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The initial-boundary value problems for the heat operator in non-cylindrical domains

Dedicated to Professor M. Nakai on the occasion of his 60th birthday

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1. Introduction.

Let Ω be a bounded C^1-domain in $\boldsymbol{R}^n$ and set

$$\Omega_T = \Omega\times(0, T), \quad S_T = \partial\Omega\times(0, T).$$

E. B. Fabes and M. N. Rivière proved a Fatou type theorem for the heat operator in [**FR**]; If $1<p<+\infty$, then for each $f\in L^p(S_T)$ there exists a function $g\in L^p(S_T)$ such that the double layer heat potential u of g is caloric in Ω_T and has the limit $f(P)$, nontangentially on the hyperplane $t=t_1$, at almost every point $P=(p, t_1)\in S_T$ with respect to the surface measure of S_T.

Moreover in the case $p>2-\varepsilon$ it has been known that this result is still valid even if Ω is a bounded Lipschitz domain and an approach region is replaced by a parabolic nontangential one (cf. [**FS**], [**Br1**], [**Br2**]).

R. M. Brown also considered the initial-Neumann problems for the heat operator in the above Lipschitz cylinder and proved that if $1<p<2+\varepsilon$, then a solution with lateral data in $L^p(S_T)$ exists and is represented as a single layer heat potential ([**Br1**], [**Br2**]).

On the other hand, for a bounded open subset Ω of $\boldsymbol{R}^{n+1}$ and a continuous real-valued function g on $\partial\Omega$ there exists a generalized solution u to the Dirichlet problem

$$\triangle u-\frac{\partial u}{\partial t} = 0 \quad \text{on } \Omega,$$

$$u_{|\partial\Omega} = g$$

by the Perron-Wiener-Brelot-Bauer method, but it is not true in general that u attains continuously the boundary value g (cf. [**Ba**]). A boundary point (x_0, t_0) is regular, i.e.,

$$\lim_{(x, t)\to(x_0, t_0),\, (x, t)\in\Omega} u(x, t) = g(x_0, t_0)$$

for every continuous function g on $\partial\Omega$ if and only if it satisfies Wiener's criterion (cf. [L], [EG]). Such characterizations for boundary points to be regular have also been studied for more general parabolic operators (cf. [GL], [FGL]).

R. Kaufman and J-M Wu considered, for $\tau\in C^2(\boldsymbol{R})$, a domain $\Omega=\{(x, t): t>\tau(x)\}$ in $\boldsymbol{R}^2$ and proved that the solution u to the Dirichlet problem for the heat operator in Ω with boundary data $f\in L^p$ exists and the parabolic nontangential limit of u is equal to f at almost every boundary point (cf. [KW]).

Suppose D is a bounded domain in $\boldsymbol{R}^{n+1}$, which is bounded by a domain B_D on the hyperplane $t=0$, a domain T_D on the hyperplane $t=T$ and a smooth manifold lying in the strip $0<t<T$. We know that under additional assumptions the initial-Dirichlet problem

$$\triangle u-\frac{\partial u}{\partial t}=0 \quad \text{on } D, \qquad u=f \quad \text{on } S_D, \qquad u=0 \quad \text{on } B_D \tag{1.1}$$

has a unique classical solution on $\bar{D}$ for all functions f on S_D such that

$$|f(X)-f(Y)| \leqq c_f \delta(X, Y)^{\lambda}$$

for all $X, Y\in S_D$ and for some positive real numbers c_f, λ (cf. [Fr, Theorem 7, p. 65]). Recall that the parabolic distance $\delta(X, Y)$ is defined by

$$\delta(X, Y)=(|x-y|^2+|t-s|)^{1/2},$$

for $X=(x, t)$ and $Y=(y, s)$.

But, for such a non-cylindrical domain D, the existence and boundary behavior of the solution to the problem (1.1) with L^p-lateral data f has not been discussed, even if S_D is smooth.

In this paper, we consider the initial-Dirichlet and initial-Neumann problems for the heat equation in such a domain with L^p-boundary data, and study boundary behavior of solutions.

More precisely it is assumed that a bounded domain D in $\boldsymbol{R}^{n+1}$ lies in the strip $0<t<T$ and ∂D is the union of three closed sets, B_D, T_D and S_D such that

$$B_D=\partial D\cap\{t=0\}, \quad T_D=\partial D\cap\{t=T\}, \quad S_D=\overline{\partial D\cap\{0<t<T\}}$$

and S_D has the following property (s).

(s) For each $Z=(z, s_0)\in S_D$ there correspond a system of orthogonal coordinates in $\boldsymbol{R}^n$ and an open ball $B(Z, \delta)$ in $\boldsymbol{R}^{n+1}$ such that each coordinate axis is orthogonal to the time-axis and with respect to the coordinates

$$S_D \cap B(Z, \delta)$$
$$= \{(\xi', \xi_n, t) : \xi' \in \boldsymbol{R}^{n-1},\ \xi_n \in \boldsymbol{R},\ \xi_n = \phi(\xi', t),\ 0 \leqq t \leqq T\} \cap B(Z, \delta)$$

and

$$D \cap B(Z, \delta)$$
$$= \{(\xi', \xi_n, t) : \xi' \in \boldsymbol{R}^{n-1},\ \xi_n \in \boldsymbol{R},\ \xi_n > \phi(\xi', t),\ 0 < t < T\} \cap B(Z, \delta).$$

Here ϕ is a C^1-function defined on $\boldsymbol{R}^{n-1} \times \boldsymbol{R}$ with compact support and satisfies

$$(1.2) \qquad |\nabla\phi| \leqq M,\ |\nabla\phi(\xi', t) - \nabla\phi(\zeta', s)| \leqq M(|\xi' - \zeta'|^{\alpha} + |t-s|^{\alpha/2}),$$
$$Z = (0, 0, s_0),\quad \frac{\partial\phi}{\partial\xi_j}(0, s_0) = 0 \quad (j=1, \cdots, n-1)$$

for some positive real number $\alpha < 1$ and some constant $M \geqq 1$. We note that in the case $n=1$, ϕ is a function defined on $\boldsymbol{R}$ with variable t and $\phi(\xi', t) = \phi(t)$.

Furthermore, we assume that D satisfies the following conditions:

(d_1) The set

$$(1.3) \qquad I_s := D \cap \{(x, s) : x \in \boldsymbol{R}^n\}$$

is a domain in the hyperplane $t=s$ and each component of $\{(x, s) : x \in \boldsymbol{R}^n\} \setminus I_s$ is unbounded for each s, $0 < s < T$.

(d_2) There exists a simple continuous curve in D connecting some point of B_D to some point of T_D along which the t-coordinate is nondecreasing.

In this domain we consider the initial-Dirichlet problem (1.1) with lateral boundary data $f \in L^p(\sigma)$, where σ is the surface measure of S_D. For this purpose, setting

$$W(X) = W(x, t) = \begin{cases} \dfrac{\exp(-|x|^2/4t)}{(4\pi t)^{n/2}} & \text{if } t>0 \\ 0 & \text{otherwise,} \end{cases}$$

we introduce a mixed layer potential kernel: for $X = (x, t) \in \boldsymbol{R}^{n+1}$ and $Y = (y, s) \in S_D$,

$$(1.4) \qquad k(X, Y) := -\langle \nabla_y W(X-Y), N_y \rangle - \frac{1}{2} W(X-Y) N_s$$
$$= \frac{\exp(-|x-y|^2/4(t-s))}{2(4\pi)^{n/2}(t-s)^{n/2+1}} (Y-X) \cdot \boldsymbol{n}_Y$$

if $t > s$ and $k(X, Y) = 0$ otherwise, where $\boldsymbol{n}_Y = (N_y, N_s)$ is the unit outward normal to S_D at Y, $\langle , \rangle$ is the inner product in $\boldsymbol{R}^n$ and $(Y-X) \cdot \boldsymbol{n}_Y$ is the inner product of $Y-X$ and $\boldsymbol{n}_Y$ in $\boldsymbol{R}^{n+1}$.

Using this kernel, we define a mixed layer potential, for $f \in L^p(\sigma)$ and $X \in \boldsymbol{R}^{n+1}$,

$$\Phi f(X) = \int k(X, Y) f(Y) d\sigma(Y) \tag{1.5}$$

if it is well-defined and $\Phi f(X)=0$ otherwise.

To study the boundary behavior of the potential Φf, we consider a parabolic approach region at $Z=(z, s)\in S_D$

$$\Gamma_\tau(Z) = \{X\in D : (Z-X)\cdot \boldsymbol{n}_Z > \tau\delta(X, Z)\}$$

for τ, $0<\tau<(1+M^2)^{-1/2}$, where M is the positive real number in (1.2).

Using these mixed layer potentials and estimating them by parabolic maximal functions, we will prove the following theorem in § 8.

THEOREM 1. *Let $p>1$, $0<\tau<(1+M^2)^{-1/2}$ and $f\in L^p(\sigma)$. Then there exists a function $g\in L^p(\sigma)$ such that the mixed layer potential $u\equiv\Phi g$ has the following properties:*

$$\triangle u - \frac{\partial u}{\partial t} = 0 \quad \text{in } D,$$

$$\lim_{X\to Z,\, X\in D} u(X) = 0 \quad \text{for all } Z \in B_D\backslash S_D$$

and

$$\lim_{X\to Z,\, X\in\Gamma_\tau(Z)} u(X) = f(Z)$$

for σ-almost every point $Z\in S_D$.

Next, let us define, for $g\in L^p(\sigma)$, $X\in \boldsymbol{R}^{n+1}$ and $Y\in S_D$, the single layer potential u_g by

$$u_g(X) = -\int W(X-Y) g(Y) d\sigma(Y) \tag{1.6}$$

if it is well-defined and $u_g(X)=0$ otherwise.

The initial-Neumann problem for the heat operator is solved as follows.

THEOREM 2. *Let $p>1$, $0<\tau<(1+M^2)^{-1/2}$ and $f\in L^p(\sigma)$. Then there exists a function $g\in L^p(\sigma)$ such that the single layer potential $u\equiv u_g$ has the following properties:*

$$\triangle u - \frac{\partial u}{\partial t} = 0 \quad \text{in } D,$$

$$\lim_{X\to Z,\, X\in D} u(Y) = 0 \quad \text{for all } Z \in B_D\backslash S_D$$

and

$$\lim_{X\to Z,\, X\in\Gamma_\tau(Z)} \left(\langle \nabla_x u(X), N_z\rangle - \frac{1}{2} u(X) N_s\right) = f(Z)$$

for σ-almost every point $Z=(z, s)\in S_D$.

REMARK. In this paper we considered only the heat operator. But our method is applicable to a more general parabolic operator, for example, for a uniformly parabolic operator, by considering $k(X, Y)$ corresponding to the fundamental solution of the operator.

