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Graph decompositions without isolated vertices II

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1. Introduction.

All graphs considered in this paper are finite, undirected and without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of $V(G)$, the neighbourhood of v in G, denoted by $N_G(v)$, is the set of vertices of G adjacent to v, and the degree $d_G(v)$ of v in G is $|N_G(v)|$. We denote by $\delta(G)$ and $\kappa(G)$ the minimum degree and the connectivity of G, respectively. For a subset S of $V(G)$, let $\langle S\rangle_G$ denote the subgraph of G induced by S. For standard terms or notation not defined here, see [**1**] or [**2**].

Given a graph G of order n and a partition $n=\sum_{i=1}^k a_i$ with $a_i\geqq 1$, S. B. Maurer [**10**] conjectured that if $\kappa(G)\geqq k$, then $V(G)$ can be decomposed as $V(G)=\bigcup_{i=1}^k A_i$ with the conditions $|A_i|=a_i$ and $\kappa(\langle A_i\rangle_G)>0$ (i.e., $\langle A_i\rangle_G$ is connected) for all i, $1\leqq i\leqq k$. A. Frank [**7**], on the other hand, conjectured the following stronger form of this, which was settled independently by L. Lovász [**9**] and E. Győri [**8**].

THEOREM A [**9**, **8**]. *Let G be a graph of order n, and $n=\sum_{i=1}^k a_i$ be a partition of n with $a_i\geqq 1$. Suppose that $\kappa(G)\geqq k$. Then for any distinct k vertices $v_1, \cdots, v_k$ of $V(G)$, $V(G)$ can be decomposed as $V(G)=\bigcup_{i=1}^k A_i$ with the conditions $|A_i|=a_i$, $v_i\in A_i$ and $\kappa(\langle A_i\rangle_G)>0$ for all i, $1\leqq i\leqq k$.*

Turning his attention from "connectedness" to "no isolation", Frank also conjectured the following as an analogue of Maurer's conjecture, in which the conditions on the connectivity are replaced by those on the minimum degree. (Note that $\delta(\langle A_i\rangle_G)>0$ implies that $\langle A_i\rangle_G$ contains no isolated vertices.) Thereafter some partial results on this came out in a row, while a complete proof was finally given by H. Enomoto [**4**].

THEOREM B [**4**]. *Let G be a connected graph of order n, and $n=\sum_{i=1}^k a_i$ be a partition of n with $a_i\geqq 2$. Suppose that $\delta(G)\geqq k$. Then $V(G)$ can be decomposed as $V(G)=\bigcup_{i=1}^k A_i$ with the conditions $|A_i|=a_i$ and $\delta(\langle A_i\rangle_G)>0$ for all i, $1\leqq i\leqq k$.*

In the present paper, we shall prove the following generalization of this, which was conjectured by Y. Egawa [3].

THEOREM 1. *Let G be a connected graph of order n, and $n=\sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geqq 2$. Then $V(G)$ can be decomposed as $V(G)=\bigcup_{i=1}^{k} A_i$ with the conditions $|A_i|=a_i$ and "if $d_G(v)\geqq k$ and $v \in A_i$, then v is not isolated in $\langle A_i \rangle_G$" for all i, $1 \leqq i \leqq k$.*

Here we should like to remark that Theorem 1 not only generalizes Theorem B but plays an important role in establishing the following analogue of Theorem A. This will be proved in a forthcoming paper.

THEOREM [6]. *Let G be a graph of order n, and $n=\sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geqq 2$. Suppose that $\delta(G)\geqq 3k-2$. Then for any distinct k vertices $v_1, \cdots, v_k$ of $V(G)$, $V(G)$ can be decomposed as $V(G)=\bigcup_{i=1}^{k} A_i$ with the conditions $|A_i|=a_i$, $v_i \in A_i$ and $\delta(\langle A_i \rangle_G)>0$ for all i, $1 \leqq i \leqq k$.*

The rest of the paper is organized as follows. In the next section, we introduce some specialized terms and notation and briefly show our basic strategy to prove Theorem 1. In Section 3, with the help of some key proposition and lemmas, we prove Theorem 1. Sections 4-6 contain the proofs of those key results used in Section 3.

§ 2. Preliminaries.

Let n be a positive integer. A sequence $\boldsymbol{a}=(a_1, \cdots, a_k)$ of positive integers is called a *k-partition of n* if $n=\sum_{i=1}^{k} a_i$, and a k-partition $\boldsymbol{a}$ is said to be *non-singular* if $a_i \geqq 2$ for all i, $1 \leqq i \leqq k$. Given a graph G and a k-partition $\boldsymbol{a}$ of $|V(G)|$, a sequence $\mathcal{A}=(A_1, \cdots, A_k)$ of subsets of $V(G)$ is called an *$\boldsymbol{a}$-decomposition* if the following conditions (D1)-(D3) are satisfied:

(D1) $V(G)=\bigcup_{i=1}^{k} A_i$;
(D2) $|A_i|=a_i$ for all i, $1 \leqq i \leqq k$;
(D3) $\delta(\langle A_i \rangle_G)>0$ for all i, $1 \leqq i \leqq k$.

On the other hand, $\mathcal{A}$ is called an *$\boldsymbol{a}^*$-decomposition* if the following weaker condition (D3)′ replaces (D3) in the above:

(D3)′ If $d_G(v)\geqq k$ and $v \in A_i$, then v is not isolated in $\langle A_i \rangle_G$.

Now we can restate Theorem B and Theorem 1 in a more simple style as Theorem C and Theorem 2 below, respectively. In proving Theorem 1, we therefore give the proof of Theorem 2.

THEOREM C [4]. *Let G be a connected graph of order n with $\delta(G)\geqq k$. Then G has an $\boldsymbol{a}$-decomposition for any non-singular k-partition $\boldsymbol{a}$ of n.*

THEOREM 2. *Let G be a connected graph of order n. Then G has an $\boldsymbol{a}^*$-decomposition for any non-singular k-partition $\boldsymbol{a}$ of n.*

A subset W of $V(G)$ is *dominating* if $G-W$ contains no edges. We say that W is *k-dominating* if W is dominating and $d_G(x)\geqq k$ for all $x\in V(G)-W$. A subgraph H of G is interpreted to be *dominating* (resp. *k-dominating*) if $V(H)$ is dominating (resp. k-dominating).

A tree T is called a *fork* if there exists a vertex $v\in V(T)$ satisfying $d_T(v)=3$ and $d_T(x)\leqq 2$ for all $x\in V(T)-\{v\}$. Figure 1 illustrates a 2-dominating fork (the subgraph consisting of the black vertices and the thick edges).

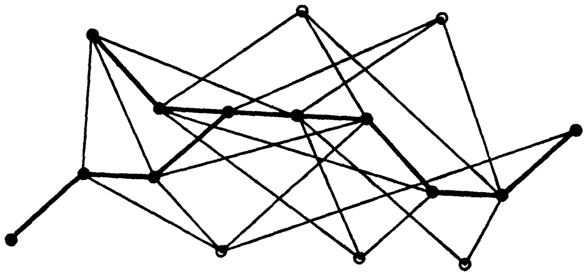

Figure 1. A graph to illustrate a 2-dominating fork.

A k-partition $\boldsymbol{a}$ is said to be *small* if $a_i\leqq 4$ for all i, $1\leqq i\leqq k$. As a special case of small k-partitions, we say that $\boldsymbol{a}$ is *exceptional* if the a_i's are all two or all three.

The following two lemmas, which are of great importance in our proof of Theorem 2, describe a certain connection between the concept of domination and our question of decompositions calling for "no isolation".

LEMMA D [5]. *Let G be a graph of order n, and $\boldsymbol{a}$ a non-singular k-partition of n. If G has a k-dominating path, then G has an $\boldsymbol{a}$-decomposition.*

LEMMA E [4]. *Let G be a graph of order n, and $\boldsymbol{a}$ a non-singular k-partition of n. Suppose that $\boldsymbol{a}$ is not exceptional and that G has a k-dominating fork. Then G has an $\boldsymbol{a}$-decomposition.*

Our approach to the proof of Theorem 2 is through extraction of a special structure of paths. Let $\mathcal{P}=(P_0, \cdots, P_r)$ be a sequence of paths in G, with $P_i=(v_1^{(i)}, \cdots, v_{m_i}^{(i)})$, $1\leqq i\leqq r$. For each P_i, let end(P_i) denote $\{v_1^{(i)}, v_{m_i}^{(i)}\}$, and define $W_i:=\bigcup_{j=0}^{i} V(P_j)$ and $S:=V(G)-W_r$. $\mathcal{P}$ is called a *path-system of degree k* if the conditions (PS0)-(PS12) below are satisfied. (The term "path-system" first appeared in [4], and the definition we give here is its refinement.) Here we consider all i, $1\leqq i\leqq r$, for (PS1)-(PS10), and assume, without loss of generality, $N_G(v_1^{(i)})\cap W_{i-1}\neq\emptyset$ for (PS4)-(PS10).

