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Modified Nash triviality theorem for a family of zero-sets of weighted homogeneous polynomial mappings

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

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§ 0. Introduction.

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be an n-tuple of positive integers. Assume that the greatest common divisor of the integers α_j is 1. Let $\boldsymbol{N}$ denote the set of positive integers, and let $f:(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}, 0)$ be a polynomial function defined by

$$f(x)=\sum_{\beta} A_\beta x_1^{\beta_1}\cdots x_n^{\beta_n}\ (A_\beta\neq 0,\ \beta_1, \cdots, \beta_n\in \boldsymbol{N}\cup\{0\}).$$

We say that f is *weighted homogeneous of type* $(\alpha_1, \cdots, \alpha_n; L)$ $(\alpha_1, \cdots, \alpha_n, L\in \boldsymbol{N})$, if

$$\alpha_1\beta_1+\cdots+\alpha_n\beta_n=L \qquad \text{for any} \quad \beta=(\beta_1, \cdots, \beta_n).$$

Let J be an open interval, and $t_0\in J$. Let $f_t:(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}^p, 0)$ be a polynomial mapping where each $f_{t,i}$ is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n; L_i)$ $(1\leqq i\leqq p)$ for $t\in J$. We define a mapping $F:(\boldsymbol{R}^n\times J, \{0\}\times J)\rightarrow(\boldsymbol{R}^p, 0)$ by $F(x, t)=f_t(x)$. Assume that F is a polynomial mapping (or of class C^2). It is well-known that the following fact holds under these assumptions:

FACT. If $f_t^{-1}(0)\cap\Sigma f_t=\{0\}$ for any $t\in J$ (where Σf_t denotes the singular points set of f_t), then $(\boldsymbol{R}^n\times J, F^{-1}(0))$ is topologically trivial i.e. there exists a t-level preserving homeomorphism $\sigma:(\boldsymbol{R}^n\times J, \{0\}\times J)\rightarrow(\boldsymbol{R}^n\times J, \{0\}\times J)$ such that

$$\sigma((\boldsymbol{R}^n\times J, F^{-1}(0)))=(\boldsymbol{R}^n\times J, f_{t_0}^{-1}(0)\times J).$$

REMARK 1. Results generalizing this fact have been obtained in [2], [5]. But it seems that the fact itself was recognized by many mathematicians a good while ago.

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Since we consider the weighted homogeneous case with an isolated singularity, it seems natural that stronger triviality than topological holds. In fact, we show that such triviality called "modified Nash triviality" holds under the above assumptions (see Theorem in § 2). On the other hand, we have introduced the notion of "strong C^0 triviality" for a family of analytic functions in [**6**]. Roughly speaking, strong C^0 equivalence is a C^0 equivalence which preserves the tangency of analytic arcs at $0 \in \boldsymbol{R}^n$. In § 4, we discuss the relation between modified Nash triviality and strong C^0 triviality for a family of zero-sets of weighted homogeneous polynomials.

Main results in this paper have been announced in [**7**].

The author would like to thank Professor M. Shiota for helpful suggestions concerning Proposition 2 in § 3 (semi-algebraic triviality theorem).

§ 1. Some properties of Nash manifolds.

In this section, we recall some important results on Nash manifolds. A *semi-algebraic set* of $\boldsymbol{R}^n$ is a finite union of sets of the form

$$\{x \in \boldsymbol{R}^n \mid f_1(x) = \cdots = f_k(x) = 0,\ g_1(x) > 0,\ \cdots,\ g_m(x) > 0\},$$

where $f_1, \cdots, f_k, g_1, \cdots, g_m$ are polynomial functions on $\boldsymbol{R}^n$. Let $r = 0, 1, 2, \cdots, \infty, \omega$. A semi-algebraic set of $\boldsymbol{R}^n$ is called a C^r (affine) *Nash manifold* if it is a regular C^r submanifold of $\boldsymbol{R}^n$. Let $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ be C^r Nash manifolds. A C^s mapping $f: M \to N$ $(s \leqq r)$ is called a C^s *Nash mapping* if the graph of f is semi-algebraic in $\boldsymbol{R}^m \times \boldsymbol{R}^n$.

THEOREM 1 (B. Malgrange [**14**]). (1) *A C^∞ Nash manifold is a C^ω Nash manifold.*

(2) *A C^∞ Nash mapping between C^ω Nash manifolds is a C^ω Nash mapping.*

After this, a Nash manifold and a Nash mapping mean a C^ω Nash manifold and a C^ω Nash mapping, respectively.

THEOREM 2 (M. Shiota [**15**]). *Let $M_1 \supset N_1$, $M_2 \supset N_2$ be compact Nash manifolds and compact Nash submanifolds. If the pairs (M_1, N_1) and (M_2, N_2) are C^∞ diffeomorphic, then they are Nash diffeomorphic.*

REMARK 2. In Theorem 2, we can replace the assumption of "C^∞ diffeomorphic" by "C^1 diffeomorphic" ([**16**]).

THEOREM 3 (M. Shiota [**16**]). *There exist two (affine) Nash manifolds which are C^ω diffeomorphic but not Nash diffeomorphic.*

In general, Nash diffeomorphism is stronger than the notion of C^ω diffeomorphism. Consequently, modified Nash triviality (cf. § 2) is stronger than modified analytic triviality in the sense of T.C. Kuo ([**11**], [**12**], [**13**]).

§ 2. Modified Nash triviality theorem.

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be an n-tuple of positive integers. Put $\rho=\alpha_1 \cdots \alpha_n$ and $\rho_i=\rho/\alpha_i\,(1\leqq i\leqq n)$. For $\gamma\geqq 0$, set

$$S_\gamma(\alpha)=\{(X_1, \cdots, X_n)\in \boldsymbol{R}^n \,|\, X_1{}^{2\rho_1}+\cdots+X_n{}^{2\rho_n}=\gamma^{2\rho}\}.$$

We define $\pi_\alpha : S_1(\alpha)\times \boldsymbol{R}\rightarrow \boldsymbol{R}^n$ by

$$\pi_\alpha(X_1, \cdots, X_n\,;\, u)=(u^{\alpha_1}X_1, \cdots, u^{\alpha_n}X_n).$$

Put $E=S_1(\alpha)\times \boldsymbol{R}$ and $E_0=\pi_\alpha^{-1}(0)=S_1(\alpha)\times\{0\}$. Then E is a Nash manifold and E_0 is a Nash submanifold. The restricted mapping $\pi_{\alpha|E-E_0}: E-E_0\rightarrow \boldsymbol{R}^n-\{0\}$ is a 2: 1 mapping. Therefore $\pi_\alpha : (E, E_0)\rightarrow(\boldsymbol{R}^n, 0)$ is a finite modification. Let J be an open interval and $t_0\in J$, and let $f_t : (\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}^p, 0)$ $(t\in J)$ be a weighted homogeneous polynomial mapping. We define $F : (\boldsymbol{R}^n\times J, \{0\}\times J)\rightarrow(\boldsymbol{R}^p, 0)$ by $F(x, t)=f_t(x)$.

