

Involution and anti involution on Hopf spaces

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1. Introduction.

A pointed space X with a base point preserving map $\mu: X \times X \rightarrow X$ is said to be a Hopf space if the restriction of μ to $X \vee X$ is homotopic to the folding map ∇ . A pointed $\mathbf{Z}/2\mathbf{Z}$ space X is said a Hopf space with involution if X is a Hopf space and the structure map $\mu: X \times X \rightarrow X$ is equivariant and the restriction of μ to $X \vee X$ is equivariantly homotopic to the folding map ∇ .

The concept of a Hopf space came from that of a topological group and a topological group G has an involution τ defined by $\tau(g) = g^{-1}$ for $g \in G$. The product of G is not equivariant but it has the property

$$\tau(xy) = \tau(y)\tau(x).$$

This leads us to the definition of Hopf space with anti involution. That is, a $\mathbf{Z}/2\mathbf{Z}$ space X which is a Hopf space is said to be a Hopf space with anti involution if the structure map $\mu: X \times X \rightarrow X$ is equivariant with respect to the involution $\tilde{\tau}$ defined by $\tilde{\tau}(x, y) = (\tau y, \tau x)$, and the restriction of μ to $X \vee X$ is equivariantly homotopic to ∇ .

Adams showed that S^n has a structure of Hopf space if and only if $n=0, 1, 3, 7$. In [4], Iriye showed that the unit sphere of a orthogonal representation space V of $\mathbf{Z}/2\mathbf{Z}$ admits a structure of a Hopf space with involution if and only if V is $\mathbf{R}^{1,1}$, $\mathbf{R}^{2,2}$, $\mathbf{R}^{4,4}$, $\mathbf{R}^{0,1}$, $\mathbf{R}^{0,2}$, $\mathbf{R}^{0,4}$ or $\mathbf{R}^{0,8}$.

In this paper we shall show that a $\mathbf{Z}/2\mathbf{Z}$ homology sphere with involution of which 'type' has a structure of Hopf space with involution or anti involution. From Quillen's localization theorem (see §4) we have that the fixed point set of a $\mathbf{Z}/2\mathbf{Z}$ homology sphere with involution is also a $\mathbf{Z}/2\mathbf{Z}$ homology sphere. Therefore we say that a $\mathbf{Z}/2\mathbf{Z}$ homology n sphere whose fixed point set is a $\mathbf{Z}/2\mathbf{Z}$ homology m sphere is of type (n, m) .

THEOREM. *Let $d=1, 2$ or 4 . There exists a $\mathbf{Z}/2\mathbf{Z}$ space X of type $(2d-1, p)$ which is a Hopf space with involution, if and only if $p=d-1, 2d-1$. There exists a $\mathbf{Z}/2\mathbf{Z}$ space X of type $(2d-1, p)$ which is a Hopf space with anti involution, if and only if $p=0, d$.*

This paper is organized as follows: In §2 we define Hopf space with involution or anti involution and offer examples. In §3 we introduce the Hopf construction of a Hopf space with involution or anti involution and show that its Hopf invariant is one. Also localization theorem plays an important role in this paper. Thus we refer to this theorem in §4. Then in §5, we shall show when an equivariant map from S^{4d-1} to S^{2d} with involutions has Hopf invariant one. This leads us to previous theorem.

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2. Hopf space with involution, anti involution.

Let (X, μ) be a Hopf space. Suppose that X has an involution τ , that is, X is a $\mathbf{Z}/2\mathbf{Z}$ space.

If $\tau\mu(x, y) = \mu(\tau x, \tau y)$ for all $x, y \in X$ and the restriction of μ to $X \vee X$ is equivariantly homotopic to ∇ , then we say (X, μ, τ) is a Hopf space with involution.

If $\tau\mu(x, y) = \mu(\tau y, \tau x)$ for all $x, y \in X$ and the restriction of μ to $X \vee X$ is equivariantly homotopic to ∇ where the involution of $X \times X$ is defined by

$$\tilde{\tau}(x, y) = (\tau y, \tau x),$$

then we say (X, μ, τ) is a Hopf space with anti involution.

