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On compact Kähler-Liouville surfaces

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Introduction.

It is rewarding to investigate riemannian manifolds with completely integrable geodesic flows, because the behavior of their geodesics can be observed.

In the 19th century, Jacobi investigated the 2-dimensional ellipsoid, and Liouville generalized this work to the geometry of a class of metrics, the so-called Liouville line elements, whose geodesic flows are integrable by a certain first integral. In relation to the present viewpoint, their investigations can be recognized as a local theory of differential geometry. However, in 1991 K. Kiyohara began to develop a global theory in this area [**1**]. In this work he first defined the compact Liouville surface and classified it; it is defined as a compact 2-dimensional riemannian manifold whose geodesic flow has a first integral on the cotangent bundle such that (1) the first integral is fiberwise a homogeneous polynomial of degree 2; (2) the first integral can not be expressed as a linear combination of the square of a certain vector field and its energy function. Additionally, K. Sugahara, K. Kiyohara and the author investigated noncompact Liouville surfaces [**2**]. Subsequently, Kiyohara generalized this concept to the higher dimensional manifolds (see [**3**] for detail) as follows:

A Liouville manifold is defined as a riemannian manifold which has a real vector space of the first integrals on the cotangent bundle of its geodesic flows such that (1) all the first integrals are fiberwise homogeneous polynomials of degree 2; (2) all the first integrals are simultaneously normalizable on each fiber; (3) the dimension of the vector space is equal to the dimension of the underlying riemannian manifold.

In the investigation [**3**] of Liouville manifolds, Kiyohara has assumed the condition of "properness," and has classified proper Liouville manifolds of rank one; he has concluded that a proper 4-dimensional real Liouville manifold of rank one is diffeomorphic with the sphere S^4, the real projective space $\boldsymbol{RP}^4$ or the euclidean space $\boldsymbol{R}^4$.

It is known that the geodesic flow of the n-dimensional complex projective space $\boldsymbol{CP}^n$ $(n\geqq 1)$ equipped with the standard metric is completely integrable (cf. [**4**], [**5**]). The author was informed by private communication with Prof. K. Kiyohara that there is a family of Kähler metrics on $\boldsymbol{CP}^n$ whose geodesic flows are completely integrable. These facts motivated the author to study "Liouville structures" on compact Kähler manifolds.

The subject of this paper is Liouville structures on compact Kähler surfaces. We say that a quadruplet $(M, g, J; \mathcal{F})$ is a Kähler-Liouville surface if (M, g, J) is a complete, connected Kähler surface and if $\mathcal{F}$ is a 2-dimensional real vector space of first integrals on the cotangent bundle T^*M of its geodesic flows such that (1) the vector space $\mathcal{F}$ contains the energy function on T^*M; (2) all the first integrals contained in the vector space $\mathcal{F}$ are fiberwise homogeneous polynomials of degree 2 and are hermitian with respect to the complex structure J. For each $p \in M$, we put $F_p \equiv F|_{T_p^*M}$ for $F \in \mathcal{F}$ and set $\mathcal{F}_p = \{F_p | F \in \mathcal{F}\}$. We call the points p of M such that $\dim \mathcal{F}_p = 1$ the singular points of $(M, g, J; \mathcal{F})$ and denote the set of them by M_{sing}.

In this paper we will study the compact Kähler-Liouville surface under the assumption of "properness" analogous to the condition of properness in the investigation [**3**] of Liouville manifolds, but not identical to it. We say that a Kähler-Liouville surface $(M, g, J; \mathcal{F})$ is proper if $M_{\text{sing}} \neq \emptyset$ and if, for any $F \in \mathcal{F}$ and $p \in M$ such that $F_p = 0$, there exists a covector $w \in T_p^*M$ such that $(dF)_w \neq 0$.

We remark that any compact proper Kähler-Liouville surface can not admit the structure of the 4-dimensional Liouville manifold of rank one.

The purpose of this paper is to study the structure of the compact proper Kähler-Liouville surface.

The main results in this paper can be stated as follows:

Let $(M, g, J; \mathcal{F})$ be a compact proper Kähler-Liouville surface. Then

(1) The geodesic flow of (M, g) is completely integrable.

(2) (M, J) is bi-holomorphic with the complex projective plane $\boldsymbol{C}P^2$.

(3) (M, g, J) has three points q_0, q_1, q_2 and three totally geodesic 1-dimensional complex submanifolds H_0, H_1, H_2 which are bi-holomorphic with the complex projective line $\boldsymbol{C}P^1$ such that

  (i) $H_1 \cap H_2 = \{q_0\}$, $H_0 \cap H_1 = \{q_2\}$, $H_0 \cap H_2 = \{q_1\}$;

  (ii) H_0 coincides with the subset of M on which $F \in \mathcal{F}$ such that $F_p = 0$ for $p \in M_{\text{sing}}$ is degenerate, and hence includes M_{sing};

  (iii) $H_0 \cup H_1 \cup H_2$ is the subset of M on which $F \in \mathcal{F}$ described in (ii) can be said to be critical in one sense.

(4) M_{sing} forms a compact real submanifold of H_0, and hence also of M, and is diffeomorphic with the circle S^1.

(5) There exists an effective action Φ of the 2-dimensional real torus $S^1 \times S^1$ on M (as automorphisms of $(M, g, J; \mathcal{F})$) such that

  (i) Φ leaves the three points q_0, q_1, q_2 fixed;

  (ii) Φ leaves the three submanifolds H_0, H_1, H_2 invariant.

(6) There exists a family $\mathfrak{S}$ of compact totally geodesic 2-dimensional real submanifolds of (M, g) such that

  (i) Each compact real submanifold $S \in \mathfrak{S}$ associated with the Liouville structure inherited from $\mathcal{F}$ forms a compact real Liouville surface which is diffeomorphic with the real projective plane $\boldsymbol{R}P^2$;

  (ii) The compact real Liouville surfaces belonging to $\mathfrak{S}$ are isomorphically transferred to each other by the action Φ on M.

This paper is organized as follows:

In §1 we will define the Kähler-Liouville surface and demonstrate its condition of properness. In §2 we will discuss local structure of the compact proper Kähler-Liouville surface and present lemmas, propositions and formulas, which will be used is subsequent sections. In §3 we will construct the compact complex submanifold H_0 on which $F \in \mathcal{F}$ described in (3) (ii) in the above main results is degenerate. We will subsequently show that H_0 is bi-holomorphic to the complex projective line $\boldsymbol{C}P^1$ and is totally geodesic, and that M_{sing} is a real submanifold of H_0 which is diffeomorphic with the circle S^1. In §4 we will study the S^1-actions naturally generated by the prescribed infinitesimal automorphisms of $(M, g, J; \mathcal{F})$ and present some lemmas and propositions which will be needed in subsequent sections. In §5 we will first construct the compact complex submanifolds H_1 and H_2, which may be described as the sets of critical points, in one sense, and show that H_1 and H_2 are bi-holomorphic to the complex projective line $\boldsymbol{C}P^1$ and are totally geodesic. Second, we will establish the complete integrability of the geodesic flows of (M, g). Third, we will define an effective action Φ of the 2-dimensional real torus $S^1 \times S^1$ on M. In §6 we will prove that (M, J) is bi-holomorphic to the standard complex projective plane $(\boldsymbol{C}P^2, J_0)$. In §7 we will first discover a family $\mathfrak{S}$ of compact real Liouville surfaces naturally imbedded in M which are diffeomorphic with the real projective plane $\boldsymbol{R}P^2$. Finally, we will see that the torus action Φ on M induces a transitive action $\tilde{\Phi}$ of the real torus $S^1 \times S^1$ on the family $\mathfrak{S}$ and that the compact real Liouville surfaces belonging to $\mathfrak{S}$ are isomorphically transferred to each other by the action Φ on M.

Throughout this paper, we assume the differentiability of class C^∞ unless otherwise stated.

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§ 1. The definition of Kähler-Liouville surfaces.

Let (M, g, J) be a complete, connected Kähler surface and let E be its energy function on T^*M. We define the complex structure J^* of T^*M by $J^*=\flat\circ(-J)\circ\flat^{-1}$, where $\flat$ is the identification map of TM onto T^*M induced by the metric g in an obvious manner.

A quadruplet $(M, g, J\,;\mathcal{F})$ will be called a Kähler-Liouville surface if $\mathcal{F}$ is a 2-dimensional real vector space of $\boldsymbol{R}$-valued functions on T^*M satisfying the following four conditions (KL1), $\cdots$, (KL4):

(KL1) $E\in\mathcal{F}$;

(KL2) For any $F\in\mathcal{F}$ and for any point $p\in M$,

$$F_p\equiv F|_{T_p^*M}: T_p^*M\longrightarrow \boldsymbol{R}$$

is a homogeneous polynomial of degree 2;

(KL3) For any $F\in\mathcal{F}$, F is hermitian, i.e., $F\circ J^*=F$;

(KL4) For any $F_1, F_2\in\mathcal{F}$,

$$\{F_1, F_2\}=0,$$

where $\{*, *\}$ is the canonical Poisson bracket on T^*M.

Two Kähler-Liouville surfaces $(M, g, J\,;\mathcal{F})$ and $(M', g', J'\,;\mathcal{F}')$ will be called mutually isomorphic if there exists a holomorphic isometry $\Psi:(M, g, J)\rightarrow(M', g', J')$ such that mapping $\Psi^\#$ defined by, for $F\in\mathcal{F}$, $\Psi^\#(F)=F\circ\Psi^*$ maps $\mathcal{F}$ into $\mathcal{F}'$.

Note that a vector field Y on M is an infinitesimal automorphism of a Kähler-Liouville surface $(M, g, J\,;\mathcal{F})$ if and only if Y satisfies the following conditions:

(IA1) $\{Y, E\}=0$, or equivalently, $\mathcal{L}_Y g=0$;

(IA2) $\mathcal{L}_Y J=0$;

(IA3) $\{Y, F\}\in\mathcal{F}$ for any $F\in\mathcal{F}$,

where Y is considered as both a vector field on M and a fiberwise linear function on T^*M, and where $\mathcal{L}_Y$ is the Lie derivation with respect to the vector field Y.

Let $(M, g, J\,;\mathcal{F})$ be a Kähler-Liouville surface. For each point $p\in M$, we set $\mathcal{F}_p=\{F_p=F|_{T_p^*M}\,|\,F\in\mathcal{F}\}$; it can be also regarded as a real vector space. It follows that $1\leqq\dim\mathcal{F}_p\leqq 2$ for any point $p\in M$.

A point p of M such that $\dim\mathcal{F}_p=1$ will be called the singular point of $(M, g, J\,;\mathcal{F})$, and the set of them will be denoted by M_{sing}. A point p of M such that $\dim\mathcal{F}_p=2$ will be called the regular point of $(M, g, J\,;\mathcal{F})$, and the set of them will be denoted by M_{reg}.

A Kähler-Liouville surface $(M, g, J; \mathcal{F})$ will be called proper if $\mathcal{F}$ satisfies the following conditions:

(PKL1) $M_{\text{sing}} \neq \emptyset$;

(PKL2) For any $F \in \mathcal{F} \setminus \{0\}$ and $p \in M$ such that $F_p = 0$, there exists a covector $w \in T_p^* M$ which satisfies $(dF)_w \neq 0$.

We know from [1] that, for a Kähler-Liouville surface $(M, g, J; \mathcal{F})$ such that $M_{\text{sing}} \neq \emptyset$, the condition (KL4) in the definition of the Kähler-Liouville surface yields the following property:

For each $F \in \mathcal{F}$, there exists a real constant r such that the equality $F_p = rE_p$ holds for every point $p \in M_{\text{sing}}$.

Thus, for any compact proper Kähler-Liouville surface $(M, g, J; \mathcal{F})$, we can always find $F \in \mathcal{F}$ such that

(N1) $F \neq rE$ for any $r \in \boldsymbol{R}$;

(N2) $F_p = 0$ if and only if $p \in M_{\text{sing}}$.

In the following sections the element F of $\mathcal{F}$ will be always assumed to satisfy the conditions (N1) and (N2) above.

§ 2. Local structure of compact proper Kähler-Liouville surfaces.

Let $(M, g, J; \mathcal{F})$ be a compact, connected proper Kähler-Liouville surface and let F be an element of $\mathcal{F}$; as mentioned at the end of § 1, F is assumed to satisfy the conditions (N1) and (N2) in § 1.

Let $\pi^*: T^*M \to M$ be the cotangent bundle over M and let g^* be the contravariant metric tensor corresponding to g. For each $p \in M$, we define the endomorphism F_p^e of $T_p^* M$ by $F_p(w) = g^*(w, F_p^e(w))$ for $w \in T_p^* M$. This induces the bundle endomorphism F^e of T^*M. The cotangent bundle T^*M is regarded as a complex vector bundle over M by the complex structure J^*. Because F is hermitian, F^e can be regarded as the complex bundle endomorphism of T^*M.

Let $\tilde{\Omega}$ be an arbitrary open subset of M which is equipped with a local unitary coframe $\tilde{V}_1^*$, $\tilde{V}_2^*$ on M. Then, F^e can be represented on $\tilde{\Omega}$ in the following form:

$$(F^e(\tilde{V}_1^*), F^e(\tilde{V}_2^*)) = (\tilde{V}_1^*, \tilde{V}_2^*) \begin{pmatrix} a_{11} & \kappa \\ \bar{\kappa} & a_{22} \end{pmatrix}, \tag{2.1}$$

where a_{11}, a_{22} are $\boldsymbol{R}$-valued functions on $\tilde{\Omega}$ and κ is a $\boldsymbol{C}$-valued function on $\tilde{\Omega}$. We put $A \equiv \begin{pmatrix} a_{11} & \kappa \\ \bar{\kappa} & a_{22} \end{pmatrix}$ on $\tilde{\Omega}$.

We can define the $\boldsymbol{R}$-valued continuous functions f_1 and f_2 on $\tilde{\Omega}$ by

$$(2.2)\qquad \begin{cases} f_1=\dfrac{a_{11}+a_{22}+\sqrt{(a_{11}-a_{22})^2+4\bar{\kappa}\kappa}}{2} \\ f_2=\dfrac{-a_{11}-a_{22}+\sqrt{(a_{11}-a_{22})^2+4\bar{\kappa}\kappa}}{2}. \end{cases}$$

Notice that $-f_2 \leqq f_1$ and that $-f_2$ and f_1 assign the eigen values of the matrix A to each point of $\tilde{\Omega}$. We can see that f_1 and f_2 are globally defined on the whole of M. However, we can not conclude that f_1 and f_2 are smooth at a point of M_{sing}.

