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Ineffability and partition property on $\mathcal{P}_\kappa\lambda$

By Shizuo KAMO

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1. Introduction.

Magidor [11] proved that if part*(κ, λ) holds, then κ is λ-ineffable. Abe [1] proved that the reverse implication also holds under the assumption of that λ is ineffable. In this paper, we shall prove the following two theorems.

THEOREM 1. *If κ is completely $\lambda^{<\kappa}$-ineffable, then* part*$(\kappa, \lambda^{<\kappa})$ *holds.*

THEOREM 2. *Assume that there exists an $\alpha<\kappa$ such that $2^\delta \leqq \delta^{+\alpha}$ for all $\delta<\kappa$. Then, if κ is $\lambda^{<\kappa}$-ineffable, then* part*$(\kappa, \lambda^{<\kappa})$ *holds.*

In order to prove Theorem 1, we need to study a hierarchy of ideals which are associated with partition property and ineffability, and the correspondence between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa\lambda^{<\kappa}$.

The hierarchy of ideals will be dealt in sections 4 and 5 and the correspondence in section 6. The theorems will be proved in section 7.

2. Notation and basic facts.

Throughout this paper, κ denotes a regular uncountable cardinal. Let $\mathcal{I}$ be an ideal on a set S. $\mathcal{I}^*$ denotes the dual filter of $\mathcal{I}$ and $\mathcal{I}^+$ the set $\mathcal{P}(S)\setminus\mathcal{I}$. A subset W of $\mathcal{I}^+$ is *$\mathcal{I}$-disjoint,* if $X\cap Y\in\mathcal{I}$ for all distinct $X, Y\in W$. An $\subset$-maximal $\mathcal{I}$-disjoint subset is called an *$\mathcal{I}$-partition.* For any set $X\subset S$, $\mathcal{I}^+|X$ denotes $\mathcal{I}^+\cap\mathcal{P}(X)$. For any $f: S\to T$, $f_*(\mathcal{I})$ denotes the ideal $\{Y\subset T \mid f^{-1}Y\in\mathcal{I}\}$ on T.

Let A be a set such that $\kappa\subset A$. $\mathcal{P}_\kappa A$ denotes the set $\{x\subset A \mid |x|<\kappa\}$. Let Y be a subset of $\mathcal{P}_\kappa A$. $[Y]^2$ denotes the set $\{(x, y)\in Y^2 \mid x\subset y$ and $x\neq y\}$. For any function $F: [Y]^2\to 2$, a subset H of Y is said to be *homogeneous for F,* if $|F''[H]^2|\leqq 1$. For any $B\supset A$, the function $p: \mathcal{P}_\kappa B\to\mathcal{P}_\kappa A$ which is defined by $p(y)=y\cap A$ is called *the projection.* For each $x\in\mathcal{P}_\kappa A$, $\hat{x}$ denotes the set $\{y\in\mathcal{P}_\kappa A \mid x\subset y$ and $x\neq y\}$ and Q_x the set $\{t\subset x \mid |t|<|x\cap\kappa|\}$. $I_{\kappa,A}$ denotes the ideal $\{X\subset\mathcal{P}_\kappa A \mid X\cap\hat{y}=\emptyset$, for some $y\in\mathcal{P}_\kappa A\}$. An element of $I^+_{\kappa,A}$ is called *unbounded.* A subset of $\mathcal{P}_\kappa A$ is called *club,* if it is unbounded and closed under unions of $\subset$-increasing chains with length $<\kappa$. A subset X of $\mathcal{P}_\kappa A$ is called *stationary*, if $X \cap C \neq \emptyset$ for any club subset C of $\mathcal{P}_\kappa A$. $NS_{\kappa,A}$ denotes the ideal $\{X \subset \mathcal{P}_\kappa A \mid X \text{ is non-stationary}\}$.

For any indexed family $\{X_a \mid a \in A\}$ of subset $\mathcal{P}_\kappa A$, *the diagonal union* $\nabla_{a\in A} X_a$ (*the diagonal intersection* $\Delta_{a\in A} X_a$) denotes the set $\{x \in \mathcal{P}_\kappa A \mid x \in X_a, \text{ for some } a \in x\}$ ($\{x \in \mathcal{P}_\kappa A \mid x \in X_a, \text{ for all } a \in x\}$). Similarly, for any indexed family $\{X_u \mid u \in \mathcal{P}_\kappa A\}$ of subset $\mathcal{P}_\kappa A$, *the strong diagonal union* $\nabla_{u \in \mathcal{P}_\kappa A} X_u$ and *the strong diagonal intersection* $\Delta_{u \in \mathcal{P}_\kappa A} X_u$ denote the set $\{x \in \mathcal{P}_\kappa A \mid x \in X_u, \text{ for some } u \in Q_x\}$ and $\{x \in \mathcal{P}_\kappa A \mid x \in X_u, \text{ for all } u \in Q_x\}$, respectively. A κ-complete ideal on $\mathcal{P}_\kappa A$ is said to be *normal* (*strongly normal*), if it contains $I_{\kappa,A}$ and closed under diagonal unions (strong diagonal unions). For any ideal $\mathcal{J}$, $\nabla^2(\mathcal{J})$ ($S(\mathcal{J})$) denote the smallest normal (strongly normal) ideal which includes $\mathcal{J}$. For any $\tau: A \times A \to \mathcal{P}_\kappa A$, cl($\tau$) denotes the set $\{x \in \mathcal{P}_\kappa A \mid \tau'' x \times x \subset x\}$. Similarly, for any $\tau: \mathcal{P}_\kappa A \to \mathcal{P}_\kappa A$, cl($\tau$) denotes $\{x \in \mathcal{P}_\kappa A \mid \forall u \in Q_x (\tau(u) \subset x)\}$.

Menas [**12**] proved that, for any club subset C of $\mathcal{P}_\kappa A$, there exists a $\tau: A \times A \to \mathcal{P}_\kappa A$ such that cl(τ)$\subset C$. It is known [3] that $NS_{\kappa,A} = \nabla^2(I_{\kappa,A})$. The notion of strong normality was introduced by Carr [**5**]. Carr and Pelletier [**6**] gave structural characterizations of strongly normal ideals. There is another characterization of strongly normal ideals in [**10**].

3. Operations *NI*, *NSI* and *NP*.

Jech [8] introduced the notion of λ-ineffability and almost λ-ineffability and partition property. After that, Carr [4] gave the ideal theoretic characterizations of λ-ineffability and almost λ-ineffability. She introduced the ideals $NIn_{\kappa,\lambda}$ and $NAIn_{\kappa,\lambda}$. It is known that the partition property also has the ideal theoretic characterization. These ideals were obtained from $NS_{\kappa,\lambda}$ and $I_{\kappa,\lambda}$ by using some operations. In order to treat these ideals uniformly, we fixed these operations.

DEFINITION 3.1. Let $\mathcal{J}$ be an ideal on $\mathcal{P}_\kappa A$. Define the ideals $NI(\mathcal{J})$, $NSI(\mathcal{J})$ and $NP(\mathcal{J})$ by

$$NI(\mathcal{J}) = \{X \subset \mathcal{P}_\kappa A \mid \exists s_x \subset x \quad (\text{for } x \in X) \\ \forall S \subset A(\{x \in X \mid s_x = S \cap x\} \in \mathcal{J})\},$$

$$NSI(\mathcal{J}) = \{X \subset \mathcal{P}_\kappa A \mid \exists s_x \subset Q_x \quad (\text{for } x \in X) \\ \forall S \subset \mathcal{P}_\kappa A \{x \in X \mid s_x = S \cap Q_x\} \in \mathcal{J}\},$$

$$NP(\mathcal{J}) = \{X \subset \mathcal{P}_\kappa A \mid \exists F: [X]^2 \to 2 \ \forall H \in \mathcal{J}^+ \restriction X \\ (H \text{ is not homogeneous for } F)\}.$$

Following Carr [4], we denote $NI(NS_{\kappa,A})$ by $NIn_{\kappa,A}$ and $NI(I_{\kappa,A})$ by $NAIn_{\kappa,A}$. κ is said to be *A-ineffable*, (*almost A-ineffable*), if the ideal $NI(NS_{\kappa,A})$ ($NI(I_{\kappa,A})$) is proper. We denote by part*(κ, A) the statement "$NP(NS_{\kappa,A})$ is proper". It is easy to see that $NI(\mathcal{I})\subset NSI(\mathcal{I})$. Magidor [11] proved that if part*(κ, λ) holds, then κ is λ-ineffable. The same proof yields a proof of that, if $\mathcal{I}$ is normal, then $NI(\mathcal{I})\subset NP(\mathcal{I})$. Carr [4, Theorem 1.2] proved that, if an ideal $\mathcal{I}$ contains $I_{\kappa,A}$, then $NI(\mathcal{I})$ is normal. By a similar argument in her proof, it holds that if an ideal $\mathcal{I}$ contains $I_{\kappa,A}$, then $NSI(\mathcal{I})$ is strongly normal. It is easy to see that, if $\mathcal{I}$ is normal, then $NP(\mathcal{I})$ is normal. I do not know whether $NP(\mathcal{I})$ is normal or not without the assumption of that $\mathcal{I}$ is normal.