EXAMPLE 2.1. The unit spheres of $\mathbf{R}^{1,1}$, $\mathbf{R}^{2,2}$, $\mathbf{R}^{4,4}$ are Hopf spaces with involution. See Iriye [4].

EXAMPLE 2.2. Let G be a Lie group. Then the ordinary product of G makes G a Hopf space. G has an involution $\tau: G \rightarrow G$ defined by $\tau(x) = x^{-1}$. Thus G with τ is a Hopf space with anti involution.

EXAMPLE 2.3. $GL(n, K)$ with the involution $\tau: GL(n, K) \rightarrow GL(n, K)$ defined by $\tau(A) = {}^t A$ is a Hopf space with anti involution by the ordinary product.

EXAMPLE 2.4. We regard S^3 (resp. S^7) as the unit sphere in \mathbf{H} (resp. the Cayley numbers \mathbf{O}). Then the involution τ defined by $\tau(x) = \bar{x}$ makes S^3 (resp. S^7) a Hopf space with anti involution by the ordinary product in \mathbf{H} (resp. \mathbf{O}).

3. Hopf construction with involution.

Given a map $\mu: A \times B \rightarrow C$, the Hopf construction $H(\mu): A * B \rightarrow \Sigma C$ is defined by $H(\mu)(a, t, b) = (t, \mu(a, b))$, where $a \in A$, $b \in B$, $t \in [0, 1]$.

Suppose that (X, μ, τ) is a Hopf space with involution or anti involution. We introduce involutions to $X * X$, ΣX so as to make $H(\mu): X * X \rightarrow \Sigma X$ equi-

variant. We define involutions $\tau'_0, \tau'_1: X * X \rightarrow X * X$ and $\tau''_0, \tau''_1: \Sigma X \rightarrow \Sigma X$ as follows:

$$\begin{aligned} \tau'_0(x, t, y) &= (\tau x, t, \tau y) & \tau'_1(x, t, y) &= (\tau y, 1-t, \tau x) & x, y \in X, t \in [0, 1] \\ \tau''_0(t, x) &= (t, \tau x) & \tau''_1(t, x) &= (1-t, \tau x) & x \in X, t \in [0, 1]. \end{aligned}$$

If (X, μ, τ) is a Hopf space with involution then $H(\mu): (X * X, \tau'_0) \rightarrow (\Sigma X, \tau''_0)$ is equivariant. If (X, μ, τ) is a Hopf space with anti involution then $H(\mu): (X * X, \tau'_1) \rightarrow (\Sigma X, \tau''_1)$ is equivariant.

Now (X, μ) is a Hopf space and let $f=H(\mu)$. In the following we use $\mathbf{Z}/2\mathbf{Z}$ as the coefficient ring of cohomology rings unless mentioned.

THEOREM 3.5 (E. Thomas [6]). *Let (X, μ) be a Hopf space and C_f be the mapping cone of $f=H(\mu)$. Consider the next exact sequence:*

$$\longrightarrow H^*(\Sigma(X * X)) \longrightarrow H^*(C_f) \longrightarrow H^*(\Sigma X) \longrightarrow H^*(X * X).$$

If $u, v \in H^*(X)$ are classes such that $\Sigma u, \Sigma v \in H^*(\Sigma X)$ pull back to $H^*(C_f)$ then $\Sigma u \cup \Sigma v \in H^*(C_f)$ comes from $\Sigma(u * v) \in H^*(\Sigma(X * X))$ which is isomorphic to $H^*(C_f, \Sigma X)$.

THEOREM 3.6. *If σ^{n-1} is a $\mathbf{Z}/2\mathbf{Z}$ homology $n-1$ sphere and (σ^{n-1}, μ) is a Hopf space, then*

$$H^*(C_f) = \begin{cases} \mathbf{Z}/2\mathbf{Z}[x]/(x^3) & n > 1 \\ \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \text{ or } \mathbf{Z}/2\mathbf{Z} & n = 1. \end{cases}$$

PROOF. Consider the next exact sequence:

$$\longrightarrow \tilde{H}^*(\Sigma(\sigma^{n-1} * \sigma^{n-1})) \longrightarrow \tilde{H}^*(C_f) \longrightarrow \tilde{H}^*(\Sigma \sigma^{n-1}) \longrightarrow \tilde{H}^*(\sigma^{n-1} * \sigma^{n-1}).$$