From the assumption (N2) of F in § 1 we obtain the following

LEMMA 2.1. *A point p of M belongs to M_{sing} if and only if $f_1(p)=f_2(p)=0$. In particular, M_{sing} is a closed subset of M.*

Moreover, we define the $\boldsymbol{R}$-valued functions h_1 and h_2 on $\tilde{\Omega}$ as follows:

$$(2.3)\qquad \begin{cases} h_1 \equiv -\det A=-a_{11}a_{22}+\bar{\kappa}\kappa \\ h_2 \equiv \operatorname{trace} A=a_{11}+a_{22}. \end{cases}$$

We can also see that h_1 and h_2 are globally defined on the whole of M. From (2.2) and (2.3), we immediately obtain the following relations:

$$(2.4)\qquad h_1=f_1\cdot f_2, \quad h_2=f_1-f_2 \quad \text{on } M.$$

Here we have another expression of the properness condition for $(M, g, J; \mathcal{F})$.

LEMMA 2.2. *Let $(M, g, J; \mathcal{F})$ be a Kähler-Liouville surface satisfying that $M_{\text{sing}} \neq \emptyset$. The condition* (PKL2) *in the definition of properness for $(M, g, J; \mathcal{F})$ in* § 1 *is equivalent to the following condition*:

$$(2.5)\qquad (dh_2)_p \neq 0 \quad \textit{for any } p \in M_{\text{sing}}.$$

PROOF. Let p be an arbitrary point of M_{sing} and $\tilde{\Omega}$ an open neighborhood of p in M which is equipped with a local unitary coframe $\tilde{V}_1^*$, $\tilde{V}_2^*$. We set the orthonormal coframe V_1^*, V_3^*, V_2^*, V_4^* on $\tilde{\Omega}$ corresponding to the unitary coframe $\tilde{V}_1^*$, $\tilde{V}_2^*$ and the orthonormal frame V_1, V_3, V_2, V_4 on $\tilde{\Omega}$ which is dual to the coframe V_3^*, V_1^*, V_4^*, V_2^*. Notice that $J^*V_1^*=V_3^*$, $J^*V_2^*=V_4^*$ and $JV_1=V_3$, $JV_2=V_4$. Recalling (2.1), we have

$$(2.6)\qquad \begin{aligned} F=\ & a_{11}\cdot((V_1)^2+(V_3)^2)+a_{22}\cdot((V_2)^2+(V_4)^2) \\ & +2a_{12}\cdot(V_1V_2+V_3V_4)+2a_{21}\cdot(V_1V_4-V_2V_3) \quad \text{on } \tilde{\Omega}, \end{aligned}$$

where a_{12} and a_{21} are the $\boldsymbol{R}$-valued functions on $\tilde{\Omega}$ such that $\kappa=a_{12}+\sqrt{-1}\,a_{21}$. We notice that $a_{11}(p)=a_{22}(p)=a_{12}(p)=a_{21}(p)=0$.

We now verify the desired equivalence.

(I) The implication (2.5)→(PKL2). Observing (2.6), the condition that $(dF)_w=0$ implies that $(da_{11})_p=(da_{22})_p=0$. From (2.3), we thus have $(dh_2)_p=(da_{11})_p+(da_{22})_p=0$.

(II) The implication (PKL2)→(2.5). Take a certain $w\in T_p^*M$ so that $\|w\|\neq 0$ and $(dF)_w\neq 0$. We can assume that $(V_1^*)_p=w/\|w\|$. Then, using (2.6), we obtain $(da_{11})_p\neq 0$. From the condition (KL4) $\{E, F\}=0$ in §1 we can see that $(da_{11})_p$ can not be obtained by the scalar multiple of $(da_{22})_p$. Thus, we have $(dh_2)_p=(da_{11})_p+(da_{22})_p\neq 0$. □

Going back to the argument for the compact, connected proper Kähler-Liouville surface $(M, g, J; \mathcal{F})$ given at the beginning of this section, we have the following

PROPOSITION 2.3. *The subset M_{reg} of M is a non-empty open subset of M.*

PROOF. Let $p\in M_{\mathrm{sing}}$. The condition (2.5) in Lemma 2.2 implies that there exists a point q close to the point p such that $h_2(q)=f_1(q)-f_2(q)\neq 0$. The openness is immediate from Lemma 2.1. □

For each point p of M_{reg}, there exist both an open neighborhood Ω of p and an orthonormal frame V_1, V_3, V_2, V_4 on Ω which satisfy the following conditions:

(1) $\Omega\subset M_{\mathrm{reg}}$;

(2) $JV_1=V_3$ and $JV_2=V_4$;

(3) F and $2E$ are expressed as

$$\left\{\begin{aligned} F &= -f_2((V_1)^2+(V_3)^2)+f_1((V_2)^2+(V_4)^2) \\ 2E &= (V_1)^2+(V_3)^2+(V_2)^2+(V_4)^2 \qquad \text{on } \Omega. \end{aligned}\right. \tag{2.7}$$

Such an orthonormal frame V_1, V_3, V_2, V_4 on Ω will be called the F-adapted orthonormal frame on Ω.

Using the neighborhood Ω and the F-adapted orthonormal frame V_1, V_3, V_2, V_4 on Ω, we can define the complex subbundles D_1 and D_2 of the tangent bundle $T(M_{\mathrm{reg}})$ over M_{reg} by

$$D_1|_\Omega=\langle\langle V_1, V_3\rangle\rangle, \quad D_2|_\Omega=\langle\langle V_2, V_4\rangle\rangle, \tag{2.8}$$

where $\langle\langle V_i, V_{i+2}\rangle\rangle$ being the distribution on Ω generated by V_i and V_{i+2}, $i=1, 2$. It follows that $T(M_{\mathrm{reg}})=D_1\oplus D_2$.

Now, we have

LEMMA 2.4. *The functions f_1, f_2 and h_1 are non-negative on M.*

PROOF. (1) We recall that $-f_2 \leqq f_1$. Assume that there exists a point p of M_{reg} such that $0 < -f_2(p) \leqq f_1(p)$. Then, observing (2.7), we see that F_p is positive definite. Joining p and a point of M_{sing} by a minimizing geodesic segment, we can find $v \in T_p M$ such that $F_p(v)=0$, which is a contradiction. Likewise, we obtain a contradiction under the assumption that $-f_2(p) \leqq f_1(p) < 0$. Thus, we conclude that $-f_2(p) \leqq 0 \leqq f_1(p)$. Non-negativity of h_1 follows immediately from the fact that $h_1 = f_1 f_2$ in (2.4). □

Combining Lemma 2.1 and Lemma 2.4, we can immediately obtain the following

PROPOSITION 2.5. (1) *The subset M_{sing} of M is characterized as the set of points on which $f_1 + f_2$ vanishes.*

(2) *The subset M_{reg} of M is characterized as the set of points on which $f_1 + f_2$ is positive.*

Moreover, we have the following

PROPOSITION 2.6. *The subset M_{reg} of M is an open dense connected subset of M.*

PROOF. (denseness) We take $p_0 \in M_{\mathrm{reg}}$. Let $S^*_{p_0}M$ be the unit sphere of the cotangent space $T^*_{p_0}M$. From (2.7), we can easily see that the set Θ of unit covectors at p_0 where $dE \wedge dF \neq 0$ is dense in $S^*_{p_0}M$. Let $\pi: T^*M \to M$ be the natural projection, let $\{\zeta_t\}_{t \in \boldsymbol{R}}$ be the geodesic flow on (M, g), let $d\alpha$ be the canonical symplectic structure on T^*M and let X_F be the symplectic vector field on T^*M defined by

$$i(X_F)d\alpha = -dF,$$

where $i(X_F)$ is the interior derivation with respect to X_F. We take $w \in \Theta$. We then consider the geodesic $\gamma(t) = \pi(\zeta_t(w))$ and the Jacobi field

$$B(t) = \pi_*(\zeta_t)_*(X_F)_w$$

along it. From the condition (KL4) $\{E, F\} = 0$ in § 1, we can see that

$$g(\dot{\gamma}(t), \nabla_{(\partial/\partial t)} B(t)) = 0. \tag{2.9}$$

Assume that $\dot{\gamma}(t_0)$, $B(t_0)$ are linearly dependent for some t_0. Replacing $B(t)$ with a linear combination of $\dot{\gamma}(t)$ and $B(t)$, we may assume that $B(t_0)=0$. Then, we can see from (2.9) that $\nabla_{(\partial/\partial t)} B(t_0)$ is a non-zero vector perpendicular to $\dot{\gamma}(t_0)$ and hence that $B(t)$ is a non-zero normal Jacobi field along $\gamma(t)$. Hence, we see that the times t such that $\dot{\gamma}(t)$, $B(t)$ are linearly dependent appear discretely in $\boldsymbol{R}$. Thus, $dF \wedge dE \neq 0$ in a dense subset in T^*M. This implies that E, F are linear independent in a dense subset in M.

(connectedness) Take any two points p_1, p_2 of M_{reg}. We put $F^\flat = F \circ \flat$, where $\flat$ is the identification map of TM onto T^*M induced by g. We join the two points p_1 and p_2 by the minimizing geodesic segment γ; we put $\gamma(0) = p_1$, $\gamma(t_2) = p_2$. When $F(\dot{\gamma}) \neq 0$, the segment γ does not pass through M_{sing}. When $F(\dot{\gamma}) = 0$, observing (2.7), Lemma 2.4 and Proposition 2.5 (2), we can take a geodesic segment γ_1 such that (i) $\gamma_1(0) = p_1$; (ii) $\gamma_1(t_2)$ is in a sufficiently small neighborhood of p_2 which is included in M_{reg}; (iii) $F^\flat(\dot{\gamma}_1) \neq 0$. Thus, we conclude that M_{reg} is arcwise connected. □

Let Ω be an open subset of M_{reg} equipped with a F-adapted orthonormal frame V_1, V_3, V_2, V_4. For $k=1, 2$, we put

$$W_k = \sqrt{f_1+f_2} \cdot V_k, \quad W_{k+2} = \sqrt{f_1+f_2} \cdot V_{k+2} \quad \text{on } \Omega.$$

Then, we obtain an orthogonal frame W_1, W_3, W_2, W_4 on Ω and the following relations:

$$\text{(2.10)} \qquad \begin{cases} -F+f_1 \cdot 2E = (W_1)^2+(W_3)^2 \\ \;\;\, F+f_2 \cdot 2E = (W_2)^2+(W_4)^2 \quad \text{on } \Omega. \end{cases}$$

Such a frame will be called the F-adapted orthogonal frame on Ω.

We take an arbitrary open subset Ω of M_{reg} equipped with a F-adapted orthogonal frame W_1, W_3, W_2, W_4. Henceforth in this section, we will use this open subset and this frame on it without further notice.

From the condition (KL4) $\{E, F\} = 0$ in § 1, we can compute

$$\text{(2.11)} \qquad W_1 f_2 = W_3 f_2 = 0, \quad W_2 f_1 = W_4 f_1 = 0 \quad \text{on } \Omega$$

and the following

LEMMA 2.7. *There exist functions σ_{12}, σ_{21}, τ_{12} and τ_{21} on Ω such that, for $(i, j) \in \{(1, 2), (2, 1)\}$,*

$$\begin{aligned} [W_i, W_j] &= -\sigma_{ji} W_{i+2} + \sigma_{ij} W_{j+2}, \\ [W_i, W_{j+2}] &= -\tau_{ji} W_{i+2} + \sigma_{ij} W_j, \\ [W_{i+2}, W_{j+2}] &= -\tau_{ji} W_i + \tau_{ij} W_j \quad \text{on } \Omega. \end{aligned}$$

Let ω be the Kähler form on (M, g, J) defined by $\omega(X, Y) \equiv g(X, JY)$ for any point p, and for any vectors, X and Y, tangent to M at p. The Kähler condition $d\omega = 0$ yields the following

LEMMA 2.8. *There exist functions ξ_1^1, ξ_1^3, ξ_2^2 and ξ_2^4 on Ω such that, for $(i, j) \in \{(1, 2), (2, 1)\}$,*

$$[W_i, W_{i+2}] = \xi_i^i \cdot W_i + \xi_i^{i+2} \cdot W_{i+2} + \frac{W_{j+2} f_j}{f_1+f_2} \cdot W_j - \frac{W_j f_j}{f_1+f_2} \cdot W_{j+2} \quad \text{on } \Omega.$$

Let $(i, j)\in\{(1, 2), (2, 1)\}$. Using Lemma 2.7 and Lemma 2.8, we have two expressions of $[W_j, [W_i, W_{i+2}]]$ and two expressions of $[W_{j+2}, [W_i, W_{i+2}]]$. Comparing the two expressions of $[W_j, [W_i, W_{i+2}]]$ and comparing those of $[W_{j+2}, [W_i, W_{i+2}]]$, we obtain the following formulas on Ω which will be used in this section and subsequent sections:

(FML1) $$(W_j f_j)\xi_j^j = W_j W_{j+2} f_j - \frac{(W_j f_j)(W_{j+2} f_j)}{f_1+f_2},$$

(FML2) $$\sigma_{ij}\xi_i^i + \tau_{ij}\xi_i^{i+2} + \frac{W_j f_j}{f_1+f_2}\cdot\xi_j^{j+2} = -\frac{(W_j)^2 f_j}{f_1+f_2} + \frac{(W_j f_j)^2}{(f_1+f_2)^2} - W_{i+2}\sigma_{ij} + W_i\tau_{ij},$$

(FML3) $$(W_{j+2} f_j)\xi_j^{j+2} = -W_{j+2} W_j f_j + \frac{(W_j f_j)(W_{j+2} f_j)}{f_1+f_2},$$

(FML4) $$\sigma_{ij}\xi_i^i + \tau_{ij}\xi_i^{i+2} - \frac{W_{j+2} f_j}{f_1+f_2}\cdot\xi_j^j = -\frac{(W_{j+2})^2 f_j}{f_1+f_2} + \frac{(W_{j+2} f_j)^2}{(f_1+f_2)^2} - W_{i+2}\sigma_{ij} + W_i\tau_{ij}$$

for $(i, j)\in\{(1, 2), (2, 1)\}$ and on Ω.

Now, we define the vector fields U_1, U_2, U_3 and U_4 on M_{reg} by

(2.12) $$\begin{cases} i\left(\dfrac{U_3}{f_1+f_2}\right)\omega = df_1, & U_1 = -JU_3, \\ i\left(\dfrac{U_4}{f_1+f_2}\right)\omega = df_2, & U_2 = -JU_4, \end{cases}$$

where $i(U)$ is the interior derivation with respect to U. We set

(2.13) $$M_{\mathrm{REG}} \equiv \{p\in M_{\mathrm{reg}} \mid (df_1)_p\neq 0 \text{ and } (df_2)_p\neq 0\}.$$

It follows that U_1, U_3, U_2 and U_4 form an orthogonal frame on M_{REG} and that $D_i=\langle\langle U_i, U_{i+2}\rangle\rangle$, $i=1, 2$, on M_{REG}. Here we define a real subbundle D_+ of the tangent bundle $T(M_{\mathrm{REG}})$ over M_{REG} by

(2.14) $$D_+ = \langle\langle U_1, U_2\rangle\rangle.$$

Using (FML1), (FML2), (FML3), (FML4) and Lemma 2.7, we obtain the following

LEMMA 2.9. *For* $(i, j)\in\{(1, 2), (2, 1)\}$,

$$[U_i, U_{i+2}] = \frac{U_i f_i}{f_1+f_2}(U_{i+2}-U_{j+2}),$$

$$[U_i, U_j] = [U_i, U_{j+2}] = [U_{i+2}, U_{j+2}] = 0 \quad on\ M_{\mathrm{reg}}.$$

Let ∇ denote the riemannian connection on M with respect to g. A simple calculation leads to the following

LEMMA 2.10. *For* $(i, j)\in\{(1, 2), (2, 1)\}$, *we have*

$$\nabla_{U_i}U_i = \frac{1}{2}\Big(\frac{(U_i)^2 f_i}{U_i f_i}+\frac{U_i f_i}{f_1+f_2}\Big)\cdot U_i - \frac{U_i f_i}{2(f_1+f_2)}\cdot U_j$$

on the domain of M_{reg} *in which* $(df_i)\neq 0$ *holds, and*

$$\nabla_{U_i}U_j = \frac{U_j f_j}{2(f_1+f_2)}\cdot U_i - \frac{U_i f_i}{2(f_1+f_2)}\cdot U_j \quad on\ M_{\mathrm{reg}}.$$

We now define the vector fields Y_1 and Y_2 on M by the following equations:

(2.15) $$i(Y_1)\omega = dh_1, \quad i(Y_2)\omega = dh_2,$$

where $i(Y)$ is the interior derivation with respect to Y.