QUESTION 1. Let $\mathcal{I}$ be an ideal containing $I_{\kappa,A}$.
1. Does it hold that $NS_{\kappa,A}\subset NP(\mathcal{I})$?
2. Is $NP(\mathcal{I})$ normal?

Using the normality of an ideal $NI(\mathcal{I})$, Carr [4, Corollary 1.3] proved

LEMMA 3.1. *Assume that* $I_{\kappa,A}\subset\mathcal{I}$. *Then, for any* $X\in NI(\mathcal{I})^+$ *and any* f_x: $x\to x$ (*for* $x\in X$), *there exists an* $f: A\to A$ *such that* $\{x\in X\mid f_x\subset f\}\in\mathcal{I}^+$. □

A similar argument gives

LEMMA 3.2. *Assume that* $I_{\kappa,A}\subset\mathcal{I}$. *Then, for any* $X\in NSI(\mathcal{I})^+$ *and any* f_x: $Q_x\to Q_x$ (*for* $x\in X$), *there exists an* $f: \mathcal{P}_\kappa A\to\mathcal{P}_\kappa A$ *such that* $\{x\in X\mid f_x\subset f\}\in\mathcal{I}^+$. □

The following lemma can be easily proved.

LEMMA 3.3. *Let* $A\subset B$ *and* $p: \mathcal{P}_\kappa B\to\mathcal{P}_\kappa A$ *the projection. Then, for any ideal* $\mathcal{I}$ *on* $\mathcal{P}_\kappa B$, $NI(p_*(\mathcal{I}))\subset p_*(NI(\mathcal{I}))$ *and* $NSI(p_*(\mathcal{I}))\subset p_*(NSI(\mathcal{I}))$. □

QUESTION 2. Let $A\subset B$, $p: \mathcal{P}_\kappa B\to\mathcal{P}_\kappa A$ the projection and $\mathcal{I}$ an ideal on $\mathcal{P}_\kappa B$. Does it hold that $NP(p_*(\mathcal{I}))\subset p_*(NP(\mathcal{I}))$?

LEMMA 3.4. *Let* $\mathcal{I}$ *be an ideal containing* $NS_{\kappa,A}$ *and* $X\in NI(\mathcal{I})^+$. *For each* $a\in A$, *let* W_a *be an* $\mathcal{I}$*-partition of* X *which satisfies that* $|W_a|\leqq|A|$. *Then, there exists* $g\in\prod_{a\in A}W_a$ *such that* $\Delta_{a\in A}\, g(a)\in\mathcal{I}^+$.

PROOF. First, we dealt the case that

$$W_a=\{X_{a,0},\ X_{a,1}\}\ \text{is a partition of } X, \text{ for all } a\in A.$$

For each $x\in X$, define $f_x: x\to 2$ by

$$f_x(a)=\begin{cases}0, & \text{if } x\in X_{a,0},\\ 1, & \text{if } x\in X_{a,1}.\end{cases}$$

Since $X\in NI(\mathcal{I})^+$, there exists a $g: A\to 2$ such that

$$Y = \{x \in X \mid f_x \subset g\} \in \mathcal{I}^+.$$

By the choice of f_x (for $x \in X$), we have that $Y \subset \Delta_{a\in A} X_{a,g(a)}$. Hence $\Delta_{a\in A} X_{a,g(a)} \in \mathcal{I}^+$.

Now, we deal the general case. Assume that

$$W_a = \{X_{a,b} \mid b \in A\} \text{ is an } \mathcal{I}\text{-partition of } X, \text{ for each } a \in A.$$

Take a bijection $\tau: A \times A \to A$. Since $NS_{\kappa,A} \subset \mathcal{I}$, without loss of generality, we may assume that $\forall x \in X$ $(\tau'' x \times x = x)$. For each $c = \tau(a, b) \in A$, set

$$Y_{c,0} = X_{a,b}, \quad Y_{c,1} = X \backslash X_{a,b} \quad \text{and} \quad V_c = \{Y_{c,0}, Y_{c,1}\}.$$

By the virtue of the previous case, there exists $g: A \to 2$ such that

$$Y = \Delta_{c\in A} Y_{c,g(c)} \in \mathcal{I}^+.$$

CLAIM. $\forall a \in A \exists b \in A (g(\tau(a, b)) = 0)$.

PROOF OF CLAIM. To get a contradiction, assume that

$$a \in A \text{ and } g(\tau(a, b)) = 1, \quad \text{for all } b \in A.$$

Set $Z = \Delta_{b\in A}(X \backslash X_{a,b})$. Then, it holds that

$$Y \cap \widehat{\{a\}} \subset Z \text{ and } Z \cap X_{a,b} \in \mathcal{I}, \quad \text{for all } b \in A.$$

This contradicts that W_a is an $\mathcal{I}$-partition of X. QED of Claim.

By the Claim, take $h: A \to A$ such that, for all $a \in A$, $g(\tau(a, h(a))) = 0$. Define $g' \in \prod_{a\in A} W_a$ by

$$g'(a) = X_{a,h(a)}, \quad \text{for all } a \in A.$$

Then, it is easy to check that

$$\{x \in Y \mid h'' x \subset x\} \subset \Delta_{a\in A} g'(a).$$

Since $\{x \in \mathcal{P}_\kappa A \mid h'' x \subset x\} \in \mathcal{I}^*$, we have that $\Delta_{a\in A} g'(a) \in \mathcal{I}^+$. □

By a similar argument, we have

LEMMA 3.5. *Let $\mathcal{I}$ be an ideal containing $\boldsymbol{S}(NS_{\kappa,A})$ and $X \in NSI(\mathcal{I})^+$. For each $u \in \mathcal{P}_\kappa A$, let W_u be an $\mathcal{I}$-partition of X which satisfies that $|W_u| \leq |A|^{<\kappa}$. Then, there exists $g \in \prod_{u\in\mathcal{P}_\kappa A} W_u$ such that $\Delta_{u\in\mathcal{P}_\kappa A} g(u) \in \mathcal{I}^+$.* □

4. The ideals $\mathcal{I}_\alpha(\kappa, A)$ and $\mathcal{J}_\alpha(\kappa, A)$.

DEFINITION 4.1. By induction, on $\alpha \in \boldsymbol{On}$ define the ideals $\mathcal{I}_\alpha(\kappa, A)$, $\mathcal{J}_\alpha(\kappa, A)$ on $\mathcal{P}_\kappa A$ as follows:

$$\mathcal{I}_0(\kappa, A) = NS_{\kappa, A},$$

$$\mathcal{I}_{\alpha+1}(\kappa, A) = NI(\mathcal{I}_\alpha(\kappa, A)),$$

$$\mathcal{I}_\alpha(\kappa, A) = \bigcup_{\xi<\alpha} \mathcal{I}_\xi(\kappa, A), \quad \text{for limit } \alpha,$$

$$\mathcal{J}_0(\kappa, A) = \boldsymbol{S}(NS_{\kappa, A}),$$

$$\mathcal{J}_{\alpha+1}(\kappa, A) = NSI(\mathcal{J}_\alpha(\kappa, A)),$$

$$\mathcal{J}_\alpha(\kappa, A) = \bigcup_{\xi<\alpha} \mathcal{J}_\xi(\kappa, A), \quad \text{for limit } \alpha.$$

Note that $\bigcup_{\alpha\in On} \mathcal{I}_\alpha(\kappa, A)$ and $\bigcup_{\alpha\in On} \mathcal{J}_\alpha(\kappa, A)$ are normal ideals on $\mathcal{P}_\kappa A$.

Following Johnson [9], κ is said to be *completely A-ineffable*, if *the completely ineffable ideal* $\bigcup_{\alpha\in On} \mathcal{I}_\alpha(\kappa, A)$ is proper. She proved that this ideal is the smallest normal ideal which satisfies the $(\lambda, 2)$-distributive law. In this section, we shall prove two theorems concerning relations on these ideals. First, we show

THEOREM 4.1. *Let $\kappa<\delta<\lambda$ and δ be a regular cardinal. Then, for any $\alpha\leqq\lambda$, it holds that*

$$\forall X \in \mathcal{I}_\alpha(\kappa, \lambda) \ (\{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X\cap\mathcal{P}_\kappa y \in \mathcal{I}_{ot(\alpha\cap y)}(\kappa, y)^+\} \in \mathcal{I}_\alpha(\delta, \lambda)\ ).$$

We will prove this theorem by induction on α. The following two lemmas serve in the cases that $\alpha=0$ and that α is a successor ordinal.

