha+\beta,-\gamma}(bA_{\alpha+\beta-\gamma}+aB_{\alpha+\beta-\gamma})$$
$$+N_{\alpha,-\beta}N_{\alpha-\beta,\gamma}(-bA_{\alpha-\beta+\gamma}+aB_{\alpha-\beta+\gamma})$$
$$-N_{\alpha,-\beta}N_{\alpha-\beta,-\gamma}(bA_{\alpha-\beta-\gamma}+aB_{\alpha-\beta-\gamma}).$$

In particular, we note that $R(A_\alpha, B_\beta)(aA_\gamma+bB_\gamma)\in\mathfrak{g}_{\alpha+\beta\pm\gamma}\oplus\mathfrak{g}_{\alpha-\beta\pm\gamma}$.

For $\delta\in\{\alpha_1, \cdots, \alpha_m\}(\delta\neq\pm\alpha, \pm\beta, \pm\gamma)$, we see that

$$R(A_\alpha, B_\beta)(aA_\gamma+bB_\gamma)\perp R(A_\alpha, B_\beta)\mathfrak{g}_\delta.$$

(For example, if $\alpha+\beta+\gamma=\alpha-\beta+\delta$, then $\delta-\gamma=2\beta\notin\Delta$. This contradicts $\delta-\gamma\in\Delta_{\mathfrak{k}}$). Hence $R(A_\alpha, B_\beta)(aA_\gamma+bB_\gamma)=0$ by (2.10) or equivalently by $R(V, V)\nu=0$. Consequently, (5.13) is derived from (5.14). □

Obviously, an equation $N_{\alpha,\beta}=0$ is equivalent to a relation $\alpha+\beta\notin\Delta$. Therefore (5.13) is equivalent to the following:

$$\alpha\pm\beta\pm\gamma\notin\Delta. \tag{5.15}$$

Put $\nu=a_\beta A_\beta+a_\alpha A_\alpha+\sum_{\alpha_i\neq\alpha,\beta}X_{\alpha_i}$, $(X_{\alpha_i}\in\mathfrak{g}_{\alpha_i})$. Then it follows from (5.15) that $[\mathfrak{g}_{\alpha\pm\beta}, \sum_{\alpha_i\neq\alpha,\beta}X_{\alpha_i}]=\{0\}$. Therefore it is easy to see that

$$\begin{aligned}\nu_t &= \mathrm{Ad}(\exp tB_{\alpha+\beta})(\nu)\\ &= \sum_{\alpha_i\neq\alpha,\beta}X_{\alpha_i}+\cos N_{\alpha,\beta}t\cdot(a_\beta A_\beta+a_\alpha B_\alpha)\\ &\quad -\sin N_{\alpha,\beta}t\cdot(a_\alpha A_\beta-a_\beta B_\alpha).\end{aligned}$$

Let t_0 be a real number such that $a_\beta\cos N_{\alpha,\beta}t_0=a_\alpha\sin N_{\alpha,\beta}t_0$. Then ν_{t_0} is perpendicular to $\mathfrak{g}_\beta$. Thus, by induction, we can see that for some $\alpha\in\Delta_{\mathfrak{p}}$ there exists a vector $\hat{\nu}\in\mathfrak{g}_\alpha$ such that $\hat{\nu}^\perp$ satisfies (2.10). Therefore, for each $\beta\in\Delta_{\mathfrak{p}}$, one of (1), (2) and (3) of Lemma 5.3 holds.

If there is $\beta\in\Delta_{\mathfrak{p}}$ satisfying (2) (or (3)) in Lemma 5.3, then Remark 5.5 and Lemma 5.6 imply that G/K has constant sectional curvature. Thus we may assume that (1) of Lemma 5.3 holds for any $\beta\in\Delta_{\mathfrak{p}}$. Then, by Corollary 5.2, the pair $(\mathfrak{g}, \mathfrak{k})$ is symmetric and G/K has constant sectional curvature.

We have thus proved Theorem B.

6. Some Lemmas.

In this section we give some lemmas which help us to prove the theorem.

From now on, we assume that $\mathrm{rk}(G)>\mathrm{rk}(K)$, unless otherwise stated. Let ν be a vector in $\mathfrak{p}$ such that it satifies condition (T-G).

If $\nu\perp\sqrt{-1}\mathfrak{h}_2$ and $\nu\not\perp\mathfrak{g}_\alpha(\alpha\in\Delta')$, then there exists $\sqrt{-1}H\in\sqrt{-1}\mathfrak{h}_2$ such that $[\sqrt{-1}H, \nu]\neq0$. Therefore by Lemma 2.2, there exists $\hat{\nu}\in\mathfrak{p}$ such that $\hat{\nu}$ satisfies condition (T-G) and $\hat{\nu}\not\perp\sqrt{-1}\mathfrak{h}_2$. (Take $\mathrm{Ad}(\exp t[X, \nu]_{\mathfrak{k}})(\nu)$ or $e^{-\varphi_t[X,\nu]_{\mathfrak{p}}}(\nu)$.)

Similarly, if $\nu \not\perp \mathfrak{g}_\beta$ for some $\beta \in \Delta_\mathfrak{p}$ with $\beta(\mathfrak{h}_2) \neq 0$, then there exists $\hat{\nu} \in \mathfrak{p}$ such that it satisfies condition (T-G) and $\hat{\nu} \not\perp \sqrt{-1}\mathfrak{h}_2$. Hence we have the following.

LEMMA 6.1. *If there is a vector* ν *in* $\mathfrak{p}$ *satisfying condition (T-G), then there exists* $\hat{\nu} \in \mathfrak{p}$ *satisfying condition (T-G) such that*

$$\hat{\nu} \not\perp \sqrt{-1}\mathfrak{h}_2 \quad or \quad \hat{\nu} \in \bigoplus_{\substack{\alpha \in \Delta_\mathfrak{p} \\ \alpha(\mathfrak{h}_2)=0}} \mathfrak{g}_\alpha .$$

From the above lemma, we suppose $\nu \not\perp \sqrt{-1}\mathfrak{h}_2$ or $\nu \in \bigoplus_{\alpha, \alpha(\mathfrak{h}_2)=0} \mathfrak{g}_\alpha$. In the case where $\nu \not\perp \sqrt{-1}\mathfrak{h}_2$, we put $\nu_{|\sqrt{-1}\mathfrak{h}_2} = \sqrt{-1}H_0$. Then the following holds.

LEMMA 6.2. *If* $\sqrt{-1}H_\alpha\,(\alpha \in \Delta_\mathfrak{p})$ *is not perpendicular to* $\sqrt{-1}H_0$, *then* $\sqrt{-1}H_\alpha$ *is parallel to* $\sqrt{-1}H_0$.

PROOF. Even if $\nu \perp \mathfrak{g}_\alpha$, there is a number t such that $e^{-\varphi t A_\alpha}(\nu) \not\perp \mathfrak{g}_\alpha$. Therefore we may assume $\nu \not\perp \mathfrak{g}_\alpha$ and $\nu_{|\mathfrak{g}_\alpha}$ is parallel to B_α.

For any $\sqrt{-1}H \in \sqrt{-1}\mathfrak{h}_2$ which is perpendicular to $\sqrt{-1}H_0$, we get

$$\begin{aligned}\langle R(\sqrt{-1}H, \nu)\sqrt{-1}H, \sqrt{-1}H_0\rangle &= \langle R(\sqrt{-1}H, \sqrt{-1}H_0)\sqrt{-1}H, \nu\rangle \\ &= 0 .\end{aligned}$$

Hence $R(-1H, \nu)\sqrt{-1}H=0$ from (2.10). Therefore

$$R(\sqrt{-1}H, B_\alpha)\sqrt{-1}H = 0 ,$$

i.e. $\alpha(H)=0$. This implies that $(\sqrt{-1}H_\alpha)_\mathfrak{p}$ is parallel to $\sqrt{-1}H_0$. Put

$$(\sqrt{-1}H_\alpha)_\mathfrak{p} = c\sqrt{-1}H_0 .$$

Then

$$\begin{aligned}\langle \sqrt{-1}H_\alpha - c\sqrt{-1}H_0, \sqrt{-1}H_0\rangle &= \alpha(H_0) - c\|\sqrt{-1}H_0\|^2 \\ &= 0 .\end{aligned} \tag{6.1}$$

Since

$$\begin{aligned}R(A_\alpha, B_\alpha)A_\alpha &= -[A_\alpha, [A_\alpha, B_\alpha]_\mathfrak{k}] - \frac{1}{4}[A_\alpha, [A_\alpha, B_\alpha]_\mathfrak{p}]_\mathfrak{p} \\ &= 2\sqrt{-1}[H_\alpha - cH_0, A_\alpha] + \frac{c\sqrt{-1}}{2}[H_0, A_\alpha] \\ &= \left\{2\alpha(H_\alpha) - \frac{3c\alpha(H_0)}{2}\right\} B_\alpha\end{aligned}$$

and

$$\begin{aligned}R(A_\alpha, \sqrt{-1}H_0)A_\alpha &= -\frac{1}{4}[A_\alpha, [A_\alpha, \sqrt{-1}H_0]]_\mathfrak{p} \\ &= \frac{c\alpha(H_0)}{2}\sqrt{-1}H_0 ,\end{aligned}$$

it follows that

$$\alpha(H_\alpha) = c\alpha(H_0). \tag{6.2}$$

From (6.1) and (6.2), we obtain $\alpha(H_\alpha)=c^2\|\sqrt{-1}H_0\|^2$, that is,

$$\|\sqrt{-1}H_\alpha\|^2 = \|(\sqrt{-1}H_\alpha)_{\mathfrak{p}}\|^2.$$

As a result, a vector $\sqrt{-1}H_\alpha$ is parallel to $\sqrt{-1}H_0$. □

LEMMA 6.3. *If ν is not perpendicular to $\mathfrak{g}_\alpha$ for some $\alpha\in\Delta_{\mathfrak{p}}$ satisfying $\alpha(\mathfrak{h}_2)=0$, then for any $\beta\in\Delta'$ and for any $\gamma\in\Delta_{\mathfrak{p}}$ satisfying $\gamma(\mathfrak{h}_2)\neq 0$, the following holds.*

$$[\mathfrak{g}_\alpha, \mathfrak{g}(\beta)] = \{0\} \quad \textit{and} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\gamma] = \{0\}$$

PROOF. At first, we note that $\nu\in\bigoplus_{\lambda,\lambda(\mathfrak{h}_2)=0}\mathfrak{g}_\lambda$. Let X be a vector in $\mathfrak{g}_\alpha$ such that it is perpendicular to ν. For some $Y\in\mathfrak{g}_\gamma$, if $[Y, \nu]$ is not zero, then by the same method as in the early part of this section, we see that there exists $\hat{\nu}$ satisfying condition (T-G) such that $\hat{\nu}\not\perp\sqrt{-1}\mathfrak{h}_2$ and $\hat{\nu}\not\perp\mathfrak{g}_\alpha$. Since $[\mathfrak{g}_\alpha, \sqrt{-1}\mathfrak{h}_2]=\{0\}$, we get

$$R(\hat{X}, \hat{\nu})\hat{X} = 0 \ (\hat{X}\in\mathfrak{g}_\alpha, \ \hat{X}\perp\hat{\nu}).$$

However, this contradicts the equation

$$R(\hat{X}, \hat{\nu}_{|\mathfrak{g}_\alpha})\hat{X} = 2\alpha(H_\alpha)\hat{\nu}_{|\mathfrak{g}_\alpha}.$$

Therefore we obtain $[Y, \nu]=0$ for any $Y\in\mathfrak{g}_\gamma$.

