J. Math. Soc. Japan
Vol. 49, No. 1, 1997

The tightness about sequential fans and combinatorial properties

By Katsuya EDA, Masaru KADA and Yoshifumi YUASA

(Received July 15, 1994)
(Revised Feb. 3, 1995)

1. Introduction.

Let κ be an infinite cardinal. The *sequential fan* S_κ *with* κ*-many spines* is the quotient space obtained from the disjoint union of κ-many convergent sequences by identifying all the limit points to a single point denoted by ∞. To be precise, $S_\kappa=\{\infty\}\cup(\kappa\times\omega)$ as a set, every point of $\kappa\times\omega$ is isolated, and a basic neighborhood of ∞ is of the form

$$U_\varphi=\{\infty\}\cup\{\langle\alpha, n\rangle : n\geqq\varphi(\alpha)\}$$

where $\varphi\in\omega^\kappa$.

For a topological space X, the *tightness* of X, $t(X)$, is the smallest cardinal λ such that for every point $x\in X$ and $A\subseteqq X$, if $x\in\mathrm{cl}A$ then there exists $B\subseteqq A$ with $|B|\leqq\lambda$ and $x\in\mathrm{cl}B$.

It follows immediately from the definition that $t(X)\leqq|X|$ and it is easily seen that $t(S_\kappa)=\omega$ for each κ. But the tightness of the product space of two sequential fans is more complicated.

Gruenhage [4] proved that $t(S_{\omega_1}\times S_{\omega_1})=\omega_1$, but it is an open question whether $t(S_{\omega_2}\times S_{\omega_2})=\omega_2$ holds in ZFC. Moreover, such a question whether $t(S_\kappa\times S_\kappa)=\kappa$ or not, is equivalent to another question related to the collectionwise Hausdorff property. (See [3, 8] for details.)

In this paper we shall give a combinatorial characterization of the tightness of $S_\omega\times S_\kappa$ for an infinite cardinal κ. Especially the tightness of $S_\omega\times S_{2^\omega}$ has a natural combinatorial characterization.

To begin with, let us review the definitions of two familiar cardinals with combinatorial characterizations, $\mathfrak{b}$ and $\mathfrak{d}$.

DEFINITION 1.1. For $f, g\in\omega^\omega$, $f\leqq^* g$ if for all but finitely many $n\in\omega$ we have $f(n)\leqq g(n)$. A family $\mathcal{F}\subseteqq\omega^\omega$ is *unbounded* (respectively *dominating*) if for every $f\in\omega^\omega$ there exists $g\in\mathcal{F}$ such that $g\not\leqq^* f$ (respectively $f\leqq^* g$). The *unbounding number* $\mathfrak{b}$ is the smallest size of the unbounded family of ω^ω, and the *dominating number* $\mathfrak{d}$ is the smallest size of the dominating family of ω^ω.

Now we introduce a new cardinal invariant $\mathfrak{b}^*$, which is defined with the notion of the unbounded family but differs from $\mathfrak{b}$.

DEFINITION 1.2. $\mathfrak{b}^*$ is the smallest cardinal λ such that, for every unbounded family $\mathcal{F}\subseteq\omega^\omega$, there exists a subfamily $\mathcal{G}\subseteq\mathcal{F}$ such that $|\mathcal{G}|\leqq\lambda$ and $\mathcal{G}$ is still unbounded.

Using this notion we can state our main results:

THEOREM 1.3. (1) *For* $\omega\leqq\kappa<\mathfrak{b}$, $t(S_\omega\times S_\kappa)=\omega$ *holds.*
(2) $t(S_\omega\times S_{\mathfrak{b}})=\mathfrak{b}$.
(3) *For* $\kappa\geqq\mathfrak{b}^*$, $t(S_\omega\times S_\kappa)=\mathfrak{b}^*$ *holds.*

THEOREM 1.4. (1) $\mathfrak{b}\leqq\mathfrak{b}^*\leqq\mathfrak{d}$.
(2) *Both* $\mathfrak{b}<\mathfrak{b}^*$ *and* $\mathfrak{b}^*<\mathfrak{d}$ *are consistent with* ZFC.