And remark $H^*(\sigma^{n-1} * \sigma^{n-1}) \cong H^*(S^{2n-1})$, $H^*(\Sigma \sigma^{n-1}) \cong H^*(S^n)$, $n \neq 2n-1$. Then we have

$$H^*(C_f) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & * = 0, n, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Let u be the generator of $H^{n-1}(\sigma^{n-1})$ then Σu comes from x the generator of $H^n(C_f)$ and, by Theorem 3.5, $x \cup x$ comes from $\Sigma(u * u)$ the generator of $H^{2n}(\Sigma(\sigma^n * \sigma^n))$. Hence $x \cup x$ is the generator of $H^{2n}(C_f)$.

For the latter part of the theorem, consider the same exact sequence and remark that $n=2n-1=1$:

$$\longrightarrow \tilde{H}^*(\Sigma(\sigma^{n-1} * \sigma^{n-1})) \longrightarrow \tilde{H}^*(C_f) \longrightarrow \tilde{H}^*(\Sigma \sigma^{n-1}) \longrightarrow \tilde{H}^*(\sigma^{n-1} * \sigma^{n-1}).$$

We obtain that f^* is a 0 map or an isomorphism. Hence, $H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$ or $\mathbf{Z}/2\mathbf{Z}$ respectively.

4. Localization theorem.

For a compact Lie group G and a G space X with a fixed base point $*$, we define $H_G^*(X; A)$ as follows. Let $PG \rightarrow BG$ be the universal G bundle

$$H_G^*(X; A) \equiv H^*(PG \times_G X; A)$$

where A is a ring. The reduced equivariant cohomology $\tilde{H}_G^*(X; A)$ is defined as follows:

$$\begin{aligned} \tilde{H}_G^*(X; A) &\equiv H^*(PG \times_G X, PG \times_{G*}; A) \\ &= \ker s^* \end{aligned}$$

where s is the section of $PG \times_G X \rightarrow BG$ defined by

$$s(x) = (y, *) \quad \text{where } x \in BG, y \in p^{-1}(x).$$

In the following we only consider the case $G = \mathbf{Z}/2\mathbf{Z}$ and $A = \mathbf{Z}/2\mathbf{Z}$. In that case, $H_G^* \equiv H^*(BG) = \mathbf{Z}/2\mathbf{Z}[t]$.

REMARK. If G acts on X trivially, then $PG \times_G X = BG \times X$. Hence $H_G^*(X) = H^*(X) \otimes_{\mathbf{Z}/2\mathbf{Z}} H_G^*$.

We refer to the next theorem (Quillen [3]).

THEOREM 4.7 (Localization Theorem). *If X is a compact G space, then the inclusion i of the fixed point set X^G into X induces an isomorphism*

$$\tilde{H}_G^*(X)[t^{-1}] \simeq \tilde{H}_G^*(X^G)[t^{-1}]$$

where $\tilde{H}_G^*(X)[t^{-1}]$ means the localization of $\tilde{H}_G^*(X)$ by t^{-1} .

From Localization Theorem we obtain two propositions.

PROPOSITION 4.8. *If X is a compact G space and $H^*(X; \mathbf{Z}/2\mathbf{Z}) \cong H^*(S^l; \mathbf{Z}/2\mathbf{Z})$ for some $l \geq 0$ and $X^G \neq \emptyset$, then $H^*(X^G) \cong H^*(S^m)$ for some $m \leq l$.*

PROOF. See Bredon [2].

PROPOSITION 4.9 *Let X be a compact G space. If $X^G \neq \emptyset$ and $H^*(X; \mathbf{Z}/2\mathbf{Z})$ is generated by one element as a graded $\mathbf{Z}/2\mathbf{Z}$ algebra, then*

$$i^*: \tilde{H}_G^*(X) \longrightarrow \tilde{H}_G^*(X^G) \text{ is monic.}$$

PROOF. First we prove that $\tilde{H}_G^*(X)$ is a free H_G^* module. Consider the Serre spectral sequence $E_i^{j,k}$ of the fiber space $PG \times_G X \rightarrow BG$.