2. Estimates of the kernel k.

We denote by (x, t) a point X in $\boldsymbol{R}^{n+1}$, where $x=(x_1, x_2, \cdots, x_n)$ are space variables and t is the time variable. We consider the heat operator L defined by

$$L = \triangle - \frac{\partial}{\partial t}$$

and the adjoint operator of L defined by

$$L^* = \triangle + \frac{\partial}{\partial t}.$$

Hereafter ∇ indicates the n-dimensional spatial gradient operator. Let Ω be a bounded piecewise smooth domain in $\boldsymbol{R}^{n+1}$ and u, v be smooth functions on $\bar{\Omega}$. Using the divergence theorem, we obtain

$$\begin{aligned} (2.1) \qquad & \int_{\Omega}(uL^*v - vLu)dxdt \\ & = \int_{\partial\Omega}\{\langle(u\nabla v - v\nabla u),\ N_x\rangle + uvN_t\}\,dS. \end{aligned}$$

If $Lu=L^*v=0$ in Ω, then (2.1) implies

$$(2.2) \qquad \int_{\partial\Omega}\{\langle(u\nabla v - v\nabla u),\ N_x\rangle + uvN_t\}\,dS = 0.$$

It is well-known that, for $(x_0, t_0)\in\boldsymbol{R}^{n+1}$ and the function W in §1, the function $v(x, t)=W(x_0-x, t_0-t)$ is a C^∞-function in the half space $t<t_0$ and $L^*v=0$ in this space. Furthermore $v=0$ in the half space $t>t_0$.

By virtue of the condition (s) for S_D we note that there exist a finite number of balls $B_j=B(Q_j, \delta_j)$ in $\boldsymbol{R}^{n+1}$, with $\delta_j<1$, a finite number of systems of orthogonal coordinates in $\boldsymbol{R}^n$ and a finite number of functions $\phi_j\in C_0^1(\boldsymbol{R}^d)$ $(j=1, \cdots, m)$ such that each coordinate axis is orthogonal to the time-axis and with respect to the coordinates

$$\begin{aligned} (2.3) \qquad & B(Q_j, 20M\delta_j)\cap S_D = B(Q_j, 20M\delta_j) \\ & \cap \{(\xi', \xi_n, t) : \xi'\in\boldsymbol{R}^{n-1},\ \xi_n\in\boldsymbol{R},\ \xi_n=\phi_j(\xi', t),\ 0\leqq t\leqq T\}, \end{aligned}$$

and

$$B(Q_j,\ 20M\delta_j)\cap D=B(Q_j,\ 20M\delta_j)\cap\{(\xi',\ \xi_n,\ t):\ \xi'\in \boldsymbol{R}^{n-1},\ \xi_n\in\boldsymbol{R},\ \xi_n>\phi_j(\xi',\ t),\ 0<t<T\} \tag{2.4}$$

and $\phi\equiv\phi_j$, $Z\equiv Q_j$ satisfy (1.2).

We next estimate the kernel k defined by (1.4). Let us begin with the following lemma.

LEMMA 2.1. *Let* $X=(x,\ t)$, $Y=(y,\ s)$, $t>s$ *and* $\beta>0$. *Then*

$$\frac{\exp(-|x-y|^2/4(t-s))}{(t-s)^\beta}\leqq c\delta(X,\ Y)^{-2\beta},$$

where c *is a constant independent of* X, Y.

PROOF. Noting that

$$\lim_{s\to\infty}s^{2\beta}\exp(-s^2)=0,$$

we can find $b>0$ such that

$$\frac{|x-y|^{2\beta}\exp(-|x-y|^2/4(t-s))}{(t-s)^\beta}\leqq 1\quad\text{if}\ \frac{|x-y|^2}{t-s}\geqq b.$$

Then

$$\frac{\exp(-|x-y|^2/4(t-s))}{(t-s)^\beta}\leqq\frac{1}{|x-y|^{2\beta}}\leqq\frac{2}{|x-y|^{2\beta}+b^\beta(t-s)^\beta}$$
$$\leqq c_1\delta(X,\ Y)^{-2\beta}.$$

On the other hand, if $|x-y|^2/t-s<b$, we have

$$\frac{\exp(-|x-y|^2/4(t-s))}{(t-s)^\beta}\leqq\frac{1}{(t-s)^\beta}\leqq c_2\delta(X,\ Y)^{-2\beta}.$$

Thus we have the conclusion. Q. E. D.

The kernels W and k defined in § 1 have the following properties.

LEMMA 2.2. (a) $W(X-Y)\leqq c\delta(X,\ Y)^{-n}$ *for all* $X,\ Y\in\boldsymbol{R}^{n+1}$,

(b) $|\nabla_y W(X-Y)|\leqq c\delta(X,\ Y)^{-n-1}$ *and* $|\nabla_x W(X-Y)|\leqq c\delta(X,\ Y)^{-n-1}$ *for all* $X,\ Y\in\boldsymbol{R}^{n+1}$, $X\neq Y$,

(c) $|k(X,\ Y)|\leqq c\delta(X,\ Y)^{\alpha-n-1}$ *for all* $X,\ Y\in S_D$.

PROOF. (a) and (b) follow from Lemma 2.1. (c) Let $X\in B(Q_j,\ \delta_j)\cap S_D$ for $B(Q_j,\ \delta_j)$ satisfying (2.3) and (2.4). First, let $Y\in B(Q_j,\ 2\delta_j)\cap S_D$. For $\phi\equiv\phi_j$ and with respect to the local coordinates we write $X=(\xi',\ \phi(\xi',\ t),\ t)$ and $Y=(\eta',\ \phi(\eta',\ s),\ s)$. Then

(2.5)
$$
\begin{aligned}
&|(Y-X)\cdot \boldsymbol{n}_Y| \\
&\leqq \left|\langle \eta'-\xi', \nabla_{\eta'}\phi(\eta', s)\rangle-(\phi(\eta', s)-\phi(\xi', t))+(s-t)\frac{\partial\phi}{\partial s}(\eta', s)\right| \\
&\leqq c_1(|\eta'-\xi'|+|s-t|)(|\eta'-\xi'|^{\alpha}+|s-t|^{\alpha/2}) \\
&\leqq c_2\delta(X, Y)^{1+\alpha}.
\end{aligned}
$$

This and Lemma 2.1 lead to the estimate (c).

Next, if $Y\notin B(Q_j, 2\delta_j)$, then $|X-Y|>\delta_j$ and

$$|k(X, Y)| \leqq c_3\delta(X, Y)^{-n-1} \leqq c_4\delta_i^{-\alpha}\delta(X, Y)^{\alpha-n-1} \leqq c_5\delta(X, Y)^{\alpha-n-1},$$

whence we have the conclusion. Q.E.D.

LEMMA 2.3. *Let* $0<\beta\leqq 1$ *and* $X=(x, t_1)$, $Z=(z, t_2)\in S_D$. *If* $Y=(y, s)\in S_D$, $Y\neq X$, $Y\neq Z$, *then*

(2.6)
$$
\begin{aligned}
&|k(X, Y)-k(Z, Y)| \\
&\leqq c\delta(X, Z)^{\beta}(\delta(X, Y)^{\alpha-\beta-1-n}+\delta(Z, Y)^{\alpha-\beta-1-n})
\end{aligned}
$$

and

(2.7)
$$
\begin{aligned}
&|k(Y, X)-k(Y, Z)| \\
&\leqq c\delta(X, Z)^{\beta}(\delta(X, Y)^{\alpha-\beta-1-n}+\delta(Y, Z)^{\alpha-\beta-1-n}).
\end{aligned}
$$

PROOF. To show (2.6), first assume that $\delta(X, Y)\leqq 3\delta(X, Z)$. Noting that

$$\delta(Y, Z) \leqq \delta(X, Z)+\delta(X, Y) \leqq 4\delta(X, Z)$$

and using Lemma 2.2, we obtain

$$
\begin{aligned}
&|k(X, Y)-k(Z, Y)| \\
&\leqq |k(X, Y)|+|k(Z, Y)| \\
&\leqq c_1(\delta(X, Y)^{\alpha-1-n}+\delta(Z, Y)^{\alpha-1-n}) \\
&\leqq c_2\delta(X, Z)^{\beta}(\delta(X, Y)^{\alpha-\beta-1-n}+\delta(Z, Y)^{\alpha-\beta-1-n}).
\end{aligned}
$$

We next assume that $\delta(X, Y)\geqq 3\delta(X, Z)$. Furthermore, assume that $t_1>s$ and $t_2>s$. Then

$$
\begin{aligned}
&|k(X, Y)-k(Z, Y)| \\
&\leqq \frac{1}{2(4\pi)^{n/2}(t_1-s)^{(n+2)/2}}\exp\left(-\frac{|x-y|^2}{4(t_1-s)}\right) \\
&\quad\times|\langle y-x, N_y\rangle+(s-t_1)N_s-\langle y-z, N_y\rangle-(s-t_2)N_s| \\
&\quad+|\langle y-z, N_y\rangle+(s-t_2)N_s|
\end{aligned}
$$

$$\times\left|\frac{1}{2(4\pi)^{n/2}(t_1-s)^{(n+2)/2}}\exp\left(-\frac{|x-y|^2}{4(t_1-s)}\right)\right.$$
$$\left.-\frac{1}{2(4\pi)^{n/2}(t_2-s)^{(n+2)/2}}\exp\left(-\frac{|z-y|^2}{4(t_2-s)}\right)\right|$$
$$\equiv I_1+I_2.$$

To estimate I_1, let $X\in B(Q_j, \delta_j)$ and first assume that $Z\in B(Q_j, 2\delta_j)$ and $Y\in B(Q_j, 2\delta_j)$. Noting that

$$\langle z-x, N_y\rangle=\langle z-x, N_y-N_x\rangle+\langle z-x, N_x\rangle$$

and

$$|\langle z-x, N_x\rangle+(t_2-t_1)N_{t_1}|\leqq c_3\delta(X, Z)^{1+\alpha},$$

we have, by Lemma 2.1,

$$I_1\leqq c_4\delta(X, Z)^{1+\alpha}\delta(X, Y)^{-n-2}\leqq c_5\delta(X, Z)^{\beta}\delta(X, Y)^{\alpha-\beta-1-n}.$$

Next, if $Z\notin B(Q_j, 2\delta_j)$, then $|X-Z|>\delta_j$, whence

$$I_1\leqq c_6|X-Z|\delta(X, Y)^{-2-n}\leqq c_6\delta_j^{-\alpha}|X-Z|^{1+\alpha}\delta(X, Y)^{-2-n}$$
$$\leqq c_7\delta(X, Z)^{\beta}\delta(X, Y)^{\alpha-\beta-1-n}.$$

Similarly if $Y\notin B(Q_j, 2\delta_j)$, then $|X-Y|>\delta_j$, whence

$$I_1\leqq c_8|X-Z|\delta(X, Y)^{-2-n}\leqq c_8\delta_j^{-\alpha}|X-Z|\delta(X, Y)^{-2-n+\alpha}$$
$$\leqq c_9\delta(X, Z)^{\beta}\delta(X, Y)^{\alpha-\beta-1-n}.$$

We next estimate I_2. Using (2.5), the mean-value theorem and Lemma 2.1, we have

$$I_2\leqq c_{10}\delta(Y, Z)^{1+\alpha}(|x-z|\delta(P, Y)^{-n-3}+|t_1-t_2|\delta(P, Y)^{-n-4}) \tag{2.8}$$

for some point $P=bX+(1-b)Z$ $(0<b<1)$.

We claim

$$\delta(P, Y)\geqq c_{11}\delta(Y, Z). \tag{2.9}$$

In fact, if $\max\{|y-z|, |s-t_2|^{1/2}\}=|y-z|$, then

$$\delta(P, Y)\geqq|bx+(1-b)z-y|\geqq|y-z|-b|x-z|$$
$$\geqq(2^{-1/2}-1/2)\delta(Y, Z).$$

If $\max\{|y-z|, |s-t_2|^{1/2}\}=|s-t_2|^{1/2}$, then

$$\delta(P, Y)^2\geqq|bt_1+(1-b)t_2-s|\geqq|t_2-s|-b|t_1-t_2|$$
$$\geqq\delta(Y, Z)^2/2-\delta(X, Z)^2\geqq(1/2-1/4)\delta(Y, Z)^2=(1/4)\delta(Y, Z)^2.$$

Thus (2.9) was shown. Combining (2.9) with (2.8), we have

$$I_2 \leqq c_{12}\delta(X, Z)^{\beta}\delta(Y, Z)^{\alpha-\beta-1-n}.$$

Therefore we have (2.6).