What happens about $t(S_\omega\times S_\kappa)$ for $\mathfrak{b}<\kappa<\mathfrak{b}^*$? In fact it is undecidable under ZFC, that is, both $t(S_\omega\times S_\kappa)=\kappa$ and $t(S_\omega\times S_\kappa)<\kappa$ are consistent with ZFC. To prove this, we study Hechler's result about dominating families of ω^ω in Section 4.

Our notation is standard and we refer the reader to [7] for undefined notions.

For $f\in\omega^\omega$ and $\varphi\in\omega^\kappa$ we shall use the notation $U_{f,\varphi}$ rather than $U_f\times U_\varphi$ for the neighborhood of $\langle\infty,\infty\rangle$ determined by f and φ. We shall also use $\langle k,m,\alpha,n\rangle$ instead of $\langle\langle k,m\rangle,\langle\alpha,n\rangle\rangle$ to denote points of $S_\omega\times S_\kappa$.

ACKNOWLEDGEMENT. We thank S. Taniyama for giving helpful advice and suggesting the idea of the notion of $\mathfrak{b}^*$. We also thank J. Brendle for some comments and information, mentioned in the context.

2. Characterization of the tightness of $S_\omega\times S_\kappa$

In this section, we shall give a combinatorial characterization of the tightness of $S_\omega\times S_\kappa$[1]. To state the combinatorial characterization, a part of which is due to [1], we generalize a notion in Definition 1.2.

DEFINITION 2.1. Let $\mathrm{b}(\kappa)$ be the smallest infinite cardinal λ satisfying the following: For every unbounded family $\mathcal{F}\subseteq\omega^\omega$ with $|\mathcal{F}|\leqq\kappa$ there exists a subfamily $\mathcal{G}\subseteq\mathcal{F}$ such that $|\mathcal{G}|\leqq\lambda$ and $\mathcal{G}$ is still unbounded.

Using this notion $\mathfrak{b}^*$ is defined as $\mathrm{b}(2^\omega)$.

---

1) After the submission of the first version of this paper, we have had a chance to see a preprint of Brendle and LaBerge [1]. It deals with a closely related topic and gives a nice idea to simplify the proof of Theorem 1.3. Our previous combinatorial characterization was more complicated.

THEOREM 2.2. *For any infinite cardinal* κ, $t(S_\omega \times S_\kappa)$ *is equal to* $\mathrm{b}(\kappa)$.

According to this theorem, it is easy to see Theorem 1.3.

LEMMA 2.3. *Let* κ *and* λ *be infinite cardinals. Then,* $t(S_\omega \times S_\kappa) \geqq \lambda$ *if there exists an unbounded family* $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ *such that any subfamily* $\mathcal{G} \subseteq \mathcal{F}$ *with* $|\mathcal{G}| < \lambda$ *is bounded.*

PROOF. Let $A = \{\langle k, f_\alpha(k), \alpha, k\rangle : k < \omega \wedge \alpha < \kappa\}$. We show A witnesses $t(S_\omega \times S_\kappa) \geqq \lambda$. Let $h \in \omega^\omega$, $\varphi \in \kappa^\omega$. Since $\mathcal{F}$ is unbounded, there exists $\alpha < \kappa$ such that $f_\alpha \not\leqq^* h$. We can find $k > \varphi(\alpha)$ such that $f_\alpha(k) > h(k)$ and so $\langle k, f_\alpha(k), \alpha, k\rangle \in A \cap U_{h,\varphi}$, which implies $\langle \infty, \infty\rangle \in \mathrm{cl} A$.