Let x denote the generator of $H^*(X)$ and the degree of x be m . Then

$$E_2^{p,q} = 0 \quad (q \notin m\mathbf{Z}).$$

Also, X has a fixed point and $p: \mathbf{P}G \times_G X \rightarrow \mathbf{B}G$ has a section s . Thus $p^*: H^*(\mathbf{B}G) \rightarrow H^*(\mathbf{P}G \times_G X)$ is monic. The image of p^* is $\bigoplus_{p \geq 0} E_{\infty}^{2,0}$. Therefore we have that

$$d_m^{0,m}(1 \otimes x) = 0.$$

Since x generates $H^*(X)$, the Serre spectral sequence is trivial. Therefore $H_G^*(X)$ is a free H_G^* module. $\tilde{H}_G^*(X)$ is a submodule of $H_G^*(X)$ and it follows that $\tilde{H}_G^*(X)$ is a free H_G^* module.

Then consider the following commutative diagram

$$\begin{array}{ccc} \tilde{H}_G^*(X) & \longrightarrow & \tilde{H}_G^*(X)[t^{-1}] \\ \downarrow i^* & & \cong \downarrow i^* \\ \tilde{H}_G^*(X^G) & \longrightarrow & \tilde{H}_G^*(X)[t^{-1}]. \end{array}$$

In the diagram the arrows which goes down means i^* and the arrows which goes across are monic since $\tilde{H}_G^*(X)$ and $\tilde{H}_G^*(X^G)$ are free H_G^* modules. Hence by localization theorem we have that $i^*: \tilde{H}_G^*(X) \rightarrow \tilde{H}_G^*(X^G)$ is monic.

5. Proof of main theorem.

Let $S^{n,m}$ mean the set of G isomorphism classes of all $\mathbf{Z}/2\mathbf{Z}$ homology n spheres which are compact G spaces and whose fixed point sets are $\mathbf{Z}/2\mathbf{Z}$ homology m spheres. (Proposition 4.8 says that if X is a compact $\mathbf{Z}/2\mathbf{Z}$ space and at the same time a $\mathbf{Z}/2\mathbf{Z}$ homology n sphere, then $X \in S^{n,m}$ for some m .)

THEOREM 5.10. *Let $d=1, 2$ or 4 . There exist $\sigma^{4d-1,q} \in S^{4d-1,q}$, $\sigma^{2d,q'} \in S^{2d,q'}$ and a continuous G map $f: \sigma^{4d-1,q} \rightarrow \sigma^{2d,q'}$ such that*

$$H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \quad \text{where } |x| = 2d,$$

that is, the Hopf invariant is one, if and only if $(q, q') = (4d-1, 2d)$ or $(2d-1, d)$ or $(2d-1, 0)$.

REMARK. Adams' theorem of Hopf invariant one tells us that the assumptions on dimensions in Theorems 5.10 and 5.11 are inessential.

PROOF. First we assume that there is a continuous G map $f: \sigma^{4d-1,q} \rightarrow \sigma^{2d,q'}$ such that $H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$, $|x| = 2d$, where $\sigma^{4d-1,q} \in S^{4d-1,q}$ and $\sigma^{2d,q'} \in S^{2d,q'}$.

By the proof of Proposition 4.9, $\tilde{H}_G^*(C_f)$ and $\tilde{H}_G^*(C_f^G)$ are free H_G^* modules and from the localization theorem the ranks of $\tilde{H}_G^*(C_f)$ and $\tilde{H}_G^*(C_f^G)$ are same. Note that $\tilde{H}_G^*(C_f^G) \cong \tilde{H}^*(C_f^G) \otimes H_G^*$. Therefore we have that

$$\tilde{H}^*(C_f^G) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}. \tag{1}$$