(PS0) P_0 is a longest path in G.
(PS1) $V(P_i) \subseteqq V(G)-W_{i-1}$.
(PS2) $|V(P_i)| \geqq 2$.
(PS3) $N_G(\mathrm{end}(P_i)) \cap W_{i-1} \neq \emptyset$.
(PS4) $N_G(v_j^{(i)}) \cap N_G(v_{j+1}^{(i)}) \subseteqq W_i$ for all j, $1 \leqq j \leqq m_i-1$.
(PS5) $N_G(v_{m_i}^{(i)}) \subseteqq W_i$.
(PS6) $N_G(v_{m_i-1}^{(i)}) \subseteqq W_i \cup S$, and if $x \in N_G(v_{m_i-1}^{(i)}) \cap S$, then $N_G(x) \subseteqq W_i \cup S$.
(PS7) If $m_i=2$ and $k \geqq 2$, then $N_G(v_1^{(i)}) \subseteqq W_i$.
(PS8) If $m_i=3$ and $k \geqq 3$, then $N_G(v_1^{(i)}) \subseteqq W_i$.
(PS9) If $m_i=3$ and $k \geqq 3$, then $d_G(v_1^{(i)}) \geqq k$ or $d_G(v_3^{(i)}) \geqq k$.
(PS10) If $m_i \geqq 3$ and $N_G(v_{m_i-1}^{(i)}) \cap S \neq \emptyset$, then $v_{m_i-2}^{(i)} v_{m_i}^{(i)} \notin E(G)$.
(PS11) If $r \geqq 1$, then $|V(P_0)| \geqq 2k+1$ and $|V(P_0)|+|V(P_1)| \geqq 3k+1$.
(PS12) If $r \geqq 2$ and $|V(P_2)| \geqq 3$, then $|V(P_0)|+|V(P_1)|+|V(P_2)| \geqq 4k+1$.

Stated in this term, the following proposition is essential in our proof of Theorem 2. We leave the proof to Section 4.

PROPOSITION 3. *Let G be a connected graph of order n, and $\boldsymbol{a}$ a non-singular k-partition of n. Let $(P_0, \cdots, P_r)$ be a sequence of paths in G with $P_r=(v_1, \cdots, v_m)$, and let $W_i=\bigcup_{j=0}^{i} V(P_j)$ and $S=V(G)-W_r$. Suppose that W_r is k-dominating, and that either*

(i) *$(P_0, \cdots, P_r)$ is a path-system of degree k; or*
(ii) *$\boldsymbol{a}$ is not small, $(P_0, \cdots, P_{r-1})$ is a path-system of degree k, and "$m \geqq 2$, $N_G(v_1) \cap W_{r-1} \neq \emptyset$ and $N_G(v_i) \cap N_G(v_{i+1}) \cap S=\emptyset$ for all i, $1 \leqq i \leqq m-1$".*

Then G has an $\boldsymbol{a}$-decomposition.

Let us return to Theorem 2. From the statement, one may readily notice that attention should be paid primarily to the vertices with degree not less than k. Accordingly, we say that a vertex $v \in V(G)$ is *major* if $d_G(v) \geqq k$ and define $V_{\mathrm{major}} := \{v \in V(G) \mid v \text{ is major}\}$. Also, we shall refer to any $v \in V(G)-V_{\mathrm{major}}$ as a *minor* vertex and define $V_{\mathrm{minor}} := V(G)-V_{\mathrm{major}}$. Now consider joining all the minor vertices in G, and let us denote the resulting graph by $\hat{G}$. Then it is quite evident that we may prove Theorem 2 by working with $\hat{G}$ instead of G, since deletion of those edges added to G after decomposing $V(\hat{G})$ does not affect the adjacency around the major vertices. Thus, in the remainder of this section and related Sections 3, 5 and 6, we are only concerned with $\hat{G}$. As we shall see later on, this provides a useful relaxation in the structure of a graph. We now construct a sequence of paths in $\hat{G}$ as follows. Here we again use the notation $\mathrm{end}(P_i)$ to denote the endvertices of P_i.

Step 0. Take P_0 as a longest path in $\hat{G}$. Define $W := V(P_0)$ and $i := 1$.
Step 1. If possible, take P_i in $\hat{G}-W$ such that:

(1) $V(P_i) \subseteqq V(\hat{G}) - W$, $|V(P_i) \cap V_{\text{major}}| \geqq 2$, and $N_{\hat{G}}(\text{end}(P_i)) \cap W \neq \emptyset$;

(2) $|V(P_i) \cap V_{\text{major}}|$ is as large as possible;

(3) Subject to (1) and (2), $|V(P_i)|$ is as large as possible.

Step 2. Put $W := W \cup V(P_i)$ and $i := i+1$. Apply Step 1.

For a sequence $(P_0, \cdots, P_s)$ of paths taken as above, we now observe the following two lemmas, which are crucial indeed in our later argument. (The proofs will appear in Sections 5 and 6.)

LEMMA 4. *If* $s \geqq 1$, *then* $|V(P_0)| \geqq 2k+1$ *and* $|V(P_0)| + |V(P_1)| \geqq 3k+1$.

LEMMA 5. *If* $s \geqq 2$ *and* $|V(P_2)| \geqq 3$, *then* $|V(P_0)| + |V(P_1)| + |V(P_2)| \geqq 4k+1$.

§ 3. Proof of Theorem 1.

As mentioned earlier, we give the proof of Theorem 2. Since we are only concerned with $\hat{G}$, for simplicity we write G for $\hat{G}$ throughout this section and Sections 5 and 6.

PROOF OF THEOREM 2. If $k=1$, then we are done. So we suppose $k \geqq 2$ and let $\boldsymbol{a} = (a_1, \cdots, a_k)$. To begin with, we take a sequence $\mathcal{P} = (P_0, \cdots, P_s)$ of paths in G in such a manner as shown at the end of the preceding section. Define $W_i := \bigcup_{j=0}^{i} V(P_j)$ and $S := V(G) - W_s$. We first claim the following.

CLAIM 1. $\mathcal{P}$ *is a path-system of degree* k.

PROOF OF CLAIM 1. From the construction, clearly (PS0)-(PS3) hold. For each P_i, $1 \leqq i \leqq s$, let $P_i = (v_1^{(i)}, \cdots, v_{m_i}^{(i)})$ and define $\mu_i := |V(P_i) \cap V_{\text{major}}|$; without loss of generality, we may assume $N_G(v_1^{(i)}) \cap W_{i-1} \neq \emptyset$. Again from the construction, (PS4) and (PS5) are immediate. Noting that P_i is taken maximally with respect to μ_i, one can easily verify (PS6) as well. Now suppose $m_i = 2$. In this case, $v_2^{(i)} \in V_{\text{major}}$, and so $N_G(v_2^{(i)}) \cap W_{i-1} \neq \emptyset$ (notice $d(v_2^{(i)}) \geqq k \geqq 2$). Thus by the maximality, we see (PS7). To see (PS8) and (PS9), next suppose $m_i = 3$ and $k \geqq 3$. Since $\mu_i \geqq 2$, clearly $v_1^{(i)} \in V_{\text{major}}$ or $v_3^{(i)} \in V_{\text{major}}$, readily implying (PS9). Now, if $v_3^{(i)} \in V_{\text{major}}$, then by (PS5), $N_G(v_3^{(i)}) \cap W_{i-1} \neq \emptyset$, so that by the maximality, $N_G(v_1^{(i)}) \subseteqq W_i$. If $v_3^{(i)} \in V_{\text{minor}}$, on the other hand, then $S \cap V_{\text{minor}} = \emptyset$. (Recall that all the minor vertices are adjacent.) Hence, $N_G(v_2^{(i)}) \cap (V(G) - W_i) = \emptyset$, for otherwise another path with more major vertices would exist, which contradicts the choice of P_i. By this together with $k \geqq 3$ and $v_2^{(i)} \in V_{\text{major}}$, we have $N_G(v_2^{(i)}) \cap W_{i-1} \neq \emptyset$; thus $N_G(v_1^{(i)}) \subseteqq W_i$. So (PS8) has been verified. (PS10) also follows from the maximality. Finally, (PS11) and (PS12) are immediate from Lemmas 4 and 5, respectively. □

Now, if $S \cap V_{\text{minor}} = \emptyset$, then W_s is k-dominating in G. Accordingly, in such a case, Claim 1 hints that the conclusion follows from Proposition 3. Therefore, in what follows, we assume $S \cap V_{\text{minor}} \neq \emptyset$. We next claim the following.

CLAIM 2. $|W_s| \geqq 2k+1$.

PROOF OF CLAIM 2. By (PS11) the claim holds if $s \geqq 1$. So suppose $s=0$, and let $P_0=(v_1, \cdots, v_m)$. By the maximality, clearly $v_1, v_m \in V_{\text{major}}$ and $N_G(v_1) \cup N_G(v_m) \subseteqq V(P_0)$. Now, if $v_1 v_i, v_{i-1} v_m \in E(G)$ for some $2 \leqq i \leqq m$, then $(v_1, \cdots, v_{i-1}, v_m, \cdots, v_i, v_1)$ is a cycle of length m. So, if this is the case, then since G is connected, a longer path would exist, contradicting the choice of P_0. Therefore, $M \cap (N_G(v_m) \cup \{v_m\}) = \emptyset$, where $M = \{v_{i-1} \mid v_i \in N_G(v_1) \cap V(P_0)\}$. Noting that $|M| = |N_G(v_1)|$, we soon have $|W_0| = |V(P_0)| \geqq 2k+1$. □