DEFINITION. We say that $(\boldsymbol{R}^n\times J, F^{-1}(0))$ admits a *π_α-modified Nash trivialization,* if there exists a t-level preserving Nash diffeomorphism $\phi : (E\times J, E_0\times J)\rightarrow(E\times J, E_0\times J)$ which induces a t-level preserving homeomorphism ϕ: $(\boldsymbol{R}^n\times J, \{0\}\times J)\rightarrow(\boldsymbol{R}^n\times J, \{0\}\times J)$ such that

$$\phi((\boldsymbol{R}^n\times J, F^{-1}(0)))=(\boldsymbol{R}^n\times J, f_{t_0}^{-1}(0)\times J).$$

THEOREM. *Let J be an open interval, and let $f_t : (\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}^p, 0)$ be a polynomial mapping where each $f_{t,i}$ is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n\,;\, L_i)$ $(1\leqq i\leqq p)$ for $t\in J$. Assume that $F : (\boldsymbol{R}^n\times J, \{0\}\times J)\rightarrow(\boldsymbol{R}^p, 0)$ is a polynomial (or Nash) mapping. If $f_t^{-1}(0)\cap\Sigma f_t=\{0\}$ for any $t\in J$, then $(\boldsymbol{R}^n\times J, F^{-1}(0))$ admits a π_α-modified Nash trivialization.*

REMARK 3. In the case $n\leqq p$, the condition $f_t^{-1}(0)\cap\Sigma f_t=\{0\}$ implies that $f_t^{-1}(0)=\{0\}$.

EXAMPLE 1. Let $f_t : (\boldsymbol{R}^3, 0)\rightarrow(\boldsymbol{R}, 0)$ $(t\in \boldsymbol{R})$ be a weighted homogeneous polynomial of type $\alpha=(1, 2, 3\,;\, 13)$ defined by

$$f_t(x, y, z)=x^{13}+xy^6+xz^4+ty^5z.$$

Then $(\partial f_t/\partial x)=13x^{12}+y^6+z^4$. Therefore each f_t has an isolated sigularity. It follows from the Theorem that $(\boldsymbol{R}^3\times \boldsymbol{R}, F^{-1}(0))$ admits a π_α-modified Nash trivialization.

PROBLEM 1. Several kinds of topological triviality theorems for a family of analytic varieties are known. Do modified Nash triviality theorems hold under the same assumption for a family of algebraic varieties?

§ 3. Proof of the Theorem.

By Remark 3, the Theorem holds in the case $n \leqq p$. Therefore we consider the case $n > p$.

LEMMA 1 (Euler's Theorem). *If* $f: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$ *is weighted homogeneous of type* $(\alpha_1, \cdots, \alpha_n; L)$, *then*

$$\alpha_1 x_1 \frac{\partial f}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial f}{\partial x_n} = Lf.$$

PROPERTY 1. Each algebraic variety $f_t^{-1}(0)$ is transverse to $S_1(\alpha)$ and $F^{-1}(0)$ is transverse to $S_1(\alpha) \times J$. Therefore $S_1(\alpha) \cap f_t^{-1}(0)$ $(t \in J)$ and $S_1(\alpha) \times J \cap F^{-1}(0)$ are Nash submanifolds of $S_1(\alpha)$ and $S_1(\alpha) \times J$, respectively.

PROPOSITION 1. *Let* $t_0 \in J$. *Under the same assumption as the Theorem, there exists a t-level preserving* C^∞ *diffeomorphism*

$$H: S_1(\alpha) \times J \cap F^{-1}(0) \to (S_1(\alpha) \cap f_{t_0}^{-1}(0)) \times J.$$

PROOF OF PROPOSITION 1. Many singularitists would know that such a t-level preserving C^∞ diffeomorphism exists. Here we concretely construct a C^∞ vector field on $S_1(\alpha) \times J \cap F^{-1}(0)$ whose flow gives the diffeomorphism.

For $\gamma > 0$, set

$$Z_\gamma(\alpha) = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n \mid \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 = \gamma^2\}.$$

Remark that $Z_\gamma(\alpha)$ and $Z_\gamma(\alpha) \times J$ are Nash manifolds. It follows from Lemma 1 that for any $x = (x_1, \cdots, x_n) \in Z_\gamma(\alpha) \cap f_t^{-1}(0)$, $(\alpha_1 x_1, \cdots, \alpha_n x_n) \cdot \mathrm{Grad}\, f_{t,i}(x) = L_i f_{t,i}(x) = 0$ $(1 \leqq i \leqq p)$, and for any $(x, t) \in Z_\gamma(\alpha) \times J \cap F^{-1}(0)$, $(\alpha_1 x_1, \cdots, \alpha_n x_n, 0) \cdot \mathrm{Grad}\, F_i(x, t) = L_i f_{t,i}(x) = 0$ $(1 \leqq i \leqq p)$. Therefore $Z_\gamma(\alpha)$ and $Z_\gamma(\alpha) \times J$ are perpendicular to $f_t^{-1}(0)$ $(t \in J)$ and $F^{-1}(0)$, respectively. It follows that $Z_\gamma(\alpha) \cap f_t^{-1}(0)$ $(t \in J)$ and $Z_\gamma(\alpha) \times J \cap F^{-1}(0)$ are Nash submanifolds of $Z_\gamma(\alpha)$ and $Z_\gamma(\alpha) \times J$, respectively.