Denotes the restricted map of f to G -fixed-point-set by f^G . Then we have the following exact sequence:

$$\begin{aligned} \longrightarrow \tilde{H}^*(\Sigma^{0,1}\sigma^{2d,q^G}) \xrightarrow{\Sigma f^{G*}} \tilde{H}^*(\Sigma^{0,1}\sigma^{4d-1,q^G}) \xrightarrow{k^*} \tilde{H}^*(C_{f^G}) \\ \xrightarrow{j^*} \tilde{H}^*(\sigma^{2d,q^G}) \xrightarrow{f^{G*}} \tilde{H}^*(\sigma^{4d-1,q^G}). \end{aligned}$$

Here $\Sigma^{0,1}\sigma^{2d,q^G}$, $\Sigma^{0,1}\sigma^{4d-1,q^G}$, σ^{2d,q^G} and σ^{4d-1,q^G} are $\mathbf{Z}/2\mathbf{Z}$ homology $q'+1$, $q+1$, q' , q spheres respectively and $C_{f^G}=C_f^G$. Then from (1) f^{G*} and Σf^{G*} are 0 maps. Now we obtain that

$$\tilde{H}^*(C_{f^G}) \cong \tilde{H}^{q+1}(\Sigma^{0,1}\sigma^{4d-1,q^G}) \oplus \tilde{H}^{q'}(\sigma^{2d,q^G}).$$

Let y be the image of the generator of $\tilde{H}^{q+1}(\Sigma^{0,1}\sigma^{4d-1,q^G})$ and y' be a pull back of the generator of $\tilde{H}^{q'}(\sigma^{2d,q^G})$. Therefore $\tilde{H}^*(C_{f^G})$ is a $\mathbf{Z}/2\mathbf{Z}$ vector space generated by y and y' . We consider y and y' to be elements of $\tilde{H}_\mathbb{C}^*(C_{f^G})$ by the isomorphism $\tilde{H}_\mathbb{C}^*(C_{f^G}) \cong \tilde{H}^*(C_{f^G}) \otimes H_\mathbb{C}^*$ as an algebra over the mod 2 Steenrod algebra.

And also we can consider x, x^2 to be elements of $\tilde{H}_\mathbb{C}^*(C_f)$ by the isomorphisms

$$\begin{aligned} \tilde{H}_\mathbb{C}^*(C_f) &\cong \bigoplus_{q>0, p \geq 0} E_{\mathbb{C}}^{p,q} \cong \bigoplus_{q>0, p \geq 0} E_2^{p,q} \text{ as an } H_\mathbb{C}^* \text{ module} \\ \bigoplus_{q>0} E_2^{0,q} &\cong \tilde{H}^*(C_f) \text{ as a } \mathbf{Z}/2\mathbf{Z} \text{ module.} \end{aligned}$$

Here x, x^2 are the basis of $\tilde{H}_\mathbb{C}^*(C_f)$ as an $H_\mathbb{C}^*$ module.

$$\begin{aligned} \tilde{H}_\mathbb{C}^*(C_f) &\ni \begin{cases} x & * = 2d \\ x^2 & * = 4d \end{cases} \\ \tilde{H}_\mathbb{C}^*(C_{f^G}) &\ni \begin{cases} y & * = q' \\ y' & * = q+1. \end{cases} \end{aligned}$$

First, it is easily seen that $y'^2=0$ since

$$\begin{aligned} y'^2 &= (k^*((k^*)^{-1}y))^2 \\ &= k^*((k^*)^{-1}y)^2 \\ &= k^*0 = 0. \end{aligned}$$

Now suppose $i^*(x) = at^{2d-d'}y + bt^{2d-(q+1)}y'$ where $a, b \in \mathbf{Z}/2\mathbf{Z}$, $H_\mathbb{C}^* = \mathbf{Z}/2\mathbf{Z}[t]$, $i: C_{f^G} \rightarrow C_f$. Then we have that

$$\begin{aligned} i^*(x^2) &= i^*(x)^2 \\ &= at^{4d-2q'}y^2 + bt^{4d-2(q+1)}y'^2 \\ &= at^{4d-2q'}y^2. \end{aligned}$$