If $[\mathfrak{g}_\alpha, \mathfrak{g}_\gamma]\neq\{0\}$, then there is a root $\delta(\delta\neq\pm\alpha)$ in $\Delta_{\mathfrak{p}}$ such that $\nu\not\perp\mathfrak{g}_\delta$ and $[\mathfrak{g}_\gamma, \mathfrak{g}_\alpha]\not\perp[\mathfrak{g}_\gamma, \mathfrak{g}_\delta]$. Therefore by (4.1), a root δ is one of $\{\pm2\gamma\pm\alpha\}$. Then for any δ, we have $[\mathfrak{g}_\delta, \mathfrak{g}_\alpha]=\{0\}$. However, by the argument under (5.12), we can see that this contradicts the assumption $\nu\not\perp\mathfrak{g}_\alpha$ and $\nu\not\perp\mathfrak{g}_\delta$.

Similarly we can get $[\mathfrak{g}_\alpha, \mathfrak{g}(\beta)]=\{0\}$. We have thus proved the lemma. □

In the following sections we shall prove Theorem A.

7. Type I, II.

In this section we suppose that $\mathfrak{g}$ is not of type G_2 and $\mathrm{rk}(G)>\mathrm{rk}(K)$.

At first, we consider the case $\nu\not\perp\sqrt{-1}\mathfrak{h}_2$. Let α be a root in Δ' such that $\mathfrak{g}(\alpha)_{|\mathfrak{p}}\not\perp\nu$. Put $\mathfrak{g}(\alpha)=\mathfrak{g}_{\alpha_1}\oplus\cdots\oplus\mathfrak{g}_{\alpha_m}(m\geqq 2)$ and $\nu_{|\sqrt{-1}\mathfrak{h}_2}=\sqrt{-1}H_0$ as in the previous section. Then for any $\sqrt{-1}H\in\sqrt{-1}\mathfrak{h}_2$ with $\sqrt{-1}H\perp\sqrt{-1}H_0$, we have $R(\sqrt{-1}H, \nu)\sqrt{-1}H=0$, that is,

$$[\sqrt{-1}H, \nu] = 0. \tag{7.1}$$

If $\nu\not\perp\mathfrak{g}_{\alpha_i}$ and $\nu\not\perp\mathfrak{g}_{\alpha_j}$ $(1\leqq i\neq j\leqq m)$, then (7.1) gives $\alpha_i(H)=\alpha_j(H)=0$. Hence $\sqrt{-1}(H_{\alpha_i}-H_{\alpha_j})$ is parallel to $\sqrt{-1}H_0$. Then if follows from Lemmas 3.2, 3.3

that $\mathfrak{g}(\alpha)$ has one of the following two forms:

(1) $\mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\delta_1} \oplus \cdots \oplus \mathfrak{g}_{\delta_{m-2}}$ with $\nu \perp \mathfrak{g}_{\delta_i} (i=1, \cdots, m-2)$. Moreover $\sqrt{-1}H_0$ is parallel to $\sqrt{-1}(H_\beta - H_\gamma)$.

(2) $\mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_\delta \oplus \mathfrak{g}_{\omega_1} \oplus \cdots \oplus \mathfrak{g}_{\omega_{m-3}}$ with $\nu \perp \mathfrak{g}_{\omega_i} (i=1, \cdots, m-3)$, $a_{\beta,\gamma}=1$, $a_{\gamma,\beta}=2$ and $\delta=2\beta-\gamma$. Moreover $\sqrt{-1}H_0$ is parallel to $\sqrt{-1}(H_\beta - H_\gamma)$.

We shall examine $\mathfrak{g}(\alpha)$, more precisely.

Throughout this section, we fix the notation as follows:

$$\mathfrak{g}(\alpha) = \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_m}$$

$$\mathfrak{g}(\alpha)_{|\mathfrak{t}} = \boldsymbol{R}X \oplus \boldsymbol{R}X' \quad \text{with} \quad X = \sum_{i=1}^m p_{\alpha_i} X_{\alpha_i}, \quad X' = \sum_{i=1}^m p_{\alpha_i} X'_{\alpha_i}$$

where $X_{\alpha_i} = a_{\alpha_i} A_{\alpha_i} + b_{\alpha_i} B_{\alpha_i}$ and $X'_{\alpha_i} = a_{\alpha_i} B_{\alpha_i} - b_{\alpha_i} A_{\alpha_i} (a_{\alpha_i}^2 + b_{\alpha_i}^2 = 1)$.

THE CASE (1).

If $m=3$, then for $\sqrt{-1}H \in \sqrt{-1}\mathfrak{h}_2$ perpendicular to $\sqrt{-1}H_0$, we have

$$[\sqrt{-1}H, \mathfrak{g}(\alpha)_{|\mathfrak{t}}] \subset \mathfrak{g}_{\delta_1}.$$

However, this contradicts the assumption $\delta_1 \in \Delta'$.

Next, we suppose $m \geqq 4$. For $U \in \mathfrak{k}$, we set $\nu_t = \mathrm{Ad}(\exp tU)(\nu)$. Then for any small $t \in \boldsymbol{R}$, we see that

$$\nu_t \not\perp \mathfrak{g}_\beta \quad \text{and} \quad \nu_t \not\perp \mathfrak{g}_\gamma.$$

Therefore, by the same argument as in the above, we see that $\nu \perp \mathfrak{g}_{\delta_i} (i=1, \cdots, m-2)$ and $(\nu_t)_{|\sqrt{-1}\mathfrak{h}_2}$ is parallel to $\sqrt{-1}H_0$. Hence we have

$$[\mathfrak{k}, \nu] \perp \mathfrak{g}_{\delta_i} (i=1, \cdots, m-2). \tag{7.2}$$

Since

$$X' = \sum_{i=1}^m p_{\alpha_i} X'_{\alpha_i} \in \mathfrak{k},$$

a vector $Y = p_{\delta_j} X'_{\delta_i} - p_{\delta_i} X'_{\delta_j}$ is in $\mathfrak{p}$. We assume that ν is perpendicular to Y. Take $H \in \mathfrak{h}_2$ so that $(\delta_i - \delta_j)(H) \neq 0$ and $\sqrt{-1}H \perp \sqrt{-1}H_0$. Then, since $[\nu, \sqrt{-1}H]=0$, we get

$$\begin{aligned} 0 &= R(\sqrt{-1}H, Y)\nu \\ &= [[\sqrt{-1}H, Y]_\mathfrak{t}, \nu] + \frac{1}{2}[[\sqrt{-1}H, Y]_\mathfrak{p}, \nu]_\mathfrak{p} - \frac{1}{4}[\sqrt{-1}H, [Y, \nu]_\mathfrak{p}]_\mathfrak{p} \\ &= \frac{1}{2}[[\sqrt{-1}H, Y]_\mathfrak{t}, \nu] + \frac{1}{2}[[\sqrt{-1}H, Y], \nu]_\mathfrak{p} \\ &\quad + \frac{1}{4}[\sqrt{-1}H, [Y, \nu]_\mathfrak{t}] - \frac{1}{4}[\nu, [Y, \sqrt{-1}H]]_\mathfrak{p} \end{aligned}$$

$$=\frac{1}{2}[[\sqrt{-1}H, Y]_{\mathfrak{k}}, \nu]+\frac{1}{4}[[\nu, Y], \sqrt{-1}H]_{\mathfrak{p}}+\frac{1}{4}[\sqrt{-1}H, [Y, \nu]_{\mathfrak{k}}].$$

Therefore, since $[\sqrt{-1}H, \sqrt{-1}H_0]=0$, it follows that

$$0=\langle R(\sqrt{-1}H, Y)\nu, \sqrt{-1}H_0\rangle = \left\langle\frac{1}{2}[[\sqrt{-1}H, Y]_{\mathfrak{k}}, \nu], \sqrt{-1}H_0\right\rangle. \tag{7.3}$$

On the other hand, we have

$$[\sqrt{-1}H, Y]=p_{\delta_i}\delta_j(H)X_{\delta_j}-p_{\delta_j}\delta_i(H)X_{\delta_i}$$

and

$$\langle[\sqrt{-1}H, Y], X\rangle=2p_{\delta_i}p_{\delta_j}(\delta_j-\delta_i)(H)\neq 0.$$

Therefore we get

$$\begin{aligned}&\langle\sqrt{-1}H_0, [[\sqrt{-1}H, Y]_{\mathfrak{k}}, \nu]\rangle\\ &=c\langle\sqrt{-1}H_0, [X, (p_\beta X'_\alpha-p_\alpha X'_\beta)+\cdots]\rangle\\ &=c\langle p_\beta X'_\alpha-p_\alpha X'_\beta, [\sqrt{-1}H_0, X]\rangle\\ &=c\langle p_\beta X'_\alpha-p_\alpha X'_\beta, p_\alpha\alpha(H_0)X'_\alpha+p_\beta\beta(H_0)X'_\beta\rangle\\ &=2cp_\alpha p_\beta(\alpha-\beta)(H_0). \quad (c \text{ is some non zero constant.})\end{aligned}$$