LEMMA 4.2. *Let $\kappa<\delta<\lambda$ and δ be a regular cardinal. Then,*

$$\forall X \in NS_{\kappa, \lambda}(\{y \in \mathcal{P}_\delta\lambda\cap\hat{\kappa} \mid X\cap\mathcal{P}_\kappa y \in NS^+_{\kappa, y}\} \in NS_{\delta, \lambda}).$$

PROOF. Let $X\in NS_{\kappa, \lambda}$. Take $\sigma: \lambda\times\lambda\to\mathcal{P}_\kappa\lambda$ such that $\mathrm{cl}(\sigma)\cap X=\emptyset$. Set

$$D = \{y \in \mathcal{P}_\delta\lambda\cap\hat{\kappa} \mid \sigma'' y\times y \subset \mathcal{P}_\kappa y\}.$$

Since $\delta>\kappa$ is regular, D is a club subset of $\mathcal{P}_\delta\lambda$. So, it suffices to show that

$$D\cap\{y \in \mathcal{P}_\delta\lambda\cap\hat{\kappa} \mid X\cap\mathcal{P}_\kappa y \in NS^+_{\kappa, y}\} = \emptyset.$$

If not, then there is a $y\in D$ such that $X\cap\mathcal{P}_\kappa y\in NS^+_{\kappa, y}$. Since $y\in D$, $\mathrm{cl}(\sigma)\cap\mathcal{P}_\kappa y$ is a club subset of $\mathcal{P}_\kappa y$. So, we have that $\mathrm{cl}(\sigma)\cap\mathcal{P}_\kappa y\cap X\neq\emptyset$. But this contradicts the choice of σ. □

LEMMA 4.3. *Let $\kappa<\delta<\lambda$ and δ be a regular cardinal. Let $\mathcal{I}$ and $\mathcal{H}$ be an ideal on $\mathcal{P}_\delta\lambda$ and $\mathcal{P}_\kappa\lambda$, respectively. For each $y\in\mathcal{P}_\delta\lambda$, let $\mathcal{I}_y$ be an ideal on $\mathcal{P}_\kappa y$. Suppose that*

$$\forall X \in \mathcal{H}(\{y \in \mathcal{P}_\delta\lambda\cap\hat{\kappa} \mid X\cap\mathcal{P}_\kappa y \in \mathcal{I}^+_y\} \in \mathcal{I}\ ).$$

Then, it holds that

$$\forall X \in NI(\mathcal{H})\ (\{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X \cap \mathcal{P}_\kappa y \in NI(\mathcal{I}_y)^+\} \in NI(\mathcal{I})\ ).$$

PROOF. Let $X \in NI(\mathcal{H})$. Take $s_x \subset x$ (for $x \in X$) such that

$$\forall S \subset \lambda\ (\{x \in X \mid s_x = S \cap x\} \in \mathcal{H}\ ).$$

Set

$$Y = \{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X \cap \mathcal{P}_\kappa y \in NI(\mathcal{I}_y)^+\}$$

For each $y \in Y$, take $S_y \subset y$ such that

$$\{x \in X \cap \mathcal{P}_\kappa y \mid s_x = S_y \cap x\} \in \mathcal{I}_y^+.$$

We will complete the proof by showing that

$$\forall S \subset \lambda\ (\{y \in Y \mid S_y = S \cap y\} \in \mathcal{I}\ ).$$

To show this, let $S \subset \lambda$. Set $X' = \{x \in X \mid s_x = S \cap x\}$. Since $X' \in \mathcal{H}$, by the assumption, we have that

$$Y' = \{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X' \cap \mathcal{P}_\kappa y \in \mathcal{I}_y^+\} \in \mathcal{I}.$$

By this and the fact that $\{y \in Y \mid S_y = S \cap y\} \subset Y'$, it holds that $\{y \in Y \mid S_y = S \cap y\} \in \mathcal{I}$. □

PROOF OF THEOREM 4.1. We show this theorem by induction on $\alpha \leqq \lambda$. It holds for $\alpha = 0$ and α a successor ordinal by Lemma 4.2 and Lemma 4.3, respectively. So, we assume that α is a limit ordinal. Let $X \in \mathcal{I}_\alpha(\kappa, \lambda)$. Take $\xi < \alpha$ such that $X \in \mathcal{I}_\xi(\kappa, \lambda)$. By the induction hypothesis, it holds that

$$Y = \{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X \cap \mathcal{P}_\kappa y \in \mathcal{I}_{ot(y \cap \xi)}(\kappa, y)^+\} \in \mathcal{I}_\xi(\delta, \lambda).$$

Since $\{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X \cap \mathcal{P}_\kappa y \in \mathcal{I}_{ot(y \cap \alpha)}(\kappa, y)^+\} \subset Y$, we have that

$$\{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid X \cap \mathcal{P}_\kappa y \in \mathcal{I}_{ot(y \cap \alpha)}(\kappa, y)^+\} \in \mathcal{I}_\xi(\delta, \lambda) \subset \mathcal{I}_\alpha(\delta, \lambda).$$ □

COROLLARY 4.4. *Let $\kappa \leqq \delta < \lambda$ and δ be a regular cardinal. If $\forall \xi < \delta$ ($\mathcal{I}_\xi(\kappa, \xi)$ is proper) and $\mathcal{I}_\lambda(\delta, \lambda)$ is proper, then $\mathcal{I}_\lambda(\kappa, \lambda)$ is proper.*

PROOF. The case of that $\kappa = \delta$ is clear. So, we assume that $\kappa < \delta$. To get a contradiction, assume that $\mathcal{I}_\lambda(\kappa, \lambda)$ is not proper. Then, since $\mathcal{P}_\kappa\lambda \in \mathcal{I}_\lambda(\kappa, \lambda)$, by the theorem, it holds that

$$Y = \{y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa} \mid \mathcal{P}_\kappa y \in \mathcal{I}_{ot(y)}(\kappa, y)^+\} \in \mathcal{I}_\lambda(\delta, \lambda).$$

By the assumption, we have that $\forall y \in \mathcal{P}_\delta\lambda \cap \hat{\kappa}(\mathcal{I}_{ot(y)}(\kappa, y)$ is proper). So, it holds that $Y = \mathcal{P}_\delta\lambda \cap \hat{\kappa}$. This contradicts that $\mathcal{I}_\lambda(\delta, \lambda)$ is proper. □

THEOREM 4.5. *Let $\kappa\leqq\lambda$ and $\alpha=\gamma+n<\lambda$, where γ is a limit ordinal and $n<\omega$. Then, it holds that*

$$\{x\in X\mid X\cap Q_x\in\mathcal{I}_{ot(\alpha\cap x)}(\kappa\cap x,\ x)\}\in\mathcal{I}_{\gamma+2n+1}(\kappa,\ \lambda),\quad \textit{for all } X\subset\mathcal{P}_\kappa\lambda.$$

As in the proof of the previous theorem, we first deal the cases that $\alpha=0$ and that α is a successor ordinal.

LEMMA 4.6. *$\{x\in X\mid X\cap Q_x\in NS_{\kappa\cap x,\,x}\}\in NSI(NS_{\kappa,\lambda})$, for all $X\subset\mathcal{P}_\kappa\lambda$.*

PROOF. To get a contradiction, assume that there is an $X\subset\mathcal{P}_\kappa\lambda$ such that

$$Y=\{x\in X\mid X\cap Q_x\in NS_{\kappa\cap x,\,x}\}\in NSI(NS_{\kappa,\lambda})^+.$$

For each $y\in Y$, take $\sigma_y: y\times y\to Q_y$ such that $\mathrm{cl}(\sigma_y)\cap X\cap Q_y=\emptyset$. Since $Y\in NSI(NS_{\kappa,\lambda})^+$, there is a function $\sigma:\lambda\times\lambda\to\mathcal{P}_\kappa\lambda$ such that

$$Z=\{y\in Y\mid \sigma_y=\sigma\restriction y\times y\}\in NS^+_{\kappa,\lambda}.$$

Since $\mathrm{cl}(\sigma)$ is a club subset, $\mathrm{cl}(\sigma)\cap Z$ is unbounded. So, $X\cap\mathrm{cl}(\sigma)=\emptyset$. This contradicts the fact that $X\in NS^+_{\kappa,\lambda}$. □

LEMMA 4.7. *Let $\mathcal{J}$ be an ideal on $\mathcal{P}_\kappa A$ which includes $NS_{\kappa,A}$. For each $x\in\mathcal{P}_\kappa A$, let $\mathcal{I}_x$ be an ideal on Q_x. Assume that*

$$\{x\in X\mid X\cap Q_x\in\mathcal{I}_x\}\in\mathcal{J},\quad \textit{for all } X\subset\mathcal{P}_\kappa A.$$

Then, it holds that

$$\{x\in X\mid X\cap Q_x\in NI(\mathcal{I}_x)\}\in NSI^2(\mathcal{J}),\quad \textit{for all } X\subset\mathcal{P}_\kappa A$$

where $NSI^2(\mathcal{J})$ denotes the ideal $NSI(NSI(\mathcal{J}))$.