Let $X \subseteq A$ with $|X| < \lambda$. There exists $I \subseteq \kappa$ such that $|I| < \lambda$ and $X \subseteq \{\langle k, f_\alpha(k), \alpha, k\rangle : k < \omega \wedge \alpha \in I\}$. By the assumption, there exists $h \in \omega^\omega$ such that $f_\alpha \leqq^* h$ for all $\alpha \in I$. For $\alpha \in I$, we can put $\varphi(\alpha) < \omega$ so that $f_\alpha(k) \leqq h(k)$ for any $k \geqq \varphi(\alpha)$. Then, $U_{h',\varphi} \cap X = \emptyset$, where $h'(k) = h(k) + 1$. This completes the proof. □

LEMMA 2.4. *Suppose that* $A \subseteq S_\omega \times S_\kappa$ *satisfies that* $\langle \infty, \infty\rangle \in \mathrm{cl} A$ *and* $\langle \infty, \infty\rangle \notin \mathrm{cl} C$ *for any countable* $C \subseteq A$. *Then, there exists* $B \subseteq A$ *such that* $\langle \infty, \infty\rangle \in \mathrm{cl} B$ *and for any* $k < \omega$ *and* $\alpha < \kappa$

(1) $\{n : \langle k, m, \alpha, n\rangle \in B$ *for some* $m < \omega\}$ *and*

(2) $\{m : \langle k, m, \alpha, n\rangle \in B$ *for some* $n < \omega\}$

are both finite.

PROOF. First we prove that for any $k < \omega$ there exists $M < \omega$ such that $\{n < \omega : \langle k, m, \alpha, n\rangle \in A$ for some $m > M\}$ is finite for all $\alpha < \kappa$. Suppose not, then we can take $k < \omega$ and $\alpha_M < \kappa$ for each $M < \omega$ so that $\{n < \omega : \langle k, m, \alpha_M, n\rangle \in A$ for some $m > M\}$ is infinite. Now we claim that $\langle \infty, \infty\rangle \in \mathrm{cl}(\{\langle k, m, \alpha_M, n\rangle \in A : k, m, M, n < \omega\})$, which contradicts the assumption. Fix $h \in \omega^\omega$ and $\psi \in \omega^\kappa$ arbitrarily and let $M = h(k)$. Then, by the choice of α_M, we can find $m > M$ so that there exists $n \geqq \psi(\alpha_M)$ with $\langle k, m, \alpha_M, n\rangle \in A$.

Let $f(k)$ be greater than M, then $\{n : \langle k, m, \alpha, n\rangle \in A$ for some $m \geqq f(k)\}$ is finite. Symmetrically, we get $\varphi(\alpha)$ so that $\{m : \langle k, m, \alpha, n\rangle \in A$ for some $n \geqq \varphi(\alpha)\}$ is finite. Then, $B = A \cap U_{f,\varphi}$ is the desired one. □

PROOF OF THEOREM 2.2. By Lemma 2.3, it suffices to show $t(S_\omega \times S_\kappa) \leqq \mathrm{b}(\kappa)$. Gruenhage [**4**, Lemma 1] proved $t(S_\omega \times S_\kappa) = \omega$ in case $\kappa < \mathfrak{b}$, which implies $t(S_\omega \times S_\kappa) = \mathrm{b}(\kappa)$. So, we assume $\kappa \geqq \mathfrak{b}$.

Let $A \subseteq S_\omega \times S_\kappa$ be so that $\langle \infty, \infty\rangle \in \mathrm{cl} A$ and assume that $\langle \infty, \infty\rangle \notin \mathrm{cl} C$ for any countable $C \subseteq A$. Then, by Lemma 2.4 we get $B \subseteq A$ with the properties in the lemma. Take an unbounded family $\mathcal{G}$ of strictly increasing functions with $|\mathcal{G}| = \mathfrak{b}$. We define $f_\alpha^g(k) = \max(\{0\} \cup \{m : \exists n(\langle k, m, \alpha, n\rangle \in B \wedge k \leqq g(n))\})$.

First, we show $\{f_\alpha^g : \alpha<\kappa \wedge g\in\mathcal{G}\}$ is unbounded.