Since G is connected and $\langle S \cap V_{\text{minor}} \rangle_G$ is complete, at most one major vertex in S has neighbours in $S \cap V_{\text{minor}}$, for otherwise another path could be taken in $\langle S \rangle_G$ to augment the present sequence $\mathscr{P}$, a contradiction. Let us now consider the case where no such major vertex exists. Noting Claim 2, we first assign all the vertices of $S \cap V_{\text{minor}}$ to the A_i's so that the remaining size of each A_i stays not less than two. Here it is easy to see that $\mathscr{P}$ is still a path-system of degree k in $G-(S \cap V_{\text{minor}})$. Thus we can now apply Proposition 3, only to see the conclusion, to the graph $G-(S \cap V_{\text{minor}})$ with the remaining partition. We next consider the case where such a (unique) major vertex, say v, exists. Let v' be any neighbour of v in $S \cap V_{\text{minor}}$. Assume now, without loss of generality, $(2 \leqq)\, a_1 \leqq \cdots \leqq a_k$. Note that by Claim 2, $a_k \geqq 3$. If $a_k \geqq 4$, then we assign $\{v, v'\}$ to A_k, by which the problem is clearly reduced to the above case (where no major vertex has neighbours in $S \cap V_{\text{minor}}$). By (PS11), however, this is always the case when $s \geqq 1$. Let us thus suppose $s=0$ and $a_k=3$, and let $P_0=(v_1, \cdots, v_m)$. Now, if $|S \cap V_{\text{minor}}| \geqq 2$, i.e., there exists some $v''(\neq v') \in S \cap V_{\text{minor}}$, then by letting $A_k = \{v, v', v''\}$, we can again reduce the situation to the above case. So we may now further suppose $S \cap V_{\text{minor}} = \{v'\}$. Then let $A_k = \{v', v_{m-1}, v_m\}$. By the maximality of P_0, here, $N_G(v) \cap \{v_{m-1}, v_m\} = \emptyset$ and $|N_G(x) \cap \{v', v_{m-1}, v_m\}| \leqq 1$ for all $x \in S - \{v, v'\}$, which together imply that P_0' is a $(k-1)$-dominating path in G', where $G' = G - A_k$ and $P_0' = (v_1, \cdots, v_{m-2})$. In order to conclude, we now simply apply Lemma D to G' with the remaining non-singular $(k-1)$-partition of $|V(G')|$.

This completes the proof of Theorem 2. □

§4. Proof of Proposition 3.

PROOF OF PROPOSITION 3. If $k=1$, then it is trivial. We thus suppose $k\geqq 2$. Note that the assumption (i) is stronger than (ii) when $\boldsymbol{a}$ is not small. Hence, it suffices to show the proposition with the assumption (i) for a small partition $\boldsymbol{a}$ or with (ii). The proof is by induction on n. Let $\boldsymbol{a}=(a_1, \cdots, a_k)$, and define $S_i := N_G(v_i)\cap S$ for each i, $1\leqq i\leqq m$. By taking P_r maximally under (ii), we may assume $S_m=\emptyset$.

Let us first consider the case when $r=0$ or 1. If $r=0$, then P_0 is a k-dominating path in G; so the conclusion is immediate from Lemma D. If $r=1$, on the other hand, then (P_0, P_1) forms a k-dominating fork of G. Under the assumption (i), (PS11) implies that some $a_i\geqq 4$, so that $\boldsymbol{a}$ is not exceptional in either assumption. Thus from Lemma E, the conclusion again follows. Hence, in the following, we suppose $r\geqq 2$. Note that by this, there must be some $a_i\geqq 4$.

1°) *$\boldsymbol{a}$ is small.*

We may assume $a_1=4$. Note that we may also assume $m_2=2$, for otherwise $\boldsymbol{a}$ cannot be small under the condition (PS12). We assign the vertices of $V(P_2)$ to A_1 and let $\tilde{G}=G-V(P_2)$ and $\tilde{\boldsymbol{a}}=(a_1-2, a_2, \cdots, a_k)$. Then $\tilde{\boldsymbol{a}}$ is a non-singular k-partition of $|V(\tilde{G})|$ $(=n-2)$. Moreover, by (PS5) and (PS7), we see that $\tilde{\mathcal{P}}=(P_0, P_1, P_3, \cdots, P_r)$ is a path-system of degree k, and that $W_r-V(P_2)$ is k-dominating in $\tilde{G}$. Accordingly, we can apply induction to $\tilde{G}$ with $\tilde{\boldsymbol{a}}$ and $\tilde{\mathcal{P}}$ to obtain an $\tilde{\boldsymbol{a}}$-decomposition $(\tilde{A}_1, \cdots, \tilde{A}_k)$ of $V(\tilde{G})$. Then clearly $(\tilde{A}_1\cup V(P_2), \tilde{A}_2, \cdots, \tilde{A}_k)$ is a desired $\boldsymbol{a}$-decomposition of $V(G)$.

2°) *$\boldsymbol{a}$ is not small.*

In the remainder we shall only be concerned with the case in which $\boldsymbol{a}$ is not small; we may assume $a_1\geqq 5$. We proceed principally by working with the two paths P_r and P_{r-1} along with $S_1, \cdots, S_{m-1}$. The argument goes somewhat complicated since we have to distinguish so many cases; however, for an inductive argument, our step is mostly based on the following (a) and/or (b):

(a) Find a subset A_i such that $|A_i|=a_i$, $\delta(\langle A_i\rangle_G)>0$ and W_r-A_i is $(k-1)$-dominating in $G-A_i$;

(b) Find a subset A such that $|A|\leqq a_i-2$, $\delta(\langle A\rangle_G)>0$ and W_r-A is k-dominating in $G-A$.

In the former way (a), for a non-singular $(k-1)$-partition $\boldsymbol{a}'=(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k)$, we obtain by induction an $\boldsymbol{a}'$-decomposition $(A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_k)$ of $V(G)-A_i$, which certainly provides a desired $\boldsymbol{a}$-decomposition $(A_1, \cdots, A_k)$ of $V(G)$. In the latter way (b), for a non-singular k-partition $\tilde{\boldsymbol{a}}=(a_1, \cdots, a_{i-1}, a_i-|A|, a_{i+1}, \cdots, a_k)$, we obtain an $\tilde{\boldsymbol{a}}$-decomposition $(\tilde{A}_1, \cdots, \tilde{A}_k)$ of $V(G)-A$. Then $(\tilde{A}_1, \cdots, \tilde{A}_{i-1}, \tilde{A}_i\cup A, \tilde{A}_{i+1}, \cdots, \tilde{A}_k)$ is a desired $\boldsymbol{a}$-decomposition of $V(G)$.

To simplify the proof, hereafter, if $V(P)=\emptyset$ and $(P_0, \cdots, P_i)$ is a path-system, then we shall refer to $(P_0, \cdots, P_i, P)$ also as a path-system.

Let $\boldsymbol{a}'=(a_1, \cdots, a_{k-1})$, $\alpha=a_k$ and $P_{r-1}=(u_1, \cdots, u_l)$. (Note that $\boldsymbol{a}'$ is not small.) Define $s_i:=|S_i|$ for each i, $1\leqq i\leqq m-1$.

Case 1. $m\neq 3$ and $\alpha\leqq s_{m-1}+2$.

Let $A_k=R\cup\{v_{m-1}, v_m\}$ for any subset $R\subseteqq S_{m-1}$ with $|R|=\alpha-2$, and let $P'_r=(v_1, \cdots, v_{m-2})$. Further, let

$$G' = G-A_k,$$

$$\mathcal{P}' = (P_0, \cdots, P_{r-1}, P'_r).$$

From the assumption, $|N_G(x)\cap\{v_{m-1}, v_m\}|\leqq 1$ for all $x\in S$. We can therefore apply induction to the triple $(G', \boldsymbol{a}', \mathcal{P}')$ to obtain an $\boldsymbol{a}'$-decomposition $(A_1, \cdots, A_{k-1})$ of $V(G')$, for which, as noted, we soon have a desired $\boldsymbol{a}$-decomposition $(A_1, \cdots, A_k)$.

Case 2. $m\neq 3$ and $\alpha\geqq s_{m-1}+4$.

We assign $A=S_{m-1}\cup\{v_{m-1}, v_m\}$ to A_k, and let

$$\tilde{G} = G-A,$$

$$\tilde{\boldsymbol{a}} = (a_1, \cdots, a_{k-1}, a_k-s_{m-1}-2),$$

$$\tilde{\mathcal{P}} = (P_0, \cdots, P_{r-1}, \tilde{P}_r),$$

where $\tilde{P}_r=(v_1, \cdots, v_{m-2})$. Since $\tilde{\boldsymbol{a}}$ is non-singular and also $|N_G(x)\cap\{v_{m-1}, v_m\}|\leqq 1$ for all $x\in S$, here we can apply induction to $(\tilde{G}, \tilde{\boldsymbol{a}}, \tilde{\mathcal{P}})$ to obtain an $\tilde{\boldsymbol{a}}$-decomposition $(\tilde{A}_1, \cdots, \tilde{A}_k)$ of $V(\tilde{G})$. Then $(\tilde{A}_1, \cdots, \tilde{A}_{k-1}, \tilde{A}_k\cup A)$ is a desired $\tilde{\boldsymbol{a}}$-decomposition.

Case 3. $m\neq 2, 4$ and $\alpha=s_{m-1}+3$.

Let $A_k=S_{m-1}\cup\{v_{m-2}, v_{m-1}, v_m\}$, and let $P'_r=(v_1, \cdots, v_{m-3})$. Recalling $S_m=\emptyset$, we certainly have $|N_G(x)\cap\{v_{m-2}, v_{m-1}, v_m\}|\leqq 1$ for all $x\in S$. The same argument as in Case 1 applies.

Case 4. $m=4$ and $\alpha=s_3+3$.