Observing (2.4), (2.12) and (2.15), we have

(2.16) $$\begin{cases} Y_1+f_1\cdot Y_2 = U_3 \\ Y_1-f_2\cdot Y_2 = U_4 \quad \text{on } M_{\mathrm{reg}}. \end{cases}$$

Then, we see that, for $k, j=1, 2$, $Y_k f_j=0$ on M_{reg}. Thus, from (2.4), it follows that, for $k, j=1, 2$, $Y_k h_j=0$ on M.

Using (2.16) and Lemma 2.9, we can compute the following

LEMMA 2.11. *The vector fields* U_1, U_2, Y_1 *and* Y_2 *on* M_{reg} *are mutually commutative on* M_{reg} *with respect to the Lie bracket, that is,*

$$[U_1, U_2] = 0, \quad [Y_1, Y_2] = 0,$$

$$[U_1, Y_1] = [U_1, Y_2] = [U_2, Y_1] = [U_2, Y_2] = 0 \quad on\ M_{\mathrm{reg}}.$$

Using (2.16), (FML1), (FML2), (FML3) and (FML4), we obtain the following

PROPOSITION 2.12. *For* $i, k=1, 2$, *we have*

$$\{(W_i)^2+(W_{i+2})^2, Y_k\} = 0 \quad on\ \Omega.$$

Using Proposition 2.6, (2.10) and Proposition 2.12, we obtain the following

THEOREM 2.13. *The vector fields* Y_1 *and* Y_2 *on* M *are infinitesimal automorphisms of* $(M, g, J; \mathcal{F})$. *Actually, they satisfy*

(1) $\{E, Y_1\}=\{E, Y_2\}=0$;
(2) $\mathcal{L}_{Y_1}J=\mathcal{L}_{Y_2}J=0$;
(3) $\{F, Y_1\}=\{F, Y_2\}=0$,

where $\mathcal{L}_Y$ means the Lie derivation with respect to Y on M.

§ 3. The submanifold H_0 including the semi-definite subset of M.

Let $(M, g, J; \mathcal{F})$ be a compact, connected proper Kähler-Liouville surface. We now recall the function h_1 on M defined by (2.3) in § 2. We define the subset H_0 of M by

$$H_0=\{p\in M \mid h_1(p)=0\}. \tag{3.1}$$

Since h_1 is non-negative, it follows that $(Y_1)_p=0$ for all $p\in H_0$. Since $h_1=f_1f_2$, we have $H_0\supset M_{\text{sing}}$.

The main objective of this section is to establish the following

THEOREM 3.1. *Let $(M, g, J; \mathcal{F})$ be a compact, connected proper Kähler-Liouville surface and let H_0 be the subset of M defined above. Then*

(1) *H_0 is a complex submanifold of (M, g, J) which is bi-holomorphic with the complex projective line $\boldsymbol{C}P^1$, and is totally geodesic with respect to g.*
(2) *M_{sing} is a compact real submanifold of H_0, and hence also of M, and is diffeomorphic with the circle S^1.*
(3) *H_0 is divided into the two open disks H_{01} and H_{02} by the circle M_{sing}, and the divided domains H_{01} and H_{02} are integral submanifolds of D_1 and D_2 respectively.*

We take an arbitrary point p_0 of M_{sing}. Let $\boldsymbol{R}\ni t\rightarrow\phi_t^{(2)}(p_0)\in M$ be the integral curve of Y_2 through p_0; it is assumed to be $\phi_0^{(2)}(p_0)=p_0$. Since $Y_2h_j=0$ for $j=1, 2$, we have

LEMMA 3.2. $\phi_t^{(2)}(p_0)\in M_{\text{sing}}$ *for all* $t\in\boldsymbol{R}$.

We define the vector fields X_1, X_2 on M by $X_1=-JY_1$, $X_2=-JY_2$. From (2.16) in § 2 we have

$$\begin{cases} X_1+f_1\cdot X_2=U_1 \\ X_1-f_2\cdot X_2=U_2 \quad \text{on } M_{\text{reg}}. \end{cases} \tag{3.2}$$

Using Proposition 2.6 and Lemma 2.11 in § 2, we obtain

$$\begin{aligned}[X_1, X_2] &= [Y_1, Y_2] = [X_1, Y_1] \\ &= [X_1, Y_2] = [X_2, Y_1] = [X_2, Y_2] = 0 \quad \text{on } M.\end{aligned} \tag{3.3}$$

Let $\boldsymbol{R}\ni t\rightarrow\varphi_t^{(2)}(p_0)\in M$ be the integral curve of X_2 through $p_0\in M_{\mathrm{sing}}$; it is assumed to be $\varphi_0^{(2)}(p_0)=p_0$. From the fact that $[X_2, Y_1]=0$ in (3.3) we obtain

LEMMA 3.3. $\varphi_t^{(2)}(p_0)\in H_0$ *for all* $t\in\boldsymbol{R}$.

Then, we have the following

PROPOSITION 3.4. *There exist points at which F is positive semi-definite and also points at which F is negative semi-definite.*

PROOF. Let p_0 be an arbitrary point of M_{sing} and let $\tilde{\Omega}$ be an open neighborhood of p_0 in M equipped with a local orthonormal frame V_1, V_3, V_2, V_4 such that $JV_1=V_3$ and $JV_2=V_4$. Then, we have

$$X_2h_2=\|X_2\|^2=(V_1h_2)^2+(V_3h_2)^2+(V_2h_2)^2+(V_4h_2)^2\geqq 0 \quad \text{on } \tilde{\Omega}.$$

Hence, the condition (2.5) in Lemma 2.2 in § 2 implies that the function

$$\boldsymbol{R}\ni t\longmapsto h_2(\varphi_t^{(2)}(p_0))\in\boldsymbol{R}$$

is a strictly increasing function. Since Lemma 3.3 means that $h_1(\varphi_t^{(2)}(p_0))=0$ for all $t\in\boldsymbol{R}$, we have

(i) when $t>0$, $F_{\varphi_t^{(2)}(p_0)}$ is positive semi-definite;

(ii) when $t<0$, $F_{\varphi_t^{(2)}(p_0)}$ is negative semi-definite. □

PROPOSITION 3.5. *There exist points at which F is indefinite.*

This proposition is obtained immediately from the following

LEMMA 3.6. $h_1\not\equiv 0$ *on* M.

PROOF. Assume that $h_1\equiv 0$ on M. For $i=1, 2$, we set $G_i\equiv\{p\in M_{\mathrm{reg}}\,|\,f_i(p)=0\}$. Then, G_1 and G_2 have the following three properties: (i) Both G_1 and G_2 are closed subsets of M_{reg}; (ii) $G_1\cap G_2=\emptyset$; (iii) $M_{\mathrm{reg}}=G_1\cup G_2$. Because of the connectedness of M_{reg} (Proposition 2.6 in § 2), either $G_1=\emptyset$ or $G_2=\emptyset$ holds. This contradicts Proposition 3.4. □

PROPOSITION 3.7. *The subset* H_0 *of* M *is a compact, connected* 1*-dimensional complex submanifold of* (M, g, J) *such that*

(1) H_0 *is bi-holomorphic to the complex projective line* $\boldsymbol{C}P^1$;

(2) H_0 *is totally geodesic with respect to* g.

PROOF. We take an arbitrary point p_0 of M_{sing} and a sufficiently small $\varepsilon>0$. Then, for an open interval $]a, b[$ we set

$$\Pi(]a, b[)\equiv\{\varphi_t^{(2)}(\phi_u^{(2)}(p_0))\,|\,a<t<b,\ -\varepsilon<u<\varepsilon\}.$$

By Lemma 3.2 and Lemma 3.3, we have $\Pi(]-\varepsilon, \varepsilon[)\subset H_0$. From the fact that $[X_2, Y_2]=0$ in (3.3), we see that $\Pi(]-\varepsilon, \varepsilon[)$ is a local integral surface of the distribution $\langle\langle X_2, Y_2\rangle\rangle$ generated by X_2 and Y_2.

Now, we set

$$\tilde{H}_0 = \{p\in M \mid (Y_1)_p=0\}.$$

Then, from (3.1) it follows that $H_0\subset\tilde{H}_0$. Let $(\tilde{H}_0)_0$ be the connected component of $\tilde{H}_0$ including the point p_0. It follows that $(\tilde{H}_0)_0\supset\Pi(]-\varepsilon, \varepsilon[)\ni p_0$. Since Y_1 is an infinitesimal isometry of (M, g), by a standard theory of the transformation group we can assert that $(\tilde{H}_0)_0$ is a closed totally geodesic submanifold whose codimension is even, namely, that $(\tilde{H}_0)_0$ coincides with one of the following: (i) The total space M; (ii) A certain 2-dimensional compact totally geodesic submanifold; (iii) A certain point of M. Because of Lemma 3.6 we have $(\tilde{H}_0)_0\neq M$. Since $(\tilde{H}_0)_0\supset\Pi(]-\varepsilon, \varepsilon[)$, $(\tilde{H}_0)_0$ is not one point. We thus conclude that $(\tilde{H}_0)_0$ is a compact, connected totally geodesic 2-dimensional real submanifold. Moreover, since Y_1 is an infinitesimal holomorphic transformation, $(\tilde{H}_0)_0$ is a complex submanifold of (M, J). Since $dh_1\equiv 0$ on $(\tilde{H}_0)_0$, we have $h_1\equiv 0$ on $(\tilde{H}_0)_0$. Thus, we obtain

$$(\tilde{H}_0)_0\subset H_0\subset \tilde{H}_0.$$

Here, we will verify that $(\tilde{H}_0)_0=H_0$. Assume that $H_0\setminus(\tilde{H}_0)_0\neq\emptyset$ and take $q\in H_0\setminus(\tilde{H}_0)_0$. We also take points p_- and p_+ of $\Pi(]-\varepsilon, 0[)$ and $\Pi(]0, \varepsilon[)$, respectively. Notice that $p_-, p_+\in(\tilde{H}_0)_0$. Let γ_- and γ_+ be the minimizing geodesics from q to p_- and from q to p_+ respectively; they are assumed to parameterized as $\gamma_-(0)=\gamma_+(0)=q$ and $\gamma_-(s_-)=p_-$, $\gamma_+(s_+)=p_+$. Since $(\tilde{H}_0)_0$ is compact and totally geodesic and since $q\notin(\tilde{H}_0)_0$, it follows that $\dot{\gamma}_-(s_-)$, $\dot{\gamma}_+(s_+)$ are not tangent to $(\tilde{H}_0)_0$. We put $F^\flat=F\circ\flat$, where $\flat$ is the identification map of TM onto T^*M induced by g. Using the same argument as in the proof of Proposition 3.4, we have

$$F^\flat(\dot{\gamma}_-(0)) = F^\flat(\dot{\gamma}_-(s_-)) < 0, \quad F^\flat(\dot{\gamma}_+(0)) = F^\flat(\dot{\gamma}_+(s_+)) > 0,$$

which contradicts the property that F is semi-definite at $q\in H_0$.

Thus, we conclude that H_0 is a compact, connected totally geodesic 1-dimensional complex submanifold of (M, g, J).

It remains to verify (1). Since $Y_2h_1=0$, it follows that Y_2 is tangent to H_0 at any point of H_0. Then, we see that Y_2 is a non-trivial infinitesimal holomorphic transformation of H_0 and hence that H_0 is bi-holomorphic to CP^1 or the 1-dimensional complex torus. Since Y_2 vanishes at the points at which the function $h_2|_{H_0}$ takes the maximal value or minimal value, we can conclude that H_0 is bi-holomorphic to the complex projective line CP^1. □

PROPOSITION 3.8. *The subset M_{sing} of M is a real submanifold of H_0 which is diffeomorphic with the circle S^1; actually, there exists a positive real constant c_2 such that, for any $p_0 \in M_{\mathrm{sing}}$,*

$$M_{\mathrm{sing}} = \left\{\phi_u^{(2)}(p_0) \mid u \in \left(\boldsymbol{R}\Big/\frac{2\pi}{c_2}\boldsymbol{Z}\right)\right\}.$$

PROOF. Since $h_1 = f_1 f_2$, we have $H_0 \supset M_{\mathrm{sing}}$. As in the proof of Proposition 3.7, for each $p \in M_{\mathrm{sing}}$, we take the open neighborhood $\Pi_p(]-\varepsilon, \varepsilon[)$ of p in H_0 as follows:

$$\Pi_p(]-\varepsilon, \varepsilon[) \equiv \{\varphi_t^{(2)}(\phi_u^{(2)}(p)) \mid -\varepsilon < t < \varepsilon, \ -\varepsilon < u < \varepsilon\}.$$

Because $\Pi_p(]-\varepsilon, \varepsilon[) \cap M_{\mathrm{sing}} = \{\phi_u^{(2)}(p) \mid -\varepsilon < u < \varepsilon\}$ for each $p \in M_{\mathrm{sing}}$, M_{sing} is a real 1-dimensional regular submanifold of H_0. Since M_{sing} is a closed subset of H_0, it follows that M_{sing} is a compact submanifold of H_0.