At first, we construct a C^∞ vector field $K(x, t)$ on $Z_\gamma(\alpha) \times J \cap F^{-1}(0)$ which is in the following form:

$$K(x, t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} W_i(x, t) \frac{\partial}{\partial x_i}.$$

It follows from the isolated singularity of $f_t^{-1}(0)$ that the vectors $\mathrm{Grad}\, f_{t,i}(x)$ $(1 \leqq i \leqq p)$ are linearly independent for any $x \in Z_\gamma(\alpha) \cap f_t^{-1}(0)$. Let $n_{t,i}(x)$ be a vector of $\boldsymbol{R}^n$ for which $\mathrm{Grad}\, f_{t,i}(x) - n_{t,i}(x)$ is the projection of $\mathrm{Grad}\, f_{t,i}(x)$ to the subspace spanned by the vectors Grad $f_{t,j}(x)$, for $j\neq i$. For $(x, t)\in Z_{\gamma}(\alpha)\times J\cap F^{-1}(0)$, we put

$$\mathrm{Grad}_x F(x, t)=(\mathrm{Grad}\, f_t(x), 0) \quad \text{in the case } p=1.$$

and

$$N_i(x, t)=(n_{t,i}(x), 0) \quad \text{in the case } p\geqq 2.$$

Then we define the Kuo vector field $K(x, t)$ ([**9**], [**10**]) on $Z_{\gamma}(\alpha)\times J\cap F^{-1}(0)$ as follows:

$$K(x, t)=\begin{cases} e_{n+1}-\sum_{i=1}^{p}\frac{\partial F_i}{\partial t}(x, t)\frac{N_i(x, t)}{|N_i(x, t)|^2} & (p\geqq 2) \\ e_{n+1}-\frac{\partial F}{\partial t}(x, t)\frac{\mathrm{Grad}_x F(x, t)}{|\mathrm{Grad}_x F(x, t)|^2} & (p=1), \end{cases}$$

where $e_{n+1}=(0, \cdots, 0, 1)\in \boldsymbol{R}^{n+1}$. In fact, this Kuo vector field is tangent to $Z_{\gamma}(\alpha)\times J\cap F^{-1}(0)$. We show it in the case $p\geqq 2$ only. The case $p=1$ follows similarly. From the construction, we have

$$\text{(3.1)} \qquad K(x, t)\cdot \mathrm{Grad}\, F_i(x, t)=0 \quad \text{on} \quad Z_{\gamma}(\alpha)\times J\cap F^{-1}(0) \ (1\leqq i\leqq p).$$

On the other hand, we can write

$$\begin{aligned} N_i(x, t) &= \mathrm{Grad}_x F_i(x, t)+(N_i(x, t)-\mathrm{Grad}_x F_i(x, t)) \\ &= \mathrm{Grad}_x F_i(x, t)+\sum_{j\neq i} a_j(x, t)\,\mathrm{Grad}_x F_j(x, t) \ (1\leqq i\leqq p), \end{aligned}$$

for $(x, t)\in Z_{\gamma}(\alpha)\times J\cap F^{-1}(0)$. It follows from Lemma 1 that for $(x, t)\in Z_{\gamma}(\alpha)\times J\cap F^{-1}(0)$,

$$(\alpha_1 x_1, \cdots, \alpha_n x_n, 0)\cdot N_i(x, t)=0 \quad (1\leqq i\leqq p).$$

Therefore we have

$$\text{(3.2)} \qquad K(x, t)\cdot(\alpha_1 x_1, \cdots, \alpha_n x_n, 0)=0 \quad \text{on} \quad Z_{\gamma}(\alpha)\times J\cap F^{-1}(0).$$

It follows from (3.1) and (3.2) that the Kuo vector field K is tangent to $Z_{\gamma}(\alpha)\times J\cap F^{-1}(0)$.

REMARK 4. It follows from the existence of the Kuo vector field that if $Z_{\gamma}(\alpha)\cap f_{t_1}^{-1}(0)\neq\emptyset$ for some $t_1\in J$, then $Z_{\gamma}(\alpha)\cap f_t^{-1}(0)\neq\emptyset$ for any $t\in J$.

Next we show that there exists a t-level preserving C^{∞} diffeomorphism H: $S_1(\alpha)\times J\cap F^{-1}(0)\to(S_1(\alpha)\cap f_{t_0}^{-1}(0))\times J$. By Remark 4, we may assume that $S_1(\alpha)\cap f_t^{-1}(0)\neq\emptyset$ for any $t\in J$, since f_t is weighted homogeneous. As the Kuo vector field $K(x, t)\in T_{(x,t)}F^{-1}(0)$, we have

PROPERTY 2. For any $(x, t)\in F^{-1}(0)-\{t\text{-axis}\}$, the tangent space $T_{(x,t)}F^{-1}(0)$ is not parallel to x-space.

By Properties 1, 2, we have

PROPERTY 3. For any $(x, t) \in S_1(\alpha) \times J \cap F^{-1}(0)$, the tangent space $T_{(x,t)}(S_1(\alpha)\times J\cap F^{-1}(0))$ is not parallel to x-space.

We put

$$T_{(x,t)}\boldsymbol{R}^n\times J = \boldsymbol{R}^{n+1}_{(x,t)} \quad \text{and} \quad T_{(x,t)}\{0\}\times J=\boldsymbol{R}_t.$$

For any $(x, t)\in S_1(\alpha)\times J\cap F^{-1}(0)$, there exists $\gamma>0$ such that

$$(x, t) \in Z_\gamma(\alpha)\times J\cap F^{-1}(0).$$

Let $\pi: \boldsymbol{R}^{n+1}_{(x,t)} \to T_{(x,t)}(S_1(\alpha)\times J\cap F^{-1}(0))$ be the orthogonal projection, and let $\pi_t: \boldsymbol{R}^{n+1}_{(x,t)} \to \boldsymbol{R}_t$ be the standard projection. It follows from Property 3 that for $K(x, t)\in T_{(x,t)}(Z_\gamma(\alpha))\times J\cap F^{-1}(0))\subset\boldsymbol{R}^{n+1}_{(x,t)}$, $\pi_t(d\pi(K(x, t)))\neq 0$. We define

$$\tilde{K}(x, t) = d\pi(K(x, t))/|\pi_t(d\pi(K(x, t)))|$$

for $(x, t)\in S_1(\alpha)\times J\cap F^{-1}(0)$. Then $\tilde{K}$ is a C^∞ vector field on $S_1(\alpha)\times J\cap F^{-1}(0)$ which is in the following form:

$$\tilde{K}(x, t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \tilde{W}_i(x, t)\frac{\partial}{\partial x_i}.$$

It is easy to see that the flow of this vector field $\tilde{K}$ gives the t-level preserving diffeomorphism H. This completes the proof of Proposition 1.