Since $\sqrt{-1}H_0$ is parallel to $\sqrt{-1}(H_\alpha-H_\beta)$, we obtain

$$\langle\sqrt{-1}H_0, [[\sqrt{-1}H, Y]_{\mathfrak{k}}, \nu]\rangle\neq 0.$$

However, this contradicts (7.3) and implies $m=2$, i.e.

$$\mathfrak{g}(\alpha)=\mathfrak{g}_\alpha\oplus\mathfrak{g}_\beta \quad \text{with some} \quad \beta\in\Delta'. \tag{7.4}$$

Now, we may assume

$$\mathfrak{g}(\alpha)_{\mathfrak{k}}=\boldsymbol{R}X\oplus\boldsymbol{R}X'$$

with $X=A_\alpha+pX_\beta$ for some $p(\neq 0)$. Furthermore we suppose that $\nu_{|\mathfrak{g}(\alpha)}$ is parallel to $pB_\alpha-X'_\beta(\in\mathfrak{p})$. Then Lemma 4.3 implies that

$$[A_\alpha, X'_\beta]+[X_\beta, B_\alpha]=0 \tag{7.5}$$

$$\sqrt{-1}H_\alpha+p^2\sqrt{-1}H_\beta\in\mathfrak{k}. \tag{7.6}$$

Since $\sqrt{-1}(H_\alpha-H_\beta)\in\mathfrak{p}$, (7.6) gives

$$c_\alpha+p^2c_\beta=0 \tag{7.7}$$

where $c_\alpha=\alpha(H_\alpha-H_\beta)$ and $c_\beta=\beta(H_\alpha-H_\beta)$.

It follows from (7.5), (7.6) and (7.7) that

$$R(pA_\alpha - X_\beta,\ pB_\alpha - X'_\beta)(pA_\alpha - X_\beta)$$
$$= -[pA_\alpha - X_\beta,\ 2\sqrt{-1}(p^2H_\alpha + H_\beta)_{\mathfrak{k}}] - \frac{1}{4}[pA_\alpha - X_\beta,\ 2\sqrt{-1}(p^2H_\alpha + H_\beta)_{\mathfrak{p}}]_{\mathfrak{p}}$$
$$= [2\sqrt{-1}(H_\alpha + p^2H_\beta),\ pA_\alpha - X_\beta] + \frac{1}{2}[(p^2-1)\sqrt{-1}(H_\alpha - H_\beta),\ pA_\alpha - X_\beta]_{\mathfrak{p}}$$
$$= 2p\alpha(H_\alpha + p^2H_\beta)B_\alpha - 2\beta(H_\alpha + p^2H_\beta)X'_\beta$$
$$+ \frac{p^2-1}{2}\{pc_\alpha B_\alpha - c_\beta X'_\beta\}_{\mathfrak{p}}.$$

Hence (7.7) gives

$$R(pA_\alpha - X_\beta,\ pB_\alpha - X'_\beta)(pA_\alpha - B_\beta) = \left\{2\alpha(H_\alpha + p^2H_\beta) + \frac{(p^2-1)(c_\alpha + c_\beta)}{2}\right\}(pB_\alpha - X'_\beta). \tag{7.8}$$

Similarly we have

$$\begin{aligned} &R(pA_\alpha - X_\beta,\ \sqrt{-1}(H_\alpha - H_\beta))(pA_\alpha - X_\beta) \\ &= \left\{2c_\alpha + \frac{(p^2-1)(c_\alpha + c_\beta)}{2}\right\}\sqrt{-1}(H_\alpha - H_\beta) \\ &+ \frac{c_\alpha}{p}(p^2[A_\alpha, X'_\beta] - [X_\beta, B_\alpha]). \end{aligned} \tag{7.9}$$

Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, if $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Delta_{\mathfrak{p}}$. Then by Lemma 6.2, a vector $\sqrt{-1}H_{\alpha+\beta}$ is perpendicular to $\sqrt{-1}(H_\alpha - H_\beta)$. This implies $\nu \perp \mathfrak{g}_{\alpha+\beta}$. For this reason, regardless of whether $\alpha+\beta \in \Delta$ or $\alpha+\beta \notin \Delta$, we have from (7.8) and (7.9) that

$$c_\alpha = \alpha(H_\alpha + p^2H_\beta),$$

that is,

$$\alpha(H_\beta) = 0. \tag{7.10}$$

Therefore it follows from (7.7) and (7.10) that

$$p^2 = \frac{\alpha(H_\alpha)}{\beta(H_\beta)}. \tag{7.11}$$

THE CASE (2).

In this case $\mathfrak{g}$ is necessarily of type II.

Let

$$\mathfrak{g}(\alpha) = \mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_\delta \oplus \mathfrak{g}_{\omega_1} \oplus \cdots \oplus \mathfrak{g}_{\omega_{m-3}}$$

with $\delta = 2\beta - \gamma$, $a_{\beta,\gamma} = 1$ and $a_{\gamma,\beta} = 2$. As in case (1), for each ω_i there exists $\sqrt{-1}H \in \sqrt{-1}\mathfrak{h}_2(\sqrt{-1}H \perp \sqrt{-1}H_0)$ such that $\omega_i(H) \neq 0$. At first we prove the following lemma.

LEMMA 7.1. *If $m\geqq 4$, then*

$$\omega_i(H_0)=0 \qquad \textit{for all} \quad i\,(i=1,\cdots,m-3).$$

PROOF. We note that $\beta(H_\beta-H_\gamma)=0$, i.e. $\sqrt{-1}H_\beta\in\mathfrak{k}$.

By making use of the description of the simple root systems in [4, p. 462–p. 474], we prove the lemma.

If $\mathfrak{g}$ is of type $B_l(l\geqq 2)$, then Δ can be identified with

$$\{\pm e_i(1\leqq i\leqq l)\,;\ \pm e_i\pm e_j(1\leqq i\neq j\leqq l)\} \tag{7.12}$$

where $\{e_1,\cdots,e_l\}$ is an orthonomal basis of $(\boldsymbol{R}^l,\langle\ ,\ \rangle_0)$. Moreover there exists a constant $c(\neq 0)$ such that

$$c\langle\alpha,\ \beta\rangle_0=\alpha(H_\beta) \tag{7.13}$$

for all $\alpha,\ \beta\in\Delta$. Since $a_{\beta,\gamma}=1$ and $a_{\gamma,\beta}=2$, we have $2\beta(H_\beta)=\gamma(H_\gamma)$. Hence, by (7.12), we may assume $\beta=e_i$ for some $i(1\leqq i\leqq l)$. Then $\{\gamma,\ \delta\}$ is necessarily equal to $\{e_i-e_j,\ e_i+e_j\}$ for some j.

Since $\sqrt{-1}H_\beta\in\mathfrak{k}$, we have

$$\langle\beta,\ \beta\rangle_0=\langle\beta,\ \omega_k\rangle_0$$

for each $k\,(1\leqq k\leqq m-3)$. Therefore $\omega_k=e_i+e_n$ or e_i-e_n for some $n\,(n\neq i,\ j,\ k)$. Then

$$\begin{aligned}\omega_k(H_\beta-H_\gamma)&=c\langle\omega_k,\ \beta-\gamma\rangle_0\\&=0.\end{aligned}$$

This proves the lemma in case where $\mathfrak{g}$ is of type B_l.

By the same way as in the above, we can prove the lemma for the other cases. □

By a hypothesis that $\sqrt{-1}\mathfrak{h}_1$ is a maximal abelian subalgebra of $\mathfrak{k}$, it is easy to see that $\beta-\gamma\in\Delta_{\mathfrak{p}}$. Then we assume $\nu_{|\mathfrak{g}_{\beta-\gamma}}\neq 0$(if $\nu_{|\mathfrak{g}_{\beta-\gamma}}=0$, then take $e^{-\varphi_t A_{\beta-\gamma}}(\nu)$ instead of ν) and $\nu_{\mathfrak{g}_{\beta-\gamma}}$ is parallel to $X_{\beta-\gamma}$. Then (4) of Lemma 4.2 gives

$$p_\gamma{}^2\gamma(H_\beta-H_\gamma)+p_\delta{}^2\delta(H_\beta-H_\gamma)=0.$$

Therefore, since $a_{\gamma,\beta}=2$ and $a_{\beta,\gamma}=1$, we obtain $p_\gamma{}^2=p_\delta{}^2$. Thus we may suppose $p_\gamma=p_\delta=1$. Then

$$X_\gamma-X_\delta,\qquad X'_\gamma-X'_\delta\in\mathfrak{p}$$

$$X_\gamma+X_\delta-\frac{2}{p_\beta}X_\beta,\qquad X'_\gamma+X'_\delta-\frac{2}{p_\beta}X'_\beta\in\mathfrak{p}.$$

Since $\nu\perp\mathfrak{g}_{\omega_i}(i=1,\cdots,m-3)$, then $\nu\in\mathfrak{g}_\beta\oplus\mathfrak{g}_\gamma\oplus\mathfrak{g}_\delta$. If $\nu\perp\mathfrak{g}_\beta$, then considering