Take an arbitrary $p_0 \in M_{\mathrm{sing}}$ and fix it. Let $(M_{\mathrm{sing}})_0$ be the connected component of M_{sing} including p_0. Then, it is a compact, connected 1-dimensional real manifold, and hence, is diffeomorphic with the circle S^1. Thus, there exists a positive real constant c_2 such that $(M_{\mathrm{sing}})_0$ can be expressed as

$$(M_{\mathrm{sing}})_0 = \left\{\phi_u^{(2)}(p_0) \mid u \in \left(\boldsymbol{R}\Big/\frac{2\pi}{c_2}\boldsymbol{Z}\right)\right\}.$$

We here set $\Xi_0 = \{\varphi_t^{(2)}(\phi_u^{(2)}(p_0)) \mid t \in \boldsymbol{R}, \ u \in (\boldsymbol{R}/(2\pi/c_2)\boldsymbol{Z})\}$, which forms an open submanifold of H_0 diffeomorphic with the cylinder. Using Lemma 2.10 in § 2, we can easily see that, for each $u \in (\boldsymbol{R}/(2\pi/c_2)\boldsymbol{Z})$, the curve $\boldsymbol{R} \ni t \to \varphi_t^{(2)}(\phi_u^{(2)}(p_0)) \in H_0$ coincides with a geodesic segment whose initial vector is $(X_2/\|X_2\|)_{\phi_u^{(2)}(p_0)}$ set-theoretically. Since H_0 is compact, we can find two points q_1 and q_2 of H_0 by taking the limit as $\lim_{t \to -\infty} \varphi_t^{(2)}(p_0) = q_2$, $\lim_{t \to +\infty} \varphi_t^{(2)}(p_0) = q_1$. Since Y_2 is a non-trivial infinitesimal isometry of the 2-dimensional sphere H_0, we can see that $\{q \in H_0 \mid (Y_2)_q = 0\} = \{q_1, q_2\}$. Hence, it is easy to see that $H_0 = \{q_2\} \cup \Xi_0 \cup \{q_1\}$ and therefore

$$M_{\mathrm{sing}} = (M_{\mathrm{sing}})_0. \qquad \square$$

From the proof of Proposition 3.8, together with the proof of Proposition 3.4, we obtain the following

PROPOSITION 3.9. *The circle M_{sing} divides the sphere H_0 into two domains H_{01} and H_{02} such that*

(1) *H_{0i}, $i=1, 2$, is an integral submanifold of D_i;*
(2) *F is positive semi-definite on H_{01}, and negative semi-definite on H_{02};*
(3) *H_{0i}, $i=1, 2$, is diffeomorphic to the 2-dimensional real open disk with origin q_i;*

(4) *the function* $f_i|_{H_0}$, $i=1, 2$, *takes the maximal value, say* m_i, *at* q_i, *and* $f_i=0$, $df_i=0$ *on* H_{0j}, *where* j *is the integer such that* $(i, j)\in\{(1, 2), (2, 1)\}$.

Therefore, combining Proposition 3.7, Proposition 3.8 and Proposition 3.9, we obtain Theorem 3.1.

Observing the proof of Proposition 3.8, we moreover obtain the following

PROPOSITION 3.10. *The one-parameter group* $\{\psi_u^{(2)}\}_{u\in\boldsymbol{R}}$ *of automorphisms of* $(M, g, J; \mathcal{F})$ *generated by* Y_2 *defines an* S^1*-action* ψ_2 *on* H_0 *with the least period* $2\pi/c_2$ *given by*

$$\psi_2: H_0\times\left(\boldsymbol{R}\Big/\frac{2\pi}{c_2}\boldsymbol{Z}\right)\ni(p, u)\longmapsto\psi_u^{(2)}(p)\in H_0,$$

where c_2 *is the positive real constant stated in Proposition* 3.8; *this action can be recognized as the rotation of the sphere* H_0 *which leaves the two points* q_1 *and* q_2 *fixed as its pivotal points.*

§ 4. S^1-actions on $(M, g, J; \mathcal{F})$.

For a submanifold L of M we denote by N_pL the normal vector space to L at $p\in L$ in (M, g); we will use this symbol in this section and subsequent sections. From Theorem 2.13 in § 2, we recall that the vector fields Y_1 and Y_2 defined by (2.15) in § 2 are infinitesimal automorphisms of $(M, g, J; \mathcal{F})$.

We begin with the following

LEMMA 4.1. *Let* Z *be an infinitesimal automorphism of* $(M, g, J; \mathcal{F})$ *described as* $Z=\bar{m}_1\cdot Y_1+\bar{m}_2\cdot Y_2$, *where* $\bar{m}_1, \bar{m}_2\in\boldsymbol{R}$, *and let* $\{\eta_t\}_{t\in\boldsymbol{R}}$ *be the one-parameter group of automorphisms of* $(M, g, J; \mathcal{F})$ *generated by* Z. *If a point* p *of* M_{reg} *satisfies* $Z_p=0$, *then, for* $i=1, 2$, *there exists a real number* $\hat{c}_i(p)$ *such that the mapping* $(\eta_t)_{*p}|_{(D_i)_p}$, $t\in\boldsymbol{R}$, *is expressed as*

$$(\eta_t)_{*p}v=\cos(\hat{c}_i(p)t)\cdot v+\sin(\hat{c}_i(p)t)\cdot Jv \quad \textit{for any } v\in(D_i)_p,$$

which is a $\boldsymbol{C}$*-linear isometry of the complex vector space* $(D_i)_p$ *onto itself.*

PROOF. We take a neighborhood Ω of p in M_{reg} which is equipped with an F-adapted orthogonal frame W_1, W_3, W_2, W_4. From Proposition 2.12 in § 2, it follows that, for $i=1, 2$, $\{Z, (W_i)^2+(W_{i+2})^2\}=0$ on Ω. This is equivalent to the following statement: there exists a function $\hat{c}_i$ on Ω such that $[Z, W_i]=-\hat{c}_iW_{i+2}$, $[Z, W_{i+2}]=\hat{c}_iW_i$ on Ω. This implies that, for any $v\in(D_i)_p$,

$$\frac{\partial}{\partial t}(\eta_t)_{*p}v=\hat{c}_i(p)\cdot J((\eta_t)_{*p}v), \quad t\in\boldsymbol{R}.$$

Integrating this equation, we obtain the desired equation. □

LEMMA 4.2. *Let* $(i, j)\in\{(1, 2), (2, 1)\}$. *Let* L *be a connected integral submanifold of the distribution* D_i *in* M_{reg} *such that* $(df_j)_p=0$ *for all* $p\in L$, *and let* $\tilde{m}_j$ *denote the constant value which* f_j *takes on the whole of* L. *Let* Z *be an infinitesimal automorphism of* $(M, g, J;\mathcal{F})$ *defined by* $Z=Y_1+(-1)^{j+1}\tilde{m}_jY_2$ *and let* $\{\eta_t\}_{t\in\boldsymbol{R}}$ *denote the one-parameter group of automorphisms of* $(M, g, J;\mathcal{F})$ *generated by* Z.

Then, there exists a real constant c *such that*

(1) *for each* $p\in L$, *the mapping* $(\eta_t)_{*p}|_{N_pL}$, $t\in\boldsymbol{R}$, *is a* $\boldsymbol{C}$*-linear isometry of* N_pL *onto itself given by*

$$(\eta_t)_{*p}v=\cos(ct)\cdot v+\sin(ct)\cdot Jv \quad \text{for any } v\in N_pL\,;$$

(2) *we have*

$$((W_j)^2f_j)(p)=((W_{j+2})^2f_j)(p)=c \quad \text{for all } p\in L,$$

where W_1, W_3, W_2, W_4 *is an F-adapted orthogonal frame on a certain neighborhood of* p *in* M_{reg}.

PROOF. Let p be an arbitrary point of L. From the condition that $(df_j)_p=0$ and $f_j(p)=\tilde{m}_j$ we obtain

$$Z_p=0. \tag{4.1}$$

Then, applying Lemma 4.1, we can see that $(\eta_t)_{*p}|_{N_pL}$, $t\in\boldsymbol{R}$, is a $\boldsymbol{C}$-linear isometry of N_pL onto itself and that there exists a real number $c(p)$ such that

$$\frac{\partial}{\partial t}(\eta_t)_{*p}v=c(p)\cdot J((\eta_t)_{*p}v) \quad \text{for all } t\in\boldsymbol{R} \text{ and } v\in N_pL. \tag{4.2}$$

Note that $T_pL=(D_i)_p$ and $N_pL=(D_j)_p$. Using the formulas (FML1) and (FML3) in §2, we obtain

$$\left.\frac{\partial}{\partial t}(\eta_t)_{*p}v\right|_{t=0}=\nabla_vZ=((W_j)^2f_j)(p)\cdot Jv \quad \text{for any } v\in N_pL. \tag{4.3}$$

By the formulas (FML2), (FML4) in §2, we have $((W_j)^2f_j)(p)=((W_{j+2})^2f_j)(p)$. Combining (4.2), (4.3) and this equality, we obtain

$$c(p)=((W_j)^2f_j)(p)=((W_{j+2})^2f_j)(p). \tag{4.4}$$

Using Lemma 2.7 and the formulas (FML1), (FML3) in §2, we can easily see that $(W_i)_p((W_j)^2f_j)=(W_{i+2})_p((W_j)^2f_j)=0$. This implies that $c(p)$ is independent of the choice of the point p of L; we thus obtain the desired real constant c. Therefore, (4.2) and (4.4) means the properties (1) and (2), respectively. □

Now, as an application of Lemma 4.2, we have the following

PROPOSITION 4.3. *There exists a non-zero real constant* c_1 *such that*

(1) *the one-parameter group* $\{\phi_t^{(1)}\}_{t\in\boldsymbol{R}}$ *of automorphisms of* $(M, g, J; \mathcal{F})$ *generated by* Y_1 *possesses the property that, for any* $t\in\boldsymbol{R}$ *and for any* $p\in H_0$, $(\phi_t^{(1)})_{*p}|_{N_p(H_0)}: N_p(H_0)\rightarrow N_p(H_0)$ *is a* $\boldsymbol{C}$*-linear isometry of* $N_p(H_0)$ *which satisfies*

$$\frac{\partial}{\partial t}(\phi_t^{(1)})_{*p}v = c_1\cdot J((\phi_t^{(1)})_{*p}v)$$

for any $t\in\boldsymbol{R}$ *and for any* $v\in N_p(H_0)$ *and* $p\in H_0$;

(2) *for* $(i, j)\in\{(1, 2), (2, 1)\}$, *we have*

$$((W_j)^2f_j)(p) = ((W_{j+2})^2f_j)(p) = c_1 \quad \text{for any } p\in H_{0i},$$

where W_1, W_3, W_2, W_4 *is an F-adapted orthogonal frame defined on a neighborhood of* p *in* M.

PROOF. Since $(Y_1)_p=0$ for all $p\in H_0$, we can see that, for each $t\in\boldsymbol{R}$, $\phi_t^{(1)}|_{H_0}$ is the identity transformation of H_0. Let p be an arbitrary point of H_0. We note that $T_pM=T_p(H_0)\oplus N_p(H_0)$ and that $N_p(H_0)$ is a 1-dimensional complex subspace of T_pM which is perpendicular to $T_p(H_0)$. Since $\phi_t^{(1)}$, $t\in\boldsymbol{R}$, is an automorphism of $(M, g, J; \mathcal{F})$ and hence a holomorphic isometry of (M, g, J) onto itself, we can obtain a $\boldsymbol{C}$-linear isometry $(\phi_t^{(1)})_{*p}|_{N_p(H_0)}: N_p(H_0)\rightarrow N_p(H_0)$, $t\in\boldsymbol{R}$. Hence, there exists a real number $\tilde{c}_1(p)$ such that

$$\frac{\partial}{\partial t}(\phi_t^{(1)})_{*p}v = \tilde{c}_1(p)\cdot J((\phi_t^{(1)})_{*p}v) \tag{4.5}$$

for each $t\in\boldsymbol{R}$ and $v\in N_p(H_0)$.

Notice that $\tilde{c}_1(p)$ is independent of the choice of t and v.

Let $(i, j)\in\{(1, 2), (2, 1)\}$. From Proposition 3.9 in §3 we recall that $T_p(H_{0i})=(D_i)_p$, $f_j(p)=0$ and $(df_j)_p=0$ for each $p\in H_{0i}$. Then, applying Proposition 4.2 to the case where $L=H_{0i}$ and $\tilde{m}_j=0$, we can find a real constant $c_{1,i}$ which satisfies the properties (4.6) and (4.7) as follows:

$$\frac{\partial}{\partial t}(\phi_t^{(1)})_{*p}v = c_{1,i}\cdot J((\phi_t^{(1)})_{*p}v) \tag{4.6}$$

for all $t\in\boldsymbol{R}$, $v\in N_p(H_{0i})$ and for all $p\in H_{0i}$;

$$((W_j)^2f_j)(p) = ((W_{j+2})^2f_j)(p) = c_{1,i} \quad \text{for all } p\in H_{0i}, \tag{4.7}$$

where W_1, W_3, W_2, W_4 is an F-adapted orthogonal frame defined on a neighborhood of p in M.

Observing the fact that $H_0=H_{02}\cup M_{\text{sing}}\cup H_{01}$ and $M_{\text{sing}}=\bar{H}_{02}\cap\bar{H}_{01}$, we obtain $\tilde{c}_1(p)=c_{1,1}=c_{1,2}$ for all $p\in H_0$. Thus, putting $c_1=c_{1,1}=c_{1,2}$, from (4.5) and (4.7) we obtain the properties (4.8) and (4.9) as follows:

$$\frac{\partial}{\partial t}(\phi_t^{(1)})_{*p}v = c_1\cdot J((\phi_t^{(1)})_{*p}v) \tag{4.8}$$
for all $t \in \boldsymbol{R}$, $v \in N_p(H_{0i})$ and for all $p \in H_{0i}$;

$$((W_j)^2 f_j)(p) = ((W_{j+2})^2 f_j)(p) = c_1 \quad \text{for all } p \in H_{0i}. \tag{4.9}$$

Now we verify that $c_1 \neq 0$. Assume that $c_1 = 0$. We take an arbitrary point q of $M \backslash H_0$ and the minimizing geodesic segment $\gamma(s)$, $0 \leqq s \leqq s_1$, from the point q to the compact submanifold H_0; it may be assumed to be $\gamma(s_1) = q$ and $\gamma(0) = p_1 \in H_0$. It follows that $\dot{\gamma}(0) \in N_{p_1}(H_0)$. Putting $v_1 = \dot{\gamma}(0)$, we have $\gamma(s) = \exp sv_1$, $0 \leqq s \leqq s_1$. Since $\phi_t^{(1)}$, $t \in \boldsymbol{R}$, is an isometry, we have

$$\phi_t^{(1)}(\gamma(s)) = \phi_t^{(1)}(\exp sv_1) = \exp(s\cdot(\phi_t^{(1)})_{*p}v_1) \tag{4.10}$$
for $s \in [0, s_1]$ and $t \in \boldsymbol{R}$.

Using (4.8) and the assumption that $c_1 = 0$, we obtain

$$(Y_1)_q = \frac{\partial}{\partial t}\phi_t^{(1)}(\gamma(s_1))\Big|_{t=0} = s_1\cdot(\exp)_{*s_1 v_1}\Big(\frac{\partial}{\partial t}(\phi_t^{(1)})_{*p_1}v_1\Big) = 0.$$

Since q is an arbitrary point of $M \backslash H_0$, we obtain $Y_1 \equiv 0$ on M, which contradicts Lemma 3.6 in § 3. Thus, we obtain the desired non-zero real constant c_1. Therefore, (4.8) and (4.9) prove Proposition 4.3. □

Reconsidering the argument at (4.10), we can see that the mapping $\boldsymbol{R} \ni t \mapsto \phi_t^{(1)}(q) \in M$ is periodic with the period $2\pi/c_1$ and that $\phi_t^{(1)}$ defines an S^1-action ϕ_1 (as automorphisms of $(M, g, J; \mathcal{F})$) in the following form:

$$\phi_1 : M \times \Big(\boldsymbol{R}\Big/\frac{2\pi}{c_1}\boldsymbol{Z}\Big) \ni (p, t) \longmapsto \phi_t^{(1)}(p) \in M, \tag{4.11}$$

which leaves the submanifold H_0 fixed pointwise.