Suppose $f_\alpha^g \leqq^* f$ for all $\alpha<\kappa$ and $g\in\mathcal{G}$. Since $\langle\infty, \infty\rangle\in \mathrm{cl}B$, there exists $\alpha<\kappa$ such that the set $\{n : \exists k, m(f(k)<m \wedge \langle k, m, \alpha, n\rangle\in B)\}$ is infinite. For $n<\omega$ choose k_n so that $f(k_n)<m$ and $\langle k_n, m, \alpha, n'\rangle\in B$ for some $m<\omega$, $n'\geqq n$. Since $\mathcal{G}$ is unbounded, there is $g\in\mathcal{G}$ such that $k_n\leqq g(n)$ for infinitely many n. By the first property of Lemma 2.4, the correspondence from n to k_n is finite-to-one, so we can find $n<\omega$ such that $f_\alpha^g(k_n)\leqq f(k_n)$ and also $k_n\leqq g(n)$. By the choice of k_n, there are $n'\geqq n$ and $m>f(k_n)$ such that $\langle k_n, m, \alpha, n'\rangle\in B$. Since $g(n)\leqq g(n')$ and by the definition of $f_\alpha^g(k_n)$, this implies $f_\alpha^g(k_n)\geqq m>f(k_n)$, which contradicts $f_\alpha^g(k_n)\leqq f(k_n)$.

We have shown that $\{f_\alpha^g : \alpha<\kappa\wedge g\in\mathcal{G}\}$ is unbounded. There exists $J\subseteq\kappa$ such that $|J|\leqq \mathfrak{b}(\kappa)$ and $\{f_\alpha^g : \alpha\in J\wedge g\in\mathcal{G}\}$ is unbounded. Let $D=\{\langle k, m, \alpha, n\rangle \in B : \alpha\in J\wedge k, m, n<\omega\}$. We claim that $\langle\infty, \infty\rangle\in \mathrm{cl}D$, which shows $t(S_\omega\times S_\kappa) \leqq \mathfrak{b}(\kappa)$. Take arbitrary $h\in\omega^\omega$ and $\varphi\in\omega^\kappa$. Then we can find $\alpha\in J$ and $g\in\mathcal{G}$ so that $f_\alpha^g\not\leqq^* h$. By the definition of $f_\alpha^g(k)$, $f_\alpha^g(k)>0$ implies $\langle k, f_\alpha^g(k), \alpha, n\rangle\in D$ for some n with $k\leqq g(n)$. Since $f_\alpha^g\not\leqq^* h$, there are infinitely many n such that $\langle k, f_\alpha^g(k), \alpha, n\rangle\in D$ and $h(k)<f_\alpha^g(k)$ for some k. So we can find $n\geqq\varphi(\alpha)$ and $k<\omega$ with $h(k)<f_\alpha^g(k)$ so that $\langle k, f_\alpha^g(k), \alpha, n\rangle\in D$, i.e., $U_{h,\varphi}\cap D\neq\emptyset$. □

3. Relations between $\mathfrak{b}$, $\mathfrak{d}$ and $\mathfrak{b}^*$.

In this section we shall show that $\mathfrak{b}^*$ is located between $\mathfrak{b}$ and $\mathfrak{d}$ but consistently different from both of them.

THEOREM 3.1. $\mathfrak{b}\leqq\mathfrak{b}^*\leqq\mathfrak{d}$.

PROOF. $\mathfrak{b}\leqq\mathfrak{b}^*$ follows immediately from the definition of $\mathfrak{b}^*$. To show $\mathfrak{b}^*\leqq\mathfrak{d}$, let $\mathcal{F}$ be any unbounded family and $\mathcal{D}=\{g_\beta : \beta<\mathfrak{d}\}$ a dominating family. For each $\beta<\mathfrak{d}$, we can find $f_\beta\in\mathcal{F}$ so that $f_\beta\not\leqq^* g_\beta$. Let $\mathcal{G}=\{f_\beta : \beta<\mathfrak{d}\}\subseteq\mathcal{F}$. Then, $|\mathcal{G}|\leqq\mathfrak{d}$ and $\mathcal{G}$ is still unbounded. □

Now we turn to the consistency proofs. Both of the models satisfying $\mathfrak{b}^*<\mathfrak{d}$ and $\mathfrak{b}<\mathfrak{b}^*$ are obtained by the Cohen extensions.

Before proving them, we observe a basic fact on the Cohen forcing. Let $\boldsymbol{C}_I=\mathrm{Fn}(I, 2, \omega)$ be the canonical Cohen forcing notion for an infinite set I (see [7, Chapter 7]).