We may assume $a_2=\cdots=a_{k-1}=\alpha$, for otherwise using such a_i $(\neq\alpha)$ instead of α, we can reduce this to Case 1 or Case 2. Now suppose $k=2$. Then for any $x\in S$, since $|N_G(x)\cap\{v_2, v_3, v_4\}|\leqq 1$, $N_G(x)\cap(W_{r-1}\cup\{v_1\})\neq\emptyset$; thus for $A=S_3\cup\{v_2, v_3, v_4\}$, we can take $(V(G)-A, A)$ as an $\boldsymbol{a}$-decomposition. So we may now assume $k\geqq 3$ as well. On the other hand, suppose there exists some $v\in S_3-S_1$. Then for such v, letting $A_k=(S_3-\{v\})\cup V(P_r)$, we can apply induction, as in Case 1, to $(G-A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-1}))$. (Note that $|N_G(x)\cap V(P_r)|\leqq 1$ for all $x\in(S-S_3)\cup\{v\}$.) Hence we may also assume $S_3\subseteqq S_1$ here. We now distinguish four subcases.

Subcase 4.1. $s_3 \geqq 2$ (i.e., $\alpha \geqq 5$) and $l \neq 3$.

For any $v \in S_3$, let $A_k = (S_3 - \{v\}) \cup \{v_3, v_4\} \cup \{u_{l-1}, u_l\}$ and $A = \{v, v_1, v_2\}$. Let also $\tilde{\boldsymbol{a}} = (a_1, \cdots, a_{k-2}, a_{k-1} - 3)$ and $\tilde{P}_{r-1} = (u_1, \cdots, u_{l-2})$. By (PS5) and (PS6), $|N_G(x) \cap (V(P_r) \cup \{u_{l-1}, u_l\})| \leqq 1$ for all $x \in S - S_3$. Therefore, applying induction to $(G - (A_k \cup A), \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-2}, \tilde{P}_{r-1}))$, we obtain an $\tilde{\boldsymbol{a}}$-decomposition $(\tilde{A}_1, \cdots, \tilde{A}_{k-1})$ of $V(G) - (A_k \cup A)$, for which we have an $\boldsymbol{a}$-decomposition $(\tilde{A}_1, \cdots, \tilde{A}_{k-2}, \tilde{A}_{k-1} \cup A, A_k)$.

Subcase 4.2. $s_3 \geqq 2$ and $l = 3$.

Let $A_k = (S_3 - \{u, v\}) \cup \{v_3, v_4\} \cup V(P_{r-1})$ for any $u, v \in S_3$, and $P'_r = (v_1, v_2)$. By (PS5), (PS6) and (PS8), we have $|N_G(x) \cap (\{v_3, v_4\} \cup V(P_{r-1}))| \leqq 1$ for all $x \in S$, and also $N_G(v_1) \cap V(P_{r-1}) = \emptyset$, implying $N_G(v_1) \cap W_{r-2} \neq \emptyset$. It is quite easy to see that we can now apply induction to $(G - A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-2}, P'_r))$.

Subcase 4.3. $s_3 = 1$ (i.e., $\alpha = 4$).

First suppose $a_1 \leqq s_1 + s_2 + 4$. Then let $A_1 = R \cup V(P_r)$ for any $R \subseteqq S_1 \cup S_2$ with $S_3 \subseteqq R$ and $|R| = a_1 - 4$. As before, noting $|N_G(x) \cap V(P_r)| \leqq 1$ for all $x \in S - R$, we can apply induction to $(G - A_1, \boldsymbol{a}'', (P_0, \cdots, P_{r-1}))$, where $\boldsymbol{a}'' = (a_2, \cdots, a_k)$. (Note that $\boldsymbol{a}''$ may be small, while $(P_0, \cdots, P_{r-1})$ is a path-system of degree k.) Next suppose $a_1 \geqq s_1 + s_2 + 5$. In this case, assign $A = S_1 \cup S_2 \cup \{v_1, v_2\}$ to A_1 and $A' = \{v_3, v_4\}$ to A_2, and also let $\tilde{\boldsymbol{a}} = (a_1 - s_1 - s_2 - 2, a_2 - 2, a_3, \cdots, a_k)$. Here, $N_G(x) \cap V(P_r) = \emptyset$ for all $x \in S - (S_1 \cup S_2)$. We thus now apply induction to $(G - (A \cup A'), \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-1}))$, obtaining an $\tilde{\boldsymbol{a}}$-decomposition $(\tilde{A}_1, \cdots, \tilde{A}_k)$ of $V(G) - (A \cup A')$. Then $(\tilde{A}_1 \cup A, \tilde{A}_2 \cup A', \tilde{A}_3, \cdots, \tilde{A}_k)$ is a desired $\boldsymbol{a}$-decomposition.

Subcase 4.4. $s_3 = 0$ (i.e., $\alpha = 3$).

We may assume $a_1 \geqq 7$, for otherwise $n \leqq 6 + 3(k-1) = 3k + 3$, contradicting $n \geqq |V(P_0)| + |V(P_1)| + |V(P_r)| \geqq 3k + 5$ (see (PS11)). We assign $A = \{v_3, v_4\}$ to A_1, and thereby let $\tilde{\boldsymbol{a}} = (a_1 - 2, a_2, \cdots, a_k)$ and $\tilde{P}_r = (v_1, v_2)$. (Note that $\tilde{\boldsymbol{a}}$ is not small.) Here, $N_G(x) \cap \{v_3, v_4\} = \emptyset$ for all $x \in S$. Apply induction to $(G - A, \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-1}, \tilde{P}_r))$.

Case 5. $m = 2$ and $\alpha = s_1 + 3$.

We may assume $a_2 = \cdots = a_{k-1} = \alpha$ as in Case 4. We distinguish three subcases.

Subcase 5.1. $s_1 \geqq 2$ (i.e., $\alpha \geqq 5$) and $l \neq 3$.

Let $A_k = (S_1 - \{v\}) \cup V(P_r) \cup \{u_{l-1}, u_l\}$ for any $v \in S_1$, and $P'_{r-1} = (u_1, \cdots, u_{l-2})$. As before, $|N_G(x) \cap (V(P_r) \cup \{u_{l-1}, u_l\})| \leqq 1$ for all $x \in S$, whence apply induction to $(G - A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-2}, P'_{r-1}))$.

Subcase 5.2. $s_1 \geqq 2$ and $l = 3$.

For all $x \in S_1$, by (PS5) and (PS6), $N_G(x) \cap \{u_2, u_3, v_2\} = \emptyset$, implying $N_G(x) \cap (W_{r-2} \cup \{u_1\}) \neq \emptyset$ (recall $k \geqq 2$). So, if $k = 2$, then letting $A = (S_1 - \{v\}) \cup V(P_r) \cup \{u_2, u_3\}$ for any $v \in S_1$, we can take, as required, $(V(G) - A, A)$ as an $\boldsymbol{a}$-decomposition. If $k \geqq 3$, then we let $A_k = (S_1 - \{u, v\}) \cup V(P_r) \cup V(P_{r-1})$ for any distinct

$u, v\in S_1$. Here, $N_G(u_1)\cap S=\emptyset$ by (PS8); hence $|N_G(x)\cap(V(P_r)\cup V(P_{r-1}))|\leqq 1$ for all $x\in S$. Now apply induction to $(G-A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-2}))$.

Subcase 5.3. $s_1\leqq 1$ (i.e., $\alpha\leqq 4$).

As in the latter part of Subcase 4.3, assign $A=S_1\cup V(P_r)$ to A_1 and let $\tilde{\boldsymbol{a}}=(a_1-s_1-2, a_2, \cdots, a_k)$. Then apply induction to $(G-A, \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-1}))$.

Case 6. $m=3$ and $\alpha\geqq s_2+5$.

If $S_1\neq\emptyset$, then assign $A=S_2\cup\{v_2, v_3\}$ to A_k and apply induction to $(G-A, \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-1}, \tilde{P}_r))$, where $\tilde{\boldsymbol{a}}=(a_1, \cdots, a_{k-1}, a_k-s_2-2)$ and $\tilde{P}_r=(v_1, v)$ for some $v\in S_1$. If $S_1=\emptyset$, on the other hand, then assign $A=S_2\cup V(P_r)$ to A_k and apply induction to $(G-A, \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-1}))$, where $\tilde{\boldsymbol{a}}=(a_1, \cdots, a_{k-1}, a_k-s_2-3)$.

Case 7. $m=3$ and $3\leqq\alpha\leqq s_2+3$.

Let $A_k=R\cup V(P_r)$ for any $R\subseteqq S_2$ with $|R|=\alpha-3$. Then apply induction to $(G-A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-1}))$.

Case 8. $m=3$ and $\alpha\in\{2, s_2+4\}$.