On the other hand, recalling Proposition 3.10 in § 3, we moreover have the following

PROPOSITION 4.4. *Let $i \in \{1, 2\}$. For any $v \in T_{q_i}(H_0) = (D_i)_{q_i}$ we have*

$$\frac{\partial}{\partial u}(\phi_u^{(2)})_{*q_i}v = (-1)^i c_2\cdot J((\phi_u^{(2)})_{*q_i}v), \quad u \in \Big(\boldsymbol{R}\Big/\frac{2\pi}{c_2}\boldsymbol{Z}\Big),$$

where c_2 is the positive real constant stated in Proposition 3.8 *and Proposition* 3.10 *in* § 3.

PROOF. Let $(i, j) \in \{(1, 2), (2, 1)\}$. We recall that $T_{q_i}(H_0) = (D_i)_{q_i}$, $N_{q_i}(H_0) = (D_j)_{q_i}$ and $(Y_2)_{q_i} = 0$. Then, applying Lemma 4.1 to the case where $Z = Y_2$, we can see that $(\phi_u^{(2)})_{*q_i}|_{T_{q_i}(H_0)}$ is a $\boldsymbol{C}$-linear isometry of $T_{q_i}(H_0)$ onto itself and that there exists a real number $c_{2,i}$ such that

$$\frac{\partial}{\partial u}(\psi_u^{(2)})_{*q_i}v = c_{2,i}\cdot J((\psi_u^{(2)})_{*q_i}v), \quad u \in \left(\boldsymbol{R}\Big/\frac{2\pi}{c_2}\boldsymbol{Z}\right),$$

for all $v \in T_{q_i}(H_0)$. Since q_2 and q_1 are the pivotal points of the rotation of the sphere H_0 which are antipodal to each other, it follows that $c_{2,2}=-c_{2,1}=c_2$. Thus, we obtain the desired equation. □

§5. Submanifolds H_1 and H_2.

We set $H \equiv \{p \in M \mid (dh_1)_p, (dh_2)_p$ are linearly dependent$\}$ and $Q \equiv \{q \in M_{\mathrm{reg}} \mid (df_1)_q = (df_2)_q = 0\}$. We recall from Corollary 3.9 (4) in §3 that, for $i=1, 2$, $f_i|_{H_0}$ takes the maximal value m_i at q_i.

The first objective of this section is to prove the following

THEOREM 5.1. *There exist a point q_0 of M and complex submanifolds H_1 and H_2 of M which have the following properties*:

(1) $H = H_0 \cup H_1 \cup H_2$;
(2) $Q = \{q_0, q_1, q_2\}$;
(3) $H_1 \cap H_2 = \{q_0\}$, $H_1 \cap H_0 = \{q_2\}$ *and* $H_2 \cap H_0 = \{q_1\}$;
(4) H_i, $i=1, 2$, *is a totally geodesic complex submanifold of* (M, g, J) *which is bi-holomorphic to the complex projective line* $\boldsymbol{C}P^1$;
(5) H_i, $i=1, 2$, *is a (maximal) integral submanifold of* D_i;
(6) H_i, $i=1, 2$, *is the set of the points on which* f_j *takes the maximal value* m_j, *where* j *is the integer such that* $(i, j) \in \{(1, 2), (2, 1)\}$.

PROOF. We divide the proof into several steps.

(Step 1) In this step we will construct H_1 and H_2, and study their properties. We begin with the following

LEMMA 5.2. *Let* $(i, j) \in \{(1, 2), (2, 1)\}$. *There exist both a compact, connected complex submanifold* H_i *of* (M, g, J) *including* q_j *and a point* q_{0i} *of* H_i *such that*

(i) H_i *is bi-holomorphic to the complex projective line* $\boldsymbol{C}P^1$, *and is totally geodesic with respect to* g;
(ii) H_i *is an integral submanifold of* D_i *in* M;
(iii) $f_j(p) = m_j > 0$, $(df_j)_p = 0$ *for all* $p \in H_i$;
(iv) $H_i \cap H_0 = \{q_j\}$;
(v) $f_i|_{H_i}$ *takes the maximal value, say* $m_{ii} > 0$, *at the point* q_{0i} *of* H_i, *and we have* $(df_i)_{q_{0i}} = 0$.

PROOF OF LEMMA 5.2. We will first construct the desired submanifold H_i. We define a function $\tilde{h}_j$ on M by $\tilde{h}_j \equiv (f_i + m_j)(f_j - m_j)$, and a vector field Z_j on M by $i(Z_j)\omega = d\tilde{h}_j$. It follows that

(5.1) $$Z_j = Y_1 + (-1)^{j+1} m_j \cdot Y_2.$$

Since Y_1 and Y_2 are infinitesimal automorphisms of $(M, g, J; \mathcal{F})$, so is Z_j. Here we set $\tilde{H}_i \equiv \{p \in M | (Z_j)_p = 0\}$. It is easy to see that $\tilde{H}_i \cap H_0 = \{q_1, q_2\}$ and hence that $\tilde{H}_i \subset M_{\mathrm{reg}}$. Let H_i denote the connected component of $\tilde{H}_i$ including the point q_j. Since Z_j is an infinitesimal holomorphic transformation of (M, J) and an infinitesimal isometry of (M, g), H_i is a compact, connected totally geodesic complex submanifold of (M, g, J) whose codimension is even. We note that $H_i \subset M_{\mathrm{reg}}$. We have

$$T_{q_j}(H_i) = \{X \in T_{q_j} M \mid [Z_j, X]_{q_j} = 0\},$$

where X appearing in the equality $[Z_j, X]_{q_j} = 0$ is regarded as a vector field which is extended on a certain open neighborhood of q_j in M. Since $[Z_j, X]_{q_j} = 0$ for $X \in (D_i)_{q_j}$, we have $(D_i)_{q_j} \subset T_{q_j}(H_i)$. Thus, we can conclude that H_i is a compact, connected totally geodesic 1-dimensional complex submanifold of (M, g, J) and that $(D_i)_{q_j} = T_{q_j}(H_i)$.

We will now verify the property (i). Since H_i is totally geodesic, we have

$$\exp_{q_j}(D_i)_{q_j} = H_i.$$

Let $\hat{p}$ be an arbitrary point of $H_i \setminus \{q_j\}$. We can take a geodesic $\gamma_{q_j \hat{p}}$ from q_j to $\hat{p}$ such that $\gamma_{q_j \hat{p}}(0) = q_j$ and $\dot{\gamma}_{q_j \hat{p}}(0) \in (D_i)_{q_j}$; we set $v = \dot{\gamma}_{q_j \hat{p}}(0)$ and $\gamma_{q_j \hat{p}}(\hat{s}) = \hat{p}$. We denote by $\{\psi_t^{(1)}\}_{t \in \boldsymbol{R}}$ the one-parameter group of automorphisms of $(M, g, J; \mathcal{F})$ generated by Y_1. From Proposition 4.3 (1) in §4 and the fact that $\psi_t^{(1)}$, $t \in \boldsymbol{R}$, is an isometry of (M, g), we have

$$(Y_1)_{\hat{p}} = (\exp_{q_1})_{*\hat{s}v}(\hat{s} c_1 \cdot Jv) \in T_{\hat{p}}(H_i).$$

Since $c_1 \neq 0$ and $\hat{s} \neq 0$, if $\hat{p}$ is sufficiently close to q_j, then $(Y_1)_{\hat{p}} \neq 0$. Thus, Y_1 can be regarded as a non-trivial infinitesimal holomorphic transformation of H_i. Since $(Y_1)_{q_j} = 0$, we can conclude that H_i is bi-holomorphic to the complex projective line $\boldsymbol{C}P^1$, thus proving (i).

Here we can find the point q_{0i} of H_i. Since Y_1 is a non-trivial infinitesimal isometry of the 2-dimensional sphere H_i, the set of points in H_i on which Y_1 vanishes consists of two points: one is the point q_j; the other we denote by q_{0i}.

We will verify the properties (ii), (iii), (iv) and (v). Since $(Z_j)_p = 0$ for all $p \in H_i$, we have $(df_j)_p = 0$ for all $p \in H_i$. This, together with the fact that $f_j(q_j) = m_j$ (>0), implies $f_j(p) = m_j > 0$ for all $p \in H_i$, thus proving (iii). From (iii), we can see that, for each $p \in H_i$,

$$(Y_1)_p = \frac{m_j}{f_i(p) + m_j} \cdot (U_{i+2})_p \in (D_i)_p.$$

Since Y_1 is tangent to H_i and $(Y_1)_p \neq 0$ for all $p \in H_i \setminus \{q_j, q_{0i}\}$, we can see that

$T_p(H_i)=(D_i)_p$ for all $p\in H_i$, which means (ii). From (iii) and the fact that $\tilde{H}_i\cap H_0=\{q_1, q_2\}$, we obtain (iv). Let γ_0 be a geodesic in H_i joining q_j and q_{0i}; we set $\gamma_0(0)=q_j$ and $\gamma_0(l_i)=q_{0i}$. The family $\{\gamma_t(s)\equiv\phi_t^{(1)}(\gamma_0(s))\}_{t\in G}$, where $G=(\boldsymbol{R}/(2\pi/c_1)\boldsymbol{Z})$, of the geodesic joining q_j and q_{0i} forms a polar coordinate in H_i with poles q_j and q_{0i} such that each geodesic γ_t is a meridian curve. Since $\dot{\gamma}_t(s)$ is perpendicular to $(Y_2)_{\gamma_t(s)}$ and since $f_i(\gamma_t(0))=f_i(q_j)=0$, it follows that $\dot{\gamma}_t(s)=(U_{i+2}/\|U_{i+2}\|)_{\gamma_t(s)}$ and $\dot{\gamma}_t(s)f_i>0$ for all $s\in]0, l_i[$ and $t\in(\boldsymbol{R}/(2\pi/c_1)\boldsymbol{Z})$. Hence, the function $f_i|_{H_i}$ takes the maximal value m_{ii} at the point q_{0i}. Thus, we obtain (v). This completes the proof of Lemma 5.2.

(Step 2) In this step we will verify the following

ASSERTION. *We have*

$$q_{01}=q_{02},\ \textit{denoted as}\ q_0,\ \textit{and}\ H_1\cap H_2=\{q_0\}. \tag{5.2}$$

This, together with Lemma 5.2 (iv), means the property (3) of Theorem 5.1. To establish Assertion (5.2) we need the following

LEMMA 5.3. *Let* $(i, j)\in\{(1, 2), (2, 1)\}$. *For each* $p\in H_i$, *we have*

$$((W_j)^2f_j)(p)=((W_{j+2})^2f_j)(p)=-m_jc_2\ (<0),$$

where W_1, W_3, W_2, W_4 *is an F-adapted orthogonal frame on a certain neighborhood of* p *in* M.

PROOF OF LEMMA 5.3. As in the proof of Lemma 4.2 in §4, we can see that $((W_j)^2f_j)(p)=((W_{j+2})^2f_j)(p)$ for all $p\in H_i$ and that

$$((W_j)^2f_j)(q_j)\cdot Jv=\nabla_vZ_j=(-1)^{j+1}m_j\cdot\nabla_vY_2=-m_jc_2\cdot Jv$$
$$\text{for any } v\in(D_j)_{q_j}=T_{q_j}(H_0)=N_{q_j}(H_i).$$

Notice that $m_j>0$ and $c_2>0$ (Proposition 3.8 in §3). Since H_i is an integral submanifold of D_i, we obtain

$$((W_j)^2f_j)|_{H_i}=\text{a constant function on } H_i=-m_jc_2\ (<0),$$

which proves Lemma 5.3.

We will now proceed to verify Assertion (5.2). We recall from Lemma 5.2 (v) that $f_1|_{H_1}$ takes the maximal value m_{11} at q_{01}. We define a function $\tilde{h}_{11}$ on M by $\tilde{h}_{11}=(f_2+m_{11})(f_1-m_{11})$ and a vector field Z_{11} on M by $i(Z_{11})\omega=d\tilde{h}_{11}$. It follows that Z_{11} is an infinitesimal automorphism of $(M, g, J; \mathcal{F})$. By the same argument as in the proof of Lemma 5.2, we see that the connected component of the set $\{p\in M|(Z_{11})_p=0\}$ including the point q_{01} forms a compact totally geodesic 1-dimensional complex submanifold of (M, g, J), which is denoted by H_{12}, and that H_{12} satisfies the following: (i) $\exp_{q_{01}}(D_2)_{q_{01}}=H_{12}$; (ii) $(df_1)_p=0$,

$f_1(p)=m_{11}$ (>0) for all $p\in H_{12}$; (iii) $H_1\cap H_{12}=\{q_{01}\}$.

Let $\{\eta_t^{(2)}\}_{t\in \boldsymbol{R}}$ denote the one-parameter group of automorphisms of $(M, g, J; \mathcal{F})$ generated by $Z_2=Y_1-m_2Y_2$. Applying Lemma 4.2 in §4 to the case where $L=H_1$ and $Z=Z_2$, from Lemma 5.3 we obtain the following results: the mapping $(\eta_t^{(2)})_{*q_{01}}|_{T_{q_{01}}(H_{12})}: T_{q_{01}}(H_{12})\to T_{q_{01}}(H_{12})$ is a $\boldsymbol{C}$-linear isometry of $T_{q_{01}}(H_{12})$, and, for each $v\in T_{q_{01}}(H_{12})$, we have

$$\frac{\partial}{\partial t}(\eta_t^{(2)})_{*q_{01}}v = -m_2c_2\cdot J((\eta_t^{(2)})_{*q_{01}}v), \quad t\in \boldsymbol{R}.$$

Hence, by the same argument as in the proof of Lemma 5.2, we can see that H_{12} is an integral submanifold of D_2 in M. It is easy to see that $H_{12}\cap H_0\neq\emptyset$. In fact, assuming that $H_{12}\cap H_0=\emptyset$, we will obtain a contradiction as follows: take an arbitrary point $p\in H_{01}\setminus\{q_1\}$ $(\subset H_0)$ and the minimizing geodesic γ from p to the submanifold H_{12}. We set $\gamma(0)=p$ and $\gamma(s_1)=p_1\in H_{12}$, $s_1>0$. We put $F^\flat\equiv F\circ\flat$, where $\flat$ is the identification map of TM onto T^*M induced by g. Since $\dot\gamma(0)\notin T_p(H_0)$ and $\dot\gamma(s_1)\in N_{p_1}(H_{12})=(D_1)_{p_1}$, we have $0<F^\flat(\dot\gamma(0))=F^\flat(\dot\gamma(s_1))\leqq 0$, which is a contradiction. Thus, we have $H_{12}\cap H_0=\{q_1\}$. Since H_{12} is totally geodesic, we obtain

$$H_{12}=\exp_{q_1}(D_2)_{q_1}=H_2.$$

Since the set of points of H_2 on which Y_1 vanishes consists of two points q_1 and q_{02}, we have $q_{01}=q_{02}$. Thus, putting $q_0\equiv q_{01}=q_{02}$, we have $H_1\cap H_2=\{q_0\}$. Besides, we have $m_1=m_{11}$.