Next we show a semi-algebraic triviality theorem for a family of compact Nash manifolds. In this paper, we assume that a submersion is surjective.

NOTATION. Let M be a C^1 manifold with or without boundary, and let $p: M\to\boldsymbol{R}$ be a proper C^1 submersion. Then we write $M_t=p^{-1}(t)$ for $t\in\boldsymbol{R}$.

PROPOSITION 2. *Let $M\supset N$ be a Nash manifold and a Nash (regular) submanifold such that N is closed in M. Let $p: M\to\boldsymbol{R}$ be a proper Nash submersion such that the restriction of p to N is also a proper Nash submersion. Then there exists a Nash diffeomorphism*

$$\Phi: (M, N) \longrightarrow (M_0, N_0)\times\boldsymbol{R}$$

such that $p\circ\Phi^{-1}$ is the canonical projection onto $\boldsymbol{R}$, namely, $p\circ\Phi^{-1}(m_0, t) = t$ for $(m_0, t)\in M_0\times\boldsymbol{R}$.

REMARK 5. We can replace $\boldsymbol{R}$ by an open interval J in Proposition 2.

We prepare some lemmas to show this proposition.

LEMMA 2 ([3] Nash trivialization theorem). *Let M be a Nash manifold, and let $p: M\to\boldsymbol{R}$ be a proper Nash submersion. Then there exists a Nash diffeomorphism $\phi: M\to M_0\times\boldsymbol{R}$ such that $p\circ\phi^{-1}$ is the canonical projection.*

LEMMA 3 ([**3**] Theorem 3). *Let* $0<r<\infty$. *Let* M *be a* C^r *Nash manifold with boundary* N, *and let* $p: M \to \boldsymbol{R}$ *be a proper* C^r *Nash submersion such that the restriction of* p *to* N *is also a proper* C^r *Nash submersion. Then there exists a* C^r *Nash diffeomorphism*

$$\phi: (M, N) \longrightarrow (M_0, N_0)\times \boldsymbol{R}$$

such that $p\circ\phi^{-1}$ *is the canonical projection onto* $\boldsymbol{R}$.

LEMMA 4 ([**3**] Theorem 8). *Let* $0<r<\infty$. *Let* S *be a* C^r *Nash manifold, and let* Q *be a* C^r *Nash manifold with boundary. Let* $(\pi, q): Q \to S\times\boldsymbol{R}$ *be a proper* C^r *Nash submersion such that the restriction of* (π, q) *to the boundary is also a proper* C^r *Nash submersion. Denote* $(Q)_0=Q\cap q^{-1}(0)$. *Then there exists a* C^r *Nash diffeomorphism*

$$h=(h_0, q): Q \longrightarrow (Q)_0\times \boldsymbol{R}$$

such that h_0 *is the identity on* $(Q)_0$ *and* $\pi\circ h_0=\pi$.

LEMMA 5 ([**16**] Lemma 1.3.2). *Given a* C^r *Nash manifold* M *in* $\boldsymbol{R}^n$ *for* $r>1$, *there exists a* C^{r-1} *Nash tubular neighbourhood* U *of* M *in* $\boldsymbol{R}^n$ (*i.e.* U *is a Nash manifold and the orthogonal projection* $\chi: U \to M$ *is a* C^{r-1} *Nash map*). *Here the radius of each fibre* $\chi^{-1}(x)$ *is not a constant, and it may come near* 0 *as* x *comes near the boundary of* M *or infinity.*

LEMMA 6 ([**16**] Corollary 2.5.7). *Let* $M_1\subset M$ *be Nash manifolds such that* M_1 *is closed in* M, *let* M_2 *be a Nash manifold, and let* $f: M\to M_2$ *be a* C^r *Nash map,* $r<\infty$, *such that* $f|_{M_1}$ *is of class Nash. Then we can approximate* f *by a Nash mapping* g *in the* C^r *topology so that* $f=g$ *on* M_1.

LEMMA 7 ([**16**] Lemma 2.1.7). *Let* M, M' *be* C^r *Nash manifolds for* $r>0$ *and let* $\phi: M\to M'$ *be a* C^r *Nash diffeomorphism. Then any* C^r *Nash close approximation* ϕ' *of* ϕ *in the* C^r *topology is a diffeomorphism.*

For the definition of C^r topology, see M. Shiota [**16**].

PROOF OF PROPOSITION 2. It follows from Lemma 2 that there exists a Nash diffeomorphism $\phi: N\to N_0\times\boldsymbol{R}$ such that $p|_N\circ\phi^{-1}$ is the canonical projection onto $\boldsymbol{R}$. Remark that we can take ϕ so that $\phi|_{N_0}(n_0)=(n_0, 0)$.

We identify $M(\subset\boldsymbol{R}^k)$ with the graph of p. Therefore we suppose that $N\subset M\subset\boldsymbol{R}^k\times\boldsymbol{R}$ and $p: \boldsymbol{R}^k\times\boldsymbol{R}\to\boldsymbol{R}$ is the canonical projection. By lemma 5, there exists a Nash tubular neighbourhood $v_1: T_1\to N$ in $\boldsymbol{R}^k\times\boldsymbol{R}$. Since p: $M\to\boldsymbol{R}$ is a Nash submersion such that $p|_N: N\to\boldsymbol{R}$ is also a Nash submersion, shrinking T_1 if necessary, we can assume that $B=M\cap T_1$ is a Nash manifold (with boundary) and $p|_B=p\circ v_1|_B$. Then $\beta=v_1|_B: B\to N$ is a Nash submersion such that $p\circ\beta=p|_B$. Here let $0<r<\infty$. We apply Lemma 4 with $Q=B$, $S=N_0$ and $(\pi, q)=\phi\circ\beta$. Then there exists a C^r Nash diffeomorphism

$$h=(h_0, q): B \longrightarrow B_0\times \boldsymbol{R}$$

such that $p\circ h^{-1}$ is the canonical projection onto $\boldsymbol{R}$, namely, $q=p|_B$. From the construction, we have $(\pi, q)|_N=\phi$. Let $\phi(n)=(n_0, t)$ for $n\in N$. Then we have

$$h_0(n)=\pi\circ h_0(n)=\pi(n)=n_0.$$

It follows that

$$h|_N\circ\phi^{-1}(n_0, t)=h(n)=(h_0(n), q(n))=(n_0, t).$$

namely, $h|_N\circ\phi^{-1}$ is the identity on $N_0\times\boldsymbol{R}$. Therefore $h|_N=\phi$ is a Nash diffeomorphism from N to $N_0\times\boldsymbol{R}$.