If $a_1\geqq s_1+s_2+5$, then by assigning $A=S_1\cup S_2\cup V(P_r)$ to A_1, we can apply induction to $(G-A, \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-1}))$, where $\tilde{\boldsymbol{a}}=(a_1-s_1-s_2-3, a_2, \cdots, a_k)$. Also, if $a_1\leqq s_1+s_2+3$, then by letting $A_1=R\cup V(P_r)$ for any $R\subseteqq S_1\cup S_2$ with $|R|=a_1-3$, we can apply induction, as before, to $(G-A_1, \boldsymbol{a}'', (P_0, \cdots, P_{r-1}))$, where $\boldsymbol{a}''=(a_2, \cdots, a_k)$. Thus, in what follows, we are only concerned with the case $a_1=s_1+s_2+4$. Note that if $a_i\notin\{2, s_2+4\}$ for some i, $2\leqq i\leqq k-1$, then as before, we may use such a_i for α, reducing this to Case 6 or Case 7. So we shall assume $a_i\in\{2, s_2+4\}$ for $2\leqq i\leqq k$. However, if $a_2=\cdots=a_k=2$, then $n=2(k-1)+a_1$, contradicting the following:

$$
\begin{aligned}
n &= |V(P_0)|+|V(P_r)|+|S_1|+|S_2| \\
&\geqq (2k+1)+3+s_1+s_2 \geqq 2k+a_1 .
\end{aligned}
$$

Accordingly, we may assume, in particular, $\alpha=s_2+4$. Now note that by this, if $S_1\neq\emptyset$, then as in Case 6, by taking $\tilde{P}_r=(v_1, v)$ for some $v\in S_1$, the same assignment is still in effect. Hence we now assume $S_1=\emptyset$ as well, which readily implies that $a_1=s_2+4$ along with $s_2>0$ (since $5\leqq a_1=s_1+s_2+4$). Here we again distinguish three subcases.

Subcase 8.1. $l\neq 3$.

Let $A_k=(S_2-\{v\})\cup V(P_r)\cup\{u_{l-1}, u_l\}$ for any $v\in S_2$, and $P'_{r-1}=(u_1, \cdots, u_{l-2})$. Then apply induction to $(G-A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-2}, P'_{r-1}))$.

Subcase 8.2. $l=3$ and $s_2\geqq 2$ (i.e., $\alpha\geqq 6$).

Let $A_k=(S_2-\{u, v\})\cup V(P_r)\cup V(P_{r-1})$ for any $u, v\in S_2$. Then apply induction to $(G-A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-2}))$.

Subcase 8.3. $l=3$ and $s_2=1$ (i.e., $\alpha=5$).

Note that in this case, $a_1=a_k=5$. So, if $k=2$, then $\boldsymbol{a}=(5, 5)$, and hence $n=10$, which is impossible since by (PS11), $n\geqq|V(P_0)|+|V(P_{r-1})|+|V(P_r)|\geqq 2k$

$+7=11$. We thus suppose $k \geqq 3$. Now, by (PS5) and (PS8), $N_G(x) \cap \{u_1, u_3\} = \emptyset$ for all $x \in S$. Therefore, if $N_G(u_2) \cap S = \emptyset$, then assign $A = V(P_{r-1})$ to A_k and apply induction to $(G-A, \tilde{\boldsymbol{a}}, (P_0, \cdots, P_{r-2}, P_r))$, where $\tilde{\boldsymbol{a}} = (a_1, \cdots, a_{k-1}, a_k - 3)$. (Note that as observed in Subcase 4.2, $N_G(v_1) \cap W_{r-2} \neq \emptyset$.) If $N_G(u_2) \cap S \neq \emptyset$, on the other hand, then let $A_k = V(P_r) \cup \{u_2, u_3\}$ if $d_G(u_1) \geqq k$, or $A_k = V(P_r) \cup \{u_1, u_2\}$ otherwise. In either case, by (PS9) and (PS10), W_{r-2} is $(k-1)$-dominating in $G - A_k$; so we can apply induction to $(G - A_k, \boldsymbol{a}', (P_0, \cdots, P_{r-2}))$.

This completes the proof of Proposition 3. □

§5. Proof of Lemma 4.

PROOF OF LEMMA 4. Let $P_0 = (u_1, \cdots, u_l)$ and $P_1 = (v_1, \cdots, v_m)$, and define $S := V(G) - (V(P_0) \cup V(P_1))$. Note that $l \geqq m \geqq 2$. Also, define $F := N_G(v_1) \cap V(P_0)$; without loss of generality, we may assume $F \neq \emptyset$. For any $u_\lambda \in F$, by the maximality of P_0, we have $\lambda > m$ and $l - \lambda \geqq m$, and consequently $l \geqq 2m+1$. So if $m \geqq k$, then the conclusion is immediate. We thus assume $m < k$ here, by which it suffices to show $l + m \geqq 3k+1$. We next remark that we may also assume $v_1 \in V_{\text{major}}$. To see this, suppose $v_1 \in V_{\text{minor}}$. Then we can always take v_m as a major vertex. (If $v_m \in V_{\text{minor}}$, consider by its index the first major vertex on P_1, say v_{i_0}. Since all the minor vertices are adjacent, we may use the path $(v_1, \cdots, v_{i_0-1}, v_m, \cdots, v_{i_0})$ for P_1 with its endvertex $v_{i_0} \in V_{\text{major}}$.) Accordingly, $N_G(v_m) \cap S = \emptyset$, implying that $N_G(v_m) \cap V(P_0) \neq \emptyset$ (notice $m < k$). By reversing P_1, we may now use the path $(v_m, \cdots, v_1)$ for P_1 having its initial endvertex $v_m \in V_{\text{major}}$. Thus, in the following, we also assume $v_1 \in V_{\text{major}}$.

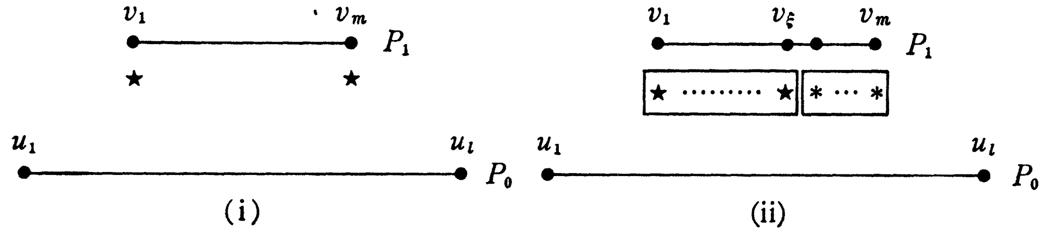

Figure 2. (i) $\xi = m$ and (ii) $\xi < m$ (★: major vertex, $*$: minor vertex).

Now set $\xi := \max\{i \mid v_i \in V_{\text{major}}\}$. Principally, we distinguish two cases as to whether (i) $\xi = m$ or (ii) $\xi < m$ (see Figure 2). We here note that if $\xi < m$, then we may assume $v_i \in V_{\text{major}}$ for $1 \leqq i \leqq \xi$ and $v_i \in V_{\text{minor}}$ for $\xi + 1 \leqq i \leqq m$ (for otherwise there must be some $v_i \in V_{\text{major}}$ $(2 < i \leqq m-1)$ with $v_{i-1} \in V_{\text{minor}}$, and for such v_i, we can take the path $(v_1, \cdots, v_{i-1}, v_m, \cdots, v_i)$ for P_1, reducing this to the first case $\xi = m$). We also note that in either case, $N_G(v_\xi) \cap S = \emptyset$. (For (i), it soon follows from the maximality of P_1. For (ii), since $S \cap V_{\text{minor}} = \emptyset$ (i.e., $S \subseteq V_{\text{major}}$), by the choice of P_1, it again follows.) Thus, defining $L := N_G(v_\xi)$

$\cap V(P_0)$, we have $L\neq\emptyset$, from which we see also $N_G(v_1)\cap S=\emptyset$ by a similar argument. (That is, $N_G(v_i)\subseteqq V(P_0)\cup V(P_1)$ for $i=1$, ξ.) Now, define further $I:=F\cap L$ and $\gamma:=|I|$, and for a subset X of $V(P_0)$, let $X^{(-i)}$ denote $\{u_{j-i}\,|\,u_j\in X(i\leqq j)\}$. By the maximality of P_0, we observe that: $F-I$, $(F-I)^{(-1)}$, $L-I$, $(L-I)^{(-1)}$, I, $I^{(-1)}$, $\cdots$, $I^{(-\xi)}$ are mutually disjoint. Let here

$$\begin{cases} H=(F-I)\cup(F-I)^{(-1)}\cup(L-I)\cup(L-I)^{(-1)}\,; \\ K=I\cup I^{(-1)}\cup\cdots\cup I^{(-\xi)}. \end{cases}$$

Also, let $F\cup L=\{u_{\lambda_1},\cdots,u_{\lambda_r}\}$ with $\lambda_1<\cdots<\lambda_r$. For later use, for the case $I=\emptyset$, we show that $l+m\geqq 4k$ ($\geqq 3k+1$).

(i) $\xi=m$.

From the above, clearly $|F|\geqq k-m+1$ ($\geqq 2$) and $|L|\geqq k-m+1$ ($\geqq 2$), and by the maximality of P_0, $\lambda_1>m$ and $l-\lambda_r\geqq m$.

Case 1. $I=\emptyset$.

Since $F\neq\emptyset$ and $L\neq\emptyset$, we can take some u_{λ_j}, $1\leqq j<r$, such that "$u_{\lambda_j}\in F$ and $u_{\lambda_{j+1}}\in L$" or "$u_{\lambda_j}\in L$ and $u_{\lambda_{j+1}}\in F$". In either case, by the maximality of P_0, $\lambda_{j+1}-\lambda_j>m$. The maximality also implies that for such u_{λ_j},

$$H\cap\{u_1,\cdots,u_{\lambda_1-2},u_{\lambda_j+1},\cdots,u_{\lambda_{j+1}-2},u_{\lambda_r+1},\cdots,u_l\}=\emptyset,$$

whence

$$\begin{aligned} l+m &\geqq 2|F|+2|L|+\{(\lambda_1-2)+(\lambda_{j+1}-\lambda_j-2)+(l-\lambda_r)\}+m \\ &\geqq 4(k-m+1)+\{2(m-1)+m\}+m=4k+2. \end{aligned}$$

Case 2. $I\neq\emptyset$.