(Step 3) In this step we will complete the verification of the properties (1), (2) and (6) of Theorem 5.1. From the arguments in (Step 1) and (Step 2), we have already obtained the following facts:

(1′) $H\supset H_0\cup H_1\cup H_2$;

(2′) $Q\supset\{q_0, q_1, q_2\}$;

(6′) $H_i\subset\{p\in M\,|\,f_j(p)=m_j\}$, where $(i, j)\in\{(1, 2), (2, 1)\}$.

To establish (1), (2) and (6) it is sufficient to verify the following

LEMMA 5.4. *For every point p of $M\setminus(H_0\cup H_1\cup H_2)$ we have*

$$(df_1)_p\neq 0 \quad \textit{and} \quad (df_2)_p\neq 0.$$

PROOF OF LEMMA 5.4. Assuming that there exists a point $p\in M\setminus(H_0\cup H_1\cup H_2)$ such that $(df_1)_p=0$ or $(df_2)_p=0$, we will derive a contradiction.

We can see that the assumption of this reductive absurdity implies the following:

There exists a point $q\in M\setminus(H_0\cup H_1\cup H_2)$ such that

$$(df_1)_q=0 \quad \text{and} \quad (df_2)_q=0. \tag{5.3}$$

In fact, let $j\in\{1, 2\}$ and assume that $(df_j)_p=0$. We put $\hat{m}_j=f_j(p)$ (>0) and set $\hat{Z}_j=Y_1+(-1)^{j+1}\hat{m}_j\cdot Y_2$, which is an infinitesimal automorphism of $(M, g, J;\mathcal{F})$. It follows that $(\hat{Z}_j)_p=0$ and moreover that $(\hat{Z}_j)_{\varphi_t^{(2)}(p)}=0$ for all $t\in\boldsymbol{R}$. Notice that $\hat{Z}_j\not\equiv 0$ on M. Let $\hat{H}_j$ be the connected component including p of the set of points on which $\hat{Z}_j$ vanishes. Since $\hat{Z}_j$ is an infinitesimal isometry and an infinitesimal holomorphic transformation, $\hat{H}_j$ forms a compact, connected totally geodesic 1-dimensional complex submanifold of (M, g, J). It follows that $\varphi_t^{(2)}(p)\in\hat{H}_j$ for all $t\in\boldsymbol{R}$. Since $\hat{H}_j$ is compact, taking a certain sequence of numbers $\{t_n\}_{n=1}^{\infty}$, we find a point $q\in\hat{H}_j$ such that the sequence of the points $\{\varphi_{t_n}^{(2)}(p)\}_{n=1}^{\infty}$ converges to q. It is easy to see that $(df_1)_q=(df_2)_q=0$. Using Assertion (5.2) in (Step 2), we can see that $Q\cap(H_0\cup H_1\cup H_2)=\{q_0, q_1, q_2\}$. Assuming that $q\in H_0\cup H_1\cup H_2$, we can conclude that $q\in\{q_0, q_1, q_2\}$ and hence that $\hat{Z}_j=Z_j$, $\hat{H}_j=H_j$, which is an inconsistency. Hence, we have $q\in M\backslash(H_0\cup H_1\cup H_2)$. These prove the desired existence.

Thus, our task is now to derive a contradiction under the condition that there exists a point $q\in M\backslash(H_0\cup H_1\cup H_2)$ with the property (5.3). Let q be a point of $M\backslash(H_0\cup H_1\cup H_2)$ with the property (5.3). Notice that $(Y_1)_q=0$. Then, using the formulas (FML1), (FML3), Lemma 2.7 in § 2 and Lemma 4.1 in § 4, we can see that, for $k\in\{1, 2\}$ and for any $v\in(D_k)_q$,

$$\text{(5.4)}\qquad \frac{f_j(q)}{(f_1+f_2)(q)}((W_k)^2f_k)(q)\cdot Jv=\nabla_v Y_1=\frac{\partial}{\partial u}(\phi_u^{(1)})_{*q}v\,|_{u=0}=\hat{c}_{1,k}(q)\cdot Jv\,,$$

where $\hat{c}_{1,k}(q)$ is a certain real number, W_1, W_3, W_2, W_4 is an F-adapted frame on a neighborhood of q in M, and j is the integer such that $(j, k)\in\{(1, 2), (2, 1)\}$. Let γ be the minimizing geodesic joining q and q_0. If $\hat{c}_{1,1}(q)=\hat{c}_{1,2}(q)=0$, then the Jacobi field $(Y_1)_{\gamma(s)}$ along γ is the 0-field, which is an inconsistency. Thus, $\hat{c}_{1,1}(q)\neq 0$ or $\hat{c}_{1,2}(q)\neq 0$. Taking $k\in\{1, 2\}$ such that $\hat{c}_{1,k}(q)\neq 0$, we have $((W_k)^2f_k)(q)=((f_1+f_2)(q)/f_j(q))\hat{c}_{1,k}(q)\neq 0$. Then, by the same argument as in (Step 2), we can see that there exists a compact, connected integral submanifold $\hat{H}_k$ of D_k in M including q which has the following properties:

(i) $H_0\cap\hat{H}_k\neq\emptyset$;

(ii) $(df_j)_p=0$ for any $p\in\hat{H}_k$, where j is the integer such that $(j, k)\in\{(1, 2), (2, 1)\}$.

It follows that $\hat{H}_k=H_1$ or H_2, which contradicts $q\in M\backslash(H_0\cup H_1\cup H_2)$. This proves Lemma 5.4 and hence establishes (1), (2) and (6) of Theorem 5.1. □

Recalling (2.13) in § 2, we immediately obtain the following

COROLLARY 5.5. *We have* $M_{\mathrm{REG}}=M\backslash H\neq\emptyset$. *In particular,* M_{REG} *is an open dense subset of* M.

We are now in position to establish the complete integrability of the geodesic flows of (M, g, J).

THEOREM 5.6. *Any compact proper Kähler-Liouville surface* $(M, g, J; \mathcal{F})$ *has the property that the geodesic flow is completely integrable, that is, regarding* Y_1 *and* Y_2 *as functions on* T^*M, *the functions* E, F, Y_1 *and* Y_2 *on* T^*M *are functionally independent almost everywhere and the following equalities hold on* M:

$$\{E, F\} = 0, \quad [Y_1, Y_2] = 0,$$

$$\{E, Y_1\} = \{E, Y_2\} = \{F, Y_1\} = \{F, Y_2\} = 0.$$

PROOF. From Corollary 5.5 we know that $M_{\mathrm{REG}} \neq \emptyset$. Then, it can be seen that the functions E_p, F_p, $((Y_1)^2)_p$ and $((Y_2)^2)_p$ on T_p^*M are linearly independent over $\boldsymbol{R}$ for each $p \in M_{\mathrm{REG}}$. This implies the functional independency of those functions on an open dense subset of T_p^*M for $p \in M_{\mathrm{REG}}$. Since M_{REG} is dense in M, it follows that dE, dF, dY_1 and dY_2 are linearly independent almost everywhere. The commutativity is immediate from Lemma 2.11 and Theorem 2.13 in § 2. □

Now, we will discuss the S^1-actions on M generated by the prescribed infinitesimal automorphisms of $(M, g, J; \mathcal{F})$.

We recall the S^1-action ϕ_1 on M from (4.11) in § 4. Then, by virtue of Theorem 5.1 and its proof, we can obtain the following

PROPOSITION 5.7. (1) *The action* ϕ_1 *on* M *leaves the point* q_0 *fixed and the submanifold* H_0 *fixed pointwise.*

(2) *The action* ϕ_1 *on* M *leaves the submanifolds* H_1 *and* H_2 *invariant, and its restriction* $\phi_1|_{H_i}$ *to* H_i, $i=1, 2$, *can be recognized as the rotation of the sphere* H_i *whose least period is* $2\pi/c_1$ *with the pivotal points* q_0 *and* q_j, *where* j *is the integer such that* $(i, j) \in \{(1, 2), (2, 1)\}$.

Then, we have the following

PROPOSITION 5.8. *The real constant* c_1 *is positive as well as* c_2.

PROOF. Let $(i, j) \in \{(1, 2), (2, 1)\}$. By Proposition 5.7 (2) and the same argument as at (5.4), we can see that, for $v \in (D_j)_{q_0}$,

$$\frac{m_i}{m_1+m_2}((W_j)^2 f_j)(q_0) \cdot Jv = \nabla_v Y_1 = \frac{\partial}{\partial u}(\phi_u^{(1)})_{*q_0} v \,|_{u=0} = -c_1 \cdot Jv.$$

Using Lemma 5.3, we have $c_2 = (1/m_1 + 1/m_2) \cdot c_1$. □

Here we demonstrate two other S^1-actions on M. Let $(i, j) \in \{(1, 2), (2, 1)\}$. From (5.1) in the proof of Lemma 5.2, we recall the infinitesimal automorphisms

Z_j on $(M, g, J; \mathcal{F})$. Let $\{\eta_t^{(j)}\}_{t\in \boldsymbol{R}}$ denote the one-parameter group of automorphisms of $(M, g, J; \mathcal{F})$ generated by Z_j. We moreover recall Lemma 5.2 (iii) and Lemma 5.3. Then, applying Lemma 4.2 in §4 to the case where $L=H_i$ and $Z=Z_j$, we see that, for each $p\in H_i$, $(\eta_t^{(j)})_{*p}|_{N_p(H_i)}$ is a $\boldsymbol{C}$-linear isometry of $N_p(H_i)$ onto itself given by

$$(5.5)\qquad (\eta_t^{(j)})_{*p}v = \cos(m_j c_2 t)\cdot v - \sin(m_j c_2 t)\cdot Jv\,, \quad v\in N_p(H_i).$$

Using the same argument as at (4.10) in §4, we can easily obtain the following S^1-actions η_j, $j=1, 2$, on M:

$$(5.6)\qquad \eta_j\colon M\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_j c_2}\boldsymbol{Z}\right)\ni(p, t)\longmapsto \eta_t^{(j)}(p)\in M.$$

We notice that by this action η_j the circle $(\boldsymbol{R}/(2\pi/m_j c_2)\boldsymbol{Z})$ acts on $(M, g, J; \mathcal{F})$ as an automorphism group. Then, we immediately have the following

PROPOSITION 5.9. (1) *The action η_j, $j=1, 2$, leaves the point q_i fixed and leaves the submanifold H_i fixed pointwise, where i is the integer such that $(i, j)\in\{(1, 2), (2, 1)\}$;*

(2) *The action η_j, $j=1, 2$, leaves the submanifolds H_0 and H_j invariant, and its restriction $\eta_j|_{H_k}$, $k=0, j$, can be recognized as a rotation of the sphere H_k whose least period is $2\pi/m_j c_2$ with pivotal points q_i and q_l, where i is the integer such that $(i, j)\in\{(1, 2), (2, 1)\}$ and l is the integer such that $(k, l)\in\{(0, j), (j, 0)\}$.*

From the fact that $[Y_1, Y_2]=0$ on M (in Theorem 5.6), it follows that $[Z_1, Z_2]=0$ on M. Hence, we can define an effective action Φ of the 2-dimensional real torus $(\boldsymbol{R}/(2\pi/m_1 c_2)\boldsymbol{Z})\times(\boldsymbol{R}/(2\pi/m_2 c_2)\boldsymbol{Z})$ on M by

$$(5.7)\qquad \Phi: M\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_1 c_2}\boldsymbol{Z}\right)\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_2 c_2}\boldsymbol{Z}\right)\ni(p, t_1, t_2)\longmapsto \eta_{t_1}^{(1)}(\eta_{t_2}^{(2)}(p))\in M.$$

We notice that by this action Φ the real torus acts on $(M, g, J; \mathcal{F})$ as an automorphism group. The effectivity of Φ follows immediately from the fact that the orbit $\{\eta_t^{(j)}(p)\,|\,t\in(\boldsymbol{R}/(2\pi/m_j c_2)\boldsymbol{Z})\}$, $j=1, 2$, through the point p of M_{REG} sufficiently close to q_0 makes the proper circle in M_{REG} with the least period $2\pi/m_j c_2$.

§6. Topology of M.

This section is devoted to the establishment of the following

THEOREM 6.1. *Let $(M, g, J; \mathcal{F})$ be a compact, connected proper Kähler-Liouville surface. Then, the complex surface (M, J) is bi-holomorphic to the standard complex projective plane $(\boldsymbol{C}P^2, J_0)$.*

Here we recall from Theorem 5.1 (2) in § 5 that $(df_1)_q=(df_2)_q=0$ if and only if $q\in\{q_0, q_1, q_2\}$. Since $h_2=f_1-f_2$ ((2.4) in § 2), it follows that $(dh_2)_q=0$ if and only if $q\in\{q_0, q_1, q_2\}$. As a well-known application of the Morse Theory (see [**6**]), we have

LEMMA 6.2. *The complex surface (M, J) is homotopy equivalent to the standard complex projective plane $(\boldsymbol{C}P^2, J_0)$.*

Thus, to establish Theorem 6.1, it is sufficient to prove the following

PROPOSITION 6.3. *If a compact Kähler surface (M, g, J) is homotopy equivalent to the standard complex projective plane $(\boldsymbol{C}P^2, J_0)$, then (M, J) is bi-holomorphic to $(\boldsymbol{C}P^2, J_0)$.*

PROOF OF PROPOSITION 6.3. Let b_1 and b_2 be the first and the second Betti numbers of M respectively. The homotopy equivalence yields

$$b_1=0 \quad \text{and} \quad b_2=1\,.$$

Let p and q be the geometric genus of (M, J) and the irregularity of (M, J) respectively. Then, we have $2p\leqq b_2=1$ and $2q=b_1=0$ and hence

$$p=0 \quad \text{and} \quad q=0\,. \tag{6.1}$$

By a Kodaira's theorem we can see from the fact that $p=0$ that (M, J) is projective algebraic.