Finally, assume that $t_1 > s \geqq t_2$. As in the proof of Lemma 2.2 we have

$$\begin{aligned} &|k(X, Y)-k(Z, Y)| = |k(X, Y)| \\ &\leqq c_{13}\frac{\exp(-|x-y|^2/4(t_1-s))}{(t_1-s)^{(n+2)/2}}\delta(X, Y)^{1+\alpha} \\ &\leqq c_{13}(t_1-t_2)^{\beta/2}(t_1-s)^{(-n-2-\beta)/2}\exp\Big(-\frac{|x-y|^2}{4(t_1-s)}\Big)\delta(X, Y)^{1+\alpha} \\ &\leqq c_{14}\delta(X, Z)^{\beta}\delta(X, Y)^{\alpha-\beta-1-n}, \end{aligned}$$

which shows that (2.6) holds.

Similarly we can show (2.7). Q.E.D.

3. Parabolic maximal functions.

Let $f \in L^p(\sigma)$. To estimate the L^p-norm of $(\Phi f)_{|S_D}$, we may suppose that $\operatorname{supp} f \subset B_j = B(Q_j, \delta_j)$ by a partition of unity, where $\operatorname{supp} f$ stands for the support of f and B_j is one of the balls satisfying (2.3) and (2.4). Putting $\phi = \phi_j$, we define, with respect to the local coordinates,

$$g(\eta', s) = \begin{cases} f(\eta', \phi(\eta', s), s) & \text{if } (\eta', \phi(\eta', s), s) \in S_D \cap B(Q_j, \delta_j) \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a function in $L^p(\boldsymbol{R}^n)$ with compact support and we can estimate, by Lemma 2.2,

$$|(\Phi f)(\xi', \phi(\xi', t), t)| \leqq c\int(|\xi'-\eta'|^2+|t-s|)^{(\alpha-n-1)/2}|g(\eta', s)|\,d\eta' ds$$

for $(\xi', \phi(\xi', t), t) \in S_D \cap B(Q_j, 20M\delta_j)$.

In this section we denote by (x, t) a point $X \in \boldsymbol{R}^{n-1}\times\boldsymbol{R}$, instead of $\xi = (\xi', t)$. Let $g \in L^p(\boldsymbol{R}^n)$ with compact support. To estimate the function

$$X \longmapsto \int \delta(X, Y)^{\alpha-1-n}g(Y)dY,$$

we introduce a maximal function with respect to parabolic cylinders in $\boldsymbol{R}^n$, instead of balls. More precisely for $X = (x, t) \in \boldsymbol{R}^n$ and $r > 0$ we denote by $C_r(X)$ the bounded cylinder

$$\{Y = (y, s) : Y \in \boldsymbol{R}^n : |y-x| < r,\ |s-t| < r^2\}$$

and call it a parabolic cylinder. We note that if $n=1$, then the x-component is not considered, i.e., $X=t$, $Y=s$ and $C_r(X)=\{s: |s-t|<r^2\}$.

Let $f\in L^1_{loc}(\boldsymbol{R}^n)$. We define the parabolic maximal function by

$$\mathcal{M}f(X)=\sup\left\{\frac{1}{|C_r(X)|}\int_{C_r(X)}|f(Y)|\,dY : r>0\right\},$$

where $|C_r(X)|$ stands for the measure of $C_r(X)$ with respect to the n-dimensional Lebesgue measure.

By the same method as in the proof of the lemma on p. 9 in [S] we can show that the following covering lemma is also valid for parabolic cylinders.

LEMMA 3.1. *Let E be a measurable subset of $\boldsymbol{R}^n$ which is covered by the union of a family of parabolic cylinders $\{C_{r_j}(X)\}_j$ with $r_j\leqq r_0$ for some r_0. Then from the family we can select a disjoint subsequence $\{C_k\}_k$ such that*

$$|E|\leqq c\sum_k |C_k|,$$

where c is a constant depending only on the dimension n.

Using this we obtain the following estimate for the parabolic maximal function by the analogous method as Theorem 1 on p. 5 in [S].

LEMMA 3.2. (a) *For $f\in L^1(\boldsymbol{R}^n)$ and $b>0$ we set*

$$E_{f,b}=\{X\in\boldsymbol{R}^n : \mathcal{M}f(X)>b\}.$$

Then there exists a constant c such that

$$|E_{f,b}|\leqq\frac{c}{b}\|f\|_1,$$

for every $f\in L^1(\boldsymbol{R}^n)$ and $b>0$.

(b) *Let $1<p\leqq\infty$. Then there exists a contant c such that*

$$\|\mathcal{M}f\|_p\leqq c\|f\|_p \quad \text{for every } f\in L^p(\boldsymbol{R}^n).$$

LEMMA 3.3. *Let $0<\beta<n+1$, $1<p\leqq+\infty$, $b>0$ and $R>0$. Furthermore define, for $g\in L^p(\boldsymbol{R}^n)$ with compact support,*

$$I_\beta(g)(X)=\int\delta(X,Y)^{\beta-n-1}|g(Y)|\,dY.$$

Then there is a constant c such that

$$I_\beta(g)(X)\leqq c\mathcal{M}g(X)$$

for every $g\in L^p(\boldsymbol{R}^n)$ with supp $g\subset B(0,R)$ and every $X\in B(0,b)$.

PROOF. Set, for an integer k and $X\in B(0, b)$,

$$E_k(X)=\{Y\in B(0, R): \delta(X, Y)^{\beta-n-1}>2^k\}.$$

If $Y\in E_k(X)$, then $\delta(X, Y)<r_k$, where $r_k=2^{-k/(n+1-\beta)}$. Writing $X=(x, t)$, $Y=(y, s)$, we have

$$|x-y|<r_k \quad \text{and} \quad |t-s|<r_k^2.$$

If $X\in B(0, b)$ and $Y\in B(0, R)$, then

$$\delta(X, Y)^{\beta-n-1}\leqq\sum_{k=m}^{\infty}2^k\chi_{E_k(X)}(Y)$$

for some integer m determined by b and R, where χ_E stands for the characteristic function of a set E. From this we deduce

$$\int\delta(X, Y)^{\beta-n-1}|g(Y)|dY\leqq c_1\sum_{k=m}^{\infty}2^k r_k^{n+1}\mathcal{M}g(X).$$

Since $\sum_{k=m}^{\infty}2^k r_k^{n+1}<\infty$, we have the conclusion. Q.E.D.

LEMMA 3.4. *If* $1<p\leqq+\infty$, *then there is a constant* $c>0$ *such that*

$$r\int_{\delta(X,Y)>r}\frac{|g(Y)|}{\delta(X, Y)^{n+2}}dY\leqq c\mathcal{M}g(X)$$

for every positive real number $r<R$, $X\in \boldsymbol{R}^n$ *and* $g\in L^p(\boldsymbol{R}^n)$.

PROOF. Set

$$E_k(X)=\{Y\in\boldsymbol{R}^n: \delta(X, Y)^{-n-2}>2^k\}$$

and $r_k=2^{-k/(n+2)}$. From the same consideration as in the proof of Lemma 3.3 we deduce

$$\int\delta(X, Y)^{-n-2}|g(Y)|dY\leqq c_1\sum_{k=-\infty}^{m}2^k r_k^{n+1}\mathcal{M}g(X),$$

where m is the least integer satisfying $r_m\leqq r$. Since

$$\sum_{k=-\infty}^{m}2^k r_k^{n+1}=\sum_{k=-\infty}^{m}2^{(1-(n+1)/(n+2))k}\leqq c_2 2^{m/(n+2)}\leqq c_3\frac{1}{r},$$

we have the conclusion. Q.E.D.

LEMMA 3.5. *Suppose that* $p>1$, $0<\beta<n+1$ *and*

$$\frac{1}{q}=\frac{1}{p}-\frac{\beta}{n+1}>0. \tag{3.1}$$

Then

$$\|I_\beta(g)\|_q \leqq c\|g\|_p \quad \text{for all } g \in L^p(\boldsymbol{R}^n).$$

PROOF. By the same method as in the proof of Theorem 1 on p. 119 in [S] we see that I_β is of weak-type (p, q) for every $p \geqq 1$ and q satisfying (3.1). Therefore I_β is of strong-type (p, q) by the Marcinkiewicz interpolation theorem (cf. [S, Appendix B]). Thus we obtain the conclusion. Q.E.D.

4. Single layer potentials.

We go back to the $(n+1)$-dimensional Euclidian space and consider the bounded domain D in §1. In this section we will show the boundedness and continuity of single layer potentials of bounded functions on S_D.

We first note that the kernel $W(X-Y)$ has the following properties.

LEMMA 4.1. *Let* $0<\beta\leqq 1$ *and* $X=(x, t_1)$, $Z=(z, t_2)\in \boldsymbol{R}^{n+1}$. *If* $Y=(y, s)\in \boldsymbol{R}^{n+1}$, $Y\neq X$, $Y\neq Z$, *then*

$$|W(X-Y)-W(Z-Y)| \leqq c\delta(X, Z)^\beta(\delta(X, Y)^{-\beta-n}+\delta(Z, Y)^{-\beta-n})$$

and

$$|W(Y-X)-W(Y-Z)| \leqq c\delta(X, Z)^\beta(\delta(X, Y)^{-\beta-n}+\delta(Z, Y)^{-\beta-n}).$$

PROOF. By the same method as in the proof of Lemma 2.3 we can prove this lemma. Q.E.D.

We next mention boundedness of single layer potentials of bounded functions. Recall that σ is the surface measure of the lateral boundary S_D of D.

LEMMA 4.2. *Let* $\beta>0$ *and* $f\in L^\infty(\sigma)$. *Then the function* h *defined by*

$$h(X) = \int \delta(X, Y)^{\beta-n-1} f(Y)d\sigma(Y)$$

is bounded on $\boldsymbol{R}^{n+1}$.

PROOF. Using a partition of unity, we may suppose $\operatorname{supp} f\subset B_j$ and $|f|\leqq 1$, where $B_j=B(Q_j, \delta_j)$ is one of the balls satisfying (2.3) and (2.4). It is obvious that the function h is bounded on $\boldsymbol{R}^{n+1}\backslash 3B_j$, where $3B_j=B(Q_j, 3\delta_j)$.

We claim that h is bounded on $S_D\cap 3B_j$. Indeed we have, by Lemma 3.3,

$$\begin{aligned} |h(X)| &\leqq c_1\int(|\xi'-\eta'|^2+|t-s|)^{(\beta-n-1)/2}d\eta' ds \\ &\leqq c_2\mathcal{M}1(\xi', t) \leqq c_2 \end{aligned}$$

for $X=(\xi', \phi_j(\xi', t), t)\in S_D\cap 3B_j$.