Let D_0 be an open Nash submanifold of M_0 such that $\bar{D}_0$ is a Nash manifold with boundary and $N_0\subset\bar{D}_0\subset\operatorname{Int}B_0$ where $\bar{D}_0$ denotes the closure of D_0 in M_0. We put $C=h^{-1}(D_0\times\boldsymbol{R})$. We further put $V=M-C$ and $W=h^{-1}(\partial D_0\times\boldsymbol{R})$. Then V is a C^r Nash manifold with boundary W and h gives a Nash diffeomorphism $h_w: W\to W_0\times\boldsymbol{R}$ such that $p\circ h_w^{-1}$ is the canonical projection onto $\boldsymbol{R}$. Since $p|_V: V\to\boldsymbol{R}$ and $p|_W: W\to\boldsymbol{R}$ are proper C^r Nash submersions, it follows from Lemma 3 that there exists a C^r Nash diffeomorphism

$$\psi=(\psi_0, p): (V, W)\longrightarrow (V_0, W_0)\times\boldsymbol{R}$$

such that $p\circ\psi^{-1}$ is the canonical projection onto $\boldsymbol{R}$. Here, by the arguments in [3] § 2, we can take ψ so that $\psi|_W=h_w$ and we can extend ψ over some semialgebraic neighbourhood U_1 of V in M. Let U_2 be a sufficiently small semialgebraic neighbourhood of W in M such that $U_2\subset U_1\cap B$. By applying a C^r Nash partition of unity (c.f. [16] Chapter 2, § 2.2.), we can construct a C^r Nash function $e: M\to\boldsymbol{R}$ with $0\leqq e\leqq 1$ such that

$$e=1 \quad \text{on} \quad \overline{V-U_2} \quad \text{and} \quad e=0 \quad \text{on} \quad \bar{C}.$$

Let $v_2: T_2\to M_0$ be a Nash tubular neighbourhood in $\boldsymbol{R}^k$. Define $\Psi=(\Psi_0, p): (M, N)\to(M_0, N_0)\times\boldsymbol{R}$ by

$$\Psi(m)=\begin{cases}\psi(m) & \text{for } m\in V-U_2,\\ (v_2(e(m)\psi_0(m)+(1-e(m))h_0(m)),\ p(m)) & \\ \qquad\qquad \text{for } m\in(V-U_2)^c \text{ with } e(m)>0, & \\ h(m) & \text{for } m\in(V-U_2)^c \text{ with } e(m)=0,\end{cases}$$

where $(V-U_2)^c$ denotes the complement of $V-U_2$ in M. Then Ψ is a C^r Nash diffeomorphism such that $\Psi|_N=\phi$ and $p: M\to\boldsymbol{R}$ is a Nash function. Therefore it follows from Lemmas 6, 7 that there exists a Nash diffeomorphism

$$\Phi: (M, N) \longrightarrow (M_0, N_0) \times \boldsymbol{R}$$

such that $p \circ \Phi^{-1}$ is the canonical projection onto $\boldsymbol{R}$. This completes the proof of Proposition 2.

Finally we show the Theorem by using Propositions 1 and 2. By Proposition 1, there exists a t-level preserving diffeomorphism

$$H: S_1(\alpha) \times J \cap F^{-1}(0) \longrightarrow (S_1(\alpha) \cap f_{t_0}^{-1}(0)) \times J.$$

Then it follows from Proposition 2 that there exists a t-level preserving Nash diffeomorphism $G: S_1(\alpha) \times J \to S_1(\alpha) \times J$ such that

$$G(S_1(\alpha) \times J \cap F^{-1}(0)) = (S_1(\alpha) \cap f_{t_0}^{-1}(0)) \times J.$$

We write $G(x, t) = (\sigma_t(x), t)$ for $(x, t) \in S_1(\alpha) \times J$. Then

$$\sigma_t = (\sigma_t^{(1)}, \cdots, \sigma_t^{(n)}):$$

$$(S_1(\alpha), S_1(\alpha) \cap f_t^{-1}(0)) \longrightarrow (S_1(\alpha), S_1(\alpha) \cap f_{t_0}^{-1}(0))$$

is a Nash diffeomorphism for each $t \in J$, where σ_{t_0} is the identity on $S_1(\alpha)$.

Let $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$ be written as follows:

$$x_i = u^{\alpha_i} P_i \quad \text{for} \quad u \geqq 0 \quad \text{and} \quad P = (P_1, \cdots, P_n) \in S_1(\alpha)$$

$(1 \leqq i \leqq n)$. Then we define a homeomorphism

$$\psi_t = (\psi_t^{(1)}, \cdots, \psi_t^{(n)}): (\boldsymbol{R}^n, 0) \longrightarrow (\boldsymbol{R}^n, 0)$$

such that $\psi_t(f_t^{-1}(0)) = f_{t_0}^{-1}(0)$ by

$$\psi_t^{(i)}(x) = u^{\alpha_i} \sigma_t^{(i)}(P) \qquad (1 \leqq i \leqq n).$$

Next we define a mapping $\phi: (\boldsymbol{R}^n \times J, \{0\} \times J) \to (\boldsymbol{R}^n \times J, \{0\} \times J)$ by

$$\phi(x, t) = (\psi_t(x), t).$$

Then ϕ is a t-level preserving homeomorphism such that

$$\phi((\boldsymbol{R}^n \times J, F^{-1}(0))) = (\boldsymbol{R}^n \times J, f_{t_0}^{-1}(0) \times J).$$

We further define a mapping $\Phi: (E \times J, E_0 \times J) \to (E \times J, E_0 \times J)$ by

$$\Phi((x; s), t) = ((\sigma_t(x); s), t).$$

Then by the construction of σ_t we can require that Φ is a t-level preserving Nash diffeomorphism such that Φ induces ϕ. Therefore $(\boldsymbol{R}^n \times J, F^{-1}(0))$ admits a π_α-modified Nash trivialization.

§ 4. Strong C^0 equivalence.

First, we define the notion of strong C^0 equivalence.