By the maximality of P_0,

$$(H\cup K)\cap\{u_{\lambda_r+1},\cdots,u_l\}=\emptyset.$$

If $\gamma\geqq k-1$, then disregarding H in the above, we have

$$\begin{aligned} l+m &\geqq (m+1)|I|+(l-\lambda_r)+m \\ &\geqq (m+1)\gamma+2m\geqq 3k+1. \end{aligned}$$

On the other hand, if $0<\gamma<k-1$, then

$$\begin{aligned} l+m &\geqq 2|F-I|+2|L-I|+(m+1)|I|+(l-\lambda_r)+m \\ &\geqq 4(k-m+1-\gamma)+(m+1)\gamma+2m \\ &=3k+(k-m+1)+(m-3)(\gamma-1). \end{aligned}$$

Now, if $m=2$,

$$l+m\geqq 3k+(k-\gamma)\geqq 3k+2\,;$$

otherwise

$$l+m \geqq 3k+(k-m+1) \geqq 3k+2.$$

(ii) $\xi<m$.

Note that if $v_1v_i \in E(G)$, $\xi+2 \leqq i \leqq m$, then we can take the path $(v_1, v_i, \cdots, v_m, v_{i-1}, \cdots, v_2)$ for P_1, reducing this to (i). So we assume $N_G(v_1) \cap V(P_1) \subseteq \{v_2, \cdots, v_{\xi+1}\}$; thus, $|F| \geqq k-\xi$ and $|L| \geqq k-m+1$ in this case. (Note that in (ii), $m>\xi \geqq 2$.)

Case 1. $I=\emptyset$.

Suppose first $u_{\lambda_1} \in F$ or $u_{\lambda_r} \in F$; without loss of generality, we may assume $u_{\lambda_1} \in F$ (reverse P_0 if necessary). By the maximality of P_0, $\lambda_1>m$ and $l-\lambda_r \geqq \xi$. As before, take any u_{λ_j}, $1 \leqq j<r$, with "$u_{\lambda_j} \in F$ and $u_{\lambda_{j+1}} \in L$" or "$u_{\lambda_j} \in L$ and $u_{\lambda_{j+1}} \in F$". Then again by the maximality, $\lambda_{j+1}-\lambda_j>\xi$, and also

$$H \cap \{u_1, \cdots, u_{\lambda_1-2}, u_{\lambda_j+1}, \cdots, u_{\lambda_{j+1}-2}, u_{\lambda_r+1}, \cdots, u_l\} = \emptyset .$$

Here we have

$$(\lambda_1-2)+(\lambda_{j+1}-\lambda_j-2)+(l-\lambda_r) \geqq m+2\xi-2 .$$

On the other hand, suppose $u_{\lambda_1} \notin F$ and $u_{\lambda_r} \notin F$. Then by the maximality, $\lambda_1>\xi$ and $l-\lambda_r \geqq m-\xi+1$. Let u_{λ_a} be, by its index, the first vertex of F, and u_{λ_b} the last. (Note that $\lambda_1<\lambda_a<\lambda_b<\lambda_r$.) Then $\lambda_a-\lambda_{a-1}>\xi$, $\lambda_{b+1}-\lambda_b>\xi$, and

$$H \cap \{u_1, \cdots, u_{\lambda_1-2}, u_{\lambda_{a-1}+1}, \cdots, u_{\lambda_a-2}, u_{\lambda_b+1}, \cdots, u_{\lambda_{b+1}-2}, u_{\lambda_r+1}, \cdots, u_l\} = \emptyset .$$

Here

$$\begin{aligned}&(\lambda_1-2)+(\lambda_a-\lambda_{a-1}-2)+(\lambda_{b+1}-\lambda_b-2)+(l-\lambda_r)\\ &\geqq 3(\xi-1)+(m-\xi+1) = m+2\xi-2 .\end{aligned}$$

Therefore, in either case,

$$\begin{aligned} l+m &\geqq 2|F|+2|L|+(m+2\xi-2)+m \\ &\geqq 2(k-\xi)+2(k-m+1)+2m+2\xi-2 = 4k .\end{aligned}$$

Case 2. $I \neq \emptyset$.

Subcase 2.1. $v_1v_{\xi+1} \notin E(G)$.

In this case, $|F| \geqq k-\xi+1$ (and $|L| \geqq k-m+1$). Now, if $u_{\lambda_r} \in F$, then by the maximality, as before, $l-\lambda_r \geqq m$, and also

$$(H \cup K) \cap \{u_{\lambda_r+1}, \cdots, u_l\} = \emptyset .$$

If $u_{\lambda_r} \notin F$, then $l-\lambda_r \geqq m-\xi+1$, and for the last vertex u_{λ_b} of F,

$$(H \cup K) \cap \{u_{\lambda_b+1}, \cdots, u_{\lambda_{b+1}-2}, u_{\lambda_r+1}, \cdots, u_l\} = \emptyset .$$

Here, $\lambda_{b+1}-\lambda_b>\xi$, and so $(\lambda_{b+1}-\lambda_b-2)+(l-\lambda_r) \geqq m$.

Thus, as in Case 2 of (i), if $\gamma \geqq k-1$, then

$$\begin{aligned} l+m &\geqq (\xi+1)|I|+m+m \\ &\geqq (\xi+1)\gamma+2m \geqq 3k+3. \end{aligned}$$

If $0<\gamma<k-1$, then

$$\begin{aligned} l+m &\geqq 2|F-I|+2|L-I|+(\xi+1)|I|+m+m \\ &\geqq 2(k-\xi+1-\gamma)+2(k-m+1-\gamma)+(\xi+1)\gamma+2m \\ &= 3k+(k-\xi+1)+(\xi-3)(\gamma-1). \end{aligned}$$

So, if $\xi=2$,

$$l+m \geqq 3k+(k-\gamma) \geqq 3k+2;$$

otherwise

$$l+m \geqq 3k+(k-\xi+1) \geqq 3k+3.$$

Subcase 2.2. $v_1 v_{\xi+1} \in E(G)$.

In this case, we may assume $v_\xi v_i \notin E(G)$ for $\xi+2 \leqq i \leqq m$, since otherwise by taking $(v_1, v_{\xi+1}, \cdots, v_{i-1}, v_m, \cdots, v_i, v_\xi, \cdots, v_2)$ for P_1, we can again reduce this to (i). Thus, $N_G(v_\xi) \cap V(P_1) \subseteqq \{v_1, \cdots, v_{\xi-1}, v_{\xi+1}\}$, so that $|L| \geqq k-\xi$. On the other hand, the maximality always requires $\lambda_1 > m$ and $l-\lambda_r \geqq m$, for we can take another path $(v_\xi, \cdots, v_1, v_{\xi+1}, \cdots, v_m)$ of order m. As before, it is easy to see

$$(H \cup K) \cap \{u_1, \cdots, u_{\lambda_1-\xi-1}, u_{\lambda_r+1}, \cdots, u_l\} = \emptyset.$$

Now, if $\gamma \geqq k-1$, then

$$\begin{aligned} l+m &\geqq (\xi+1)|I|+\{(\lambda_1-\xi-1)+(l-\lambda_r)\}+m \\ &\geqq (\xi+1)\gamma+\{(m-\xi)+m\}+m \\ &\geqq (\xi+1)\gamma+2\xi+3 \geqq 3k+4. \end{aligned}$$

If $0<\gamma<k-1$, then

$$\begin{aligned} l+m &\geqq 2|F-I|+2|L-I|+(\xi+1)|I|+\{(\lambda_1-\xi-1)+(l-\lambda_r)\}+m \\ &\geqq 4(k-\xi-\gamma)+(\xi+1)\gamma+\{(m-\xi)+m\}+m \\ &\geqq 3k+(k-\xi+1)+(\xi-3)(\gamma-1)-1. \end{aligned}$$

In view of Subcase 2.1, the conclusion clearly follows.

This completes the proof of Lemma 4. □

§6. Proof of Lemma 5.

PROOF OF LEMMA 5. Let $P_0=(u_1, \cdots, u_l)$, $P_1=(v_1, \cdots, v_m)$ and $P_2=(w_1, \cdots, w_h)$, and define $S:=V(G)-\bigcup_{j=0}^{2} V(P_j)$. Note that $l \geqq m \geqq 3$ and $l \geqq h \geqq 3$. On the other hand, we may assume $h<k$, for otherwise the conclusion is immediate from Lemma 4. Now define $F_0^1:=N_G(v_1)\cap V(P_0)$ and $F_j^2:=N_G(w_1)\cap V(P_j)$, $j=0, 1$. Without loss of generality, we may assume $F_0^1\neq\emptyset$ and $F_0^2\cup F_1^2\neq\emptyset$. By taking w_1 here for v_1 in the proof of Lemma 4, we may also assume $w_1\in V_{\text{major}}$. Setting $\xi:=\max\{i|v_i\in V_{\text{major}}\}$ and $\eta:=\max\{i|w_i\in V_{\text{major}}\}$, we next define $L_0^1:=N_G(v_\xi)\cap V(P_0)$ and $L_j^2:=N_G(w_\eta)\cap V(P_j)$, $j=0, 1$. Define further $I_0^1:=F_0^1\cap L_0^1$ (and $\gamma_0^1:=|I_0^1|$) and $I_j^2:=F_j^2\cap L_j^2$ (and $\gamma_j^2:=|I_j^2|$), $j=0, 1$. As in the preceding proof, we let $F_0^1\cup L_0^1=\{u_{\lambda_1}, \cdots, u_{\lambda_r}\}$ with $\lambda_1<\cdots<\lambda_r$. Also, we use the same notation $X^{(-i)}$ here, and similarly define $Y^{(-i)}$ to be $\{v_{j-i}|v_j\in Y\ (i\leqq j)\}$ for any subset $Y\subseteqq V(P_1)$.