Next, let $X\in 3B_j\backslash S_D$ and pick $Z\in S_D$ such that

$$\delta(X, Z)=\inf_{Y\in S_D}\delta(X, Y). \tag{4.1}$$

We have

$$|h(X)| \leqq \int_{\delta(Y,Z)\leqq 2\delta(X,Z)}\delta(X, Y)^{\beta-n-1}d\sigma(Y)$$

$$+\int_{\delta(Y,Z)>2\delta(X,Z)}\delta(X, Y)^{\beta-n-1}d\sigma(Y)\equiv I_1+I_2.$$

Noting that $\delta(Y, Z)\leqq 2\delta(X, Z)$ implies $\delta(X, Y)\leqq 3\delta(X, Z)$, we obtain, by (4.1),

$$I_1 \leqq \frac{c_1}{\delta(X, Z)^{n+1}}\int_{\delta(Y,Z)\leqq 2\delta(X,Z)}((3\delta_j)^2+3\delta_j)^{\beta}d\sigma(Y) \leqq c_3.$$

Further if $\delta(Y, Z)>2\delta(X, Z)$, then $\delta(X, Y)\geqq(1/2)\delta(Y, Z)$. Consequently we have

$$I_2 \leqq c_4\int\delta(Y, Z)^{\beta-n-1}d\sigma(Y) \leqq c_5.$$

Thus we see that h is bounded. Q.E.D.

LEMMA 4.3. *Let $f\in L^{\infty}(\sigma)$ and $0<\varepsilon<1$. Furthermore define*

$$u(X)=\int W(X-Y)f(Y)d\sigma(Y) \quad \textit{and} \quad v(Y)=\int W(X-Y)f(X)d\sigma(X).$$

Then

$$|u(X)-u(Z)| \leqq c\delta(X, Z)^{\varepsilon}\|f\|_{\infty} \quad \textit{and} \quad |v(Y)-v(Z)| \leqq c\delta(Y, Z)^{\varepsilon}\|f\|_{\infty}.$$

PROOF. We first note that $W(X-Y)\leqq c_1\delta(X, Y)^{-n}$ by Lemma 2.2. From Lemma 4.1 we deduce

$$|v(Y)-v(Z)|$$
$$\leqq c_1\delta(Y, Z)^{\varepsilon}\|f\|_{\infty}\Big(\int\delta(X, Y)^{-\varepsilon-n}d\sigma(X)+\int\delta(X, Z)^{-\varepsilon-n}d\sigma(X)\Big).$$

Similarly we see that u has the same property. Lemma 4.2 leads to the conclusion.

5. Estimates of layer potentials.

In this section we will study the boundary behavior of the potentials with respect to the kernel k. We define, for $f\in L^p(\sigma)$ and $X\in \boldsymbol{R}^{n+1}\backslash S_D$,

$$\mathcal{D}f(X)=-\int\langle\nabla_y W(X-Y), N_y\rangle f(Y)d\sigma(Y).$$

To study the boundary behavior of the functions $\mathcal{D}f$ and u_f, which is defined by (1.6), we consider a parabolic approach region at $Z=(z, s_0)\in S_D$

$$\Gamma_\tau(Z) = \{X\in D : (Z-X)\cdot \boldsymbol{n}_Z>\tau\delta(X, Z)\}$$

and

$$\Gamma_\tau^e(Z) = \{X\in \boldsymbol{R}^{n+1}\setminus\bar{D} : 0<t<T,\ (X-Z)\cdot \boldsymbol{n}_Z>\tau\delta(X, Z)\}$$

for a positive real number τ satisfying $\tau<(1+M^2)^{-1/2}$.

Let v be a function defined on $\boldsymbol{R}^{n+1}\setminus S_D$. We set

$$(v)_\varepsilon^*(Z) = \sup\{|v(X)| : X\in\Gamma_\tau(Z)\cap B(Z, \varepsilon)\}$$

and

$$(v)_\varepsilon^{**}(Z) = \mathrm{supp}\{|v(X)| : X\in\Gamma_\tau^e(Z)\cap B(Z, \varepsilon)\}$$

for $Z\in S_D$ and $\varepsilon>0$.

The layer potentials $\mathcal{D}f$ and u_f are estimated as follows.

LEMMA 5.1. *Let $1<p\leqq\infty$ and $f\in L^p(\sigma)$. Then there exist positive real numbers c and ε such that*

$$\|(\mathcal{D}f)_\varepsilon^*\|_p \leqq c\|f\|_p,\quad \|(\mathcal{D}f)_\varepsilon^{**}\|_p \leqq c\|f\|_p \tag{5.1}$$

and

$$\|(u_f)_\varepsilon^*\|_p \leqq c\|f\|_p,\quad \|(u_f)_\varepsilon^{**}\|_p \leqq c\|f\|_p, \tag{5.2}$$

where c is a constant independent of f.

PROOF. Cover S_D by a finite number of balls $B_j=B(Q_j, \delta_j)$ $(j=1, \cdots, m)$ satisfying (2.3) and (2.4). We write simply ϕ instead of ϕ_j. Recall that $|\partial\phi/\partial t|\leqq M$. Without loss of generality we may suppose

$$|\nabla_{\xi'}\phi(\xi', t)| < \frac{\tau}{18} \quad\text{and}\quad M\delta_j^{1/2} < \frac{\tau}{18}.$$

Further we may suppose $\mathrm{supp} f\subset B_j$. Set $\varepsilon=\min\{\delta_1, \cdots, \delta_m\}$. If $Z\in S_D\setminus B(Q_j, 3\delta_j)$ and $|X-Z|<\varepsilon$, then $|X-Y|\geqq\delta_j$ for all $Y\in B_j$. Hence we have, by Lemma 2.2,

$$\begin{aligned}(\mathcal{D}f)_\varepsilon^*(Z) &\leqq c_1 \sup_{X\in B(Z,\varepsilon)}\int\frac{|f(Y)|}{\delta(X, Y)^{n+1}}d\sigma(Y)\\ &\leqq c_2\frac{1}{\delta_j^{n+1}}\int|f(Y)|d\sigma(Y) \leqq c_3\|f\|_p.\end{aligned} \tag{5.3}$$

Next, let $Z=(z, s_0)=(\zeta', \zeta_n, s_0)\in B(Q_j, 3\delta_j)\cap S_D$. If $X=(x, t)=(\xi', \xi_n, t)\in \Gamma_\tau(Z)\cap B(Z, \varepsilon)$, then

$$\langle \nabla_{\zeta'}\phi(\zeta', s_0), \zeta'-\xi'\rangle-(\zeta_n-\xi_n)+\frac{\partial\phi}{\partial t}(\zeta', s_0)(s_0-t) > \tau\delta(X, Z).$$

Therefore we have

$$|\xi_n-\zeta_n| > \tau\delta(X, Z)-\frac{\tau}{9}\delta(X, Z) = \frac{8}{9}\tau\delta(X, Z). \tag{5.4}$$

Further if $Y=(y, s)=(\eta', \phi(\eta', s), s)\in B(Q_j, 3\delta_j)\cap S_D$ and $\delta(Y, Z)\leqq 3\delta(X, Z)$, then

$$\begin{aligned}|x-y| &\geqq |\xi_n-\phi(\eta', s)| \geqq |\xi_n-\zeta_n|-|\phi(\zeta', s_0)-\phi(\eta', s)| \\ &\geqq \frac{8\tau}{9}\delta(X, Z)-\left(\frac{\tau}{18}|\eta'-\zeta'|+M|s-s_0|\right) \geqq \frac{2\tau}{9}\delta(X, Z).\end{aligned} \tag{5.5}$$

From this and Lemma 2.2 it follows that

$$|-\langle\nabla_y W(X-Y), N_y\rangle| \leqq c_4\frac{1}{\delta(X, Y)^{n+1}} \leqq c_5\frac{1}{\delta(X, Z)^{n+1}},$$

whence

$$\left|\int_{\delta(Y,Z)\leqq 3\delta(X,Z)} -\langle\nabla_y W(X-Y), N_y\rangle f(Y)d\sigma(Y)\right| \leqq c_6\mathcal{M}g(\zeta', s_0), \tag{5.6}$$

where $g(\eta', s)=f(\eta', \phi(\eta', s), s)$.

We next suppose $\delta(Y, Z)>3\delta(X, Z)$. Then

$$\delta(X, Y) \geqq \delta(Y, Z)-\delta(X, Z) \geqq \frac{2}{3}\delta(Y, Z). \tag{5.7}$$

From (5.7), Lemma 2.2 and

$$\begin{aligned}-\langle\nabla_y W(X-Y), N_y\rangle = &\frac{\exp(-|x-y|^2/4(t-s))}{2(4\pi)^{n/2}(t-s)^{(n+2)/2}}\langle y-z, N_y\rangle \\ &+\frac{\exp(-|x-y|^2/4(t-s))}{2(4\pi)^{n/2}(t-s)^{(n+2)/2}}\langle z-x, N_y\rangle\end{aligned}$$

we deduce

$$\begin{aligned}|-\langle\nabla_y W(X-Y), N_y\rangle| &\leqq c_7\left(\frac{\delta(Y, Z)^{\alpha+1}}{\delta(X, Y)^{n+2}}+\frac{|X-Z|}{\delta(X, Y)^{n+2}}\right) \\ &\leqq c_8\left(\delta(Y, Z)^{\alpha-n-1}+\frac{\delta(X, Z)}{\delta(Y, Z)^{n+2}}\right).\end{aligned}$$

On account of Lemmas 3.3 and 3.4 we have

$$\left|-\int_{\delta(Y,Z)>3\delta(X,Z)}\langle\nabla_y W(X-Y), N_y\rangle f(Y)d\sigma(Y)\right| \leqq c_9\mathcal{M}g(\zeta', s_0). \tag{5.8}$$

Therefore we obtain the first estimate of (5.1) by (5.3), (5.6), (5.8) and Lemma 3.2.

Similarly we can show the second estimate of (5.1) and the estimates of (5.2). Q. E. D.

Let us now introduce a kernel h defined by

$$h(X, Y) = -\langle \nabla_y W(X-Y), N_y \rangle - W(X-Y)N_s$$
$$= k(X, Y) - \frac{1}{2}W(X-Y)N_s$$

for $X=(x, t)\in \boldsymbol{R}^{n+1}$ and $Y=(y, s)\in\partial D$, if $t>s$ and $h(X, Y)=0$ otherwise.

Furthermore we set $I_D=S_D\cup B_D$.