LEMMA 3.2 ([2, Corollary 3.5]). *For any infinite set I, if $\mathcal{F}\subseteq\omega^\omega$ is an unbounded family, then* $\Vdash_{\boldsymbol{C}_I}$ "*$\mathcal{F}$ is unbounded.*"

DEFINITION 3.3. For a forcing notion $\boldsymbol{P}$, a standard $\boldsymbol{P}$-name $\dot{f}$ for a real is a name uniquely determined by a system $\{A_{mn} : m, n<\omega\}$ with the following:

(1) $A_{mn}\subseteq\boldsymbol{P}$ is an antichain of $\boldsymbol{P}$ and $n\neq n'$ implies $A_{mn}\cap A_{mn'}=\emptyset$,

(2) $\bigcup_{n<\omega} A_{mn}$ is a maximal antichain of $\boldsymbol{P}$, and
(3) For each $p \in A_{mn}$, $p \Vdash_{\boldsymbol{P}} \dot{f}(m)=n$.

THEOREM 3.4. *Let $2^\omega=\lambda$. Then, in the Cohen extension by $\boldsymbol{C}_\kappa$ for an infinite κ, any unbounded family $\mathcal{F}$ of ω^ω has an unbounded subfamily of size less than or equal to λ.*

PROOF. For an infinite $I \subseteq \kappa$, let $X(I)$ be the collection of all standard $\boldsymbol{C}_I$-names of reals and let $\mathcal{X}=X(\kappa)$. It suffices to deal with the case $\kappa>\lambda$. Suppose that there are $p_0 \in \boldsymbol{C}_\kappa$ and a collection $\mathcal{F}$ of standard $\boldsymbol{C}_\kappa$-names for reals such that

$$p_0 \Vdash \text{“}\mathcal{F} \text{ is unbounded} \wedge \forall \mathcal{G} \subseteq \mathcal{F}(|\mathcal{G}| \leqq \lambda \rightarrow \mathcal{G} \text{ is bounded}).\text{”}$$

Let $S=\{X(I): I \in [\kappa]^\lambda \wedge \operatorname{supp}(p_0) \subseteq I\}$, then $S \subseteq [\mathcal{X}]^\lambda$. S is stationary, since it is unbounded and closed under unions of increasing ω_1-sequences. By assumption and using Lemma 3.2, for each $X=X(I) \in S$ we get a standard $\boldsymbol{C}_I$-name $\dot{g}_X$ for a real so that p_0 forces $\dot{f} \leqq^* \dot{g}_X$ for all $\dot{f} \in \mathcal{F} \cap X$. By Fodor's lemma for $[\mathcal{X}]^\lambda$ (see [**6**, Theorem 3.2]) there is a stationary set $S' \subseteq S$ such that $\dot{g}_X=\dot{g}$ for all $X \in S'$. Since S' is unbounded in $[\mathcal{X}]^\lambda$, we have $p_0 \Vdash \text{“}\dot{f} \leqq^* \dot{g}\text{”}$ for all $\dot{f} \in \mathcal{F}$, which is a contradiction. □

COROLLARY 3.5. *Assume* CH. *For a cardinal κ of uncountable cofinality, $\mathfrak{b}=\mathfrak{b}^*=\omega_1$ and $\mathfrak{d}=\kappa$ hold in the forcing model by $\boldsymbol{C}_\kappa$*[2]

Using Lemma 3.2 and Theorem 3.4, we can easily prove both the consistency of $\mathfrak{b}<\mathfrak{b}^*<\mathfrak{d}$ and that of $\mathfrak{b}<\mathfrak{b}^*=\mathfrak{d}$.