We split the following argument primarily into two pieces: when (I) $m<k$ and when (II) $m\geqq k$, in each of which, as before, we distinguish the two cases $\xi=m$ and $\xi<m$. Note that in either (I) or (II), when $\xi<m$, by the maximality of P_1, $\eta=h$ (since all the minor vertices are adjacent).

(I) $m<k$.

As before, we may assume $v_1\in V_{\text{major}}$. Clearly, it suffices to show $l+m\geqq 4k-2$. In the preceding proof, however, we have already observed it for the case $I_0^1\ (=I)=\emptyset$. Accordingly, in what follows, we also assume $I_0^1\neq\emptyset$ (i.e., $\gamma_0^1\neq 0$). As observed, $F_0^1-I_0^1$, $(F_0^1-I_0^1)^{(-1)}$, $L_0^1-I_0^1$, $(L_0^1-I_0^1)^{(-1)}$, I_0^1, $I_0^{1(-1)}$, $\cdots$, $I_0^{1(-\xi)}$ are mutually disjoint. Let now

$$\begin{cases} H=(F_0^1-I_0^1)\cup(F_0^1-I_0^1)^{(-1)}\cup(L_0^1-I_0^1)\cup(L_0^1-I_0^1)^{(-1)}\,; \\ K=I_0^1\cup I_0^{1(-1)}\cup\cdots\cup I_0^{1(-\xi)}. \end{cases}$$

(I-i) $\xi=m$.

We recall, by the maximality of P_1, $|F_0^1|\geqq k-m+1$ and $|L_0^1|\geqq k-m+1$, and also, by the maximality of P_0, $l-\lambda_r\geqq m$.

Case 1. $F_0^1=L_0^1\ (=I_0^1)$.

By the maximality, $K\cap\{u_{\lambda_r+1}, \cdots, u_l\}=\emptyset$; hence

$$\begin{aligned} l+m &\geqq (m+1)|I_0^1|+(l-\lambda_r)+m \\ &\geqq (m+1)(k-m+1)+2m \\ &= 4k+(m-3)\{k-(m+1)\}-2 \\ &\geqq 4k-2. \end{aligned}$$

Case 2. $F_0^1\neq L_0^1$.

Since $|F_0^1|\geqq 2$ and $|L_0^1|\geqq 2$, we observe:

(*) There exists some u_{λ_j}, $1\leqq j<r$, such that "$u_{\lambda_j}\in F_0^1$ and $u_{\lambda_{j+1}}\in L_0^1-I_0^1$" or "$u_{\lambda_j}\in L_0^1$ and $u_{\lambda_{j+1}}\in F_0^1-I_0^1$".

Then for such u_{λ_j}, $(H\cup K)\cap\{u_{\lambda_{j+1}}, \cdots, u_{\lambda_{j+1}-2}, u_{\lambda_r+1}, \cdots, u_l\}=\emptyset$ with $\lambda_{j+1}-\lambda_j>m$, so that

$$\begin{aligned} l+m &\geqq 2|F_0^1-I_0^1|+2|L_0^1-I_0^1|+(m+1)|I_0^1|+\{(m-1)+m\}+m \\ &\geqq 4(k-m+1-\gamma_0^1)+(m+1)\gamma_0^1+3m-1 \\ &= 4k+(m-3)(\gamma_0^1-1)\geqq 4k\,. \end{aligned}$$

(I-ii) $\xi<m$.

Recall that in this case, we may assume $N_G(v_1)\cap V(P_1)\subseteqq\{v_2, \cdots, v_{\xi+1}\}$. As before, we consider the two cases: when $v_1v_{\xi+1}\notin E(G)$ and when $v_1v_{\xi+1}\in E(G)$. (Note that throughout (I-ii), $m>\xi$ $(\geqq h)\geqq 3$.)

Case 1. $v_1v_{\xi+1}\notin E(G)$.

In this case, $|F_0^1|\geqq k-\xi+1$ and $|L_0^1|\geqq k-m+1$.

Subcase 1.1. $F_0^1=L_0^1$ $(=I_0^1)$.

By the maximality, $l-\lambda_r\geqq m$, and also $K\cap\{u_{\lambda_r+1}, \cdots, u_l\}=\emptyset$. Hence

$$\begin{aligned} l+m &\geqq (\xi+1)|I_0^1|+(l-\lambda_r)+m \\ &\geqq (\xi+1)(k-\xi+1)+2m\geqq(\xi+1)k-\xi^2+2\xi+3 \\ &= 4k+(\xi-3)\{k-(\xi+1)\}\geqq 4k\,. \end{aligned}$$

Subcase 1.2. $F_0^1\neq L_0^1$.

Suppose first $u_{\lambda_1}\in F_0^1$ or $u_{\lambda_r}\in F_0^1$. Without loss of generality, we may assume $u_{\lambda_r}\in F_0^1$; hence $l-\lambda_r\geqq m$. Now, if (*) holds, then as above, $(H\cup K)\cap\{u_{\lambda_{j+1}}, \cdots, u_{\lambda_{j+1}-2}, u_{\lambda_r+1}, \cdots, u_l\}=\emptyset$ with $\lambda_{j+1}-\lambda_j>\xi$. If (*) fails, which means that $u_{\lambda_1}\notin I_0^1$ and $u_{\lambda_j}\in I_0^1$ for all j, $2\leqq j\leqq r$ (notice $F_0^1\neq L_0^1$), then $(H\cup K)\cap\{u_1, \cdots, u_{\lambda_1-2}, u_{\lambda_r+1}, \cdots, u_l\}=\emptyset$ with $\lambda_1>\xi$. Thus, in either case,

$$\begin{aligned} l+m &\geqq 2|F_0^1-I_0^1|+2|L_0^1-I_0^1|+(\xi+1)|I_0^1|+\{(\xi-1)+m\}+m \\ &\geqq 2(k-\xi+1-\gamma_0^1)+2(k-m+1-\gamma_0^1)+(\xi+1)\gamma_0^1+2m+\xi-1 \\ &= 4k+(\xi-3)(\gamma_0^1-1)\geqq 4k\,. \end{aligned}$$

Suppose next $u_{\lambda_1}\notin F_0^1$ and $u_{\lambda_r}\notin F_0^1$. Then $\lambda_1>\xi$ and $l-\lambda_r\geqq m-\xi+1$. Moreover, (*) holds for the last vertex u_{λ_b} of F_0^1, for which we have $(H\cup K)\cap\{u_1, \cdots, u_{\lambda_1-2}, u_{\lambda_b+1}, \cdots, u_{\lambda_{b+1}-2}, u_{\lambda_r+1}, \cdots, u_l\}=\emptyset$ with $\lambda_{b+1}-\lambda_b>\xi$. Therefore here $(\lambda_1-2)+(\lambda_{b+1}-\lambda_b-2)+(l-\lambda_r)\geqq m+\xi-1$, ending in the same calculation as above.

Case 2. $v_1v_{\xi+1}\in E(G)$.

As observed, $l-\lambda_r\geqq m$, and we may assume $N_G(v_\xi)\cap V(P_1)\subseteqq\{v_1, \cdots, v_{\xi-1}, v_{\xi+1}\}$. In this case, hence, $|F_0^1|\geqq k-\xi$ and $|L_0^1|\geqq k-\xi$. We now remark that we may also assume $m\geqq\xi+2$, since otherwise (i.e., $m=\xi+1$) by taking $(v_1, v_m, \cdots, v_2)$ for P_1, we can reduce this to (I-i).

Subcase 2.1. $F_0^1=L_0^1$ $(=I_0^1)$.

As before, $K\cap\{u_{\lambda_r+1},\cdots,u_l\}=\emptyset$, and so

$$\begin{aligned}l+m &\geqq (\xi+1)|I_0^1|+(l-\lambda_r)+m\\ &\geqq (\xi+1)(k-\xi)+2m \geqq (\xi+1)k-\xi^2+\xi+4\\ &= 4k+(\xi-3)\{k-(\xi+2)\}-2 \geqq 4k-2.\end{aligned}$$

Subcase 2.2. $F_0^1\neq L_0^1$.

In this case, by reversing P_0 if necessary, we can always take such u_{λ_j} as in (*). Then for such u_{λ_j}, $(H\cup K)\cap\{u_{\lambda_j+1},\cdots,u_{\lambda_{j+1}-2},u_{\lambda_r+1},\cdots,u_l\}=\emptyset$ with $\lambda_{j+1}-\lambda_j>\xi$. Hence

$$\begin{aligned}l+m &\geqq 2|F_0^1-I_0^1|+2|L_0^1-I_0^1|+(\xi+1)|I_0^1|+\{(\xi-1)+m\}+m\\ &\geqq 4(k-\xi-\gamma_0^1)+(\xi+1)\gamma_0^1+2m+\xi-1\\ &\geqq 4k+(\xi-3)\gamma_0^1-\xi+3\\ &= 4k+(\xi-3)(\gamma_0^1-1) \geqq 4k.\end{aligned}$$

(II) $m\geqq k$.