While Proposition 4.9 says i^* is monic. Hence $a \neq 0$ and $y^2 \neq 0$. Thus we have

$$a = 1 \quad \text{and} \quad 2 \deg y = q' \text{ or } q+1.$$

First case. We consider the first case $2 \deg y = q'$ that implies $q' = 0$.
If we assume $b = 0$,

$$i^*(x) = t^{2d}y$$

$$i^*(x^2) = t^{4d}y^2 = t^{4d}y.$$

Thus $\text{Im } i^*$ is contained in the $H_{\mathcal{G}}^{\#}$ pure submodule generated by y which contradicts to the fact that $i^*: \tilde{H}_{\mathcal{G}}^*(C_f)[t^{-1}] \rightarrow \tilde{H}_{\mathcal{G}}^*(C_f^G)[t^{-1}]$ is an isomorphism. Therefore $b = 1$ and we have that

$$i^*(x) = t^{2d}y + t^{2d-(q+1)}y' \quad \text{in } H_{\mathcal{G}}^*(C_f^G).$$

Here $2d \geq q+1$, $\text{Sq}(y') = y'$, $\text{Sq}(y) = y$ and $\text{Sq}(t) = t + t^2$. Assuming that $q+1 \neq 2d$, we shall be led to a contradiction. That is

$$\begin{aligned} \text{Sq}^{2d-(q+1)}(i^*x) &= \sum_{i+j=2d-(q+1)} \text{Sq}^i(t^{2d})\text{Sq}^j(y) + t^{4d-2(q+1)}y' \\ &= t^{4d-2(q+1)}y', \quad \text{since } 2d \text{ is a power of } 2, \end{aligned}$$

$$\begin{aligned} \text{while } i^*(\text{Sq}^{2d-(q+1)}x) &= i^*(0 \text{ or } t^{2d-(q+1)}x) \\ &= 0 \text{ or } t^{4d-(q+1)}y + t^{4d-2(q+1)}y'. \end{aligned}$$

Hence we have $q+1 = 2d$, that is, $q' = 0$, $q = 2d - 1$.

Second case. Next we assume $2q' = q+1 > 0$. Then $(q+1) - q' = q' \leq 2d$, $y^2 = y'$ and $i^*(x) = t^{2d-q'}y + bt^{2d-2q'}y^2$.

a) Now assume that $b = 0$, i.e., $i^*(x) = t^{2d-q'}y$. Then

$$\begin{aligned} i^*(\text{Sq}^{q'}(x)) &= \text{Sq}^{q'}(i^*x) \\ &= \text{Sq}^{q'}(t^{2d-q'}y) \\ &= \binom{2d-q'}{q'} t^{2d}y + t^{2d-q'}y^2. \end{aligned} \tag{2}$$

Here if we suppose that $q' \neq 2d$ ($0 \leq q' < 2d$), then $\text{Sq}^{q'}(x) = 0$ or $t^{q'}x$. Thus

$$i^*(\text{Sq}^{q'}x) = 0 \text{ or } t^{2d}y$$

and this contradicts to (2). Therefore we obtain

$$q' = 2d, \quad q = 4d - 1.$$

b) The last case is that $i^*(x) = t^{2d-q'}y + t^{2d-2q'}y^2$. Here $2d - 2q' \geq 0$, that is, $d \geq q'$ and $2q' = q+1$.

In this case, let us consider the following sequence

$$\sigma^{4d-1, q^G} \longrightarrow \sigma^{2d, q'^G} \longrightarrow C_{fG} \longrightarrow \Sigma^{0,1} \sigma^{4d-1, q^G} \longrightarrow \Sigma^{0,1} \sigma^{2d, q'^G} \longrightarrow$$

$$\text{where } q=2q'-1, H^*(C_{fG})=\mathbf{Z}/2\mathbf{Z}[y]/(y^3).$$

By Adams' theorem of Hopf invariant one, we obtain that $4 \geq d \geq q' = 2^r$ for some $r \geq 0$, that is,

$$(d, q') = (4, 4), (4, 2), (4, 1), (2, 2), (2, 1) \text{ or } (1, 1). \quad (3)$$

If we suppose that $q' \neq d$ ($0 < q' < d$), then $\text{Sq}^{2q'}(x) = 0$ or $t^{2q'}x$. And it follows that

$$i^*(\text{Sq}^{2q'}(x)) = 0 \text{ or } t^{2d+q'}y + t^{2d}y^2.$$

On the other hand, $q' \neq d$ means, by (3) and the direct computation,

$$\begin{aligned} i^*(\text{Sq}^{2q'}(x)) &= \text{Sq}^{2q'}(t^{2d-q'}y + t^{2d-2q'}y^2) \\ &= t^{2d+q'}y. \end{aligned}$$

This is contradiction. Thus we obtained that

$$q' = d, \quad q = 2d - 1.$$

Now we complete the proof of the former part of theorem.