LEMMA 5.2. *The kernel h has the following properties*:

(a) *If $X\in D$, then*

$$\int_{I_D} h(X, Y)dS(Y) = 1.$$

(b) *If $X\notin \bar{D}$, then*

$$\int_{I_D} h(X, Y)dS(Y) = 0.$$

(c) *If $Z\in S_D\backslash B_D$, then*

$$\int_{I_D} h(Z, Y)dS(Y) = \frac{1}{2}.$$

PROOF. (a) Let $X=(x, t_o)$. For $0<t<t_o$ the set D_t defined by

$$D_t = \{(z, s)\in D: s<t\}$$

is a domain by the assumptions (d_1) and (d_2). Applying (2.2) to $u(Y)=1$, $v(Y)=W(X-Y)$ and $\Omega=D_t$, we obtain

$$\int_{(I_D\cap\{s<t\})\cup I_t} \{\langle \nabla_y W(X-Y), N_y\rangle + W(X-Y)N_s\}\, dS(Y) = 0,$$

where I_t is the set defined by (1.3) and N_s is taken with respect to D_t on I_t. Hence

$$\int_{I_D\cap\{s<t\}} h(X, Y)dS(Y) = \int_{I_t} W(x-y, t_o-t)dS(Y).$$

Let $I'_t=\{y\in \boldsymbol{R}^n: (y, t)\in I_t\}$ and J_t be the image of I'_t by the transformation $z=(x-y)/2(t_o-t)^{1/2}$. Then

$$\int_{I_t} W(x-y, t_o-t)dS(Y) = \int_{I'_t} W(x-y, t_o-t)dy = \frac{1}{\pi^{n/2}}\int_{J_t} \exp(-|z|^2)dz.$$

Since J_t tends to $\boldsymbol{R}^n$ as $t \to t_0$, we have

$$\int_{I_D} h(X, Y)dS(Y) = \lim_{t \to t_0} \int_{I_D \cap \{s<t\}} h(X, Y)dS(Y) = \frac{1}{\pi^{n/2}} \int_{R^n} \exp(-|z|^2)dz = 1.$$

(b) Let $X=(x, t_0) \notin \bar{D}$ and $t<t_0$. If $I_D \cap \{s<t\} = \emptyset$, then

$$\int_{I_D \cap \{s<t\}} h(X, Y)dS(Y) = 0.$$

If $\partial D \cap \{s<t\} \neq \emptyset$, then by the same arguments as in the proof of (a),

$$\int_{I_D} h(X, Y)dS(Y) = \lim_{t \to t_0} \int_{I_D \cap \{s<t\}} h(X, Y)dS(Y)$$

$$= \lim_{t \to t_0} \int_{I_t} W(X-Y)dS(Y) = 0.$$

(c) Let $Z=(z, t) \in S_D \backslash B_D$ and take positive real numbers β, τ satisfying $\beta<\alpha$ and $\tau<(1+M^2)^{-1/2}$. Further let B_j and ε be the same ball and the same number as in the proof of Lemma 5.1, respectively. Moreover we can assume that

(5.9) $$M(2\delta_j)^{\alpha-\beta} < \tau/54.$$

Set $r=\min\{\varepsilon, t/3, 1/2\}$. Further assume that $X=(x, t) \in D \cap B(Z, r)$ lies on the spatial normal to S_D at Z and $Q=(q, t)=2Z-X$.

We claim that

(5.10) $$|h(X, Y)+h(Q, Y)| \leqq c\delta(Y, Z)^{\beta-1-n}$$

for all $Y=(y, s) \in S_D$ satisfying $\delta(Y, Z)<2r$. To show (5.10), we write

(5.11) $$|h(X, Y)+h(Q, Y)|$$

$$\leqq c_1 \left| \frac{(\exp(-|x-y|^2/4(t-s))+\exp(-|q-y|^2/4(t-s)))}{(t-s)^{(n+2)/2}} \langle y-z, N_y \rangle \right|$$

$$+c_1 \left| \frac{\exp(-|x-y|^2/4(t-s))}{(t-s)^{(n+2)/2}} \langle z-x, N_y-N_z \rangle \right|$$

$$+c_1 \left| \frac{\exp(-|q-y|^2/4(t-s))}{(t-s)^{(n+2)/2}} \langle z-q, N_y-N_z \rangle \right|$$

$$+c_1 \left| \frac{\exp(-|x-y|^2/4(t-s))}{(t-s)^{(n+2)/2}} \langle z-x, N_z \rangle + \frac{\exp(-|q-y|^2/4(t-s))}{(t-s)^{(n+2)/2}} \langle z-q, N_z \rangle \right|$$

$$+|W(X-Y)|+|W(Q-Y)| \equiv I_1+I_2+I_3+I_4+I_5+I_6.$$

Note that $\delta(Y, Z)<2r<1$. Using Lemma 2.1 and (2.5), we obtain

$$I_1 \leqq c_2(\delta(X, Y)^{-(n+2)}+\delta(Q, Y)^{-(n+2)})\delta(Y, Z)^{1+\alpha},$$

$$I_2 \leqq c_3\delta(X, Y)^{-(n+2)}\delta(X, Z)\delta(Y, Z)^{\alpha}$$

and

$$I_3 \leqq c_4\delta(Q, Y)^{-(n+2)}\delta(Q, Z)\delta(Y, Z)^{\alpha}.$$

Since $X\in\Gamma_\tau(Z)\cap B(Z, r)$ and $Q\in\Gamma_\tau^e(Z)\cap B(Z, r)$, (5.5) and (5.7) show that $\delta(X, Y)\geqq c_5\delta(X, Z)$ and $\delta(X, Y)\geqq c_6\delta(Y, Z)$, as well as $\delta(Q, Y)\geqq c_7\delta(Q, Z)$ and $\delta(Q, Y)\geqq c_8\delta(Y, Z)$. Hence

(5.12) $$I_j \leqq c_9\delta(Y, Z)^{\alpha-1-n} \quad \text{for } j=1, 2, 3.$$

Similarly we have, by Lemma 2.1 and (5.5) and (5.7),

(5.13) $$I_j \leqq c_{10}\delta(Y, Z)^{-n} \quad \text{for } j=5, 6.$$

Let us estimate I_4. We can write $Y=(\eta', \eta_n, s)$ and $Z=(\zeta', \zeta_n, t)$ with respect to the same local coordinates as in the proof of Lemma 5.1 and $t=s_0$. Set

$$u=(\eta', \langle\nabla_{\zeta'}\phi(\zeta', t), \eta'-\zeta'\rangle+\zeta_n)\equiv(\eta', \iota_n)\in \boldsymbol{R}^n.$$

Noting that $|x-u|=|q-u|$ and $q=2z-x$, we have

$$\begin{aligned} I_4 &\leqq c_{11}\left|\frac{(\exp(-|x-y|^2/4(t-s))-\exp(-|x-u|^2/4(t-s)))}{(t-s)^{(n+2)/2}}\langle z-x, N_z\rangle\right| \\ &\quad +c_{11}\left|\frac{(\exp(-|q-y|^2/4(t-s))-\exp(-|q-u|^2/4(t-s)))}{(t-s)^{(n+2)/2}}\langle z-q, N_z\rangle\right| \\ &\equiv I_{41}+I_{42}. \end{aligned}$$

To estimate I_{41}, we write $X=(\xi', \xi_n, t)$. The mean-value theorem and (5.9) yield

(5.14) $$\begin{aligned} |\eta_n-\iota_n| &= |\phi(\eta', s)-\phi(\zeta', t)-\langle\nabla_{\zeta'}\phi(\zeta', t), \eta'-\zeta'\rangle| \\ &\leqq M(|\eta'-\zeta'|^{\alpha}+|s-t|^{\alpha/2})|\eta'-\zeta'|+M|s-t| \\ &\leqq 2M\delta(Y, Z)^{\alpha-\beta}\delta(Y, Z)^{1+\beta}+M\delta(Y, Z)^{1-\beta}\delta(Y, Z)^{1+\beta}\leqq\frac{\tau}{18}\delta(Y, Z)^{1+\beta} \end{aligned}$$

and

$$\begin{aligned} &\left|\exp\left(-\frac{|\xi_n-\eta_n|^2}{4(t-s)}\right)-\exp\left(-\frac{|\xi_n-\iota_n|^2}{4(t-s)}\right)\right| \\ &\leqq c_{12}\frac{|\eta_n-\iota_n|}{t-s}|\xi_n-\rho_n|\exp\left(-\frac{|\xi_n-\rho_n|^2}{4(t-s)}\right), \end{aligned}$$

where $\rho_n=(1-b)\eta_n+b\iota_n$ for some positive real number $b<1$. Putting $P=(\eta', \rho_n, s)$, we have, by Lemma 2.1.

$$I_{41} \leqq c_{13} \frac{\exp(-|\xi'-\eta'|^2/4(t-s))\delta(Y, Z)^{1+\beta}|\xi_n-\rho_n|\exp(-|\xi_n-\rho_n|^2/4(t-s))}{(t-s)^{(n+4)/2}}|x-z|$$
$$\leqq c_{14}\delta(Y, Z)^{1+\beta}\delta(X, P)^{-n-3}\delta(X, Z).$$

We claim

$$\delta(X, P)^{-n-3}\delta(X, Z) \leqq c_{15}\delta(Y, Z)^{-n-2}. \tag{5.15}$$

In fact, if $\delta(Y, Z) \leqq 2\delta(X, Z)$, then, by (5.5) and (5.14),

$$\delta(X, P) \geqq |\xi_n-\rho_n| \geqq |\xi_n-\phi(\eta', s)|-|\eta_n-\rho_n|$$
$$\geqq \frac{2\tau}{9}\delta(X, Z)-\frac{\tau}{18}\delta(Y, Z)^{1+\beta} \geqq \frac{\tau}{18}\delta(Y, Z),$$

whence (5.15) holds. Next, if $\delta(Y, Z)>2\delta(X, Z)$, then, by (5.14),

$$\delta(P, Y) = |\rho_n-\eta_n| \leqq |\eta_n-\iota_n| \leqq \frac{\tau}{18}\delta(Y, Z)^{1+\beta},$$

whence

$$\delta(X, P) \geqq \delta(Y, Z)-\delta(X, Z)-\delta(P, Y)$$
$$\geqq \delta(Y, Z)-\frac{1}{2}\delta(Y, Z)-\frac{1}{18}\delta(Y, Z) = \frac{4}{9}\delta(Y, Z),$$

whence (5.15) holds. Thus we obtain the claim and hence

$$I_{41} \leqq c_{16}\delta(Y, Z)^{\beta-n-1}.$$

Similarly we obtain

$$I_{42} \leqq c_{17}\delta(Y, Z)^{\beta-n-1},$$

whence we have

$$I_4 \leqq c_{18}\delta(Y, Z)^{\beta-1-n}. \tag{5.16}$$

Combining (5.12), (5.13) and (5.16) with (5.11), we conclude that the inequality (5.10) holds.

Finally, if $Y \in B_D \cup \{Y \in S_D : \delta(Y, Z) \geqq 2r\}$, then

$$|h(X, Y)| \leqq c_{19}r^{-1-n}, \quad |h(Q, Y)| = |h(2Z-X, Y)| \leqq c_{19}r^{-1-n}.$$

Therefore, using (a) and (b), we have

$$2\int_{I_D} h(Z, Y)dS(Y) = \lim_{X\to Z,\, X\in T_Z}\left(\int_{I_D} h(X, Y)dS(Y)+\int_{I_D} h(2Z-X, Y)dS(Y)\right) = 1,$$

where

$$T_Z = \{X=(x, t)\in D : \langle z-x, N_z\rangle=|z-x||N_z|, X\in B(Z, r)\}.$$

Thus we have the conclusion. Q. E. D.

We now define, for $X=(x, t)\in\partial D$, $Y=(y, s)\in \boldsymbol{R}^{n+1}$,

$$h^*(X, Y) = -\langle\nabla_x W(X-Y), N_x\rangle + W(X-Y)N_t$$

if $t>s$ and $h^*(X, Y)=0$ otherwise.

We also have the following lemma.

LEMMA 5.3. *The kernel h^* has the following properties*:

(a) *If $Y\in D$, then*

$$\int_{S_D\cup T_D} h^*(X, Y)dS(X) = 1.$$

(b) *If $Y\notin\bar{D}$, then*

$$\int_{S_D\cup T_D} h^*(X, Y)dS(X) = 0.$$

(c) *If $Y\in S_D\backslash T_D$, then*

$$\int_{S_D\cup T_D} h^*(X, Y)dS(X) = \frac{1}{2}.$$

PROOF. Let $Y=(y, s_0)\in D$. For $s>s_0$ we put

$$D^s = \{(z, t) : (z, t)\in D, t>s\}.$$

Applying (2.2) to $u(X)=W(X-Y)$, $v(X)=1$ and $\Omega=D^s$, we obtain (a) by the same method as in the proof of Lemma 5.2. Similarly the assertions (b) and (c) can be shown. Q. E. D.