PROOF. To get a contradiction, assume that there is an $X\subset\mathcal{P}_\kappa A$ such that

$$X_0=\{x\in X\mid X\cap Q_x\in NI(\mathcal{I}_x)\}\in NSI^2(\mathcal{J})^+.$$

Since κ becomes A-ineffable, κ is an inaccessible cardinal. For each $x\in X_0$, take $s^x_u\subset u$ (for $u\in X\cap Q_x$) such that

$$\forall S\subset x(\{u\in Q_x\cap X\mid s^x_u=S\cap u\}\in\mathcal{I}_x).$$

For each $u\in X$, set

$$Y_u(s)=\{x\in X_0\mid u\in Q_x \text{ and } s^x_u=s\},\quad \text{for } s\subset u,$$

$$W_u=\{Y_u(s)\mid s\subset u\}.$$

Then, it holds that

$$W_u \text{ is an } NSI(\mathcal{J})\text{-partition of } X_0 \text{ and } |W_u|<\kappa,\quad \text{for all } u\in X.$$

So, by Lemma 3.5, there exists a sequence $\langle s_u \mid u \in X\rangle$ such that

$$Y = \Delta_{u\in X} Y_u(s_u) \in NSI(\mathcal{I})^+ .$$

Since $Y \in NSI(\mathcal{I})^+$, take an $S \subset A$ such that

$$Z = \{u \in Y \mid s_u = S \cap u\} \in \mathcal{I}^+ .$$

By the assumption, it holds that

$$\{x \in Z \mid Z \cap Q_x \in \mathcal{I}_x\} \in \mathcal{I} .$$

So, we can take $x \in Z$ such that $Z \cap Q_x \in \mathcal{I}_x^+$. Let $s = S \cap x$.

CLAIM 1. $Z \cap Q_x \subset \{u \in X \cap Q_x \mid s_u^x = s \cap u\}$.

PROOF OF CLAIM 1. Let $u \in Z \cap Q_x$. It holds that $x \in Y_u(s_u)$. Hence $s_u^x = s_u$. By this and the fact that $u \in Z$, we have that $s_u^x = S \cap u = S \cap x \cap u = s \cap u$.

QED of Claim 1

Claim 1 contradicts the choice of s_u^x (for $u \in X \cap Q_x$). □

PROOF OF THEOREM 4.5. We prove this theorem by induction on $\alpha < \lambda$. By the virtue of the previous two lemmas, we only need to deal the case that α is a limit ordinal (i.e., $n=0$). Let

$$Z = \{x \in \mathcal{P}_\kappa\lambda \mid ot(x \cap \alpha) \text{ is a limit ordinal}\} .$$

It holds that $Z \in NS^*_{\kappa,\lambda}$. Let $X \subset \mathcal{P}_\kappa\lambda$. By induction hypothesis, it holds that

$$X_\xi = \{x \in X \mid X \cap Q_x \in \mathcal{I}_{ot(x\cap\xi)}(x \cap \kappa,\ x)\} \in \mathcal{I}_{\xi+\omega}(\kappa,\ \lambda) \subset \mathcal{I}_\alpha(\kappa,\ \lambda),$$
$$\text{for all } \xi < \alpha .$$

Hence, $Y = \nabla_{\xi<\alpha} X_\xi \in \mathcal{I}_{\alpha+1}(\kappa,\ \lambda)$. By this and the fact that

$$\{x \in X \mid X \cap Q_x \in \mathcal{I}_{ot(x\cap\alpha)}(x \cap \kappa,\ x)\} \cap Z \subset Y ,$$

we have that $\{x \in X \mid X \cap Q_x \in \mathcal{I}_{ot(x\cap\alpha)}(x \cap \kappa,\ x)\} \in \mathcal{I}_{\alpha+1}(\kappa,\ \lambda)$. □

COROLLARY 4.8. $\{x \in \mathcal{P}_\kappa\lambda \mid \mathcal{I}_{ot(x)}(x \cap \kappa,\ x)$ *is proper*$\} \in \nabla^2 \mathcal{I}_\lambda(\kappa,\ \lambda)^*$.

PROOF. Let $Z = \{x \in \mathcal{P}_\kappa\lambda \mid ot(x)$ is a limit ordinal$\}$ $(\in NS^*_{\kappa,\lambda})$. For each $\alpha < \lambda$, let

$$X_\alpha = \{x \in \mathcal{P}_\kappa\lambda \mid \mathcal{I}_{ot(x\cap\alpha)}(x \cap \kappa,\ x) \text{ is proper}\} .$$

By the theorem, it holds that $X_\alpha \in \mathcal{I}_\lambda(\kappa,\ \lambda)^*$, for all $\alpha < \lambda$. So, $\Delta_{\alpha<\lambda} X_\alpha \in \nabla^2 \mathcal{I}_\lambda(\kappa,\ \lambda)^*$. By this and the fact that

$$\Delta_{\alpha<\lambda} X_\alpha \cap Z \subset \{x \in \mathcal{P}_\kappa\lambda \mid \mathcal{I}_{ot(x)}(x \cap \kappa,\ x) \text{ is proper}\} ,$$

we have that $\{x \in \mathcal{P}_\kappa\lambda \mid \mathcal{I}_{ot(x)}(x \cap \kappa,\ x)$ is proper$\} \in \nabla^2 \mathcal{I}_\lambda(\kappa,\ \lambda)^*$. □

5. The ideals $\mathcal{K}_\alpha(\kappa, A)$ and $\mathcal{L}_\alpha(\kappa, A)$.

DEFINITION 5.1. By induction on $\alpha \in \boldsymbol{On}$, define the ideals $\mathcal{K}_\alpha(\kappa, A)$, $\mathcal{L}_\alpha(\kappa, A)$ as follows:

$$\mathcal{K}_0(\kappa, A) = NS_{\kappa, A},$$

$$\mathcal{K}_{\alpha+1}(\kappa, A) = NP(\mathcal{K}_\alpha(\kappa, A)),$$

$$\mathcal{K}_\alpha(\kappa, A) = \nabla^2(\bigcup_{\xi<\alpha} \mathcal{K}_\xi(\kappa, A)), \quad \text{for limit } \alpha,$$

$$\mathcal{L}_0(\kappa, A) = \boldsymbol{S}(NS_{\kappa, A}),$$

$$\mathcal{L}_{\alpha+1}(\kappa, A) = NP(\mathcal{L}_\alpha(\kappa, A)),$$

$$\mathcal{L}_\alpha(\kappa, A) = \boldsymbol{S}(\bigcup_{\xi<\alpha} \mathcal{L}_\xi(\kappa, A)), \quad \text{for limit } \alpha.$$

Note that $\mathcal{I}_\alpha(\kappa, A) \subset \mathcal{K}_\alpha(\kappa, A) \subset \mathcal{L}_\alpha(\kappa, A)$, for all $\alpha \in \boldsymbol{On}$. Set $NCP_{\kappa, A} = \bigcup_{\alpha \in \boldsymbol{On}} \mathcal{K}_\alpha(\kappa, A)$.

For $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(\mathcal{P}_\kappa A)$, $\mathcal{A} \overset{*}{\rightarrow} (\mathcal{B})^2$ denotes the statement:

$$\forall X \in \mathcal{A}\ \forall f : [X]^2 \rightarrow 2\ \exists H \in \mathcal{B}\ (H \subset X \text{ and } H \text{ is homogeneous for } f).$$

By the definition, it follows directly that $NP(\mathcal{I})$ is the smallest ideal $\mathcal{J}$ which satisfies $\mathcal{J}^+ \overset{*}{\rightarrow} (\mathcal{J}^+)^2$. So, $NCP_{\kappa, A}$ is the smallest normal ideal which satisfies that $NCP^+_{\kappa, A} \overset{*}{\rightarrow} (NCP^+_{\kappa, A})^2$. Menas [**13**] proved that, under the assumption of that κ is $2^{\lambda^{<\kappa}}$-supercompact, there exists a normal ultrafilter U on $\mathcal{P}_\kappa\lambda$ which satisfies $U \overset{*}{\rightarrow} (U)^2$. A similar argument in [**13**] gives a proof of that, if κ is A-supercompact, then $NCP_{\kappa, A}$ is proper.

QUESTION 3. In the result of Menas, can the assumption be weakened to that κ is λ-supercompact? I.e., does one can prove the existence of a normal ultrafilter U on $\mathcal{P}_\kappa\lambda$ which satisfies $U \overset{*}{\rightarrow} (U)^2$, under the assumption of that κ is λ-supercompact?

Now we give two results concerning relations between $\mathcal{I}_\alpha(\kappa, A)$ and $\mathcal{L}_\alpha(\kappa, A)$.

THEOREM 5.1. *For any $\alpha < \lambda$, and any $X \in \mathcal{L}_\alpha(\kappa, \lambda)$,*

$$\{x \in \mathcal{P}_\kappa\lambda \mid X \cap Q_x \in \mathcal{L}_{ot(\alpha\cap x)}(\kappa\cap x, x)\} \in \mathcal{I}_{\alpha+1}(\kappa, \lambda)^*.$$

As in the proofs of theorems in the previous section, we first deal the cases of that $\alpha=0$ and that α is a successor ordinal.

LEMMA 5.2. *Let κ be an inaccessible cardinal.*

(1) $\{x \in \mathcal{P}_\kappa A \mid X \cap Q_x \in NS_{\kappa\cap x, x}\} \in NS^*_{\kappa, A}$, *for any* $X \in NS_{\kappa, A}$.