$\mathrm{Ad}(\exp t\sqrt{-1}H_\beta)(\nu)$, we can assume $\nu_{\mathfrak{g}(\alpha)}$ is parallel to $(X_\gamma - X_\delta)$. Then since

$$
\begin{aligned}
& R(X'_{\beta-\gamma}, \sqrt{-1}(H_\beta - H_\gamma))X'_{\beta-\gamma} \\
&= -\frac{1}{4}[X'_{\beta-\gamma}, [X'_{\beta-\gamma}, \sqrt{-1}(H_\beta - H_\gamma)]] \\
&= \frac{\|\sqrt{-1}(H_\beta - H_\gamma)\|^2}{2} \sqrt{-1}(H_\beta - H_\gamma) \\
&\neq 0,
\end{aligned}
$$

we see that $[X'_{\beta-\gamma}, X_\gamma - X_\delta](\in \mathfrak{g}_\beta)$ is not zero. Hence we get

$$e^{-\varphi_t X'_{\beta-\gamma}}(\nu) \not\perp \mathfrak{g}_\beta$$

for small $t \in \boldsymbol{R}$. $(e^{-\varphi_t X'_{\beta-\gamma}}(\nu))_{|\sqrt{-1}\mathfrak{h}_2}$ is parallel to $\sqrt{-1}(H_\beta - H_\gamma)$). For this reason, we suppose $\nu \not\perp \mathfrak{g}_\beta$ and $\nu_{\mathfrak{g}_\beta}$ is parallel to X'_β. Set

$$Y = p_\beta X_{\omega_i} - p_{\omega_i} X_\beta (\perp \nu).$$

Then Lemma 7.1 implies that $[Y, \sqrt{-1}(H_\beta - H_\gamma)] = 0$. Hence we get

$$R(Y, \nu)Y = 0,$$

that is, $[Y, \nu] = 0$. On the other hand, we have

$$
\begin{aligned}
\langle [Y, \nu], \sqrt{-1}H_\beta \rangle &= \langle [\sqrt{-1}H_\beta, Y], \nu \rangle \\
&= \langle \omega_i(H_\beta) p_\beta X'_{\omega_i} - \beta(H_\beta) p_{\omega_i} X'_\beta, \nu \rangle \\
&= -\beta(H_\beta) p_{\omega_i} \langle X'_\beta, \nu \rangle \neq 0.
\end{aligned}
$$

This contradicts the fact $[Y, \nu] = 0$, and we have

$$\mathfrak{g}(\alpha) = \mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_\delta \tag{7.14}$$

with $a_{\beta,\gamma} = 1$, $a_{\gamma,\beta} = 2$ and $\delta = 2\beta - \gamma$.

In the following, we shall investigate the case (1), (2) in an early part of this section, more precisely.

THE CASE $\mathfrak{g}(\alpha) = \mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_\delta$.

From (4) of Lemma 4.2, we can set

$$\mathfrak{g}(\alpha)_t = \boldsymbol{R}(pA_\beta + X_\gamma + X_\delta) \oplus \boldsymbol{R}(pB_\beta + X'_\gamma + X'_\delta) \quad (p \neq 0)$$

where $X_\gamma = aA_\gamma + bB_\gamma$ and $X_\delta = a_\delta A_\delta + b_\delta B_\delta$ $(a^2 + b^2 = a_\delta^2 + b_\delta^2 = 1)$. Then Lemma 4.3 and (7.14) give

$$
\begin{aligned}
& [pA_\beta + X_\gamma + X_\delta, pB_\beta + X'_\gamma + X'_\delta] \\
&= 2\sqrt{-1}(p^2 H_\beta + H_\gamma + H_\delta) \\
&\quad + p([A_\beta, X'_\gamma + X'_\delta] - [B_\beta, X_\gamma + X_\delta])
\end{aligned}
$$

$$\in \sqrt{-1}\mathfrak{h}_2 .$$

Hence $[A_\beta, X'_\gamma+X'_\delta]=[B_\beta, X_\gamma+X_\delta]$, that is

$$\begin{aligned}&(aN_{\beta,-\gamma}+a_\delta N_{\beta,-\delta})B_{\beta-\gamma}+(bN_{\beta,-\gamma}-b_\delta N_{\beta,-\delta})A_{\beta-\gamma}\\ &\quad=-(aN_{\beta,-\gamma}+a_\delta N_{\beta,-\delta})B_{\beta-\gamma}+(b_\delta N_{\beta,-\delta}-bN_{\beta,-\gamma})A_{\beta-\gamma} .\end{aligned}$$

Hence we obtain

$$aN_{\beta,-\gamma}=-a_\delta N_{\beta,-\delta}, \qquad bN_{\beta,-\gamma}=b_\delta N_{\beta,-\delta} . \tag{7.15}$$

Then by (7.15), we can see that

$$\left[X_\gamma-X_\delta,\, X_\gamma+X_\delta-\frac{2}{p}A_\beta\right]=0 \tag{7.16}$$

$$\left[X'_\gamma-X'_\delta,\, X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\right]=0 . \tag{7.17}$$

If $\nu\perp(X_\gamma-X_\delta)$ and $\nu\perp(X'_\gamma-X'_\delta)$, then (suppose $\nu_{|\mathfrak{g}(\alpha)}$ is parallel to $X_\gamma+X_\delta-(2/p)A_\beta$) from (7.16) we get

$$R(X_\gamma-X_\delta, \nu)(X_\gamma-X_\delta)=0 ,$$

that is, $[X_\gamma-X_\delta, \nu]=0$. However,

$$\begin{aligned}\langle[X_\gamma-X_\delta, \nu], X'_\gamma+X'_\delta\rangle &= -\langle[X_\gamma-X_\delta, X'_\gamma+X'_\delta], \nu\rangle\\ &= -\langle 2\sqrt{-1}(H_\gamma-H_\delta), \nu\rangle\\ &\neq 0 .\end{aligned}$$

Next, if $\nu\perp(X_\gamma+X_\delta-(2/p)A_\beta)$, $(X'_\gamma+X'_\delta-(2/p)B_\beta)$, then, by using isotropy representation $\mathrm{Ad}(\exp\sqrt{-1}\mathfrak{h}_1)$, we may assume $\nu\not\perp(X_\gamma-X_\delta)$ and $\nu\not\perp(X'_\gamma-X'_\delta)$. Then, also by (7.16), we get

$$\left[X_\gamma+X_\delta-\frac{2}{p}A_\beta, \nu\right]=0 .$$

However, it follows that

$$\begin{aligned}&\left\langle\left[X_\gamma+X_\delta-\frac{2}{p}A_\beta, \nu\right], \sqrt{-1}(H_\beta-H_\gamma)\right\rangle\\ &\quad=\left\langle\left[\sqrt{-1}(H_\beta-H_\gamma), X_\gamma+X_\delta-\frac{2}{p}A_\beta\right], \nu\right\rangle\\ &\quad=-\beta(H_\beta)\langle X'_\gamma-X'_\delta, \nu\rangle\\ &\quad\neq 0 .\end{aligned}$$

Consequently, we have

$$\nu\not\perp \boldsymbol{R}(X_\gamma-X_\delta)\oplus\boldsymbol{R}(X'_\gamma-X'_\delta)$$

and

$$\nu \not\perp \boldsymbol{R}\Big(X_\gamma+X_\delta-\frac{2}{p}A_\beta\Big)\oplus\boldsymbol{R}\Big(X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\Big).$$

Furthermore we assume $\nu\perp(X_\gamma-X_\delta)$, as before.

In the following we compute

$$R_{(X_\gamma-X_\delta)}\Big(X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\Big) \quad\text{and}\quad R_{(X_\gamma-X_\delta)}(X'_\gamma-X'_\delta),$$

where $R_X(\cdot)=R(X,\ \cdot)X$.

Since $\gamma(H_\gamma)=\delta(H_\delta)$ and $\sqrt{-1}(H_\gamma+H_\delta)\in\mathfrak{k}$, it follows that

$$(7.18)\qquad \begin{aligned} R_{(X_\gamma-X_\delta)}(X'_\gamma-X'_\delta) &= -[X_\gamma-X_\delta,\ 2\sqrt{-1}(H_\gamma+H_\delta)] \\ &= 2\gamma(H_\gamma)(X'_\gamma-X'_\delta). \end{aligned}$$

From (7.15) we obtain

$$\begin{aligned} \Big[X_\gamma-X_\delta,\ X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\Big] &= 2\sqrt{-1}(H_\gamma-H_\delta)+\frac{2}{p}[B_\beta,\ X_\gamma-X_\delta] \\ &= 2\sqrt{-1}(H_\gamma-X_\delta)-\frac{2}{p}(bN_{\beta,-\gamma}+b_\delta N_{\beta,-\delta})A_{\beta-\gamma} \\ &\quad -\frac{2}{p}(aN_{\beta,-\gamma}-a_\delta N_{\beta,-\delta})B_{\beta-\gamma} \\ &= 2\sqrt{-1}(H_\gamma-H_\delta)-\frac{4}{p}(bN_{\beta,-\gamma}A_{\beta-\gamma}+aN_{\beta,-\gamma}B_{\beta-\gamma}). \end{aligned}$$

Hence (2.3) gives

$$\begin{aligned} & R_{(X_\gamma-X_\delta)}\Big(X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\Big) \\ &= -\frac{1}{4}\Big[X_\gamma-X_\delta,\ 2\sqrt{-1}(H_\gamma-H_\delta)-\frac{4N_{\beta,-\gamma}}{p}(bA_{\beta-\gamma}+aB_{\beta-\gamma})\Big]_{\mathfrak{p}} \\ &= \frac{\gamma(H_\gamma)}{2}(X'_\gamma+X'_\delta)_{\mathfrak{p}}-\frac{N_{\beta,-\gamma}N_{\beta,-\delta}}{p}(ab_\delta+a_\delta b)(A_\beta)_{\mathfrak{p}} \\ &\quad +\frac{N_{\beta,-\gamma}}{p}(-N_{\beta,-\gamma}+aa_\delta N_{\beta,-\delta}-bb_\delta N_{\beta,-\delta})(B_\beta)_{\mathfrak{p}}. \end{aligned}$$

Then it follows from (2.3), (2.4) and (7.15) that

$$(7.19)\qquad \begin{aligned} R_{(X_\gamma-X_\delta)}\Big(X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\Big) &= \Big\{\frac{\gamma(H_\gamma)}{2}(X'_\gamma+X'_\delta)-\frac{2(N_{\beta,-\gamma})^2}{p}B_\beta\Big\}_{\mathfrak{p}} \\ &= \frac{\gamma(H_\gamma)}{2}\Big(X'_\gamma+X'_\delta-\frac{2}{p}B_\beta\Big). \end{aligned}$$

Considering (7.19) together with (7.18), a vector $R_{(X_\gamma-X_\delta)}(\nu)$ is not parallel to ν. This contradicts (2.10). Consequently, ν must be normal to $\mathfrak{g}(\alpha)$.

From (7.10) and the above argument we have the following.