PROPOSITION 3.6. *Assume* $\mathrm{MA}+\omega_1<2^\omega=\lambda \leqq \kappa$ *and κ has uncountable cofinality. Then, $\mathfrak{b}=\omega_1$, $\mathfrak{b}^*=\lambda$ and $\mathfrak{d}=\kappa$ hold in the forcing model by $\boldsymbol{C}_\kappa$.*

PROOF. Since MA and $2^\omega=\lambda$ hold in the ground model, we can take an unbounded family $\mathcal{F}$ of order type λ with respect to $\leqq^*$. Then, in the forcing model $\mathcal{F}$ is still unbounded by Lemma 3.2 and every subfamily of $\mathcal{F}$ of size $<\lambda$ must be bounded, since λ is regular. This implies $\lambda \leqq \mathfrak{b}^*$. On the other hand, $\mathfrak{b}^* \leqq \lambda$ by Theorem 3.4. As is well-known, $\mathfrak{b}=\omega_1$ and $\mathfrak{d}=\kappa$ hold in the forcing model by $\boldsymbol{C}_\kappa$. □

4. More on $\mathfrak{b}^*$ and the tightness of $S_\omega \times S_\kappa$.

In this section we study Hechler's result about dominating families of ω^ω and show that $t(S_\omega \times S_\kappa)$ for $\mathfrak{b}<\kappa<\mathfrak{b}^*$ may or may not be equal to κ.

[2] J. Brendle informed us that LaBerge and Landver [8] proved this same result by another method independently. The paper was published after the submission of the present paper.

To investigate structures of dominating subfamilies of ω^ω, Hechler [5] introduced the so-called Hechler Forcing. However, his paper had been written before the simplified forcing method appeared and consequently it involves some complicated presentation. Here, we introduce a simplified notion in the current presentation. Since our final purpose is to investigate the notions around the cardinals $\mathfrak{b}$, $\mathfrak{b}^*$ and $\mathfrak{d}$, we confine ourselves only to a well-founded partially ordered set R.

DEFINITION 4.1. Let R be a well-founded partially ordered set. We define forcing notions inductively.

A member of a partially ordered set $\boldsymbol{H}_a$ for $a\in R$ is of the form $\{\langle s_b, \mathcal{T}_b\rangle : b\in F\}$ with the following:

(1) F is a finite subset of $\{b\in R: b\leqq a\}$;

(2) $s_b\in\omega^{<\omega}$ for $b\in F$;

(3) For $b\in F$, $\mathcal{T}_b$ is a finite subset of standard names for reals such that if $\dot{f}\in\mathcal{T}_b$, $\dot{f}$ is an $\boldsymbol{H}_c$-name for some $c<b$.

$\{\langle t_c, \mathcal{G}_c\rangle : c\in G\}$ extends $\{\langle s_b, \mathcal{T}_b\rangle : b\in F\}$ if the following hold:

(a) $F\subseteqq G$, and $\mathcal{T}_b\subseteqq\mathcal{G}_b$ and $s_b\subseteqq t_b$ for $b\in F$;

(b) For each $b\in F$, $c<b$, an $\boldsymbol{H}_c$-name $\dot{f}\in\mathcal{T}_b$ and $k\in\mathrm{dom}(t_b)\backslash\mathrm{dom}(s_b)$, we have

$$\{\langle t_d, \mathcal{G}_d\rangle : d\in G\wedge d\leqq c\}\Vdash_{\boldsymbol{H}_c}\dot{f}(k)\leqq t_b(k).$$

Finally, $\boldsymbol{H}_R$ is the set $\cup_{a\in R}\boldsymbol{H}_a$ with the ordering $\cup_{a\in R}\leqq_a$, where $\leqq_a$ is the ordering of $\boldsymbol{H}_a$.

Let $\boldsymbol{G}$ be the canonical name for an $\boldsymbol{H}_R$-generic filter, i.e., $p\Vdash p\in\boldsymbol{G}$ for $p\in\boldsymbol{H}_R$ and let $\dot{d}_a$ be the name for $\cup\{s_a : \langle s_a, \mathcal{T}\rangle\in p\in\boldsymbol{G}$ for some p, $\mathcal{T}\}$ for each $a\in R$.

Note that if $a<b$ we can put $\dot{d}_a$ in $\mathcal{T}_b$.