6. Compact operators.

Let us introduce operators K and K^* which map the family of all Borel measurable functions on S_D into itself. We define, for $X\in S_D$,

$$Kf(X) = \int k(X, Y)f(Y)d\sigma(Y)$$

if it is well-defined and $Kf(X)=0$ otherwise. Similarly we also define

$$K^*f(Y) = \int k(X, Y)f(X)d\sigma(X)$$

if it is well-defined and $K^*f(Y)=0$ otherwise.

LEMMA 6.1. *Let $p>1$. Then*

(a) *K and K^* are bounded operators on $L^p(\sigma)$.*

(b) *K and K^* are compact operators on $L^p(\sigma)$.*

PROOF. (a) Let $f\in L^{p}(\sigma)$ and $\{B_{j}\}$ be the family of balls satisfying (2.3) and (2.4). By a partition of unity we may assume that $\mathrm{supp}\, f\subset B_{j}$. From Lemmas 2.2 and 3.5 it follows that

$$\|(Kf)\chi_{2B_{j}}\|_{p}\leqq c_{1}\|f\|_{p},$$

where $2B_{j}=B(Q_{j}, 2\delta_{j})$. Moreover, by Lemma 2.2, we have

$$\|(Kf)\chi_{S_{D}\backslash 2B_{j}}\|_{p}\leqq c_{2}\delta_{j}^{\alpha-1-n}\|f\|_{p},$$

which leads to the conclusion.

(b) Let $\{f_{m}\}$ be a sequence in $L^{p}(\sigma)$ such that $\|f_{m}\|_{p}\leqq 1$. To show that there is a subsequence $\{f_{m_{k}}\}$ of $\{f_{m}\}$ such that $\{Kf_{m_{k}}\}$ converges in $L^{p}(\sigma)$, we may assume that $\mathrm{supp}\, f_{m}\subset B_{j}$. Since $\{(Kf_{m})\chi_{S_{D}\backslash 2B_{j}}\}_{m}$ is uniformly bounded and equicontinuous on $S_{D}\backslash 2B_{j}$, we can select $\{f_{m_{k}}\}_{k}$ such that $\{Kf_{m_{k}}\}$ converges uniformly on $S_{D}\backslash 2B_{j}$. We also denote by $\{f_{m}\}$ the subsequence instead of $\{f_{m_{k}}\}$. Therefore it suffices to see that there exist a subsequence $\{Kf_{m_{k}}\}$ of $\{Kf_{m}\}$ and a function $g_{0}\in L^{p}(\sigma)$ such that

$$\|(Kf_{m_{k}}-g_{0})\chi_{2B_{j}}\|_{p}\longrightarrow 0 \quad \text{as } k\rightarrow\infty. \tag{6.1}$$

Set $\phi=\phi_{j}$, $g_{m}(\eta', s)=f_{m}(\eta', \phi(\eta', s), s)$ with respect to the local coordinates. To simplify the notations, we use $x', y', \cdots$, instead of $\xi', \eta', \cdots$, in the proof of this lemma.

$$\hat{k}(x', t; y', s)=\frac{\exp(-(|x'-y'|^{2}+|\phi(x', t)-\phi(y', s)|^{2})/4(t-s))}{2(4\pi)^{n/2}(t-s)^{(n+2)/2}}$$
$$\times(\langle y'-x', \nabla_{y'}\phi(y', s)\rangle-(\phi(y', s)-\phi(x', t))+(s-t)\phi_{s}(y', s))$$

if $t>s$ and $\hat{k}(x', t; y', s)=0$ otherwise.

Furthermore we define

$$\hat{K}g(x', t)=\int\hat{k}(x', t; y', s)g(y', s)dy'ds.$$

Note that

$$\int k(X, Y)f_{m}(Y)d\sigma(Y)=\int\hat{k}(x', t; y', s)g_{m}(y', s)dy'ds$$

for $X=(x', \phi(x', t), t)\in 2B_{j}$.

We next consider a class $\{v_{\varepsilon}\}_{\varepsilon>0}$ of approximations to the identity consisting of functions $v_{\varepsilon}(x', t)=\varepsilon^{-n-1}v(x'/\varepsilon, t/\varepsilon^{2})$, where

$$v(x', t)=\begin{cases}\gamma\exp\left(-\dfrac{1}{1-(|x'|^{2}+|t|)}\right) & \text{if } |x'|^{2}+|t|<1,\\ 0 & \text{otherwise}\end{cases}$$

and $\gamma>0$ is so chosen that $\int v(x', t)dx'dt=1$. Using this, we define a kernel and an operator by

$$\hat{k}_n(x', t; y', s)=\int \hat{k}(x'-w', t-r; y', s)v_{1/n}(w', r)dw'dr$$

and

$$\hat{K}_n g(x', t)=\int \hat{k}_n(x', t; y', s)g(y', s)dy'ds,$$

respectively. Set $Q_j=(0, 0, t_0)$ and $C_0=\{(x', t): |x'|\leqq 2\delta_j, |t-t_0|\leqq 2\delta_j\}$. Since $\hat{k}_n$ is continuous and bounded on $C_0\times C_0$, $\{\hat{K}_n g_m\}_m$ is equicontinuous and uniformly bounded on C_0. By the diagonal method we can select a subsequence $\{g_{m_k}\}$ of $\{g_m\}$, independent of n, such that $\{\hat{K}_n g_{m_k}\}_k$ converges to u_n uniformly on C_0. Therefore we have

$$\int |\hat{K}_n g_{m_k}-u_n|^p \chi_{C_0} dx'dt \longrightarrow 0 \quad \text{as } k\to\infty.$$

So, to see (6.1) it suffices to show that there exists a sequence $\{a_n\}$ of positive real numbers such that $a_n\to 0$ as $n\to\infty$ and

$$\int |\hat{K}_n g-\hat{K}g|^p \chi_{C_0} dx'dt \leqq a_n \int |g|^p dx'dt \tag{6.2}$$

for every $g\in L^p(\boldsymbol{R}^n)$ with $\operatorname{supp} g\subset C_0$. In fact, using (6.2), we can easily show that $\{\hat{K}g_{m_k}\}_k$ is a Cauchy sequence with respect to the L^p-norm with weight χ_{C_0}.

To show (6.2), we choose a positive real number ρ with $2\rho<\alpha$. Lemma 2.3 yields

$$\begin{aligned} &|\hat{k}(x'-w', t-r; y, s)-\hat{k}(x', t; y', s)| \\ &\leqq c_3(|w'|^2+|r|)^{\rho/2}\{(|x'-w'-y'|^2+|t-r-s|)^{(\alpha-\rho-n-1)/2} \\ &\quad +(|x'-y'|^2+|t-s|)^{(\alpha-\rho-n-1)/2}\}. \end{aligned} \tag{6.3}$$

If $|w'|<1$ and $|r|<1$, then, by Lemma 3.3,

$$\begin{aligned} &\int (|x'-w'-y'|^2+|t-r-s|)^{(\alpha-\rho-n-1)/2}|g(y', s)|dy'ds \\ &\leqq c_4 \mathcal{M}g(x'-w', t-r), \end{aligned}$$

whence, together with Lemma 3.3 and (6.3),

$$
\begin{aligned}
&|\hat{K}_n g(x', t) - \hat{K} g(x', t)| \\
&\leqq c_5 \int v_{1/n}(w', r) \left(\frac{1}{n}\right)^{\rho} (\mathcal{M} g(x'-w', t-r) + \mathcal{M} g(x', t)) \\
&\leqq c_6 \left(\frac{1}{n}\right)^{\rho} (\mathcal{M}(\mathcal{M} g)(x', t) + \mathcal{M} g(x', t)).
\end{aligned}
$$

Using Lemma 3.2, we obtain

$$
\left(\int |(\hat{K}_n g - \hat{K} g) \chi_{C_0}|^p dx' dt\right)^{1/p} \leqq c_7 \left(\frac{1}{n}\right)^{\rho} \left(\int |g|^p dy' ds\right)^{1/p},
$$

which shows (6.2). Thus we see that K is a compact operator.

Similarly we can show that K^* is also a compact operator. Q.E.D.

To solve the initial-Neumann problem we introduce another kernel j. Define, for $X=(x, t)$ and $Y=(y, s)$,

$$
j(X, Y) = -\langle \nabla_x W(X-Y), N_x \rangle + \frac{1}{2} W(X-Y) N_t
$$

if $t>s$ and $j(X, Y)=0$ otherwise. Furthermore define, for $f \in L^p(\sigma)$ and for $X \in S_D$,

$$
Jf(X) = \int j(X, Y) f(Y) d\sigma(Y)
$$

if it is well-defined and $Jf(X)=0$ otherwise.

If $t>s$, then

$$
j(X, Y) = \frac{\exp(-|x-y|^2/4(t-s))}{2(4\pi)^{n/2}(t-s)^{n/2+1}} (\langle x-y, N_x \rangle + (t-s) N_t).
$$

Therefore the kernel j has the same properties as those of k in Lemma 2.2, (c) and Lemma 2.3. So we can also prove the following lemma by the same method as in the proof of Lemma 6.1.

LEMMA 6.2. *Let $p>1$. Then J is a compact operator on $L^p(\sigma)$.*

7. Parabolic limits.

In this section we consider the parabolic limits of layer potentials Φf for the heat kernel and of spatial normal derivatives of single layer potentials for the adjoint heat kernel.

We first study the boundary behavior of Φf.

LEMMA 7.1. *Let $p>1$ and $f \in L^p(\sigma)$. Then*

(7.1) $$\lim_{X\to Z,\, X\in\Gamma_\tau(Z)} \Phi f(X) = Kf(Z)+\frac{1}{2}f(Z)$$

for σ-almost every point $Z\in S_D$ and

(7.2) $$\lim_{X\to Z,\, X\in\Gamma_\tau^e(Z)} \Phi f(X) = Kf(Z)-\frac{1}{2}f(Z)$$

for σ-almost every point $Z\in S_D$.

PROOF. First, suppose $f\in C^1(S_D)$. Let $Z=(z, s_0)\in S_D\backslash B_D$ and $X=(x, t)\in \Gamma_\tau(Z)\cap B(Z, \varepsilon)$. On account of Lemma 2.2 and (5.5), (5.7) we have

$$|k(X, Y)(f(Y)-f(Z))| \leqq c_1\delta(X, Y)^{-(n+1)}|Y-Z| \leqq c_2\delta(Y, Z)^{-n}$$

and $|W(X, Y)| \leqq c_3\delta(Y, Z)^{-n}$ for $Y\in S_D$. Noting that $N_y=0$ on B_D, we can write

$$\begin{aligned}\Phi f(X) &= \int_{S_D} k(X, Y)(f(Y)-f(Z))d\sigma(Y)\\ &\quad +f(Z)\int_{I_D} h(X, Y)dS(Y)+\frac{f(Z)}{2}\int_{S_D} W(X-Y)N_s dS(Y)\\ &\quad +f(Z)\int_{B_D} W(X-Y)N_s dS(Y).\end{aligned}$$

From Lemma 5.2, (a) we deduce that the second term on the right-hand side is equal to $f(Z)$. Using the dominated convergence theorem and Lemma 5.2, (c) we obtain

$$\begin{aligned}\lim_{X\to Z,\, X\in\Gamma_\tau(Z)} \Phi f(X) &= Kf(Z)-f(Z)\int_{I_D} h(Z, Y)dS(Y)+f(Z)\\ &= Kf(Z)+\frac{1}{2}f(Z).\end{aligned}$$

Thus we have (7.1) for all functions $f\in C^1(S_D)$.