LEMMA 7.2. *Let* $\alpha\in\Delta'$. *If there exists* $\nu\in\mathfrak{p}$ *satisfying condition (T-G) such that* $\nu\not\perp\sqrt{-1}\mathfrak{h}_2$ *and* $\nu\not\perp\mathfrak{g}(\alpha)$, *then* $\mathfrak{g}(\alpha)$ *has the following form.*

$$\mathfrak{g}(\alpha)=\mathfrak{g}_\alpha\oplus\mathfrak{g}_\beta$$

for some $\beta\in\Delta'$ *satisfying the property* $\alpha\pm\beta\notin\Delta$.

At first we assume that there are α, $\beta\in\Delta'$ such that $\mathfrak{g}(\alpha)=\mathfrak{g}_\alpha\oplus\mathfrak{g}_\beta$, $\alpha(H_\alpha)\neq\beta(H_\beta)$ and $\nu\not\perp\mathfrak{g}(\alpha)$. Then G is necessarily of type II. We determine the pair $(\mathfrak{g}, \mathfrak{k})$ by case by case check.

We call that for γ, $\delta\in\Delta$ the pair (γ, δ) *satisfies condition* (*) if

$$\gamma(H_\gamma)\geqq\delta(H_\delta) \quad\text{and}\quad \gamma\pm\delta\notin\Delta.$$

$C_l(l\geqq3)$.

In this case we can identify Δ with

$$\{\pm 2e_i,\ \pm e_i\pm e_j\}.$$

(see [4]).

Without loss of generality, we suppose $\alpha(H_\alpha)>\beta(H_\beta)$ and put $\alpha=2e_i$. Then β is one of $\{\pm e_j\pm e_k\}(i\neq j,\ k)$ from Lemma 7.2. It is easy to see that (α, β) is the only pair (up to sign) which satisfies condition (*) and the difference is propotional to $\alpha-\beta$. Then ν must be in $\boldsymbol{R}\sqrt{-1}(H_\alpha-H_\beta)+\mathfrak{g}(\alpha)_\mathfrak{p}$. In fact, if there is $\gamma\in\Delta'$ such that $\nu\not\perp\mathfrak{g}(\gamma)$, then by Lemma 7.2 there is $\delta\in\Delta'$ such that $\gamma\pm\delta\notin\Delta$ and $\gamma-\delta$ is parallel to $\alpha-\beta$. For example, set $\beta=e_j+e_k$. It can be easily seen that $\{\gamma, \delta\}$ is equal to $\{e_i-e_j,\ e_k-e_i\}$ or to $\{e_j-e_i,\ e_i-e_k\}$. However, this contradicts the condition $\gamma+\delta\notin\Delta$.

Since $2e_j$ (and $2e_k$) is not normal to $\alpha-\beta$, the root $2e_j$ must be in Δ' (see Lemma 6.2). Thus, since $[\sqrt{-1}(H_\alpha-H_\beta),\ \mathfrak{g}_{2e_j})]\neq 0$, there exists $X\in\mathfrak{g}(2e_j)$ such that $[X, \nu]\neq 0$. Therefore the same argument as in the early part of section 6 implies that there is $\hat{\nu}\in\mathfrak{p}$ such that $\hat{\nu}\not\perp\mathfrak{g}(\alpha)$ and $\hat{\nu}\not\perp\mathfrak{g}(2e_j)$. However, there is no $\gamma\in\Delta$ such that $2e_j-\gamma$ is proportional to $\alpha-\beta$.

Consequently, this case is not possible.

F_4.

$$\Delta=\left\{\pm e_i,\ \pm e_i\pm e_j,\ \frac{1}{2}(\pm e_1\pm e_2\pm e_3\pm e_4)\right\}.$$

Let (α, β) be a pair satisfying condition (*). Then we may suppose α is one of $\{e_i\pm e_j\}$. Moreover β is one of the following.

$$\pm e_k(k\neq i,\ j), \qquad \pm\frac{1}{2}(e_1\pm e_2\pm e_3\pm e_4).$$

In the following we show that there is no vector satisfying condition (T-G) in the case where $\alpha=e_i+e_j$. (Similarly we can prove it when $\alpha=e_i-e_j$).

(1) The case where $\beta=e_k$ or $-e_k$.

Set $(\alpha, \beta)=(e_i+e_j, e_k)$. Then the pair satisfying condition (*), whose difference is propotional to $\alpha-\beta$, is one of the following.

$$\pm(e_i+e_j, e_k), \qquad \pm(e_k-e_i, e_j), \qquad \pm(e_k-e_j, e_i). \tag{7.20}$$

Suppose $\nu \not\perp \boldsymbol{R}(H_\alpha-H_\beta)$ and $\nu \not\perp \mathfrak{g}(\alpha)$ $(\mathfrak{g}(\alpha)=\mathfrak{g}_{e_k}\oplus\mathfrak{g}_{e_i+e_j})$. Then by Lemma 7.2

$$\nu \in \mathfrak{g}(\alpha)_{\mathfrak{p}}\oplus\mathfrak{g}_{e_j}\oplus\mathfrak{g}_{e_k-e_i}\oplus\mathfrak{g}_{e_i}\oplus\mathfrak{g}_{e_k-e_j}\oplus\boldsymbol{R}\sqrt{-1}(H_\alpha-H_\beta). \tag{7.21}$$

If $e_l\in\Delta_{\mathfrak{p}}(\{i, j, k, l\}=\{1, 2, 3, 4\})$, then since

$$\begin{aligned}\langle R(A_{e_l}, \nu)A_{e_l}, \sqrt{-1}(H_\alpha-H_\beta)\rangle &= \langle R(A_{e_l}, \sqrt{-1}(H_\alpha-H_\beta))A_{e_l}, \nu\rangle \\ &= 0,\end{aligned}$$

it follows that $[A_{e_l}, \nu]=0$. On the other hand, by (7.21)

$$[A_l, \nu]\not\perp\mathfrak{g}_{e_l\pm e_k}.$$

We have thus $e_l\notin\Delta_{\mathfrak{p}}$.

Next, suppose $e_l\in\Delta_{\mathfrak{k}}$. Set

$$\mathfrak{g}(\alpha)_{\mathfrak{k}} = \boldsymbol{R}(A_{e_k}+pX_{e_i+e_j})\oplus\boldsymbol{R}(B_{e_k}+pX'_{e_i+e_j}).$$

Then $[A_{e_l}, A_{e_k}+pX_{e_i+e_j}]=[A_{e_l}, A_{e_k}]$ is in $\mathfrak{k}$. However, since $[\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p}$,

$$[A_{e_l}, A_{e_k}] = \left[A_{e_l}, A_{e_k}-\frac{1}{p}X_{e_i+e_j}\right]\in\mathfrak{p}.$$

Consequently $e_l\notin\Delta_{\mathfrak{k}}$.

Assume that $e_l\in\Delta'$. If $[\mathfrak{g}(e_l)_{\mathfrak{p}}, \nu]\neq 0$, then there exists $\hat{\nu}\in\mathfrak{p}$ satisfying condition (T-G) such that $\hat{\nu}\not\perp\mathfrak{g}(\alpha)$, $\hat{\nu}\not\perp\mathfrak{g}(e_l)$ and $\hat{\nu}_{|\sqrt{-1}\mathfrak{h}_2}$ is parallel to $\sqrt{-1}(H_\alpha-H_\beta)$. However, this contradicts (7.20). Therefore $[\nu, \mathfrak{g}(e_l)_{\mathfrak{p}}]=0$. Since

$$[\mathfrak{g}_{e_l}, \nu]\subset\mathfrak{g}_{e_l\pm e_i}\oplus\mathfrak{g}_{e_l\pm e_j}\oplus\mathfrak{g}_{e_l\pm e_k}$$

and

$$\langle[\mathfrak{g}_{e_l}, \nu], \mathfrak{g}_{e_l\pm e_k}\rangle\neq 0,$$

there is $\gamma\in\Delta$ such that $\mathfrak{g}_\gamma\subset\mathfrak{g}(e_l)$ and

$$\langle[\mathfrak{g}_\gamma, \nu], \mathfrak{g}_{e_l\pm e_k}\rangle\neq 0. \tag{7.22}$$

Since $\mathfrak{g}_\gamma\subset\mathfrak{g}(e_l)$ and $\sqrt{-1}(H_\alpha+2H_\beta)\in\mathfrak{k}$ (by (7.11)) it follows that

$$\langle\gamma, e_i+e_j+2e_k\rangle = \langle e_l, e_i+e_j+2e_k\rangle = 0.$$

Therefore γ is one of

$$\left\{\pm(e_i-e_j);\ \pm\frac{1}{2}(e_k-e_i-e_j\pm e_l)\right\}.$$

Therefore by (7.21) it is easy to check that there is no γ satisfying (7.22).

Thus, this case is not possible.

(2) The case where $\beta\in\{\pm 1/2(e_i-e_j\pm e_k\pm e_l)\}$.

For example, set $\beta=1/2(e_i-e_j+e_k+e_l)$. Then the pair satisfying condition $(*)$, whose difference is propotional to $\alpha-\beta$, is one of the following.

$$\pm\left(e_i+e_j,\ \frac{1}{2}(e_i-e_j+e_k+e_l)\right),\qquad \pm\left(e_j-e_k,\ \frac{1}{2}(e_l-e_j-e_k-e_i)\right)$$

$$\pm\left(e_j-e_l,\ \frac{1}{2}(e_k-e_j-e_i-e_l)\right).$$

Then $\sqrt{-1}H_{e_i}$ is not normal to $\sqrt{-1}(H_\alpha-H_\beta)$. Therefore by Lemma 6.2 the root e_i is not in $\Delta_{\mathfrak{p}}$. Moreover $e_i\notin\Delta_{\mathfrak{k}}$ since $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$. Thus we have $e_i\in\Delta'$. Let $\gamma\in\Delta$ such that $\mathfrak{g}_\gamma\subset\mathfrak{g}(e_i)$. Since $\sqrt{-1}(H_\alpha+2H_\beta)$ is in $\mathfrak{k}$, we have

$$\begin{aligned}\langle\gamma,\ \alpha+2\beta\rangle &= \langle e_i,\ \alpha+2\beta\rangle\\ &= \langle e_i,\ 2e_i+e_k+e_l\rangle = 2.\end{aligned}$$

Therefore γ is one of

$$\left\{e_k+e_l,\ \frac{1}{2}(e_i+e_k+e_l+e_j),\ \beta\right\}.$$

By the same method as in (1) we get

$$[\mathfrak{g}(e_i)_{\mathfrak{p}},\ \nu]=0. \tag{7.23}$$

Since

$$\begin{gathered}\nu\in\boldsymbol{R}(H_\alpha-H_\beta)\oplus\mathfrak{g}(\alpha)_{\mathfrak{p}}\oplus\mathfrak{g}_{e_j-e_k}\oplus\mathfrak{g}_{e_j-e_l}\\ \oplus\mathfrak{g}_{1/2(e_l-e_j-e_k-e_i)}\oplus\mathfrak{g}_{1/2(e_k-e_j-e_i-e_l)},\\ [\mathfrak{g}_{e_i},\ \nu]\not\perp\mathfrak{g}_{e_i}.\end{gathered}$$

However, regardless of whether $\gamma=e_k+e_l$ or $1/2(e_i+e_j+e_k+e_l)$, we can see

$$[\gamma,\ \nu]\perp\mathfrak{g}_{e_i}.$$

This contradicts (7.23).