(2) $\{x \in \mathcal{P}_\kappa A \mid X \cap Q_x \in \boldsymbol{S}(NS_{\kappa\cap x, x})\} \in \boldsymbol{S}(NS_{\kappa, A})^*$, *for any* $X \in \boldsymbol{S}(NS_{\kappa, A})$.

PROOF. (1) and (2) can be proved by a similar argument. So, we deal only (1). Let $X \in NS_{\kappa, A}$. Take $\sigma: A \times A \to \mathcal{P}_\kappa A$ such that $\mathrm{cl}(\sigma) \cap X = \emptyset$. Take $\tau: A \times A \to \mathcal{P}_\kappa A$ such that

$$\sigma(a, b) \in Q_{\tau(a, b)}, \quad \text{for all } a, b \in A.$$

We claim that

$$\mathrm{cl}(\tau) \subset \{x \in \mathcal{P}_\kappa A \mid X \cap Q_x \in NS_{\kappa \cap x, x}\}.$$

To show this, let $x \in \mathrm{cl}(\tau)$. Then, for any $a, b \in x$, since $\sigma(a, b) \in Q_{\tau(a, b)}$ and $\tau(a, b) \subset x$, $\sigma \restriction (x \times x)$ is a function from $x \times x$ to Q_x. So, $\mathrm{cl}(\sigma) \cap Q_x \in NS^*_{\kappa \cap x, x}$. By this, since $\mathrm{cl}(\sigma) \cap X = \emptyset$, we have that $X \cap Q_x \in NS_{\kappa \cap x, x}$. □

LEMMA 5.3. *Let $\mathcal{I}$, $\mathcal{K}$ be ideals on $\mathcal{P}_\kappa A$ which include $NS_{\kappa, A}$. For each $x \in \mathcal{P}_\kappa A$, let $\mathcal{K}_x$ be an ideal on Q_x. Suppose that*

$$\forall X \in \mathcal{K}(\{y \in \mathcal{P}_\kappa A \mid X \cap Q_y \in \mathcal{K}_y\} \in \mathcal{I}^*).$$

Then, it holds that

$$\forall X \in NP(\mathcal{K})(\{y \in \mathcal{P}_\kappa A \mid X \cap Q_y \in NP(\mathcal{K}_y)\} \in NSI(\mathcal{I})^*).$$

PROOF. To get a contradiction, suppose that there is an $X \in NP(\mathcal{K})$ such that

$$Y = \{y \in \mathcal{P}_\kappa A \mid X \cap Q_y \in NP(\mathcal{K}_y)^+\} \in NSI(\mathcal{I})^+.$$

Take $F: [X]^2 \to 2$ such that

$$\forall H \in \mathcal{K}^+ \restriction X \ (H \text{ is not homogeneous for } F).$$

For each $y \in Y$, take $H_y \in \mathcal{K}_y^+ \restriction (X \cap Q_y)$ and $e_y < 2$ such that $F''[H_y]^2 = \{e_y\}$. Since $Y \in NSI(\mathcal{I})^+$, there exist $H \subset \mathcal{P}_\kappa A$ and $e < 2$ such that

$$Y' = \{y \in Y \mid H_y = H \cap Q_y \text{ and } e_y = e\} \in \mathcal{I}^+.$$

Since $I_{\kappa, A} \subset \mathcal{I}$, it holds that

$$H \subset X \text{ and } H \text{ is homogeneous for } F.$$

By the choice of F, $H \in \mathcal{K}$. So, by the assumption, we have that

$$Z = \{y \in \mathcal{P}_\kappa A \mid H \cap Q_y \in \mathcal{K}_y\} \in \mathcal{I}^*.$$

By this, there is a $y \in Z \cap Y'$, since $Y' \in \mathcal{I}^+$. Then, it holds that

$$H \cap Q_y \in \mathcal{K}_y \quad \text{and} \quad H_y = H \cap Q_y.$$

This contradicts the choice of H_y. □

COROLLARY 5.4. *Assume that* part*(κ, A) *fails. Then, it holds that*

$$\{x \in \mathcal{P}_\kappa A \mid \text{part}^*(x \cap \kappa,\ x)\ \textit{fails}\} \in NSI(NS_{\kappa, A})^*.$$

PROOF. The case that κ is not inaccessible is trivial. We assume that κ is inaccessible. So, by Lemma 5.2(1) and Lemma 5.3, we have that

$$\{x \in \mathcal{P}_\kappa A \mid X \cap Q_x \in NP(NS_{x\cap\kappa, x})\} \in NSI(NS_{\kappa, A})^*, \quad \text{for all } X \in NP(NS_{\kappa, A}).$$

Assume that part*(κ, A) fails. Then, since $\mathcal{P}_\kappa A \in NP(NS_{\kappa, A})$, we have that

$$\{x \in \mathcal{P}_\kappa A \mid Q_x \in NP(NS_{x\cap\kappa, x})\} \in NSI(NS_{\kappa, A})^*,$$

$$\text{i.e.,} \quad \{x \in \mathcal{P}_\kappa A \mid \text{part}^*(x \cap \kappa,\ x)\ \text{fails}\} \in NSI(NS_{\kappa, A})^*. \qquad \square$$

PROOF OF THEOREM 5.1. By induction on $\alpha < \lambda$.

Case I. $\alpha = 0$.

To get a contradiction, suppose that there is an $X \in \mathcal{L}_0(\kappa, \lambda)$ $(= S(NS_{\kappa, \lambda}))$ such that

$$Y = \{x \in \mathcal{P}_\kappa \lambda \mid X \cap Q_x \in \mathcal{L}_0(\kappa \cap x,\ x)\} \notin \mathcal{J}_1(\kappa, \lambda)^*.$$

Then, κ becomes λ-ineffable. So, κ is inaccessible. By Lemma 5.2, $Y \in S(NS_{\kappa, \lambda})^*$. This contradicts that $S(NS_{\kappa, \lambda}) \subset \mathcal{J}_1(\kappa, \lambda)$.

Case II. α is a successor ordinal.

This case follows from the induction hypothesis and Lemma 5.3.

Case III. α is a limit ordinal.

Let $Z = \{x \in \mathcal{P}_\kappa \lambda \mid ot(x \cap \alpha) \text{ is a limit ordinal}\}$. Let $X \in \mathcal{L}_\alpha(\kappa, \lambda)$. Take $X_u \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta(\kappa, \lambda)$ (for $u \in \mathcal{P}_\kappa \lambda$) and $X' \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta(\kappa, \lambda)$ such that $X \subset \nabla_{u \in \mathcal{P}_\kappa \lambda} X_u \cup X'$. It holds that

$$X \cap Q_x \subset \nabla_{u \in Q_x}(X_u \cap Q_x) \cup (X' \cap Q_x), \quad \text{for all } x \in \mathcal{P}_\kappa \lambda.$$

By the induction hypothesis, it holds that

$$Y_u = \{x \in \mathcal{P}_\kappa \lambda \mid X_u \cap Q_x \in \textstyle\bigcup_{\beta \in x\cap\alpha} \mathcal{L}_{ot(x\cap\beta)}(\kappa \cap x,\ x)\} \in \mathcal{J}_\alpha(\kappa, \lambda)^*, \quad \text{for all } u \in \mathcal{P}_\kappa \lambda,$$

and

$$Y' = \{x \in \mathcal{P}_\kappa \lambda \mid X' \cap Q_x \in \mathcal{L}_{ot(x\cap\alpha)}(x \cap \kappa,\ x)\} \in \mathcal{J}_\alpha(\kappa, \lambda)^*.$$

So, $\Delta_{u \in \mathcal{P}_\kappa \lambda} Y_u \in \mathcal{J}_{\alpha+1}(\kappa, \lambda)^*$. By this and the fact that

$$Y' \cap Z \cap \Delta_{u \in \mathcal{P}_\kappa \lambda} Y_u \subset \{x \in \mathcal{P}_\kappa \lambda \mid X \cap Q_x \in \mathcal{L}_{ot(x\cap\alpha)}(x \cap \kappa,\ x)\},$$

we have that $\{x \in \mathcal{P}_\kappa \lambda \mid X \cap Q_x \in \mathcal{L}_{ot(x\cap\alpha)}(x \cap \kappa,\ x)\} \in \mathcal{J}_{\alpha+1}(\kappa, \lambda)^*$. $\square$

COROLLARY 5.5. *For any* $X \in \bigcup_{\alpha<\lambda} \mathcal{L}_\alpha(\kappa, \lambda)$,

$$\{x \in \mathcal{P}_\kappa \lambda \mid X \cap Q_x \in \textstyle\bigcup_{\xi < ot(x)} \mathcal{L}_\xi(x \cap \kappa,\ x)\} \in \mathcal{J}_\lambda(\kappa, \lambda)^*. \qquad \square$$

Carr [5, Theorem 4.2(1)] proved that, under the assumption of that $\lambda^{<\kappa}=\lambda$,

$$\forall X \in \mathcal{I}_1(\kappa, \lambda)^+ \text{ (if } \forall(x, y) \in [X]^2\ (x \in Q_y), \text{ then } X \in \mathcal{K}_1(\kappa, \lambda)^+).$$

The same argument gives a proof of that

$$\forall X \in \mathcal{I}_1(\kappa, \lambda)^+ \text{ (if } \forall(x, y) \in [X]^2\ (x \in Q_y), \text{ then } X \in \mathcal{L}_1(\kappa, \lambda)^+).$$

We generalize her result as the following theorem.