NOTATION. (1) By an analytic arc at $0 \in \boldsymbol{R}^n$, we mean the germ of an analytic map $\lambda: [0, \varepsilon) \to \boldsymbol{R}^n$ with $\lambda(0)=0$, $\lambda(s) \neq 0$, $s>0$. The set of all such arcs is denoted by $\mathcal{A}(\boldsymbol{R}^n, 0)$.

(2) For $\lambda, \mu \in \mathcal{A}(\boldsymbol{R}^n, 0)$, $O(\lambda, \mu)>1$ (resp. $O(\lambda, \mu)=1$) means that arcs λ, μ are *tangent* (resp. *crossing without touching*) at $0 \in \boldsymbol{R}^n$.

Let $\mathcal{E}_{[\omega]}(n, 1)$ be the set of analytic function germs: $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$, and let $\mathcal{S}(\boldsymbol{R}^n, 0)$ be the set of set germs at $0 \in \boldsymbol{R}^n$.

DEFINITION. Given $f, g \in \mathcal{E}_{[\omega]}(n, 1)$, we say that $(\boldsymbol{R}^n, f^{-1}(0))$, $(\boldsymbol{R}^n, g^{-1}(0)) \in \mathcal{S}(\boldsymbol{R}^n, 0)$ are *strongly C^0 equivalent*, if there exists a local homeomorphism σ: $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that

(I) $\sigma(f^{-1}(0))=g^{-1}(0)$,

(II) if $\lambda \in \mathcal{A}(\boldsymbol{R}^n, 0)$ with $\lambda \subset f^{-1}(0)$ (resp. $g^{-1}(0)$), then $\sigma(\lambda)$ (resp. $\sigma^{-1}(\lambda)) \in \mathcal{A}(\boldsymbol{R}^n, 0)$, and

(III) for any $\lambda, \mu \in \mathcal{A}(\boldsymbol{R}^n, 0)$ with $\lambda, \mu \subset f^{-1}(0)$, $O(\lambda, \mu)=1$ if and only if $O(\sigma(\lambda), \sigma(\mu))=1$.

EXAMPLE 2. Let $f_k: (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ $(k \in \boldsymbol{N})$ be a polynomial function defined by

$$f_k(x, y) = x^2 - y^{2k+1}.$$

Then $(\boldsymbol{R}^2, f_i^{-1}(0))$, $(\boldsymbol{R}^2, f_j^{-1}(0)) \in \mathcal{S}(\boldsymbol{R}^2, 0)$ are strongly C^0 equivalent for any i, $j \in \boldsymbol{N}$.

Let J be an open interval, and let $f_t: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$ $(t \in J)$ be a weighted homogeneous polynomial of type $\alpha=(\alpha_1, \cdots, \alpha_n)$ with an isolated singularity. In this section, we discuss the relation between π_α-modified Nash triviality and strong C^0 triviality of the family $\{(\boldsymbol{R}^n, f_t^{-1}(0))\}_{t \in J}$.

(A) Consider the homogeneous case: $\alpha_1 = \cdots = \alpha_n = 1$. Recall the notations $E=S_1(\alpha) \times \boldsymbol{R}$ and $E_0=S_1(\alpha) \times \{0\}$. We say that $(X; u)=(X_1, \cdots, X_n; u)$, $(Y, s)=(Y_1, \cdots, Y_n; s) \in E$ are *equivalent*. if

(i) $X_i = Y_i$ $(1 \leqq i \leqq n)$ and $u=s$, or

(ii) $X_i = -Y_i$ $(1 \leqq i \leqq n)$ and $u=-s$.

Then this relation is an equivalence relation. We denote by $\tilde{E}$ and $\tilde{E}_0$ the quotient sets of E and E_0 by the relation $\sim$, respectively. Let $\pi: (E, E_0) \to (\tilde{E}, \tilde{E}_0)$ be the quotient map, and let $\tilde{\pi}_\alpha: (\tilde{E}, \tilde{E}_0) \to (\boldsymbol{R}^n, 0)$ be the blow up at $0 \in \boldsymbol{R}^n$. Then the following diagram commutes:

$$\begin{array}{ccc} (E,\ E_0) & & \\ \downarrow \pi & \searrow^{\pi_\alpha} & \\ (\tilde{E},\ \tilde{E}_0) & \xrightarrow{\tilde{\pi}_\alpha} & (\boldsymbol{R}^n,\ 0). \end{array}$$

By the Theorem, $\{(\boldsymbol{R}^n,\ f_t^{-1}(0))\}_{t\in J}$ admits a π_α-modified Nash trivialization i.e. there exists a t-level preserving Nash diffeomorphism Φ: $(E\times J,\ E_0\times J)\to(E\times J,\ E_0\times J)$ which induces a t-level preserving homeomorphism ϕ: $(\boldsymbol{R}^n\times J,\ \{0\}\times J)\to(\boldsymbol{R}^n\times J,\ \{0\}\times J)$ such that

$$\phi((\boldsymbol{R}^n\times J,\ F^{-1}(0)))=(\boldsymbol{R}^n\times J,\ f_{t_0}^{-1}(0)\times J) \qquad \text{for} \quad t_0\in J.$$

In this case, the Nash diffeomorphism Φ induces a t-level preserving Nash diffeomorphism $\tilde{\Phi}$: $(\tilde{E}\times J,\ \tilde{E}_0\times J)\to(\tilde{E}\times J,\ \tilde{E}_0\times J)$ such that the following diagram commutes:

$$\begin{array}{ccccc} (E\times J,\ E_0\times J) & \xrightarrow{(\pi,\ id)} & (\tilde{E}\times J,\ \tilde{E}_0\times J) & \xrightarrow{(\tilde{\pi}_\alpha,\ id)} & (\boldsymbol{R}^n\times J,\ \{0\}\times J) \\ \downarrow \Phi & & \downarrow \tilde{\Phi} & & \downarrow \phi \\ (E\times J,\ E_0\times J) & \xrightarrow{(\pi,\ id)} & (\tilde{E}\times J,\ \tilde{E}_0\times J) & \xrightarrow{(\tilde{\pi}_\alpha,\ id)} & (\boldsymbol{R}^n\times J,\ \{0\}\times J). \end{array}$$

Therefore π_α-modified Nash triviality implies strong C^0 triviality.

(B) Consider the case where $n=3$ and $\alpha_1<\alpha_2<\alpha_3$.