LEMMA 4.2. (1) *$\boldsymbol{H}_R$ satisfies c.c.c.*

(2) *For $a\leqq b$, the inclusion from $\boldsymbol{H}_a$ to $\boldsymbol{H}_b$ is a complete embedding and so is the inclusion from $\boldsymbol{H}_a$ to $\boldsymbol{H}_R$.*

(3) *For a, $b\in R$, $a\leqq b$ implies $\Vdash\dot{d}_a\leqq^*\dot{d}_b$ and $a\nleqq b$ implies $\Vdash\dot{d}_a\nleqq^*\dot{d}_b$.*

(4) *If any countable subset of R has a strict upper bound in R, $\Vdash$ "$\{\dot{d}_a : a\in R\}$ is a dominating family."*

Now it is easy to see the following:

PROPOSITION 4.3. *Let $R=\omega_1\times\omega_2\times\omega_3$ with the product ordering. Then $\mathfrak{b}=\omega_1$, $\mathfrak{b}^*=\mathfrak{d}=\omega_3$, and $t(S_\omega\times S_{\omega_2})=\omega_2$ hold in the forcing model by $\boldsymbol{H}_R$.*

PROPOSITION 4.4. *Let $R=\omega_1\times\omega_3$ with the product ordering. Then $\mathfrak{b}=\omega_1$, $\mathfrak{b}^*=\mathfrak{d}=\omega_3$, and $t(S_\omega\times S_{\omega_2})=\omega_1$ hold in the forcing model by $\boldsymbol{H}_R$.*

PROOF. By Lemma 4.2 there exists a dominating family $\{d_a : a \in R\}$ such that $d_a \leqq^* d_b$ iff $a \leqq b$ in the product ordering. Now, the first two statements are clear. To show the last one, let $\mathcal{F}$ be an unbounded family of size ω_2. For $f \in \mathcal{F}$ and $\alpha < \omega_1$, let $\beta(f, \alpha) < \omega_3$ such that $f \leqq^* d_{\langle \alpha, \beta(f, \alpha) \rangle}$ if such $\beta(f, \alpha)$ exists and $\beta(f, \alpha) = 0$ otherwise. Let $\beta_0 = \sup\{\beta(f, \alpha) : f \in \mathcal{F} \wedge \alpha < \omega_1\} < \omega_3$ and take $\mathcal{G} \subseteqq \mathcal{F}$ so that $|\mathcal{G}| = \omega_1$ and $d_{\langle \alpha, \beta_0 \rangle}$ does not bound $\mathcal{G}$ for any $\alpha < \omega_1$. Then, $\mathcal{G}$ is unbounded. □

References

[1] J. Brendle and T. LaBerge, Forcing tightness in products of fans, Fund. Math., **150** (1996), 211–226.
[2] J. Cichoń and J. Pawlikowski, On ideals of subsets of the plane and on Cohen reals, J. Symbolic Logic, **51** (1986), 560–569.
[3] K. Eda, G. Gruenhage, P. Koszmider, K. Tamano and S. Todorčević, Sequential fans in topology, (submitted).
[4] G. Gruenhage, k-spaces and products of closed images of metric spaces, Proc. Amer. Math. Soc., **80** (1980), 478–482.
[5] S.H. Hechler, On the existence of certain cofinal subsets of ${}^{\omega}\omega$, Axiomatic Set Theory, (ed. T. Jech), Amer. Math. Soc., 1974, pp. 155–173.
[6] T. Jech, Some combinatorial problems concerning uncountable cardinals, Ann. Math. Logic, **5** (1973), 165–198.
[7] K. Kunen, Set theory, North Holland, 1980.
[8] T. LaBerge and A. Landver, Tightness in products of fans and pseudo-fans, Topology Appl., **65** (1995), 237–255.

Katsuya EDA
Waseda University
Tokyo 169
Japan
(e-mail: eda@logic.info.waseda.ac.jp)

Masaru KADA
University of Osaka Prefecture
Sakai, Osaka 591
Japan
(e-mail: kada@center.osakafu-u.ac.jp)

Yoshifumi YUASA
Waseda University
Tokyo 169
Japan
(e-mail: yuasa@logic.info.waseda.ac.jp)