$B_l(l\geqq 3)$.

$$\Delta=\{\pm e_i\pm e_j;\ \pm e_k\}.$$

In this case α is one of $\{\pm e_i\pm e_j\}$ and $\beta=e_k$ or $-e_k(k\neq i,j)$. For example set $(\alpha,\beta)=(e_i+e_j,\ e_k)$. Then the pair satisfying condition $(*)$, whose difference is propotional to $\alpha-\beta$, is one of the following.

$$(e_i+e_j,\ e_k),\qquad (e_k-e_j,\ e_i),\qquad (e_k-e_i,\ e_j).$$

Then since $\nu\not\perp\mathfrak{g}(\alpha)_{\mathfrak{p}}$ and $\nu\not\perp\boldsymbol{R}(H_\alpha-H_\beta)$, Lemma 7.2 implies

$$\nu \in \boldsymbol{R}\sqrt{-1}(H_\alpha - H_\beta) \oplus \mathfrak{g}(\alpha)_{\mathfrak{p}} \oplus \mathfrak{g}_{e_i} \oplus \mathfrak{g}_{e_j} \oplus \mathfrak{g}_{e_k - e_j} \oplus \mathfrak{g}_{e_k - e_i}. \tag{7.24}$$

At first we suppose $l \geqq 4$.

If $e_n \in \Delta_{\mathfrak{p}}$, then since $[\mathfrak{g}_{e_n}, \sqrt{-1}(H_\alpha - H_\beta)] = 0$ we have

$$[\mathfrak{g}_{e_n}, \nu] = 0.$$

However, we see that $[\mathfrak{g}_{e_n}, \nu] \not\perp \mathfrak{g}_{e_n \pm e_k}$. Thus $e_n \notin \Delta_{\mathfrak{p}}$.

Suppose $e_n \in \Delta_{\mathfrak{k}}$. Put $\nu_t = \mathrm{Ad}(\exp tA_{e_n})(\nu)$. Since

$$[A_{e_n}, \sqrt{-1}\mathfrak{h}_2] = 0 \quad \text{and} \quad [A_{e_n}, \mathfrak{g}(\alpha)_{\mathfrak{p}}] = \mathfrak{g}_{e_n \pm e_k},$$

there is $t \in \boldsymbol{R}$ such that $\nu \not\perp \mathfrak{g}(\alpha)_{\mathfrak{p}}$, $\nu_t \not\perp \mathfrak{g}_{e_n \pm e_k}$ and $\sqrt{-1}\mathfrak{h}_2$-component of ν_t is parallel to $\sqrt{-1}(H_\alpha - H_\beta)$. This contradicts (7.24). Therefore $e_n \notin \Delta_{\mathfrak{k}}$.

Take $\gamma \in \Delta$ so that $\mathfrak{g}_\gamma \subset \mathfrak{g}(e_n)$. Since $\sqrt{-1}(H_{e_i+e_j} + 2H_{e_k})$ is in $\mathfrak{k}$, we obtain

$$\langle \gamma, e_i + e_j + 2e_k \rangle = \langle e_n, e_i + e_j + 2e_k \rangle = 0.$$

Therefore γ is one of

$$\{\pm e_p \pm e_q;\ \pm(e_i - e_j);\ \pm e_r\} \quad (p, q, r \notin \{i, j, k\}). \tag{7.25}$$

As before, since

$$[\mathfrak{g}(e_n)_{\mathfrak{p}}, \nu] = 0 \quad \text{and} \quad [\mathfrak{g}_{e_n}, \nu] \not\perp \mathfrak{g}_{e_n \pm e_k},$$

it follows that

$$[\mathfrak{g}_\gamma, \nu] \not\perp \mathfrak{g}_{e_n \pm e_k}. \tag{7.26}$$

However, by (7.24) we can easily check that there is no root in (7.25) satisfying (7.26).

Finally we investigate the case where $\mathfrak{g}$ is of type B_3. Since $e_i, e_j \not\perp \alpha - \beta$, we get $e_i, e_j \in \Delta'$. Moreover, by similar argument as above, we obtain $[\mathfrak{g}(e_i)_{\mathfrak{p}}, \nu] \neq 0$ and $[\mathfrak{g}(e_j)_{\mathfrak{p}}, \nu] \neq 0$. Therefore, it follows from Lemma 7.2 that

$$\mathfrak{g}(e_i) = \mathfrak{g}_{e_i} \oplus \mathfrak{g}_{e_k - e_j} \quad \text{and} \quad \mathfrak{g}(e_j) = \mathfrak{g}_{e_j} \oplus \mathfrak{g}_{e_k - e_i}.$$

Then by (7.11) we get

$$\sqrt{-1}(H_{e_i+e_j} + 2H_{e_k}), \quad \sqrt{-1}(H_{e_k - e_i} + 2H_{e_j}), \quad \sqrt{-1}(H_{e_k - e_j} + 2H_{e_i}) \in \mathfrak{k}.$$

(Then $\sqrt{-1}H_{e_i - e_j}$, $\sqrt{-1}H_{e_k + e_i}$, $\sqrt{-1}H_{e_k + e_j} \in \mathfrak{k}$). Therefore $e_i - e_j$, $e_k + e_i$, $e_k + e_j \notin \Delta'$, since

$$\langle e_i - e_j, e_i - e_j \rangle = 2$$

$$\langle e_i - e_j, e_k + e_i \rangle = 1$$

$$\langle e_i - e_j, e_k + e_j \rangle = -1.$$

Finally we prove $e_i-e_j,\ e_k+e_i,\ e_k+e_j\in\Delta_{\mathfrak{k}}$. We assume that

$$\nu\not\perp\mathfrak{g}(\alpha) \quad\text{and}\quad \nu\not\perp\mathfrak{g}(e_i)\oplus\mathfrak{g}(e_j).$$

If $e_i-e_j\in\Delta_{\mathfrak{k}}$, then $[\mathfrak{g}_{e_i-e_j},\nu]=0$ since $\langle e_i-e_j,\ \alpha-\beta\rangle=0$. However, since

$$[\mathfrak{g}_{e_i-e_j},\ \mathfrak{g}(e_i)]=\mathfrak{g}(e_j),\qquad [\mathfrak{g}_{e_i-e_j},\ \mathfrak{g}(e_j)]=\mathfrak{g}(e_i)$$
$$[\mathfrak{g}_{e_i-e_j},\ \mathfrak{g}(\alpha)]=\{0\},$$

we can see $[\mathfrak{g}_{e_i-e_j},\nu]\neq 0$. Thus $e_i-e_j,\ e_k+e_i,\ e_k+e_j\in\Delta_{\mathfrak{k}}$ and

$$\mathfrak{k}=\boldsymbol{R}\sqrt{-1}H_{e_i-e_j}+\boldsymbol{R}\sqrt{-1}H_{e_k+e_i}+\mathfrak{g}_{e_i-e_j}+\mathfrak{g}_{e_k+e_i}$$
$$+\mathfrak{g}_{e_k+e_j}+\mathfrak{g}(e_i)_{\mathfrak{k}}+\mathfrak{g}(e_j)_{\mathfrak{k}}+\mathfrak{g}(e_k)_{\mathfrak{k}}.$$

Then $(G/K, \langle\ ,\ \rangle)$ has constant sectional curvature. (The pair $(\mathfrak{g}, \mathfrak{k})$ coincides with (B_3, G_2) in [**1**, p. 220]).

Consequently, if there is $\alpha\in\Delta'$ such that $\nu\not\perp\mathfrak{g}(\alpha)$ with $\alpha(H_\alpha)\neq\beta(H_\beta)$ $(\mathfrak{g}(\alpha)=\mathfrak{g}_\alpha+\mathfrak{g}_\beta)$, then G/K is a space with constant sectional curvature.

Next we assume that $\alpha(H_\alpha)=\beta(H_\beta)$. Then by (7.11) and Lemma 4.2 we may put

$$\mathfrak{g}(\alpha)=\boldsymbol{R}(A_\alpha+X_\beta)\oplus\boldsymbol{R}(B_\alpha+X'_\beta).$$

(Then $A_\alpha-X_\beta,\ B_\alpha-X'_\beta\in\mathfrak{p}$).

As before, suppose $\nu_{|\mathfrak{g}(\alpha)}$ is parallel to $(B_\alpha-X'_\beta)$.