PROPOSITION 3. *Let* f, g: $(\boldsymbol{R}^3,\ 0)\to(\boldsymbol{R},\ 0)$ *be weighted homogeneous polynomials of type* $(\alpha_1,\ \alpha_2,\ \alpha_3;\ L)$ $(\alpha_1<\alpha_2<\alpha_3)$ *with an isolated singularity. Assume that two set germs* $(\boldsymbol{R}^3,\ f^{-1}(0))$, $(\boldsymbol{R}^3,\ g^{-1}(0))\in\mathcal{S}(\boldsymbol{R}^3,\ 0)$ *are strongly* C^0 *equivalent. If* $f^{-1}(0)\supset\{x_1=0\}$, *then* $g^{-1}(0)\supset\{x_1=0\}$.

We shall show this proposition in the next section.

EXAMPLE 1. Let f_t: $(\boldsymbol{R}^3,\ 0)\to(\boldsymbol{R},\ 0)$ $(t\in\boldsymbol{R})$ be a weighted homogeneous polynomial defined in § 2. Then, by Proposition 3, $\{(\boldsymbol{R}^3,\ f_t^{-1}(0))\}_{t\in\boldsymbol{R}}$ is not strongly C^0 trivial at $0\in\boldsymbol{R}$.

EXAMPLE 3 (Briançon-Speder family [**1**]). Let f_t: $(\boldsymbol{R}^3,\ 0)\to(\boldsymbol{R},\ 0)$ be a weighted homogeneous polynomial defined by

$$f_t(x,\ y,\ z)=z^5+tzy^6+y^7x+x^{15}$$

for $|t|<1+\varepsilon$, where ε is a sufficiently small positive number. Then each f_t has an *algebraically* isolated singularity. But two set germs $(\boldsymbol{R}^3,\ f_0^{-1}(0))$, $(\boldsymbol{R}^3,\ f_{-1}^{-1}(0))\in\mathcal{S}(\boldsymbol{R}^3,\ 0)$ are not strongly C^0 equivalent (Theorem A in [**6**]).

It follows from the Theorem and the above examples that modified Nash triviality does not imply strong C^0 triviality in the non-homogeneous case.

REMARK 6. In [8], the author formulated a necessary condition for a family of weighted homogeneous polynomials of three variables to be strongly C^0 trivial. Recently T. Fukui has given a new approach to strong C^0 triviality of a family of polynomial functions of three variables, by using toric resolution ([4]).

§5. Proof of Proposition 3.

In this section, let $\alpha=(\alpha_1, \alpha_2, \alpha_3)$ be a 3-tuple of positive integers with $\alpha_1<\alpha_2<\alpha_3$, and let x, y, z be the coordinates of $\boldsymbol{R}^3$. Then we have the following lemma.

LEMMA 8. *Let* $f:(\boldsymbol{R}^3, 0)\to(\boldsymbol{R}, 0)$ *be a weighted homogeneous polynomial of type* $(\alpha_1, \alpha_2, \alpha_3; L)$ *with an isolated singularity, and let* $g:(\boldsymbol{R}^3, 0)\to(\boldsymbol{R}, 0)$ *be weighted homogeneous of type* $(\alpha_1, \alpha_2, \alpha_3; L)$. *If* $f^{-1}(0)\supset\{x=0\}$, *then we have* $g^{-1}(0)\supset\{x=y=0\}\cup\{x=z=0\}$.

PROOF. Since $f^{-1}(0)\supset\{x=0\}$, we can write $f(x, y, z)=x\phi(x, y, z)$ where ϕ is a polynomial. It follows from the isolated singularity of f that

(*) $\phi^{-1}(0)\cap\{x=0\}=\{0\}$.

As $0\in\boldsymbol{R}^3$ is a singular point of f, $\phi(0)=0$.

Now assume that there is no term of the form $cz^n(c\neq 0)$ in ϕ. Then we have

$$\phi^{-1}(0)\cap\{x=0\}\supset\{x=y=0\}.$$

This contradicts (*). Therefore ϕ contains the term $cz^n(c\neq 0)$ for some n. Similarly ϕ contains the term $by^m(b\neq 0)$ for some m. Therefore f contains both terms $bxy^m(b\neq 0)$ and $cxz^n(c\neq 0)$.

Since f and g are weighted homogeneous of type $(\alpha_1, \alpha_2, \alpha_3; L)$ and $\alpha_1<\alpha_2<\alpha_3$, g contains neither term of the form $dy^s(d\neq 0)$ nor term of the form $ez^t(e\neq 0)$. Therefore we can write

$$g(x, y, z)=x\psi(x, y, z)+yzh(y, z).$$

It follows that $g^{-1}(0)\supset\{x=y=0\}\cup\{x=z=0\}$.

In order to show Proposition 3, assume that f, $g:(\boldsymbol{R}^3, 0)\to(\boldsymbol{R}, 0)$ are weighted homogeneous polynomials of type $(\alpha_1, \alpha_2, \alpha_3; L)$ with an isolated singularity, from now on. In addition, assume that $(\boldsymbol{R}^3, f^{-1}(0))$, $(\boldsymbol{R}^3, g^{-1}(0))\in\mathcal{S}(\boldsymbol{R}^3, 0)$ are strongly C^0 equivalent. Then there exists a local homeomorphism $\sigma:(\boldsymbol{R}^3, 0)\to(\boldsymbol{R}^3, 0)$ which gives their strong C^0 equivalence.

PROPERTY 4 (H. Whitney [17]). The number of connected components of $f^{-1}(0)-\{0\}$ (or $g^{-1}(0)-\{0\}$) is finite.

Therefore we can write

$$f^{-1}(0)-\{0\} = \{C_1, \cdots, C_d\}$$
$$g^{-1}(0)-\{0\} = \{C'_1, \cdots, C'_d\} \qquad (d \in \boldsymbol{N}\cup\{0\}),$$

where C_i(resp. C'_i) $(1 \leqq i \leqq d)$ is a connected component of $f^{-1}(0)-\{0\}$ (resp. $g^{-1}(0)-\{0\}$). For simplicity, we assume that $C'_i=\sigma(C_i)$ $(1\leqq i\leqq d)$ as germs at $0\in \boldsymbol{R}^3$.

We shall show a more precise statement of Proposition 3.

PROPOSITION 3′. *If* $\bar{C}_1=\{x=0\}$, *then* $\bar{C}'_1=\{x=0\}$, *where* $\bar{C}$ *denotes the closure of* C *in* $\boldsymbol{R}^3$.