On the other hand, from Lemma 5.1 we deduce the following weak-type estimate:

(7.3) $$\sigma(\{Y\in S_D : (\Phi f)_\varepsilon^*(Y)>b\}) \leqq c\left(\frac{\|f\|_p}{b}\right)^p$$

for every $b>0$ and $f\in L^p(\sigma)$. Since $C^1(S_D)$ is dense in $L^p(\sigma)$ and $K+(1/2)I$ is bounded on $L^p(\sigma)$, we see by the usual method that (7.1) holds for all $f\in L^p(\sigma)$ (cf. [S, PROOF OF THEOREM 1, COROLLARY 1 in Chapter 1]).

To see (7.2), let $f\in C^1(S_D)$. From the same consideration as in the proof of (7.1) we deduce

$$\lim_{X\to Z,\, X\in\Gamma_\eta^e(Z)} \Phi f(X) = Kf(Z) - \frac{1}{2}f(Z)$$

by using Lemma 5.2, (b). Therefore we have also (7.2). Q. E. D.

Next, let us define, for $g\in L^p(\sigma)$, $Y\in \boldsymbol{R}^{n+1}$ and $X\in S_D$, the adjoint single layer potential v_g by

$$v_g(Y) = -\int W(X-Y)g(X)d\sigma(X)$$

if it is well-defined and $v_g(Y)=0$ otherwise.

The adjoint single layer potentials have the following properties.

LEMMA 7.2. *Let $p>1$ and $g\in L^p(\sigma)$. Then for σ-almost every point $Z=(z, s_0)\in S_D$ we have*

$$\lim_{Y\to Z,\, Y\in\Gamma_\tau(Z)}\left(\langle\nabla_y v_g(Y),\, N_z\rangle+\frac{1}{2}N_{s_0}v_g(Y)\right) = K^*g(Z)-\frac{1}{2}g(Z) \tag{7.4}$$

and

$$\lim_{Y\to Z,\, Y\in\Gamma_\tau^e(Z)}\left(\langle\nabla_y v_g(Y),\, N_z\rangle+\frac{1}{2}N_{s_0}v_g(Y)\right) = K^*g(Z)+\frac{1}{2}g(Z). \tag{7.5}$$

PROOF. We will prove only (7.4). First, assume that $g\in C^1(S_D)$ and $Z\in S_D\backslash T_D$. Noting that $\partial W(X-Y)/\partial y_j = -\partial W(X-Y)/\partial x_j$ and $N_x=0$ on T_D, we write

$$\begin{aligned}
&-\int_{S_D}\langle\nabla_y W(X-Y),\, N_z\rangle g(X)d\sigma(X)\\
&= -\int_{S_D}\langle\nabla_y W(X-Y),\, N_z\rangle (g(X)-g(Z))d\sigma(X)\\
&\quad +g(Z)\int_{S_D}\langle\nabla_x W(X-Y),\, N_z-N_x\rangle d\sigma(X)\\
&\quad +g(Z)\int_{S_D\cup T_D}\{\langle\nabla_x W(X-Y),\, N_x\rangle - N_t W(X-Y)\}\, dS(X)\\
&\quad +g(Z)\int_{S_D\cup T_D} W(X-Y)N_t dS(X).
\end{aligned}$$

By Lemma 5.3, (a), the third term on the right-hand side is equal to $-g(Z)$. Thus, using the dominated convergence theorem (cf. the proof of Lemma 7.1) and Lemma 4.3, and then Lemma 5.3, (c), we obtain

$$\lim_{Y\to Z,\, Y\in\Gamma_\tau(Z)}\Big(\langle\nabla_y v_g(Y),\, N_z\rangle+\frac{1}{2}N_{s_0}v_g(Y)\Big)$$

$$=K^*g(Z)-g(Z)\int_{S_D}\langle\nabla_x W(X-Z),\, N_x\rangle d\sigma(X)$$

$$-g(Z)+g(Z)\int_{S_D\cup T_D}W(X-Z)N_t dS(X)$$

$$=K^*g(Z)-\frac{1}{2}g(Z).$$

Thus we see that (7.4) holds for all functions $g\in C^1(S_D)$.

Let $g\in L^p(\sigma)$ and set

$$(\Psi g)^*_\varepsilon(Z)=\sup\{|\langle\nabla_y v_g(Y),\, N_z\rangle| : Y\in\Gamma_\tau(Z)\cap B(Z,\,\varepsilon)\}$$

$$=\sup\Big\{\Big|-\int\langle\nabla_y W(X-Y),\, N_z\rangle g(X)d\sigma(X)\Big| : Y\in\Gamma_\tau(Z)\cap B(Z,\,\varepsilon)\Big\}$$

and

$$(v_g)^*_\varepsilon(Z)=\sup\{|v_g(Y)| : Y\in\Gamma_\tau(Z)\cap B(Z,\,\varepsilon)\}.$$

Then we can show, by the same method as in the proof of Lemma 5.1, that

$$\|(\Psi g)^*_\varepsilon\|_p\leqq c_1\|g\|_p \quad\text{and}\quad \|(v_g)^*_\varepsilon\|_p\leqq c_2\|g\|_p. \tag{7.6}$$

Since the operator $K^*-(1/2)I$ is bounded on $L^p(\sigma)$, we see that (7.4) holds for all $g\in L^p(\sigma)$. Q. E. D.

Similarly we can show the following lemmas 5.1 and 5.2.

LEMMA 7.3. *Let $p>1$ and $g\in L^p(\sigma)$. Then*

$$\lim_{X\to Z,\, X\in\Gamma_\tau(Z)}\Big(\langle\nabla_x u_g(X),\, N_z\rangle-\frac{1}{2}N_{s_0}u_g(X)\Big)=Jg(Z)-\frac{1}{2}g(Z)$$

for σ-a.e. $Z\in S_D$.

8. Proofs of Theorem 1 and Theorem 2.

We begin with the following lemma.

LEMMA 8.1. *Let $f\in L^\infty(\sigma)$ and set*

$$\Omega=\{X=(x,\, t)\notin\bar{D} : 0<t<T\} \tag{8.1}$$

and

$$v(Y)=-\int W(X-Y)f(X)d\sigma(X).$$

Then

$$\int_{\Omega}|\nabla v|^2 dY=\int_{S_D} v\Big(K^*f+\frac{1}{2}f\Big)d\sigma-\frac{1}{2}\int_{\partial\Omega\cap\{t=0\}} v^2 dS. \tag{8.2}$$

PROOF. Without loss of generality we may suppose $\|f\|_\infty=1$. Choose r_0 satisfying $\bar{D}\subset B(0, r_0/2)$ and consider the set

$$A=\{X=(x_1, \cdots, x_n, t): |x_j|<r_0 \ (j=1, \cdots, n),\ 0<t<T\}.$$

By dividing the intervals $\{x_j: |x_j|<r_0\}$ and $\{t: 0<t<T\}$ into m intervals with the same length, respectively, we obtain a mesh $\mathcal{N}_m$ which is a collection of m^{n+1} open intervals in $\boldsymbol{R}^{n+1}$. Denote by $\mathcal{I}_m$ the collection of all intervals I in $\mathcal{N}_m$ such that $\bar{I}\cap\Omega\neq\emptyset$. For a sufficiently large m we may assume that for each $I\in\mathcal{I}_m$ with $S_D\cap\bar{I}\neq\emptyset$, $S_D\cap\bar{I}\subset(n+1)^{-1/2}B_j$ for some j, where $B_j=B(Q_j, \delta_j)$ is one of the balls satisfying (2.3), (2.4) and $bB_j=B(Q_j, b\delta_j)$ for $b>0$. We note that $\phi\equiv\phi_j$ satisfies (1.2) for $Z=Q_j=(0, 0, s_0)$.

Moreover we may assume that

$$|\nabla_{\eta'}\phi|<\frac{1}{3} \quad\text{and}\quad (2\delta_j)^{1/2}M<\frac{1}{3}. \tag{8.3}$$

We claim that

$$\delta(X, Y)\geqq\frac{1}{3}|\phi(\eta', s)-\eta_n| \tag{8.4}$$

for every $X=(\xi', \phi(\xi', t), t)\in S_D\cap 2B_j$ and $Y=(\eta', \eta_n, s)\in\Omega\cap B_j$. In fact, since

$$|\phi(\xi', t)-\phi(\eta', s)|\leqq\frac{1}{3}(|\xi'-\eta'|+|t-s|^{1/2})$$

by the mean-value theorem and (8.3), we have

$$\begin{aligned}\delta(X, Y)&\geqq\frac{1}{3}(|\xi'-\eta'|+|\phi(\xi', t)-\eta_n|+|t-s|^{1/2})\\ &\geqq\frac{1}{3}|\phi(\eta', s)-\eta_n|.\end{aligned}$$

Let I be one of the intervals in $\mathcal{I}_m$ and set $J=I\cap\Omega$. First, assume that $\bar{J}\cap S_D=\emptyset$. Since $L^*v=0$ on $\bar{J}$, the divergence theorem yields

$$\int_J|\nabla v|^2 dY=\int_{\partial J} v\Big(\langle\nabla v, N_y\rangle+\frac{1}{2}vN_s\Big)dS(Y). \tag{8.5}$$

Next, assume that $\bar{I}\cap S_D\neq\emptyset$. Let $\bar{I}\cap S_D\subset B_j$ and $\phi=\phi_j$. Take a sufficiently small positive number ε_0 and set

$$G=J\cap\{(\eta', \eta_n, s): \eta_n>\phi(\eta', s)-\varepsilon_0\}.$$

We define

$$G_\varepsilon = G \cap \{(\eta', \eta_n, s) : \eta_n < \phi(\eta', s) - \varepsilon\}$$

for $0 < \varepsilon < \varepsilon_0$. By the divergence theorem we have

$$\int_{G_\varepsilon} |\nabla v|^2 dY = \int_{\partial G_\varepsilon} v\Big(\langle \nabla v, N_y\rangle + \frac{1}{2} v N_s\Big) dS(Y). \tag{8.6}$$

Choose β satisfying $0 < 2\beta < 1$. On account of Lemma 4.2 and (8.4) we have, for $Y \in G_\varepsilon$,

$$\begin{aligned} |\nabla v(Y)| &\leqq c_1 \int_{(S_D \cap 2B_j) \cup (S_D \setminus 2B_j)} \delta(X, Y)^{-1-n} |f(X)| d\sigma(X) \\ &\leqq c_2(|\phi(\eta', s) - \eta_n|^{-\beta} + \delta_j^{-1-n}). \end{aligned} \tag{8.7}$$

Since

$$\int_G |\phi(\eta', s) - \eta_n|^{-2\beta} dY \leqq \int_{|\eta'| < \delta_j, |s - s_0| < \delta_j} d\eta' ds \int_0^{\varepsilon_0} r^{-2\beta} dr \leqq c_3 \varepsilon_0^{1-2\beta},$$

we have

$$\int_G |\nabla v(Y)|^2 dY < \infty,$$

whence

$$\lim_{\varepsilon \to 0} \int_{G_\varepsilon} |\nabla v|^2 dY = \int_G |\nabla v|^2 dY. \tag{8.8}$$