THEOREM 5.6. *For any $\alpha \in \boldsymbol{On}$ and $X \in \mathcal{I}_{\alpha+1}(\kappa, A)^+$, if $\forall(x, y) \in [X]^2\ (x \in Q_y)$, then $X \in \mathcal{L}_\alpha(\kappa, A)^+$.*

PROOF. By induction on $\alpha \in \boldsymbol{On}$.

Case I. $\alpha=0$.

This case is clear.

Case II. $\alpha=\beta+1$.

Let $X \in \mathcal{I}_{\alpha+1}(\kappa, A)^+$ satisfy $\forall(x, y) \in [X]^2\ (x \in Q_y)$. Let $F: [X]^2 \to 2$. For each $x \in X$, set

$$s_x = \{u \in X \cap Q_x \mid F(u, x) = 0\}.$$

Since $X \in \mathcal{I}_{\alpha+1}(\kappa, A)^+$, we can take an $S \subset X$ such that

$$Y = \{x \in X \mid s_x = S \cap Q_x\} \in \mathcal{I}_\alpha(\kappa, A)^+.$$

By the induction hypothesis, $Y \in \mathcal{L}_\beta(\kappa, A)^+$. It is easy to see that $F''[S \cap Y]^2 = \{0\}$ and $F''[Y \setminus S]^2 = \{1\}$. Since $Y \cap S \in \mathcal{L}_\beta(\kappa, A)^+$ or $Y \setminus S \in \mathcal{L}_\beta(\kappa, A)^+$, we have that $X \in \mathcal{L}_\alpha(\kappa, A)^+$.

Case III. α is a limit ordinal.

Let $X \in \mathcal{I}_{\alpha+1}(\kappa, A)^+$ satisfy $\forall(x, y) \in [X]^2\ (x \in Q_y)$. To get a contradiction, assume that $X \in \mathcal{L}_\alpha(\kappa, A)$. Take $X_u \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta(\kappa, A)$ (for $u \in \mathcal{P}_\kappa A$) and $X' \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta(\kappa, A)$ such that $X \subset \nabla_{u \in \mathcal{P}_\kappa A} X_u \cup X'$. Without loss of generality, we may assume that

$$X_u \subset X, \text{ for all } u \in \mathcal{P}_\kappa A \text{ and } X' \subset X.$$

By the induction hypothesis, it holds that

$$X_u \in \bigcup_{\beta<\alpha} \mathcal{I}_{\beta+1}(\kappa, A) = \mathcal{I}_\alpha(\kappa, A), \text{ for all } u \in \mathcal{P}_\kappa A \text{ and } X' \in \mathcal{I}_\alpha(\kappa, A).$$

By this and the fact that $\mathcal{I}_{\alpha+1}(\kappa, A)$ is strongly normal, it holds that $\nabla_{u \in \mathcal{P}_\kappa A} X_u \in \mathcal{I}_{\alpha+1}(\kappa, A)$. So, $X \in \mathcal{I}_{\alpha+1}(\kappa, A)$. This is a contradiction. □

COROLLARY 5.7. *For any $X \in \mathcal{I}_\lambda(\kappa, \lambda)^+$, if $\forall(x, y) \in [X]^2\ (x \in Q_y)$, then $X \in (\bigcup_{\alpha<\lambda} \mathcal{L}_\alpha(\kappa, \lambda))^+$.* □

6. Correspondence between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa\lambda^{<\kappa}$.

This section is a proof of the following theorem.

THEOREM 6.1. *Let* $\kappa \leqq \lambda$, $\theta = \lambda^{<\kappa}$ *and* $p : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ *the projection. Then, for any ordinal* $\alpha > 0$,

(1) $\mathcal{I}_\alpha(\kappa, \lambda) = p_*(\mathcal{I}_\alpha(\kappa, \theta))$.

(2) $\mathcal{L}_\alpha(\kappa, \lambda) = p_*(\mathcal{K}_\alpha(\kappa, \theta))$.

Let $\kappa \leqq \lambda$, $\theta = \lambda^{<\kappa}$ and $p : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ the projection. Theorem is trivial, if κ is not inaccessible. So, we assume that κ is inaccessible. In order to prove this theorem, we need the canonical correspondence between $\mathcal{P}_\kappa\theta$ and $\mathcal{P}_\kappa\lambda$ which was introduced by Abe [**2**].

Take a bijection $h : \theta \to \mathcal{P}_\kappa\lambda$. Define $\rho : \mathcal{P}_\kappa\theta \to \mathcal{P}_\kappa\lambda$ and $\pi : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\theta$ by

$$\rho(y) = \bigcup h''y\,, \quad \pi(x) = h^{-1}Q_x$$

and set

$$D = \{x \in \mathcal{P}_\kappa\lambda \mid 2 \subset x\}\,, \quad E = \pi''D\,.$$

The following can be easily checked (see [**10**, Lemma 3.6] for (2)).

(1) $\pi \restriction D$ is a bijection from D to E and $\rho\pi \restriction D$ is an identity function.

(2) $X \in \boldsymbol{S}(NS_{\kappa,\lambda})$ iff $\pi''X \in NS_{\kappa,\theta}$, for all $X \subset \mathcal{P}_\kappa\lambda$.

LEMMA 6.2. *Let* $\mathcal{I}$ *be an ideal on* $\mathcal{P}_\kappa\theta$ *such that* $E \in \mathcal{I}^*$. *Then,*

(1) $NSI(\rho_*(\mathcal{I})) = \rho_*(NI(\mathcal{I}))$.

(2) $NP(\rho_*(\mathcal{I})) = \rho_*(NP(\mathcal{I}))$.

(3) $\boldsymbol{S}(\rho_*(\mathcal{I})) = \rho_*(\nabla^2(\mathcal{I}))$.

PROOF. Let $X \subset \mathcal{P}_\kappa\lambda$. Note that $D \in \rho_*(\mathcal{I})^*$, since $E = \pi''D \subset \rho^{-1}D$. So, we may assume that $X \subset D$. Set $Y = \pi''X$ $(= E \cap \rho^{-1}X)$.

($1\subset$) Let $X \in NSI(\rho_*(\mathcal{I}))$. Take $f_x : Q_x \to 2$ (for $x \in X$) such that

$$\{x \in X \mid f_x \subset f\} \in \rho_*(\mathcal{I})\,, \quad \text{for any } f : \mathcal{P}_\kappa\lambda \to 2.$$

For each $y \in Y$, define $g_y : y \to 2$ by

$$g_y(\alpha) = f_{\rho(y)}(h(\alpha))\,.$$

CLAIM 1. $\{y \in Y \mid g_y \subset g\} \in \mathcal{I}$, *for all* $g : \theta \to 2$.

PROOF OF CLAIM 1. Let $g : \theta \to 2$. Define $f : \mathcal{P}_\kappa\lambda \to 2$ by

$$f(u) = g(h^{-1}(u))$$

and set

$$X' = \{x \in X \mid f_x \subset f\}\,.$$

Since $X' \in \rho_*(\mathcal{I})$, it holds that $\rho^{-1}X' \in \mathcal{I}$. Since $\{y \in Y \mid g_y \subset g\} \subset \rho^{-1}X'$, we have that $\{y \in g \mid g_y \subset \mathcal{I}\} \in \mathcal{I}$. QED of Claim 1

By Claim 1, $\rho^{-1}X \cap E = Y \in NI(\mathcal{I})$. $\therefore$ $\rho^{-1}X \in NI(\mathcal{I})$. $\therefore$ $X \in \rho_*(NI(\mathcal{I}))$.

(1$\supset$) Let $X\in\rho_*(NI(\mathcal{J}))$. Since $Y\in NI(\mathcal{J})$, we can pick $g_y: y\to 2$ (for $y\in Y$) such that

$$\{y \in Y \mid g_y \subset g\} \in \mathcal{J}, \quad \text{for all } g: \theta \to 2.$$

Define $f_x: Q_x\to 2$ (for $x\in X$) by

$$f_x(u) = g_{\pi(x)}(h^{-1}(u)).$$

CLAIM 2. *$\{x\in X \mid f_x\subset f\}\in\rho_*(\mathcal{J})$, for all $f: \mathcal{P}_\kappa\lambda\to 2$.*

PROOF OF CLAIM 2. Let $f: \mathcal{P}_\kappa\lambda\to 2$. Define $g: \theta\to 2$ by

$$g(\alpha) = f(h(\alpha)).$$

Then, it holds that

$$E\cap\rho^{-1}\{x \in X \mid f_x \subset f\} \subset \{y \in Y \mid g_y \subset g\}.$$

So $\rho^{-1}\{x\in X \mid f_x\subset f\}\in\mathcal{J}$. Hence $\{x\in X \mid f_x\subset f\}\in\rho_*(\mathcal{J})$. QED of Claim 2

By Claim 2, it holds that $X\in NSI(\rho_*(\mathcal{J}))$.