Then, we consider Noether's formula:

$$(c_1)^2+c_2=12(p-q+1)\,, \tag{6.2}$$

where c_1 and c_2 denote the first and the second Chern classes of (M, J) respectively. Since c_2 is the Euler characteristic of M, we have

$$(c_1)^2=9 \quad \text{and} \quad c_2=3\,. \tag{6.3}$$

Since $b_2=1$, one of the following holds: $c_1<0$; $c_1=0$; $c_1>0$. Obviously, the case where $c_1=0$ is impossible. Assume that $c_1<0$. Then (M, J) admits a unique Einstein-Kähler metric (see [**7**], [**8**]). By applying Chen and Ogiue's result ([**9**] Theorem 2), we can see from (6.3) that (M, J) is of constant holomorphic sectional curvature. Hence, because M is compact and simply connected, (M, J) is bi-holomorphic to $(\boldsymbol{C}P^2, J_0)$, which contradicts the fact that $c_1<0$. Thus, we obtain $c_1>0$. Denoting by K the canonical bundle of (M, J), we have $K<0$. Since (M, J) is an algebraic surface, (M, J) is bi-holomorphic to $(\boldsymbol{C}P^2, J_0)$ (see [**10**] Lemma, p. 487).

§7. The compact real Liouville surfaces imbedded in M.

For the sake of simplicity, in this section we will use the terms "torus action" to refer to the action of the 2-dimensional real torus $(\boldsymbol{R}/(2\pi/m_1c_2)\boldsymbol{Z})\times(\boldsymbol{R}/(2\pi/m_2c_2)\boldsymbol{Z})$ and the terms "real surface" to refer to a 2-dimensional real submanifold of $(M, g, J; \mathcal{F})$.

The main objective of this section is to establish the two theorems and corollary which follow.

THEOREM 7.1. *There exists a family* $\mathfrak{S}$ *of compact, connected totally geodesic 2-dimensional real submanifolds of* $(M, g, J; \mathcal{F})$ *such that, for each* $S\in\mathfrak{S}$, $(S, g_S; F|_{T^*S})$ *forms a compact real Liouville surface diffeomorphic with the real projective plane* $\boldsymbol{RP}^2$, *where* g_S *means the induced metric on* S *from* g *and where* F *is an element of* $\mathcal{F}$ *assumed to satisfy* $(N1)$ *and* $(N2)$ *in* §1.

THEOREM 7.2. *A transitive torus action* $\tilde{\Phi}$ *on* $\mathfrak{S}$ *can be naturally induced from the torus action* Φ *on* M.

In particular, from Theorem 7.2, we deduce the following

COROLLARY 7.3. *Any two compact real Liouville surfaces belonging to* $\mathfrak{S}$ *are isomorphically transferred from one onto the other by* Φ.

Recalling from (2.14) in §2 the distribution D_+ on M_{REG}, we have

PROPOSITION 7.4. *For each point* p_0 *of* M_{REG}, *we can construct the maximal integral real surface* Σ_{p_0} *of* D_+ *through* p_0 *in* M_{REG}, *which has the following properties*:

(1) Σ_{p_0} *has the coordinate system* (x_1, x_2) *with origin* p_0 *generated by* U_1 *and* U_2, *and this coordinate mapping is a diffeomorphism of* Σ_{p_0} *onto* $\boldsymbol{R}^2$;

(2) Σ_{p_0} *is totally geodesic with respect to* g.

PROOF. (1) Since $U_1f_2=0$ and $U_2f_1=0$, we can define the one-parameter groups $\{\xi_t^{(1)}\}_{t\in\boldsymbol{R}}$ and $\{\xi_t^{(2)}\}_{t\in\boldsymbol{R}}$ of transformations of M_{REG} generated by U_1 and U_2, respectively. Note that U_1 and U_2 are linearly independent at each point of M_{REG} and satisfy $[U_1, U_2]=0$ (Lemma 2.9 in §2). Then, we have a coordinate (x_1, x_2) in Σ_{p_0} with origin p_0 such that $(\partial/\partial x_1)=U_1$, $(\partial/\partial x_2)=U_2$. This defines a diffeomorphism of Σ_{p_0} onto $\boldsymbol{R}^2$.

(2) Recalling Lemma 2.10 in §2, we can see that, for $(i, j)\in\{(1, 2), (2, 1)\}$, $\nabla_{U_i}U_i$ and $\nabla_{U_i}U_j$ can be written as a linear combination of U_1 and U_2 for each point of M_{REG}. This implies (2). □

We denote by $\tilde{\mathfrak{S}}$ the set of such real surfaces in M_{REG}, that is, $\tilde{\mathfrak{S}}=\{\Sigma_p\,|\,p\in M_{\mathrm{REG}}\}$. It follows that

$$\bigcup_{\Sigma\in\tilde{\mathfrak{S}}}\Sigma = M_{\mathrm{REG}}.$$

For any $\Sigma\in\mathfrak{S}$ we set $\partial\Sigma=\bar{\Sigma}\setminus\Sigma$, where $\bar{\Sigma}$ is the closure of Σ in M. Then, we can immediately see that $\partial\Sigma=\bar{\Sigma}\cap H$, where H denotes the subset of M defined in the beginning of § 5. For $\Sigma\in\tilde{\mathfrak{S}}$, we put $\sigma_i(\Sigma)=\partial\Sigma\cap H_i$, $i=0, 1, 2$. Since $H=H_0\cup H_1\cup H_2$ (Theorem 5.1 (1) in § 5), we have $\partial\Sigma=\sigma_0(\Sigma)\cup\sigma_1(\Sigma)\cup\sigma_2(\Sigma)$.

PROPOSITION 7.5. *For each $\Sigma\in\tilde{\mathfrak{S}}$, the family $\sigma_0(\Sigma)$, $\sigma_1(\Sigma)$, $\sigma_2(\Sigma)$ of the subsets of M forms a geodesic triangle whose vertices are q_0, q_1 and q_2.*

PROOF. It is sufficient to verify that each $\sigma_i(\Sigma)$, $i=0, 1, 2$, forms a geodesic segment joining q_j and q_k, where j and k are the integers determined by $(i, j, k)\in\{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Let $(i, j)\in\{(1, 2), (2, 1)\}$ and let $p\in\Sigma$. With the same notation as in the proof of Proposition 7.4, we observe the following integral curves of U_i:

$$\boldsymbol{R}\ni t\longmapsto \xi_t^{(i)}(\xi_u^{(j)}(p))\in\Sigma,\quad u\in\boldsymbol{R}.$$

Taking the limit as $u\to+\infty$, we obtain the integral curve segment $\xi_t^{(i)}(p_{+\infty})$ of U_i in H_i which joins q_j and q_0, where $p_{+\infty}$ is the point of H_i which $\xi_u^{(j)}(p)$ converges as $u\to+\infty$. Using Lemma 2.10 in § 2, we see that the integral curve segment $\xi_t^{(i)}(p_{+\infty})$ of U_i in H_i forms a geodesic segment which coincides with $\sigma_i(\Sigma)$ set-theoretically. Similarly, taking the limits of $\xi_t^{(2)}(\xi_u^{(1)})$ and $\xi_t^{(1)}(\xi_u^{(2)})$ as $u\to-\infty$, we can obtain a geodesic segment in H_0 which coincides with $\sigma_0(\Sigma)$ set-theoretically. □

For each $\Sigma\in\tilde{\mathfrak{S}}$, we can obtain three pairs of unit vectors

$$(v_{1i}(\Sigma),\ v_{2i}(\Sigma))\in(D_1)_{q_i}^{\mathrm{unit}}\times(D_2)_{q_i}^{\mathrm{unit}},\quad i=0,\ 1,\ 2$$

by the following method:

Denoting by $\sigma_i^j(s)$ the geodesic segment starting from q_j which coincides with $\sigma_i(\Sigma)$ set-theoretically, we can define the vectors $v_{ij}(\Sigma)$ by

$$v_{10}(\Sigma)\equiv\frac{d}{dt}\sigma_1^0(0)\Big|_{t=0},\quad v_{20}(\Sigma)\equiv\frac{d}{dt}\sigma_2^0(0)\Big|_{t=0},$$

$$v_{11}(\Sigma)\equiv\frac{d}{dt}\sigma_0^1(0)\Big|_{t=0},\quad v_{21}(\Sigma)\equiv\frac{d}{dt}\sigma_2^1(0)\Big|_{t=0},$$

$$v_{12}(\Sigma)\equiv\frac{d}{dt}\sigma_1^2(0)\Big|_{t=0},\quad v_{22}(\Sigma)\equiv\frac{d}{dt}\sigma_0^2(0)\Big|_{t=0}.$$

LEMMA 7.6. *For each $\Sigma\in\tilde{\mathfrak{S}}$, we can define the vector fields R_0^Σ, R_1^Σ and R_2^Σ along the geodesic segments $\sigma_0(\Sigma)$, $\sigma_1(\Sigma)$ and $\sigma_2(\Sigma)$, respectively, such that*

(1) $R_i^{\Sigma}(q_0)=v_{j0}(\Sigma)$, $R_i^{\Sigma}(q_j)=v_{jj}(\Sigma)$ *for* $(i, j)\in\{(1, 2), (2, 1)\}$, *and* $R_0^{\Sigma}(q_1)=v_{21}(\Sigma)$, $R_0^{\Sigma}(q_2)=v_{12}(\Sigma)$;

(2) R_i^{Σ}, $i=0, 1, 2$, *is normal to the geodesic segment* $\sigma_i(\Sigma)$;

(3) R_i^{Σ}, $i=0, 1, 2$, *is tangent to* $\bar{\Sigma}$ *and pointing into* Σ;

(4) R_i^{Σ}, $i=0, 1, 2$, *is parallel along the geodesic segments* $\sigma_i(\Sigma)$.

PROOF. Let $(i, j)\in\{(1, 2), (2, 1)\}$. We put $\sigma_{0i}(\Sigma)\equiv\sigma_0(\Sigma)\cap H_{0i}$. Let p_0 be an arbitrary point of Σ. Using the same notation as in the proofs of Proposition 7.4 and Proposition 7.5, we define a vector field R_i^{Σ} along $\sigma_i(\Sigma)$ and a vector field R_{0i}^{Σ} along $\sigma_{0i}(\Sigma)$ by

$$(R_i^{\Sigma})_{\xi_t^{(i)}(p_{+\infty}^{(j)})} = \lim_{u\to+\infty}\left(-\frac{U_j}{\|U_j\|}\right)_{\xi_u^{(j)}\xi_t^{(i)}(p_0)},$$

$$(R_{0i}^{\Sigma})_{\xi_t^{(i)}(p_{-\infty}^{(j)})} = \lim_{u\to-\infty}\left(\frac{U_j}{\|U_j\|}\right)_{\xi_u^{(j)}\xi_t^{(i)}(p_0)},$$

where $p_{+\infty}^{(j)}=\lim_{u\to+\infty}\xi_u^{(j)}(p_0)\in\sigma_i(\Sigma)$ and $p_{-\infty}^{(j)}=\lim_{u\to-\infty}\xi_u^{(j)}(p_0)\in\sigma_{0i}(\Sigma)$.

These vector fields are determined independently of the choice of the reference point $p_0\in\Sigma$. From the very above definition we can easily see that R_i^{Σ} and R_{0i}^{Σ} are normal to $\sigma_i(\Sigma)$ and $\sigma_{0i}(\Sigma)$ respectively and that they are tangent to $\bar{\Sigma}$ and pointing into Σ. We note that $\sigma_0(\Sigma)\cap M_{\mathrm{sing}}$ consists of one point, say $q_{\mathrm{sing}}(\Sigma)$. Since these facts imply that $\lim_{q_1\to q_{\mathrm{sing}}(\Sigma)}(R_{01}^{\Sigma})_{q_1}=\lim_{q_2\to q_{\mathrm{sing}}(\Sigma)}(R_{02}^{\Sigma})_{q_2}$, where $q_1\in\sigma_1(\Sigma)$, $q_2\in\sigma_2(\Sigma)$, we can construct the vector field R_0^{Σ} by combining R_{01}^{Σ} and R_{02}^{Σ}. The properties (1), (2) and (3) are now obvious. Since Σ is totally geodesic, it is easy to verify (4). □

Here, from (5.6) in §5, we recall the S^1-actions η_j, $j=1, 2$, on M. We denote by *id* the identity automorphism of $(M, g, J; \mathcal{F})$. We put

$$\tau_j = \eta_{\pi/m_j c_2}^{(j)} \quad j = 1, 2.$$

It follows that $\tau_j^2=id$, $j=1, 2$ and $\tau_1\tau_2=\tau_2\tau_1$. Hence, $\{id, \tau_1, \tau_2, \tau_1\tau_2\}$ forms an automorphism group of $(M, g, J; \mathcal{F})$.

PROPOSITION 7.7. *For any* $\Sigma\in\tilde{\mathfrak{S}}$, *the compact subset*

$$S_{\Sigma} \equiv \bar{\Sigma}\cup\overline{\tau_1(\Sigma)}\cup\overline{\tau_2(\Sigma)}\cup\overline{\tau_1\tau_2(\Sigma)}$$

of M forms a compact, connected 2-dimensional real submanifold of M without a boundary, which satisfies

(1) S_{Σ} *is totally geodesic with respect to* g;

(2) S_{Σ} *is diffeomorphic with the real projective plane* $\boldsymbol{RP}^2$;

(3) $T_{q_0}(S_{\Sigma})=\langle\langle v_{10}(\Sigma), v_{20}(\Sigma)\rangle\rangle$, *where the symbol* $\langle\langle v_1, v_2\rangle\rangle$ *means the vector space spanned by* v_1 *and* v_2.

PROOF. Let $(i, j)\in\{(1, 2), (2, 1)\}$ and let $\Sigma\in\tilde{\mathfrak{S}}$. From Lemma 2.11 in §2, we see that $[Z_j, U_1]=[Z_j, U_2]=0$. This implies that $(\tau_j)_*D_+=D_+$. Hence, we can see that the automorphism group $\{id, \tau_1, \tau_2, \tau_1\tau_2\}$ yields four real surfaces Σ, $\tau_1(\Sigma)$, $\tau_2(\Sigma)$ and $\tau_1\tau_2(\Sigma)=\tau_2\tau_1(\Sigma)$ which belong to $\tilde{\mathfrak{S}}$. Using Proposition 5.9, we can see that $\tau_j(\tilde{\partial}\Sigma)=\tilde{\partial}(\tau_j(\Sigma))$ and $\tau_j(\sigma_i(\Sigma))=\sigma_i(\tau_j(\Sigma))$. We notice that $\overline{\tau_j(\Sigma)}=\tau_j(\bar{\Sigma})$, which is a real surface with a boundary.