As differs from the case (I), we shall explicitly show $l+m+h\geqq 4k+1$ by working with all the paths P_0, P_1 and P_2. Since $w_1, w_\eta\in V_{\mathrm{major}}$ and $h<k$, by the maximality of P_2, we have $|F_0^2\cup F_1^2|\geqq 2$ and $|L_0^2\cup L_1^2|\geqq 2$. However, if $F_1^2=\emptyset$ and $L_1^2=\emptyset$, then the same argument as in the proof of Lemma 4 to the paths P_0 and P_2 gives $l+h\geqq 3k+1$, bringing us to the conclusion. (This can be observed since in the preceding proof, we have only been concerned with the degrees of v_1 and v_ξ.) Therefore, in what follows, we assume $F_1^2\cup L_1^2\neq\emptyset$, and thereby let $v_p\in F_1^2\cup L_1^2$. Note that in (II) we cannot determine whether $v_1\in V_{\mathrm{major}}$ or $v_1\in V_{\mathrm{minor}}$.

(II-i) $\xi=m$.

We first note that we may assume $|L_0^1|\leqq 1$. (If not, then we can take some distinct two vertices $u_\alpha\in F_0^1$ and $u_\beta\in L_0^1$ (we may assume $\alpha<\beta$); by the maximality, each of the subpaths $(u_1,\cdots,u_{\alpha-1})$, $(u_{\alpha+1},\cdots,u_{\beta-1})$ and $(u_{\beta+1},\cdots,u_l)$ must have order at least m, showing that $l+m\geqq 4m+2\geqq 4k+2$.) Now, define $L_1^1:=N_G(v_m)\cap V(P_1)$, and let $L_1^1=\{v_{\zeta_1},\cdots,v_{\zeta_s}\}$ with $\zeta_1<\cdots<\zeta_s$. Since $|L_1^1|\geqq k-|L_0^1|\geqq k-1$, we have $\zeta_1\leqq m-k+1$, $m-\zeta_s\leqq m-k+1$ and $\zeta_{j+1}-\zeta_j<m-k+1$ for $1\leqq j\leqq s-1$.

Case 1. $\eta\geqq k/3$.

We assume $v_p\in L_1^2$ here, since the argument for the case $v_p\notin L_1^2$ (i.e., $v_p\in F_1^2$) results in essentially the same. First suppose $p>\zeta_1$. Then clearly:

(**) Either there exists some ζ_j $(1\leqq j<s)$ satisfying $\zeta_j<p\leqq\zeta_{j+1}$, or $\zeta_s<p$.

Now, for such ζ_j, consider the path $P=(v_1, \cdots, v_{\zeta_j}, v_m, \cdots, v_p, w_\eta, \cdots, w_1)$ of order at least $\{m-(m-k)\}+\eta=k+\eta$. Then by the maximality of P_0,

$$\begin{aligned} l+m+h &\geqq (2|P|+1)+m+\eta \\ &\geqq 2k+m+3\eta+1 \geqq 4k+1. \end{aligned}$$

Next suppose $p\leqq\zeta_1$. If $F_0^2\neq\emptyset$, then by taking the path $P=(w_1, \cdots, w_\eta, v_p, \cdots, v_m)$, we can conclude with the same calculation as above (since $|P|\geqq\eta+k$). So now further suppose $F_0^2=\emptyset$. Then $|F_1^2|\geqq 2$, and hence we can take some $v_q\in F_1^2$ distinct from v_p. Now, if $q>\zeta_1$, then by interchanging the roles of w_1 and w_η in the above, we are done. On the other hand, if $q\leqq\zeta_1$, take as P, $(v_1, \cdots, v_p, w_\eta, \cdots, w_1, v_q, \cdots, v_m)$ when $p<q$, or $(v_1, \cdots, v_q, w_1, \cdots, w_\eta, v_p, \cdots, v_m)$ when $p\geqq q$. Since $|P|\geqq k+\eta+1$ in either case, the conclusion again follows.

Case 2. $\eta<k/3$.

We first claim we may assume $|F_0^2\cup F_1^2|\geqq 2k/3+1$. Since $|F_0^2\cup F_1^2|\geqq k-h+1$, this is true when $\eta=h-1$ or $\eta=h$. So suppose $\eta<h-1$. Now, if $w_1w_i\in E(G)$ for some $\eta+1<i\leqq h$, then we can take the path $(w_1, w_i, \cdots, w_h, w_{i-1}, \cdots, w_2)$ of order h with its both endvertices $w_1, w_2\in V_{\text{major}}$, which is the very case $\eta=h$. If not (i.e., $N_G(w_1)\cap V(P_2)\subseteqq\{w_2, \cdots, w_{\eta+1}\}$), then clearly $|F_0^2\cup F_1^2|\geqq 2k/3+1$. The claim is thus verified. We now recall that $\mu_i=|V(P_i)\cap V_{\text{major}}|\geqq 2$ ($i=1, 2$) and that P_1 is taken in $G-V(P_0)$ such that μ_1 is as large as possible. So, $F_1^{2(-2)}$, $F_1^{2(-1)}$, $(L_1^1\cup\{v_m\})$ must be mutually disjoint; thus, $m\geqq 2|F_1^2|+(|L_1^1|+1)\geqq 2|F_1^2|+k$, implying that $|F_1^2|\leqq(m-k)/2<k/6$. Consequently, we have $|F_0^2|\geqq k/2+1$. We again assume $v_p\in L_1^2$ here; however, we see the conclusion also for the case $v_p\notin L_1^2$ by simply replacing the subpath $(w_1, \cdots, w_\eta)$ or $(w_\eta, \cdots, w_1)$ by w_1 in each P below. As in Case 1, if $p>\zeta_1$, then take $P=(v_1, \cdots, v_{\zeta_j}, v_m, \cdots, v_p, w_\eta, \cdots, w_1)$, otherwise $P=(w_1, \cdots, w_\eta, v_p, \cdots, v_m)$. Here, as observed, $|P|\geqq\eta+k$ in either case. Now let u_f be (by its index) the first vertex of F_0^2, and u_g the last. Then by the maximality of P_0, $f>|P|$ and $l-g\geqq|P|$. The maximality also implies that F_0^2, $F_0^{2(-1)}$, $\{u_1, \cdots, u_{f-2}, u_{g+1}, \cdots, u_l\}$ are mutually disjoint; thus

$$\begin{aligned} l+m+h &\geqq (2|F_0^2|+2|P|-1)+m+h \\ &\geqq 3k+m+3\eta+1 \geqq 4k+7. \end{aligned}$$

(II-ii) $\xi<m$.

Recall that in this case, $\eta=h$ (i.e., $w_h\in V_{\text{major}}$), and also that $L_0^2\cup L_1^2\neq\emptyset$.

Case 1. $F_0^2=\emptyset$ or $L_0^2=\emptyset$.

Without loss of generality, we may assume $F_0^2=\emptyset$. Then $|F_1^2|\geqq k-h+1$ (>0). Let now $v_{g'}$ be the last vertex of F_1^2. Then by the choice of P_1. $\xi-g'\geqq h$; hence

$$|P_1| \geqq (2|F_1^2|-1)+h+(m-\xi) \geqq 2k-h+2,$$

so that

$$
\begin{aligned}
l+m+h &\geqq (2m+1)+|P_1|+h \\
&\geqq (2m+1)+2k-h+2+h \\
&\geqq 4k+3.
\end{aligned}
$$

Case 2. $F_1^2 \neq \emptyset$ and $L_0^2 \neq \emptyset$.

Suppose first that $|F_0^2| \leqq |F_1^2|$ or $|L_0^2| \leqq |L_1^2|$; we may here assume $|F_0^2| \leqq |F_1^2|$. Let $v_{f'}$ be the first vertex of F_1^2. Then for the path $P=(w_1, v_{f'}, \cdots, v_m)$, as above,

$$|P| \geqq |\{w_1\}|+(2|F_1^2|-1)+h+(m-\xi) \geqq 2|F_1^2|+h+1,$$

whence by the maximality,

$$
\begin{aligned}
l &\geqq 2|F_0^2|+2|P|-1 \geqq 2(|F_0^2|+|F_1^2|)+2|F_1^2|+2h+1 \\
&\geqq 3(|F_0^2|+|F_1^2|)+2h+1 \geqq 3(k-h+1)+2h+1 \\
&= 3k-h+4.
\end{aligned}
$$

Suppose next that $|F_0^2|>|F_1^2|$ and $|L_0^2|>|L_1^2|$. Then clearly $|F_0^2| \geqq (k-h+1)/2$ and $|L_0^2| \geqq (k-h+1)/2$. As remarked, $F_1^2 \cup L_1^2 \neq \emptyset$ in (II); in particular, we may assume $L_1^2 \neq \emptyset$ with $v_p \in L_1^2$. Now, if $p \geqq m/2$, then consider the path $(w_1, \cdots, w_h, v_p, \cdots, v_1)$, otherwise $(w_1, \cdots, w_h, v_p, \cdots, v_m)$. In either case, the path has order at least $h+m/2$. Therefore by a similar argument to that we have applied in the proof of Lemma 4 or in (I), it soon follows that

$$
\begin{aligned}
l &\geqq 2|F_0^2-I_0^2|+2|L_0^2-I_0^2|+\{(h+m/2)-h\}+(h+m/2) \\
&\geqq 4\{(k-h+1)/2-\gamma_0^2\}+(h+1)\gamma_0^2+m+h \\
&\geqq 2k+m+(h-3)\gamma_0^2-h+2 \geqq 3k-h+2.
\end{aligned}
$$

The above argument together with the assumption $m \geqq k$ readily leads to the conclusion.

This completes the proof of Lemma 5. □

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