(2, 3) Similar to (1). □

Define $C\subset\mathcal{P}_\kappa\theta$ by

$$C = \{y \in \mathcal{P}_\kappa\theta \mid \forall\alpha \in y\ (h(\alpha) \in Q_{p(y)}) \text{ and } p(y) = \rho(y)\}.$$

Note that C is a club subset of $\mathcal{P}_\kappa\theta$ and $\forall y\in C\ (y\subset\pi p(y))$.

LEMMA 6.3. *$E \in \boldsymbol{S}(NS_{\kappa,\theta})^*$.*

PROOF. To get a contradiction, assume that $E\notin\boldsymbol{S}(NS_{\kappa,\theta})^*$. Then, it holds that

$$Y = C\setminus E \in \boldsymbol{S}(NS_{\kappa,\theta})^+ \text{ and } \forall y \in Y\ (y \subset \pi p(y) \text{ and } \pi p(y) \neq y).$$

For each $y\in Y$, take $u_y\in Q_{p(y)}$ such that $h^{-1}(u_y)\notin y$. Since $\forall y\in Y\ (u_y\in Q_{p(y)} \subset Q_y)$, we can choose $u\in\mathcal{P}_\kappa\lambda$ such that

$$Z = \{y \in Y \mid u_y = u\} \in \boldsymbol{S}(NS_{\kappa,\theta})^+.$$

Set $\alpha=h^{-1}(u)$. Since Z is unbounded, there is a $y\in Z$ such that $\alpha\in y$. Since $u_y=u$, we have $\alpha=h^{-1}(u)=h^{-1}(u_y)\notin y$. This is a contradiction. □

LEMMA 6.4.

(1) $\rho_*(NI(NS_{\kappa,\theta})) = NSI(\boldsymbol{S}(NS_{\kappa,\lambda}))$. *I.e.*, $\rho_*(\mathcal{J}_1(\kappa, \theta)) = \mathcal{J}_1(\kappa, \lambda)$.

(2) $\rho_*(NP(NS_{\kappa,\theta})) = NP(\boldsymbol{S}(NS_{\kappa,\lambda}))$. *I.e.*, $\rho_*(\mathcal{K}_1(\kappa, \theta)) = \mathcal{L}_1(\kappa, \lambda)$.

PROOF. Define the ideal $\mathcal{H}$ on $\mathcal{P}_\kappa\theta$ by

$$\mathcal{H} = \{Y \subset \mathcal{P}_\kappa\theta \mid Y\cap E \in NS_{\kappa,\theta}\}.$$

CLAIM 1. $\rho_*(\mathcal{H})=\boldsymbol{S}(NS_{\kappa,\lambda})$.

PROOF OF CLAIM 1. Let $X\subset\mathcal{P}_\kappa\lambda$. Then, it holds that

$$X\in\rho_*(\mathcal{H}) \text{ iff } \rho^{-1}X\in\mathcal{H} \text{ iff } \rho^{-1}X\cap E\in NS_{\kappa,\theta} \text{ iff } \pi''X\in NS_{\kappa,\theta} \text{ iff } X\in\boldsymbol{S}(NS_{\kappa,\lambda}).$$

QED of Claim 1

CLAIM 2. *$NI(NS_{\kappa,\theta})=NI(\mathcal{H})$ and $NP(NS_{\kappa,\theta})=NP(\mathcal{H})$.*

PROOF OF CLAIM 2. Since $\theta^{<\kappa}=\theta$, by a result of Carr [**5**, Theorem 3.6(2)], $NI(NS_{\kappa,\theta})$ is strongly normal. So, $E\in NI(NS_{\kappa,\theta})^*\subset NP(NS_{\kappa,\theta})^*$. This claim directly follows from this. QED of Claim 2

By Claims 1, 2 and Lemma 6.2, we have that

$$\rho_*(NI(NS_{\kappa,\theta}))=\rho_*(NI(\mathcal{H}))=NSI(\rho_*(\mathcal{H}))=NSI(\boldsymbol{S}(NS_{\kappa,\lambda}))$$

and that

$$\rho_*(NP(NS_{\kappa,\theta}))=\rho_*(NP(\mathcal{H}))=NP(\rho_*(\mathcal{H}))=NP(\boldsymbol{S}(NS_{\kappa,\lambda})). \qquad \square$$

By induction on $\alpha>0$, using Lemmas 6.2, 6.4, we can prove that

$$\mathcal{I}_\alpha(\kappa,\lambda)=\rho_*(\mathcal{I}_\alpha(\kappa,\theta)) \text{ and } \mathcal{L}_\alpha(\kappa,\lambda)=\rho_*(\mathcal{K}_\alpha(\kappa,\theta)).$$

So, the following fact completes the proof of Theorem 6.1.

FACT. For any ideal $\mathcal{I}$ on S and any $f, g: S\to T$, if $\{s\in S\mid f(x)=g(x)\}\in\mathcal{I}^*$, then $f_*(\mathcal{I})=g_*(\mathcal{I})$.

COROLLARY 6.5. *Let $\kappa\leqq\lambda<\delta$ and $p:\mathcal{P}_\kappa\delta\to\mathcal{P}_\kappa\lambda$ the projection. Then, it holds that*

$$p_*(\mathcal{I}_\alpha(\kappa,\delta))\supset\mathcal{I}_\alpha(\kappa,\lambda), \quad \textit{for all } \alpha>0.$$

PROOF. Let $\theta=\lambda^{<\kappa}$ and $r:\mathcal{P}_\kappa\theta\to\mathcal{P}_\kappa\lambda$ the projection. The case of that κ is not δ-ineffable is trivial. We may assume that κ is δ-ineffable. Then, by a result of Johnson [**9**, Corollary 2.6], it holds that $\theta=\lambda^{<\kappa}\leqq\lambda^+\leqq\delta$. So, let $q:\mathcal{P}_\kappa\delta\to\mathcal{P}_\kappa\theta$ be the projection.

Let $\alpha>0$. Then, it holds that $\mathcal{I}_\alpha(\kappa,\theta)\subset q_*(\mathcal{I}_\alpha(\kappa,\delta))$. By Theorem 6.1, it holds that $\mathcal{I}_\alpha(\kappa,\theta)=r_*(\mathcal{I}_\alpha(\kappa,\theta))$. By this and the fact that $p_*=r_*q_*$, we have that $p_*(\mathcal{I}_\alpha(\kappa,\delta))\supset\mathcal{I}_\alpha(\kappa,\lambda)$. $\square$

7. Main Theorem.

This section is devoted to the proofs of Theorems 1, 3 which were mentioned in section 1. In the proofs, we need an ω-Jónsson function. For any set A, $F: {}^\omega A\to A$ is called an *ω-Jónsson function for A*, if $\forall S\subset A$ (if $|S|=|A|$, then $F''^{\omega}S=A$). Erdös and Hajnal [7] showed that every infinite set has an ω-Jónsson function. Johnson [9, Lemma 2.3] proved that, if F is an ω-Jónsson function for A, then $\{x\in\mathcal{P}_{\kappa}A \mid F\restriction^{\omega}x$ is an ω-Jónsson function for $x\}\in NI(I_{\kappa,A})^*$. (In fact, she proved a stronger result.)

Theorem 1 follows directly from the following theorem.

THEOREM 7.1. *If $\nabla^2\mathcal{G}_{\lambda}(\kappa, \lambda)$ is a proper ideal, then $\bigcup_{\xi<\lambda}\mathcal{L}_{\xi}(\kappa, \lambda)$ is proper.*

PROOF. To get a contradiction, assume that there exists κ such that

$$\exists\lambda\geqq\kappa\ (\nabla^2\mathcal{G}_{\lambda}(\kappa, \lambda) \text{ is proper and } \bigcup_{\xi<\lambda}\mathcal{L}_{\xi}(\kappa, \lambda) \text{ is not proper}).$$

Take the least such κ, and let $\lambda\geqq\kappa$ such that

$$\nabla^2\mathcal{G}_{\lambda}(\kappa, \lambda) \text{ is proper and } \bigcup_{\xi<\lambda}\mathcal{L}_{\xi}(\kappa, \lambda) \text{ is not proper.}$$

We denote $\nabla^2\mathcal{G}_{\lambda}(\kappa, \lambda)$ by $\mathcal{M}$. Note that $\{x\in\mathcal{P}_{\kappa}\lambda \mid x\cap\kappa$ is a regular cardinal$\}\in S(NS_{\kappa,\lambda})^*\subset\mathcal{M}^*$. Take an ω-Jónsson function F for λ. Define $X\subset\mathcal{P}_{\kappa}\lambda$ by

$$X=\{x\in\mathcal{P}_{\kappa}\lambda \mid x \text{ satisfies } (0)\sim(2)\}, \text{ where}$$

(0) $\forall x\in X$ (ot(x) and $x\cap\kappa$ are cardinals).
(1) $\forall x\in X$ ($F\restriction^{\omega}x$ is ω-Jónsson for x).
(2) $\forall\alpha, \beta<\lambda$ (if $\mathrm{cof}(\alpha)=\beta$, then $\forall x\in X\cap\widehat{\{\alpha\}}$ ($\beta\in x$ and $\mathrm{cof}(x\cap\alpha)=\mathrm{ot}(x\cap\beta)$)).