We will now establish that the closures $\bar{\Sigma}$, $\overline{\tau_1(\Sigma)}$, $\overline{\tau_2(\Sigma)}$ and $\overline{\tau_1\tau_2(\Sigma)}$ of these real surfaces are united into one compact, connected real surface S_Σ without a boundary. The surface S_Σ is constructed as follows:

Using Proposition 5.9 in §5, (5.5) in §5 and Lemma 7.6, we obtain

$$\sigma_i(\Sigma) = \sigma_i(\tau_j(\Sigma)),$$

$$R_i^\Sigma+R_i^{\tau_j(\Sigma)} = 0 \quad \text{along } \sigma_i(\Sigma).$$

From these facts, together with the totally geodesicity of Σ and $\tau_j(\Sigma)$ (Proposition 7.4 (2)), we see that the two real surfaces $\bar{\Sigma}$ and $\tau_j(\bar{\Sigma})$ with boundaries are smoothly joined with joint $\sigma_i(\Sigma)=\tilde{\partial}\Sigma\cap\tilde{\partial}(\tau_j(\Sigma))$. We can see that $\sigma_0(\Sigma)\cup\sigma_0(\tau_j(\Sigma))$ forms a geodesic circle in H_0 through the two points q_1, q_2 and hence that

$$\sigma_0(\Sigma) = \sigma_0(\tau_1\tau_2(\Sigma)), \quad \sigma_0(\tau_1\Sigma) = \sigma_0(\tau_2(\Sigma)). \tag{7.1}$$

We can also see that $\sigma_j(\Sigma)\cup\sigma_j(\tau_j(\Sigma))$ forms a geodesic circle in H_j through the points q_0, q_i and hence that $v_{ji}(\Sigma)+v_{ji}(\tau_1\tau_2(\Sigma))=0$. Hence, the parallelism of R_0^Σ, $R_0^{\tau_1(\Sigma)}$, $R_0^{\tau_2(\Sigma)}$ and $R_0^{\tau_1\tau_2(\Sigma)}$ along $\sigma_0(\Sigma)$, $\sigma_0(\tau_1(\Sigma))$, $\sigma_0(\tau_2(\Sigma))$ and $\sigma_0(\tau_1\tau_2(\Sigma))$ respectively implies

$$R_0^\Sigma+R_0^{\tau_1\tau_2(\Sigma)} = 0, \quad R_0^{\tau_1(\Sigma)}+R_0^{\tau_2(\Sigma)} = 0. \tag{7.2}$$

Since $\Sigma\in\tilde{\mathfrak{S}}$ is totally geodesic, we see from (7.1) and (7.2) that $\bar{\Sigma}$ and $\overline{\tau_1\tau_2(\Sigma)}$ are smoothly joined with joint $\sigma_0(\Sigma)$, and likewise $\overline{\tau_1(\Sigma)}$ and $\overline{\tau_2(\Sigma)}$ with joint $\sigma_0(\tau_1(\Sigma))=\sigma_0(\tau_2(\Sigma))$. These arguments establish that $S_\Sigma\setminus\{q_0, q_1, q_2\}$ forms a totally geodesic smooth real surface. The totally geodesicity of Σ also ensures the smoothness of S_Σ at the points q_0, q_1 and q_2. Thus, we conclude that S_Σ forms a compact, connected 2-dimensional real submanifold of M without a boundary.

From the very construction of S_Σ, the properties (1), (2) and (3) are now obvious. □

Here, we set

$$\mathfrak{S} = \{S_\Sigma \mid \Sigma\in\tilde{\mathfrak{S}}\}.$$

Then, we moreover have the following

PROPOSITION 7.8. *For any $S\in\mathfrak{S}$, the triplet $(S, g_S; F|_{T^*S})$ forms a compact real Liouville surface with two singular points, which is diffeomorphic with the real projective plane $\boldsymbol{R}P^2$, where g_S means the induced metric on S from g and where F is an element of $\mathcal{F}$ assumed to satisfy* $(N1)$ *and* $(N2)$ *in* § 1.

PROOF. We notice that each $S\in\mathfrak{S}$ is almost completely composed of four real surfaces belonging to $\tilde{\mathfrak{S}}$. For each $\Sigma\in\tilde{\mathfrak{S}}$, we denote by $\tilde{E}_\Sigma$ the energy function on $T^*\Sigma$ with respect to the induced metric g_Σ on Σ. Recalling from [**1**] the definition of the compact real Liouville surface, we can see that it is sufficient for the verification that, for any $S\in\mathfrak{S}$, $(S, g_S; F|_{T^*S})$ forms a compact real Liouville surface to verify the following conditions for each $\Sigma\in\tilde{\mathfrak{S}}$:

(i) $\{\tilde{E}_\Sigma, F|_{T^*\Sigma}\}=0$ on Σ;

(ii) For each $p\in\Sigma$, $F|_{T_p^*\Sigma}$ is a homogeneous polynomial on $T_p^*\Sigma$ of degree 2;

(iii) $F|_{T_p^*\Sigma}$ is not of the form $r_1\cdot V^2+r_2\cdot\tilde{E}_\Sigma$, where $r_1, r_2\in\boldsymbol{R}$ and V is a vector field on Σ.

Since F is a homogeneous polynomial on T_p^*M of degree 2 for each $p\in M$, (ii) is obvious. We define the vector fields $\tilde{W}_1$, $\tilde{W}_2$, $\tilde{W}_3$ and $\tilde{W}_4$ on $M_{\mathbf{REG}}$ by

$$\tilde{W}_k=\frac{U_k}{\sqrt{U_kf_k}},\quad \tilde{W}_{k+2}=\frac{U_{k+2}}{\sqrt{U_kf_k}},\quad k=1,\,2.$$

Then, from (2.10) and (2.12) in § 2, we can see that $\tilde{W}_1$, $\tilde{W}_2$, $\tilde{W}_3$ and $\tilde{W}_4$ form an F-adapted orthogonal frame on $M_{\mathbf{REG}}$. We notice that the vector fields $\tilde{W}_1$ and $\tilde{W}_2$ are tangent to the surface Σ for each $\Sigma\in\tilde{\mathfrak{S}}$. Then, $E|_{T^*\Sigma}$ and $F|_{T^*\Sigma}$ can be expressed as

$$\begin{cases} E|_{T^*\Sigma}=\dfrac{1}{(f_1+f_2)|_\Sigma}((\tilde{W}_1|_\Sigma)^2+(\tilde{W}_2|_\Sigma)^2) \\[2ex] F|_{T^*\Sigma}=-\dfrac{f_2|_\Sigma}{(f_1+f_2)|_\Sigma}(\tilde{W}_1|_\Sigma)^2+\dfrac{f_1|_\Sigma}{(f_1+f_2)|_\Sigma}(\tilde{W}_2|_\Sigma)^2, \end{cases}$$

on Σ. From this expression of $F|_{T^*\Sigma}$, it is easy to check (iii). We can see that $(\tilde{W}_1/\sqrt{f_1+f_2})|_\Sigma$, $(\tilde{W}_2/\sqrt{f_1+f_2})|_\Sigma$ form an orthonormal frame on Σ and hence that $\tilde{E}_\Sigma=E|_{T^*\Sigma}$. Hence, we can easily compute $\{\tilde{E}_\Sigma, F|_{T^*\Sigma}\}=0$ for each $\Sigma\in\tilde{\mathfrak{S}}$, which completes the verification of (i). Thus, we conclude that, for each $S\in\mathfrak{S}$, $(S, g_S; F|_{T^*S})$ forms a compact real Liouville surface.

From the very construction of $S\in\mathfrak{S}$ in the proof of Proposition 7.7, we can see that the set of the singular points of $(S, g|_S; F|_{T^*S})$ is $S\cap M_{\mathrm{sing}}$, which consists of two points. Recalling Proposition 7.7 (2) or using the classification of the compact real Liouville surfaces in [**1**], we see that S is diffeomorphic with $\boldsymbol{R}P^2$. □

Thus, by virtue of Proposition 7.7 and Proposition 7.8, we establish Theorem 7.1.

We will now proceed to establish Theorem 7.2. Let $P_{q_0}M$ be the set of the 2-dimensional real vector subspace $\langle\langle v_1, v_2\rangle\rangle$ of $T_{q_0}M$ spanned by $v_1\in(D_1)_{q_0}^{\text{unit}}$ and $v_2\in(D_2)_{q_0}^{\text{unit}}$. We need the following

PROPOSITION 7.9. *There exists a one to one correspondence between* $\mathfrak{S}$ *and* $P_{q_0}M$ *as follows*:

$$\mathfrak{S}\ni S\longleftrightarrow T_{q_0}(S)\in P_{q_0}M.$$

PROOF. From Proposition 7.7 (3), for each $S\in\mathfrak{S}$, we can assign $T_{q_0}S\in P_{q_0}M$.

Conversely, we can see that, for any $K\in P_{q_0}(M)$, there exists a unique $S_K\in\mathfrak{S}$ such that $T_{q_0}(S_K)=K$ as follows:

Take a pair of unit vectors $(v_1, v_2)\in(D_1)_{q_0}^{\text{unit}}\times(D_2)_{q_0}^{\text{unit}}$ at q_0 such that $K=\langle\langle v_1, v_2\rangle\rangle$. We put $v_0=(1/\sqrt{2})v_1+(1/\sqrt{2})v_2\in S_{q_0}M$, where $S_{q_0}M$ denotes the unit sphere in the tangent vector space $T_{q_0}M$. We define a geodesic γ_0 by $\gamma_0(s)=\exp(sv_0)$. Taking a sufficiently small $s_0>0$, we may assume that the geodesic segment $\gamma_0|_{[0,s_0]}$ is a minimizing geodesic segment from q_0 to $p_0=\gamma_0(s_0)$. We denote by Σ_0 the real surface belonging to $\mathfrak{S}$ through the point $p_0=\gamma_0(s_0)$. It is easy to see that $\gamma_0(]0, s_0[)\subset\Sigma_0$. In fact, from the fact that Y_1 and Y_2 are infinitesimal isometries of (M, g), we can see that the vector fields $(Y_1)_{\gamma_0(s)}$ and $(Y_2)_{\gamma_0(s)}$, $0\leqq s\leqq s_0$, are non-zero normal Jacobi fields along $\gamma_0|_{[0,s_0]}$ and hence that $\dot{\gamma}_0(s)\in(D_+)_{\gamma_0(s)}$ for all $s\in]0, s_0]$. As in the proof of Proposition 7.7, we can construct $S_{\Sigma_0}\in\mathfrak{S}$ by setting $S_{\Sigma_0}=\overline{\Sigma_0}\cup\overline{\tau_1(\Sigma_0)}\cup\overline{\tau_2(\Sigma_0)}\cup\overline{\tau_1\tau_2(\Sigma_0)}$. We note that the geodesic $\gamma_0(s)$ lies on S_{Σ_0}. Using Proposition 7.7 (3), we have

$$\frac{1}{\sqrt{2}}v_1+\frac{1}{\sqrt{2}}v_2=v_0=\dot{\gamma}_0(0)=a_1\cdot v_{10}(\Sigma_0)+a_2\cdot v_{20}(\Sigma_0), \tag{7.3}$$

where a_1 and a_2 are real numbers such that $(a_1)^2+(a_2)^2=1$. Since $T(M_{\text{reg}})=D_1\oplus D_2$ (direct sum), comparing D_1-component of (7.3) and comparing D_2-component of (7.3), we have $v_{10}(\Sigma_0)=\pm v_1$, $v_{20}(\Sigma_0)=\pm v_2$. Thus, from Proposition 7.7 (3), we obtain

$$T_{q_0}(S_{\Sigma_0})=\langle\langle v_1, v_2\rangle\rangle=K.$$

Since $\exp_{q_0}K=S_{\Sigma_0}$, the uniqueness of this surface is obvious. □

We denote by $S(K)$ the surface belonging to $\mathfrak{S}$ which corresponds to $K\in P_{q_0}M$.

Recalling the effective torus action Φ on M from (5.7) in § 5, we denote by $\Phi_{(t_1,t_2)}$ the automorphism of $(M, g, J; \mathfrak{F})$ defined by $M\ni p\mapsto\Phi(p, t_1, t_2)\in M$. Observing (5.5), (5.6) and (5.7) in § 5, we can obtain the transitive torus action $\hat{\Phi}$ on $P_{q_0}M$ expressed as

$$\hat{\Phi}: P_{q_0}(M)\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_1c_2}\boldsymbol{Z}\right)\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_2c_2}\boldsymbol{Z}\right)\ni(K,\ t_1,\ t_2)$$
$$\longmapsto(\Phi_{(t_1,t_2)})_{*q_0}K\in P_{q_0}M.$$

Hence, by virtue of Proposition 7.9, we immediately obtain the following transitive torus action $\tilde{\Phi}$ on $\mathfrak{S}$:

$$\tilde{\Phi}: \mathfrak{S}\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_1c_2}\boldsymbol{Z}\right)\times\left(\boldsymbol{R}\Big/\frac{2\pi}{m_2c_2}\boldsymbol{Z}\right)\ni(S(K),\ t_1,\ t_2)$$
$$\longrightarrow S(\hat{\Phi}_{(t_1,t_2)}(K))\in\mathfrak{S},$$

thereby establishing Theorem 7.2.

Finally, we verify Corollary 7.3 as follows:

From the definition of the torus action $\tilde{\Phi}$ on $\mathfrak{S}$, we can obtain

$$\tilde{\Phi}(S(K),\ t_1,\ t_2)=\Phi_{(t_1,t_2)}(S(K))$$

for any $K\in P_{q_0}M$ and for any $(t_1,\ t_2)\in(\boldsymbol{R}/(2\pi/m_1c_2)\boldsymbol{Z})\times(\boldsymbol{R}/(2\pi/m_2c_2)\boldsymbol{Z})$. This, together with the transitivity of the action $\tilde{\Phi}$ on $\mathfrak{S}$, implies that any two surfaces belonging to $\mathfrak{S}$ are transferred diffeomorphically from one onto the other by Φ. Since $[Z_j,\ U_k]=0$, $j,\ k=1,\ 2$ (Lemma 2.11 in §2), we have $(\Phi_{(t_1,t_2)})_*(D_+)=D_+$. Hence, we see that, for each $(t_1,\ t_2)\in(\boldsymbol{R}/(2\pi/m_1c_2)\boldsymbol{Z})\times(\boldsymbol{R}/(2\pi/m_2c_2)\boldsymbol{Z})$, $\Phi_{(t_1,t_2)}$ maps diffeomorphically a surface $\Sigma\in\tilde{\mathfrak{S}}$ onto a certain surface $\Sigma'\in\tilde{\mathfrak{S}}$. Then, using Theorem 2.13 (3), we have

$$F|_{T^*_{\Phi_{(t_1,t_2)}(p)}\Sigma'}\circ(\Phi_{(t_1,t_2)})^*|_{T^*_p\Sigma}=F|_{T^*_p\Sigma},\quad p\in\Sigma,$$

where F is an element of $\mathcal{F}$ assumed to satisfy (N1) and (N2) in §1. This implies that, for any $S\in\mathfrak{S}$ and for any $(t_1,\ t_2)\in(\boldsymbol{R}/(2\pi/m_1c_2)\boldsymbol{Z})\times(\boldsymbol{R}/(2\pi/m_2c_2)\boldsymbol{Z})$, the mapping $\Phi_{(t_1,t_2)}|_S: S\to\Phi_{(t_1,t_2)}(S)$ is an isomorphism of the compact real Liouville surface $(S,\ g|_S\,;\ F|_{T^*S})$ belonging to $\mathfrak{S}$ onto the compact real Liouville surface $(\Phi_{(t_1,t_2)}(S),\ g|_{\Phi_{(t_1,t_2)}(S)}\,;\ F|_{T^*(\Phi_{(t_1,t_2)}(S))})$ belonging to $\mathfrak{S}$.

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