If there is a number d such that $d(\alpha-\beta)\in\Delta$ (then necessarily $d(\alpha+\beta)\in\Delta$ from (7.10)), then from (7.13) we get

$$\|\sqrt{-1}H_{d(\alpha-\beta)}\|^2=d^2(\alpha(H_\alpha)+\beta(H_\beta))$$
$$=2d^2e\quad(\alpha(H_\alpha)=\beta(H_\beta)=e).$$

Since $\mathfrak{g}$ is either of type I or type II, then $2d^2=1$, 2 or 1/2, that is, $d=\pm(\sqrt{2}/2)$, ±1 or $\pm(1/2)$. However $\pm(\alpha-\beta)$ and $\pm(\sqrt{2}/2)(\alpha-\beta)$ are not roots. Thus we have $d=\pm(1/2)$. As mentioned under the proof of Lemma 7.1, a root $(\alpha-\beta)/2$ is in $\Delta_{\mathfrak{p}}$ and we assume $\nu_{|\mathfrak{g}(\alpha-\beta)/2}=X'_{(\alpha-\beta)/2}$. Moreover, since $[A_\alpha+X_\beta,\ \mathfrak{g}_{(\alpha-\beta)/2}]$ is contained in $\alpha_{(\alpha+\beta)/2}$, a root $(\alpha+\beta)/2$ is necessarily in $\Delta_{\mathfrak{p}}$. Hence we get

$$[\mathfrak{g}_{(\alpha+\beta)/2},\ \nu]=\{0\}\quad\text{and}\quad \nu\perp\mathfrak{g}_{(\alpha+\beta)/2}$$

because $[\mathfrak{g}_{(\alpha+\beta)/2},\ \sqrt{-1}(H_\alpha-H_\beta)]=\{0\}$. Furthermore Lemma 2.2 implies

$$[X_{(\alpha-\beta)/2},\ B_\alpha-X'_\beta]=0. \tag{7.27}$$

(If not, then $e^{-\varphi_t X_{(\alpha-\beta)/2}}(\nu)$ is not perpendicular to $\mathfrak{g}_{(\alpha+\beta)/2}$ and its $\sqrt{-1}\mathfrak{h}_2$-component is parallel to $\sqrt{-1}(H_\alpha-H_\beta)$. This contradicts Lemma 6.3). Therefore

$$R(X_{(\alpha-\beta)/2},\ \nu)X_{(\alpha-\beta)/2}=0. \tag{7.28}$$

On the other hand, we can see that $R(X_{(\alpha-\beta)/2}, \sqrt{-1}(H_\alpha-H_\beta))X_{(\alpha-\beta)/2}$ is parallel to $\sqrt{-1}(H_\alpha-H_\beta)$ and not zero. This contradicts (7.28). Hence we conclude

$$(7.29) \qquad d(\alpha\pm\beta)\notin\Delta \quad \text{for all} \quad d\in \boldsymbol{R}.$$

Now, suppose α, $\beta(\alpha, \beta\in\Delta')$ satisfy (7.29) and $\nu\not\perp\mathfrak{g}(\alpha)=\mathfrak{g}_\alpha\oplus\mathfrak{g}_\beta$. Then since $\alpha\pm\beta\notin\Delta$ and $\alpha(H_\alpha)=\beta(H_\beta)$, it is easy to check that

$$[\sqrt{-1}(H_\alpha-H_\beta), \mathfrak{g}(\alpha)_\mathfrak{p}]\subset\mathfrak{k}.$$

Then we shall prove $[\sqrt{-1}(H_\alpha-H_\beta), \mathfrak{p}]\subset\mathfrak{k}$ in the following.

If $[\mathfrak{g}(\delta)_\mathfrak{p}, \nu]\neq\{0\}$ for $\delta\in\Delta'$, then, as before, we may assume $\nu\not\perp\mathfrak{g}(\delta)_\mathfrak{p}$. (Since $\nu\not\perp\mathfrak{g}(\alpha)_\mathfrak{p}$, then $\nu_{|\sqrt{-1}\mathfrak{h}_2}$ must be parallel to $\sqrt{-1}(H_\alpha-H_\beta)$). Therefore, in this case, we have

$$[\sqrt{-1}(H_\alpha-H_\beta), \mathfrak{g}(\delta)_\mathfrak{p}]\subset\mathfrak{k}$$

as above.

Suppose $[\nu, \mathfrak{g}(\delta)_\mathfrak{p}]=\{0\}\,(\mathfrak{g}(\delta)=\mathfrak{g}_{\delta_1}\oplus\cdots\oplus\mathfrak{g}_{\delta_m})$. If there exist δ_i and δ_j such that

$$\delta_i(H_\alpha-H_\beta)\neq 0 \quad\text{and}\quad \delta_j(H_\alpha-H_\beta)\neq 0,$$

then

$$(7.30) \qquad \begin{aligned}&[p_{\delta_j}X_{\delta_i}-p_{\delta_i}X_{\delta_j}, \sqrt{-1}(H_\alpha-H_\beta)]\\ &\quad= p_{\delta_i}\delta_j(H_\alpha-H_\beta)X'_{\delta_j}-p_{\delta_j}\delta_i(H_\alpha-H_\beta)X'_{\delta_i}\\ &\quad\neq 0.\end{aligned}$$

Therefore, since $[p_{\delta_j}X_{\delta_i}-p_{\delta_i}X_{\delta_j}, \nu]=0$, there exists a unique (up to sign) root ω such that $\nu\not\perp\mathfrak{g}_\omega$ and $[\mathfrak{g}_\omega, \mathfrak{g}_{\delta_i}]\not\perp\mathfrak{g}_{\delta_j}$. In fact, if there exists another root ω' such that $\nu\not\perp\mathfrak{g}_{\omega'}$ and $[\mathfrak{g}_{\omega'}, \mathfrak{g}_{\delta_i}]\not\perp\mathfrak{g}_{\delta_j}$, then ω (and ω') is one of $\{\pm\delta_i\pm\delta_j\}$. On the other hand, since $[\nu, \sqrt{-1}H]=0$ for $H\in\mathfrak{h}_2$ with $\sqrt{-1}H\perp\sqrt{-1}(H_\alpha-H_\beta)$, we get $\omega(H)=\omega'(H)=0$. This implies that $\delta_i(H)=\delta_j(H)=0$, i.e. $\sqrt{-1}(H_{\delta_i}-H_{\delta_j})$ is parallel to $\sqrt{-1}(H_\alpha-H_\beta)$. Howe,ver since $\delta_i-\delta_j$ is a root (it is one of $\{\pm\omega, \pm\omega'\}$), this contradicts Lemma 7.2.

Now we write

$$\nu=\sqrt{-1}(H_\alpha-H_\beta)+aB_\omega+\cdots, \qquad (a\neq 0).$$

Then from the above argument and the assumption $[\nu, \mathfrak{g}(\delta)_\mathfrak{p}]=\{0\}$, it follows that

$$[\sqrt{-1}(H_\alpha-H_\beta)+\alpha B_\omega, p_{\delta_i}X_{\delta_j}-p_{\delta_j}X_{\delta_i}]\perp\mathfrak{g}_{\delta_i}\oplus\mathfrak{g}_{\delta_j}$$

and, by considering isotropy representation $\mathrm{Ad}(\exp\sqrt{-1}\mathfrak{h}_1)$, we also get

$$[\sqrt{-1}(H_\alpha-H_\beta)+aA_\omega, p_{\delta_i}X_{\delta_j}-p_{\delta_j}X_{\delta_i}]\perp\mathfrak{g}_{\delta_i}\oplus\mathfrak{g}_{\delta_j}.$$

Therefore we get

$$(7.31) \qquad [A_\omega-B_\omega, p_{\delta_i}X_{\delta_j}-p_{\delta_j}X_{\delta_i}]\perp\mathfrak{g}_{\delta_i}\oplus\mathfrak{g}_{\delta_j}.$$

However, by a straightforward computation, we can see that (7.31) implies $p_{\delta_i}=p_{\delta_j}=0$ (ω is one of $\{\pm\delta_i\pm\delta_j\}$). This is a contradiction. As a result, we obtain

$$[\sqrt{-1}(H_\alpha-H_\beta),\ \mathfrak{g}(\delta)_{\mathfrak{p}}]=\{0\}.$$

Consequently, in this case, we have

$$[\sqrt{-1}(H_\alpha-H_\beta),\ \mathfrak{p}]\subset\mathfrak{k}$$

and hence $(\mathfrak{g}, \mathfrak{k})$ is symmetric by Corollary 5.2. Therefore G/K is a space of constant sectional curvature.

Finally, in the case where $\nu\not\perp\sqrt{-1}\mathfrak{h}_2$, we suppose that $\nu\not\perp\mathfrak{g}_\gamma$ for some $\gamma\in\Delta_{\mathfrak{p}}$. Then $\nu_{|\sqrt{-1}\mathfrak{h}_2}$ must be parallel to $\sqrt{-1}H_\gamma$ by Lemma 6.2. Furthermore ν is contained in $\boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma$ from Lemma 6.2, (7.29) and the argument below Lemma 7.2. Set

$$\nu=a\sqrt{-1}H_\gamma+bA_\gamma+cB_\gamma,\qquad X=bB_\gamma-cA_\gamma$$

$(a\neq 0,\ b^2+c^2=1)$. Then we have by a straightforward computation that

$$e^{-\varphi_{tX}}(\nu)=\cos\sqrt{\frac{\gamma(H_\gamma)}{2}}t\cdot\nu -\frac{1}{\sqrt{2\gamma(H_\gamma)}}\sin\sqrt{\frac{\gamma(H_\gamma)}{2}}t\cdot\{a\gamma(H_\gamma)(bA_\gamma+cB_\gamma)-2\sqrt{-1}H_\gamma\}. \tag{7.32}$$

As before, for all $Y(\in\mathfrak{p})$ perpendicular to $\boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma$, we get $[Y,\nu]=0$, and also we have

$$[Y,\ e^{-\varphi_{tX}}(\nu)]=0\qquad\text{for all}\quad t\in\boldsymbol{R}. \tag{7.33}$$

Then (7.32) and (7.33) give

$$[\boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma,\ Y]=\{0\}. \tag{7.34}$$

Since $\langle\ ,\ \rangle$ is biinvariant, (7.34) gives

$$[\mathfrak{k},\ \boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma]\subset\boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma.$$

Therefore Lemma 5.1 implies that

$$\mathfrak{p}=\boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma.$$

Since $\mathfrak{g}$ is simple, we find that

$$\mathfrak{g}=\boldsymbol{R}\sqrt{-1}H_\gamma\oplus\mathfrak{g}_\gamma(\cong\mathfrak{su}(2)),$$

and $(G,\langle\ ,\ \rangle)$ has constant sectional curvature.

This completes the proof of the case $\nu\not\perp\sqrt{-1}\mathfrak{h}_2$.