Let S^2 denote the two dimensional unit sphere. For $P=(P_1, P_2, P_3)\in S^2$, let $L(P)$, $a(P)$: $[0, \delta)\to \boldsymbol{R}^3$ $(\delta>0)$ be mappings defined by

$$L(P)(t)=(P_1t, P_2t, P_3t) \quad \text{and} \quad a(P)=(P_1t^{\alpha_1}, P_2t^{\alpha_2}, P_3t^{\alpha_3}),$$

respectively. Then $L(P)$, $a(P)\in \mathcal{A}(\boldsymbol{R}^3, 0)$. For any $\lambda\in\mathcal{A}(\boldsymbol{R}^3, 0)$, there exists unique $P\in S^2$ such that $O(\lambda, L(P))>1$. Then we write $L(P)=T(\lambda)$.

LEMMA 9. *For a weighted homogeneous polynomial of type* $(\alpha_1, \alpha_2, \alpha_3)$ h, *let* $h^{-1}(0)-\{0\}=\{C_1, \cdots, C_d\}$. *Assume that* $C_i\subset\{x>0\}$ (*resp.* $C_i\subset\{x<0\}$). *Then*

$$T(\lambda)=L((1, 0, 0)) \ (\textit{resp. } T(\lambda)=L((-1, 0, 0)))$$

for any $\lambda\in\mathcal{A}(\boldsymbol{R}^3, 0)$ *with* $\lambda\subset\bar{C}_i$.

PROOF. Since $C_i\cap\{x=0\}=\emptyset$, there exists $K>0$ such that

$$\bar{C}_i\subset\{x\geqq0\}\cap\{K(|y|^{\alpha_1/\alpha_2}+|z|^{\alpha_1/\alpha_3})\leqq|x|\}$$
$$(\text{resp. } \bar{C}_i\subset\{x\leqq0\}\cap\{K(|y|^{\alpha_1/\alpha_2}+|z|^{\alpha_1/\alpha_3})\leqq|x|\}).$$

Therefore the statement of the lemma immediately follows.

PROOF OF PROPOSITION 3′. At first we show the case $d=1$. Therefore let $f^{-1}(0)=\bar{C}_1$ and $g^{-1}(0)=\bar{C}'_1$. Assume that $\bar{C}'_1\neq\{x=0\}$. Then we have

$$\#(S^2\cap g^{-1}(0)\cap\{x=0\})<\infty.$$

Since $S^2\cap f^{-1}(0)$ is homeomorphic to a circle, so is $S^2\cap g^{-1}(0)$. We write

$$S^2\cap g^{-1}(0)\cap\{x\neq0\}=\{D_1, \cdots, D_k\} \qquad (k\in\boldsymbol{N}).$$

where D_j $(1\leqq j\leqq k)$ is a connected component. Put

$$\begin{cases} Q_+ = (0,\, 0,\, 1), & Q_- = (0,\, 0,\, -1), \\ R_+ = (0,\, 1,\, 0), & R_- = (0,\, -1,\, 0). \end{cases}$$

By Lemma 8, we have

$$\{Q_+, Q_-, R_+, R_-\} \subset S^2 \cap g^{-1}(0) \cap \{x = 0\}.$$

Therefore one of the following conditions holds:

(i) $\#\{D_j \mid D_j \subset S^2 \cap g^{-1}(0) \cap \{x>0\}\} \geqq 2$,

(ii) $\#\{D_j \mid D_j \subset S^2 \cap g^{-1}(0) \cap \{x<0\}\} \geqq 2$.

Now assume that (i) (resp. (ii)) holds. Let $D_1, D_2 \subset S^2 \cap g^{-1}(0) \cap \{x>0\}$ (resp. $S^2 \cap g^{-1}(0) \cap \{x<0\}$). Pick $P_1 \in D_1$ and $P_2 \in D_2$. We have $T(a(P_1)) = T(a(P_2))$. We write

$$S^2 \cap g^{-1}(0) - D_1 \cup D_2 = E_1 \cup E_2,$$

where E_j ($j=1, 2$) is a connected component. Then $E_j \cap \{x=0\} \neq \emptyset$ for $j=1, 2$. Pick $P_3 \in E_1 \cap \{x=0\}$ and $P_4 \in E_2 \cap \{x=0\}$. Then we have

$$T(a(P_1)) = T(a(P_2)) \neq T(a(P_j)) \quad (j = 3, 4).$$

Since σ is a strong homeomorphism, it follows that

$$T(\sigma^{-1}(a(P_1))) = T(\sigma^{-1}(a(P_2))) \neq T(\sigma^{-1}(a(P_j))) \quad (j = 3, 4).$$

This is a contradiction, because $\sigma^{-1}(g^{-1}(0)) = \{x=0\}$. Therefore we deduce that $\bar{C}_1' = \{x=0\}$.

Next we show the case $d \geqq 2$. Since $\bar{C}_1 = \{x=0\}$, $C_i \cap \{x=0\} = \emptyset$ $(2 \leqq i \leqq d)$. Assume that $C_j' \cap \{x=0\} \neq \emptyset$ for some $j \geqq 2$. In the case where $\bar{C}_j' = \{x = 0\}$, $a((0, 1, 0)), a((0, 0, 1)) \subset C_j'$. Then we have

$$O(a((0,\, 1,\, 0)),\, a((0,\, 0,\, 1))) = 1.$$

On the other hand, by Lemma 9, we have

$$O(\sigma^{-1}(a((0,\, 1,\, 0))),\, \sigma^{-1}(a((0,\, 0,\, 1)))) > 1.$$

This contradicts the strong C^0 equivalence. In the case where $\bar{C}_j' \neq \{x = 0\}$, pick

$$P \in C_j' \cap S^2 \cap \{x = 0\} \quad \text{and} \quad Q \in C_j' \cap S^2 \cap \{x \neq 0\}.$$

Then we have $O(a(P), a(Q)) = 1$. Similarly to the above, we have $O(\sigma^{-1}(a(P)), \sigma^{-1}(a(Q))) > 1$. This also contradicts the strong C^0 equivalence. Therefore $C_i' \cap \{x=0\} = \emptyset$ for $2 \leqq i \leqq d$. By Lemma 8, we have

$$\bar{C}_1' \supset \{x = y = 0\} \cup \{x = z = 0\}.$$

Then we deduce $\bar{C}_1' = \{x=0\}$ from the same argument as the case $d=1$.

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