It holds that $X\in\mathcal{I}_1(\kappa, \lambda)^*\subset\mathcal{M}^*$. Set $X_1=\{x\in X \mid \mathcal{I}_{\mathrm{ot}(x)}(x\cap\kappa, x)$ is proper$\}$. By Corollary 4.8, it holds that $X_1\in\mathcal{M}^*$. Set $X_2=\{x\in X_1 \mid \bigcup_{\xi<\mathrm{ot}(x)}\mathcal{L}_{\xi}(x\cap\kappa, x)$ is not proper$\}$. Since $\mathcal{P}_{\kappa}\lambda\in\bigcup_{\alpha<\lambda}\mathcal{L}_{\alpha}(\kappa, \lambda)$, by Corollary 5.5, we have that $X_2\in\mathcal{M}^*$. By the leastness of κ, we have that

$$\forall x\in X_2\ (\nabla^2\mathcal{G}_{\mathrm{ot}(x)}(x\cap\kappa, x) \text{ is not proper}).$$

CLAIM 1. $\forall(x, y)\in[X_2]^2$ $(x\in Q_y)$.

PROOF OF CLAIM 1. Let $(x, y)\in[X_2]^2$. Since $F\restriction^{\omega}x$, $F\restriction^{\omega}y$ are ω-Jónsson for x, y, respectively, it holds that $|x|<|y|$. To get a contradiction, assume that $y\cap\kappa\leqq\mathrm{ot}(x)$. Then, $\mathcal{I}_{y\cap\kappa}(x\cap\kappa, y\cap\kappa)$ is proper. Since, $\mathcal{I}_{\mathrm{ot}(y)}(y\cap\kappa, y)$ is proper, by Corollary 4.4, $\mathcal{I}_{\mathrm{ot}(y)}(x\cap\kappa, y)$ is proper. So, by Corollary 6.5, $\mathcal{G}_{\mathrm{ot}(y)}(x\cap\kappa, x)$ is proper. This is a contradiction, since $\nabla^2\mathcal{G}_{\mathrm{ot}(x)}(x\cap\kappa, x)\subset\mathcal{G}_{\mathrm{ot}(y)}(x\cap\kappa, x)$. QED of Claim 1

Since $X_2\in\mathcal{M}^*\subset\mathcal{M}^+$, by the Claim and Corollary 5.7, it holds that $X_2\in\bigcup_{\xi<\lambda}\mathcal{L}_{\xi}(\kappa, \lambda)^+$. This contradicts the fact that $\bigcup_{\xi<\lambda}\mathcal{L}_{\xi}(\kappa, \lambda)$ is not proper. □

COROLLARY 7.2 (Theorem 1). *If κ is completely $\lambda^{<\kappa}$-ineffable, then* part*$(\kappa, \lambda^{<\kappa})$ *holds.*

PROOF. Assume that κ is completely $\lambda^{<\kappa}$-ineffable. Then, it holds that

$$\mathcal{J}_\alpha(\kappa, \lambda^{<\kappa}) \text{ is proper, for all } \alpha \in \boldsymbol{On}.$$

So, by Theorem 6.1(1), it holds that

$$\mathcal{I}_\alpha(\kappa, \lambda) \text{ is proper, for all } \alpha \in \boldsymbol{On}.$$

So, by the previous theorem, $\bigcup_{\xi<\lambda} \mathcal{L}_\xi(\kappa, \lambda)$ is proper. So, by Theorem 6.1(2), we have that

$$\bigcup_{\xi<\lambda} \mathcal{K}_\xi(\kappa, \lambda^{<\kappa}) \text{ is proper.}$$

Especially, $\mathcal{K}_1(\kappa, \lambda^{<\kappa})$ is proper. I.e., part*$(\kappa, \lambda^{<\kappa})$ holds. □

DEFINITION 7.1. For any cardinal δ and ordinal α, $\delta^{+\alpha}$ denotes the cardinal which is defined by induction on α as follows:

$$\delta^{+0} = \delta,$$

$$\delta^{+(\alpha+1)} = (\delta^{+\alpha})^+,$$

$$\delta^{+\alpha} = \sup\{\delta^{+\beta} \mid \beta < \alpha\}, \quad \text{for any limit ordinal } \alpha.$$

Recall the statement of Theorem 2 from section 1.

THEOREM 2. *Assume that there is an $\alpha<\kappa$ such that $\forall\delta<\kappa$ $(2^\delta \leqq \delta^{+\alpha})$. Then, if κ is $\lambda^{<\kappa}$-ineffable, then* part*$(\kappa, \lambda^{<\kappa})$ *holds.*

In order to prove this theorem, we need the following Lemma.

LEMMA 7.3. *Under the same assumption in Theorem 2, there exists an $\mathcal{S} \subset \mathcal{P}(\kappa)$ such that*

(1) *$\mathcal{S}$ is a partition of the set of infinite cardinals below κ and $|\mathcal{S}|<\kappa$.*

(2) *For any $S \in \mathcal{S}$ and any δ, $\eta \in S$, if $\delta<\eta$, then $2^{\delta^+}<\eta$.*

PROOF. Easy. □

PROOF OF THEOREM 2. Since $(\lambda^{<\kappa})^{<\kappa}=\lambda^{<\kappa}$, it suffices to show that

(*) $\forall\lambda \geqq \kappa$ (if κ is $\lambda^{<\kappa}$-ineffable, then part*(κ, λ) holds).

To get a contradiction, assume that (*) does not hold.

Take $\lambda \geqq \kappa$ such that κ is $\lambda^{<\kappa}$-ineffable and part*(κ, λ) fails. Take an ω-Jónsson function F for λ. Define $X \subset \mathcal{P}_\kappa\lambda$ by

$$X = \{x \in \mathcal{P}_\kappa\lambda \mid x \text{ satisfies the following (0)}\sim\text{(3)}\}.$$

(0) $x \cap \kappa$ and $\mathrm{ot}(x)$ are cardinals.

(1) $F \restriction {}^\omega x$ is ω-Jónsson for x.

(2) $x \cap \kappa$ is almost x-ineffable.

(3) part*$(x \cap \kappa, x)$ fails.

By [**10**, Theorem 4.1] and Corollary 5.4, $X_0 \in \mathcal{I}_1(\kappa, \lambda)^*$. Take $\mathcal{S} \subset \mathcal{P}(\kappa)$ which satisfies (1), (2) in Lemma 7.3. For each $S \in \mathcal{S}$, set

$$X_S = \{x \in X \mid \mathrm{ot}(x) \in S\}.$$

Since $\mathcal{J}_1(\kappa, \lambda)$ is κ-complete, there exists an $S \in \mathcal{S}$ such that $X_S \in \mathcal{J}_1(\kappa, \lambda)^+$.

CLAIM 1. $\forall(x, y) \in [X_S]^2\ (x \in Q_y)$.

PROOF OF CLAIM 1. Let $(x, y) \in [X_S]^2$. By (1), we have $|x| < |y|$. So, $2^{|x|^+} \leqq |y|$. To get a contradiction, assume that $|y \cap \kappa| \leqq |x|$. Then, $x \cap \kappa$ is almost $y \cap \kappa$-ineffable. So, $x \cap \kappa$ is almost y-ineffable. Since $2^{|x|^+} \leqq |y|$, by a result of Carr [**4**, Theorem 3.2], $x \cap \kappa$ is x-supercompact. This contradicts that part*$(x \cap \kappa, x)$ fails. QED of Claim 1

Since $X_S \in \mathcal{J}_1(\kappa, \lambda)^+$, by Claim 1, we have that $X_S \in \mathcal{L}_1(\kappa, \lambda)^+$. This contradicts to that part*(κ, λ) fails. □

COROLLARY 7.4. *Under the same assumption in Theorem* 2, *for any* $\kappa \leqq \delta < \lambda$, *if* part*$(\kappa, \lambda)$ *holds, then* part*(κ, δ) *holds.*

PROOF. Let $\kappa \leqq \delta < \lambda$ and part*(κ, λ). By a result of Magidor [**11**], κ is λ-ineffable. So, by a result of Johnson [**9**], $\delta^{<\kappa} \leqq \delta^+ \leqq \lambda$. Hence, κ is $\delta^{<\kappa}$-ineffable. So, by Theorem 2, part*(κ, δ) holds. □

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Shizuo KAMO
Department of Mathematics
University of Osaka Prefecture
Gakuen-chou, Sakai
Japan
(e-mail: kamo@center.osakafu-u.ac.jp)