Next, we suppose that $\nu\perp\sqrt{-1}\mathfrak{h}_2$. Then from Lemma 6.1, we may assume

$$\nu \in \mathfrak{g}_{\tau_1} \oplus \cdots \oplus \mathfrak{g}_{\tau_l}$$

with $\tau_i \in \Delta_{\mathfrak{p}}$ and $\tau_i(\mathfrak{h}_2)=0(i=1, \cdots, l)$. Furthermore we assume $\nu \not\perp \mathfrak{g}_{\tau_i}(i=1, \cdots, l)$. If there exist $i, j(1 \leqq i \neq i \leqq l)$ such that $\tau_i+\tau_j \in \Delta'$, then there exist $Y \in \mathfrak{g}_{\tau_i+\tau_j}$ such that

$$\langle [X, \nu], Y \rangle = -\langle [X, Y], \nu \rangle \neq 0 (X \in \mathfrak{g}_{\tau_i}, X \perp \nu)$$

because $[X, Y] \in \mathfrak{g}_{\tau_j}$ and $\mathfrak{g}_{2\tau_i+\tau_j} \perp \nu$. (If $\mathfrak{g}_{2\tau_i+\tau_j} \not\perp \nu$, then from Lemma 6.3, we have $2\tau_i+\tau_j \in \Delta_{\mathfrak{p}}$. On the other hand, $[\mathfrak{g}_{\tau_j}, \mathfrak{g}_{2\tau_i+\tau_j}]=\{0\}$. This contradicts the assumption $\nu \not\perp \mathfrak{g}_{\tau_j}$). However, this contradicts Lemma 6.3. Hence for any i, j, we have $\tau_i \pm \tau_j \notin \Delta'$. For this reason, we may assume $\nu \in \mathfrak{g}_\tau$ for some $\tau \in \Delta_{\mathfrak{p}}$ with $\tau(\mathfrak{h}_2)=0$. (cf. the proof of Theorem B). Moreover, by the proof of Theorem B and the above argument, we have for any $\lambda \in \Delta_{\mathfrak{p}}$ and for any $\alpha \in \Delta'$ that

$$[\nu, \mathfrak{g}_\lambda]_{\mathfrak{p}} = \{0\}, \qquad [\nu, \mathfrak{g}(\alpha)_{\mathfrak{p}}] = \{0\}.$$

Thus, by Corollary 5.2 and Fact 1.1, we have proved Theorem A in the case where $\nu \perp \sqrt{-1}\mathfrak{h}_2$.

Consequently, if $\mathfrak{g}$ is of type I or of type II, then G/K has constant sectional curvature.

8. Type G_2.

In this section we shall prove the following lemma which completes the proof of Theorem A.

LEMMA 8.1. *Let* $\mathfrak{g}$ *be of type* G_2 *and* $\mathfrak{k}$ *a Lie subalgebra of* $\mathfrak{g}$ *such that* $\mathrm{rk}(\mathfrak{g}) > \mathrm{rk}(\mathfrak{k})$ (*then necessarily* $\mathrm{rk}(\mathfrak{k})=1$). *Then there is no vector satisfying condition (T-G).*

PROOF. Since $\mathrm{rk}(\mathfrak{k})=1$, $\mathfrak{k}$ is isomorphic either to $\boldsymbol{R}$ or to $\mathfrak{su}(2)$. Then there exist at most one root space $\mathfrak{g}_\alpha$ such that

$$[\sqrt{-1}\mathfrak{h}_2, \mathfrak{g}_\alpha] = \{0\}$$

because $\mathrm{rk}(G)=2$. Therefore, if $\alpha \in \Delta_{\mathfrak{p}}(\alpha(\mathfrak{h}_2)=0)$ and $\nu \not\perp \mathfrak{g}_\alpha$, then Lemma 6.3 implies that $[\mathfrak{g}_\alpha, \mathfrak{p}] \subset \mathfrak{k}$.

Next, we assume $\nu \not\perp \sqrt{-1}\mathfrak{h}_2$.

If $\mathfrak{k} \cong \boldsymbol{R}$, then $\alpha \in \Delta_{\mathfrak{p}}$ for all $\alpha \in \Delta$. However, this contradicts Lemma 6.2. Hence there is no vector in $\mathfrak{p}$ satisfying condition (T-G).

If $\mathfrak{k} \cong \mathfrak{su}(2)$, then from Proposition 4.5 we see that $\mathfrak{k}$ has one of the following forms:

(1) $\boldsymbol{R}\sqrt{-1}H_\lambda \oplus \mathfrak{g}_\lambda$ for some $\lambda \in \Delta$.

(2) $\sqrt{-1}\mathfrak{h}_1 \oplus \mathfrak{g}(\alpha)_{\mathfrak{k}}$ is one of (a), (b), (c), (d) of (3) in Proposition 4.5. Then

the above argument and Lemma 6.2 imply that the number of positive roots in $\Delta_{\mathfrak{p}}$ are at most two. Therefore in the case (a), (b), (c) and $\mathfrak{k}=\boldsymbol{R}\sqrt{-1}H_\lambda\oplus\mathfrak{g}_\lambda$, there are no vectors in $\mathfrak{p}$ satisfying condition (T-G).

Finally we examine the case (d) in the above. In this case it is easy to see that

$$\Delta=\{\pm\alpha_1,\ \pm\alpha_2,\ \pm\alpha_1\pm\alpha_2,\ \pm(2\alpha_1-\alpha_2),\ \pm(2\alpha_2-\alpha_1)\}$$

$$\mathfrak{g}(\alpha)=\mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_2}\oplus\mathfrak{g}_{2\alpha_1-\alpha_2}\oplus\mathfrak{g}_{2\alpha_2-\alpha_1}$$

$$\sqrt{-1}\mathfrak{h}_2=\boldsymbol{R}\sqrt{-1}(H_{\alpha_1}-H_{\alpha_2}).$$

Then we note that

$$[\mathfrak{g}_{\alpha_1+\alpha_2},\ \mathfrak{g}_{\alpha_1}]=\mathfrak{g}_{\alpha_2},\qquad [\mathfrak{g}_{\alpha_1+\alpha_2},\ \mathfrak{g}_{\alpha_2}]=\mathfrak{g}_{\alpha_1}$$

$$[\mathfrak{g}_{\alpha_1+\alpha_2},\ \mathfrak{g}_{2\alpha_1-\alpha_2}]=\mathfrak{g}_{2\alpha_2-\alpha_1},\qquad [\mathfrak{g}_{\alpha_1+\alpha_2},\ \mathfrak{g}_{2\alpha_2-\alpha_1}]=\mathfrak{g}_{2\alpha_1-\alpha_2}$$

$$[\mathfrak{g}_{\alpha_1+\alpha_2},\ \sqrt{-1}\mathfrak{h}_2\oplus\mathfrak{g}_{\alpha_1-\alpha_2}]=\{0\}\ .$$

Hence, for any $X\in\mathfrak{g}(\alpha)_{\mathfrak{p}}$, we obtain

$$[X,\ \mathfrak{g}_{\alpha_1+\alpha_2}]\neq\{0\}\ . \tag{8.1}$$

However, as before, (8.1) implies that there exists a vector $\hat{v}\in\mathfrak{p}$ satisfying condition (T-G) such that

$$\hat{v}\perp\sqrt{-1}\mathfrak{h}_1\oplus\mathfrak{g}_{\alpha_1-\alpha_2}\oplus\mathfrak{g}(\alpha)_{\mathfrak{p}}\,.$$

This contradicts Lemma 6.3. Consequently, for any $\mathfrak{k}$, there is no vector satisfying condition (T-G). □

9. Some Remarks.

By the Proofs of Theorems A, B we can see the following.

REMARK 9.1. Let G be a compact simple Lie group and K a closed subgroup of G. Assume that the pair (G, K) is not symmetric and a normal homogeneous space G/K has constant sectional curvature. Then the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the following (cf. [1]).

$$(\mathfrak{su}(2),\ \{0\}),\qquad (G_2,\ A_2),\qquad (B_3,\ G_2).$$

Next, we shall show that if $\mathrm{rk}(G)>\mathrm{rk}(K)$, then Theorem B is not true.

Let $(\mathfrak{g}, \mathfrak{k})$ be a Hermitian symmetric pair of compact type, where $\mathfrak{g}$ is a simple Lie algebra ($\mathfrak{g}$ is not isomorphic to $\mathfrak{su}(2)$). $\mathfrak{p}$ denotes the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. As is well-known, there exists 1-dimensional center of $\mathfrak{k}$ (denoted by $\boldsymbol{R}Z$). Then set

$$\mathfrak{p}' = \boldsymbol{R}Z \oplus \mathfrak{p}.$$

Let $\mathfrak{k}'$ be the orthogonal complement of $\mathfrak{p}'$. Then the hyperplane $\mathfrak{p}$ of $\mathfrak{p}'$ is curvature invariant. In fact, let X, Y be in $\mathfrak{p}$. Since $[x, y] \in \mathfrak{k}$ and $[Z, \mathfrak{k}'] = 0$, we have

$$\begin{aligned} R(X, Y)Z &= [[X, Y]_{\mathfrak{k}'}, Z] + \frac{1}{2}[[X, Y]_{\mathfrak{p}'}, Z]_{\mathfrak{p}'} \\ &\quad - \frac{1}{4}[X, [Y, Z]_{\mathfrak{p}'}]_{\mathfrak{p}'} + \frac{1}{4}[Y, [X, Z]_{\mathfrak{p}'}]_{\mathfrak{p}'} \\ &= -\frac{1}{4}[X, [Y, Z]_{\mathfrak{p}'}]_{\mathfrak{p}'} + \frac{1}{4}[Y, [X, Z]_{\mathfrak{p}'}]_{\mathfrak{p}'}. \end{aligned}$$

Moreover, since $[\mathfrak{p}, Z] \subset \mathfrak{g}$, we see that $[\mathfrak{g}, [\mathfrak{g}, Z]_{\mathfrak{p}'}]_{\mathfrak{p}'}$ are parallel to Z. On the other hand $R(X, Y)Z$ is perpendicular to Z. Thus $\mathfrak{p}$ is curvature invariant hyperplane of $\mathfrak{g}'$. Let G be a compact, connected simple Lie group with Lie algebra $\mathfrak{g}$ and K' be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}'$. Then by Remark 9.1, we can see that a normal homogeneous space G/K' is not a real space form.

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