Set

$$G_\varepsilon^b = G \cap \{(\eta', \eta_n, s) : \eta_n = \phi(\eta', s) - \varepsilon\}$$

and $Y_\varepsilon = (y_\varepsilon, s) = (\eta', \phi(\eta', s) - \varepsilon, s)$ for $Y = (\eta', \phi(\eta', s), s)$. Then $Y_\varepsilon \in \Gamma_{1/2}^{\varepsilon}(Y)$. Let $Y_\varepsilon \in G_\varepsilon^b$. Then

$$|\langle \nabla_y v(Y_\varepsilon), N_{y_\varepsilon}\rangle| \leqq \sup_{X \in \Gamma_\tau(Y) \cap B(Y, \varepsilon_0)} |\langle \nabla_x v(X), N_y\rangle|$$

for $\tau \leqq 1/2$, where N_{y_ε} is taken with respect to G_ε on G_ε^b. By the same method as in the proof of Lemma 5.1, we can see that the right-hand side is dominated by $c_4 \|f\|_\infty$. Noting that v is continuous on $\boldsymbol{R}^{n+1}$ and using the dominated convergence theorem and Lemma 7.2, we have

$$\lim_{\varepsilon \to 0} \int_{G_\varepsilon^b} v\Big(\langle \nabla v, N_y\rangle + \frac{1}{2} v N_s\Big) dS = \int_{\partial G \cap S_D} v\Big(K^* f + \frac{1}{2} f\Big) dS. \tag{8.9}$$

We next discuss the surface integral over $F := \partial G \cap \{y_j = a\}$, which lies on a hyperplane $\eta_k = a_1 \eta_1 + \cdots + a_{k-1}\eta_{k-1} + a_{k+1}\eta_{k+1} + \cdots + a_n \eta_n + a_{n+1} s$ $(1 \leqq k \leqq n-1)$. Set

$$F_\varepsilon=\partial G_\varepsilon\cap\{y_j=a\},\quad E'=\{(\eta',s):|\eta'|<\delta_j,\ |s-s_0|<\delta_j\}.$$

Every point in F_ε can be written as Y_r for some $Y=(\eta',\phi(\eta',s),s)$ and $(\eta',s)\in E'$. From (8.7) we deduce

$$|v(Y_r)||\langle\nabla v(Y_r),N_{y_r}\rangle|\leqq c_5(r^{-\beta}+\delta_j^{-n-1}),$$

whence

$$\int_{F_\varepsilon}|v(Y)||\langle\nabla v(Y),N_y\rangle|dS(Y)$$

$$\leqq c_6\int_{E'_k}d\eta_1,\cdots,d\eta_{k-1}d\eta_{k+1},\cdots,d\eta_{n-1}ds\int_\varepsilon^{\varepsilon_0}(r^{-\beta}+\delta_j^{-n-1})dr,$$

where

$$E'_k=\{(\eta_1,\cdots,\eta_{k-1},\eta_{k+1},\cdots,\eta_{n-1},s):|\eta_j|<\delta_j,\ |s-s_0|<\delta_j\}.$$

Therefore we have

$$\lim_{\varepsilon\to0}\int_{F_\varepsilon}v\Big(\langle\nabla v,N_y\rangle+\frac{1}{2}vN_s\Big)dS=\int_F v\Big(\langle\nabla v,N_y\rangle+\frac{1}{2}vN_s\Big)dS.$$

Moreover note that $\langle\nabla v,N_y\rangle=0$ on $\partial G_\varepsilon\cap\{s=\tau\}$; we have, together with (8.6), (8.8) and (8.9),

$$\int_G|\nabla v|^2dY=\int_{\partial G\cap S_D}v\Big(K^*f+\frac{1}{2}f\Big)dS+\int_{\partial G\setminus S_D}v\Big(\langle\nabla v,N_y\rangle+\frac{1}{2}vN_s\Big)dS.$$

Since $|\nabla v|$ is continuous on $\overline{J\setminus G}$, we also obtain the same equality as (8.5) in which J is replaced by $J\setminus\overline{G}$. Hence

$$\text{(8.10)}\quad \int_J|\nabla v|^2dY=\int_{S_D\cap\partial J}v\Big(K^*f+\frac{1}{2}f\Big)dS+\int_{(\partial J)\setminus S_D}v\Big(\langle\nabla v,N_y\rangle+\frac{1}{2}vN_s\Big)dS$$

for each $J=I\cap\Omega$ satisfying $I\cap S_D\neq\emptyset$.

Next, to estimate the integral of $|\nabla v|^2$ over $\Omega\setminus\overline{A}$, set

$$\Omega_r=\{Y=(y,s)\in\Omega\setminus\overline{A}:|y|<r\}$$

for $r>\sqrt{n}\,r_0$. By the divergence theorem we obtain

$$\text{(8.11)}\quad \int_{\Omega_r}|\nabla v|^2dY=\int_{\partial\Omega_r}v\Big(\langle\nabla v,N_y\rangle+\frac{1}{2}vN_s\Big)dS.$$

If $Y\in\Omega_r$, then $|\nabla v(Y)|\leqq c_7|y|^{-n-1}$. This implies

$$\int_{\Omega_r}|\nabla v|^2dY\leqq c_8\int_0^T ds\int_{r_1\leq|y|<r}\frac{1}{|y|^{2(n+1)}}dy\quad\text{for some } r_1>0,$$

whence

$$\int_{\Omega \setminus A} |\nabla v|^2 dY < +\infty .$$

Similarly we have

$$\int_{\partial\Omega_T \cap \{|y|=r\}} \left| v\left(\langle \nabla v,\ N_y \rangle + \frac{1}{2} v N_s\right) \right| dS \leqq c_9 \int_0^T ds \int_{|y|=r} \frac{1}{|y|^n} \cdot \frac{1}{|y|^{n+1}} dy$$

whence

$$\lim_{r\to\infty} \int_{\partial\Omega_T \cap \{|y|=r\}} \left| v\left(\langle \nabla v,\ N_y \rangle + \frac{1}{2} v N_s\right) \right| dS = 0 .$$

Thus we see, by (8.11), that

$$\int_{\Omega \setminus \bar{A}} |\nabla v|^2 dY = \int_{\partial(\Omega \setminus \bar{A})} v\left(\langle \nabla v,\ N_y \rangle + \frac{1}{2} v N_s\right) dS . \tag{8.12}$$

The equalities (8.5), (8.10) and (8.12) lead to the equality

$$\int_{\Omega} |\nabla v|^2 dY = \int_{S_D} v\left(K^* f + \frac{1}{2} f\right) d\sigma$$
$$+ \int_{\partial\Omega \cap (\{s=0\} \cup \{s=T\})} v\left(\langle \nabla v,\ N_y \rangle + \frac{1}{2} v N_s\right) dS .$$

Since $\langle \nabla v, N_y \rangle = 0$ on $\partial\Omega \cap (\{s=0\} \cup \{s=T\})$, $v=0$ on $\partial\Omega \cap \{s=T\}$ and $N_s=-1$ on $\partial\Omega \cap \{s=0\}$, we have (8.2). Q. E. D.

We now show that $K^* + (1/2)I$ is injective on $L^p(\sigma)$.

LEMMA 8.2. *Let $p>1$ and $f \in L^p(\sigma)$. If $K^* f + (1/2) f = 0$, then $f=0$ σ-a.e.*

PROOF. First, we claim that $K^* f$ is bounded on S_D. Indeed, if $p\alpha > n+1$, then $K^* f$ is bounded on S_D, because $X \mapsto \delta(0, X)^{\alpha-n-1}$ belongs to $L^{p'}(\sigma)$ for $p' = p/(p-1)$. If $p\alpha \leqq n+1$, then we choose an irrational number β satisfying $0<\beta<\alpha$. Lemma 3.5 yields $K^* f \in L^q(\sigma)$ for $q = p(n+1)/(n+1-p\beta)$. Therefore $f = -2K^* f \in L^q(\sigma)$. Repeating this, we see that $K^* f$ is bounded on S_D. Note that the function f, which is equal to $-2K^* f$ σ-a.e., is essentially bounded.

Setting

$$v(Y) = -\int W(X-Y) f(X) d\sigma(X) ,$$

we see by Lemma 4.3 that v is continuous on $\boldsymbol{R}^{n+1}$. Let Ω be the set defined by (8.1). From Lemma 8.1 we deduce

$$\int_{\Omega}|\nabla_y v(Y)|^2 dY+\frac{1}{2}\int_{\partial\Omega\cap\{t=0\}} v^2 dS=\int_{\partial\Omega\cap S_D} v\Big(K^*f+\frac{1}{2}f\Big)dS=0,$$

whence $|\nabla v|=0$ in Ω and $\partial v/\partial s=-\triangle v=0$ in Ω. Since $v=0$ on $T_D\backslash S_D$, we see that $v=0$ on Ω and $v=0$ on S_D. Therefore v is equal to 0 on D by the assumption (d_2) and the maximum principle. This, together with Lemma 7.2, yields

$$\begin{aligned}0&=\lim_{Y\to Z,\,Y\in\Gamma_\tau(Z)}\Big(\langle\nabla_y v(Y),\,N_z\rangle+\frac{1}{2}N_s v(Y)\Big)\\&=K^*f(Z)-\frac{1}{2}f(Z)=K^*f(Z)+\frac{1}{2}f(Z)-f(Z)=-f(Z)\end{aligned}$$

for σ-almost every point $Z\in S_D$. Thus we have the conclusion. Q.E.D.

Furthermore we have

LEMMA 8.3. *Let* $p>1$ *and* $f\in L^p(\sigma)$. *If* $Kf+(1/2)f=0$, *then* $f=0$ *σ-a.e.*

PROOF. Set $q=p/(p-1)$. On account of Lemma 6.1 and Lemma 8.2 we see that $K^*+(1/2)I$ is surjective on $L^q(\sigma)$. Therefore we have the conclusion. Q.E.D.

LEMMA 8.4. *Let* $p>1$ *and* $f\in L^p(\sigma)$. *If* $Jf-(1/2)f=0$, *then* $f=0$ *σ-a.e.*

PROOF. Set

$$u(X)=-\int W(X-Y)f(Y)d\sigma(Y).$$

Using Lemma 7.3 and

$$\int_D|\nabla u|^2=\int_{S_D}u\Big(Jf-\frac{1}{2}f\Big)dS-\frac{1}{2}\int_{T_D}u^2 dS,$$

we can show this lemma by the same method as in the proof of Lemma 8.2. Q.E.D.

Finally we prove our theorems.

PROOF OF THEOREM 1. Let $f\in L^p(\sigma)$. Since K is a compact operator on $L^p(\sigma)$ by Lemma 6.1, $K+(1/2)I$ is invertible by Lemma 8.3. Hence there exists a function $g\in L^p(\sigma)$ such that $Kg+(1/2)g=f$. Lemma 7.1 yields

$$\lim_{X\to Z,\,X\in\Gamma_\tau(Z)}\Phi g(X)=\Big(K+\frac{1}{2}I\Big)g(Z)=f(Z)$$

for σ-almost every point $Z\in S_D$. Moreover it is obvious that Φg satisfies the heat equation in D and

$$\lim_{X \to Z,\, X \in D} \Phi g(X) = 0$$

for every $Z \in B_D \backslash S_D$. Thus we see that $u \equiv \Phi g$ is the desired function. Q.E.D.

PROOF OF THEOREM 2. Since $J-(1/2)I$ is invertible by Lemmas 6.2 and 8.4, there exists a function $g \in L^p(\sigma)$ such that $(J-(1/2)I)g=f$. We see by Lemmas 7.3 and 5.1 that g is the desired function. Q.E.D.

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