All we have to do is to show the existence of $\sigma^{4d-1, q} \in \mathcal{S}^{4d-1, q}$, $\sigma^{2d, q'} \in \mathcal{S}^{2d, q'}$ and a continuous G map $f: \sigma^{4d-1, q} \rightarrow \sigma^{2d, q'}$ such that

$$H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \quad \text{where } |x| = 2d,$$

for $(q, q') = (4d-1, 2d), (2d-1, d), (2d-1, 0)$. We construct these in the theorem 5.12, 5.13. Q. E. D.

REMARK. A part of previous theorem can be proved by using Bredon's theorem [2, pp. 425-427, Theorem 11.1]. But his proof uses a not obvious fact. Thus we offered our own proof.

THEOREM 5.11. *There exist $\sigma^{1, q} \in \mathcal{S}^{1, q}$, $\sigma^{1, q'} \in \mathcal{S}^{1, q'}$ and a continuous G map $f: \sigma^{1, q} \rightarrow \sigma^{1, q'}$ such that*

$$H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \quad \text{where } |x| = 1 \text{ or } \mathbf{Z}/2\mathbf{Z}.$$

if and only if $(q, q') = (1, 1)$ or $(0, 0)$.

PROOF. First we assume that there are $\sigma^{1, q} \in \mathcal{S}^{1, q}$, $\sigma^{1, q'} \in \mathcal{S}^{1, q'}$ and a continuous G map $f: \sigma^{1, q} \rightarrow \sigma^{1, q'}$ such that

$$H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \quad \text{where } |x| = 1.$$

a) We suppose $(q, q')=(0, 1)$. Consider the exact sequence

$$\longrightarrow \tilde{H}^*(\Sigma^{0,1}\sigma^{1,1^G}) \xrightarrow{\Sigma f^{G*}} \tilde{H}^*(\Sigma^{0,1}\sigma^{1,0^G}) \xrightarrow{k^*} \tilde{H}^*(C_f^G) \xrightarrow{j^*} \tilde{H}^*(\sigma^{1,1^G}) \xrightarrow{f^{G*}} \tilde{H}^*(\sigma^{1,0^G}).$$

It is easily seen that f^* and Σf^{G*} are 0 maps and that

$$\tilde{H}^*(C_f^G) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let y, y' be the basis of $\tilde{H}^1(C_f^G)$. Consider y and y' to be elements of $\tilde{H}_G^*(C_f^G)$ and x, x^2 to be elements of $\tilde{H}_G^*(C_f)$ as we did in the proof of Theorem 5.10.

Let $i^*(x)=ay+by'$ where $a, b \in \mathbf{Z}/2\mathbf{Z}$. Then $i^*(x^2)=0$. This contradicts to Proposition 4.9. Thus $(q, q') \neq (0, 1)$.

b) Next we suppose $(q, q')=(1, 0)$. Consider the exact sequence

$$\longrightarrow \tilde{H}^*(\Sigma^{0,1}\sigma^{1,0^G}) \xrightarrow{\Sigma f^{G*}} \tilde{H}^*(\Sigma^{0,1}\sigma^{1,1^G}) \xrightarrow{k^*} \tilde{H}^*(C_f^G) \xrightarrow{j^*} \tilde{H}^*(\sigma^{1,0^G}) \xrightarrow{f^{G*}} \tilde{H}^*(\sigma^{1,1^G}).$$

It is easily seen that f^* and Σf^{G*} are 0 maps and that

$$\tilde{H}^*(C_f^G) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Let y, y' be the generator of $\tilde{H}^0(C_f^G)$ and $\tilde{H}^2(C_f^G)$ respectively and consider y and y' to be elements of $\tilde{H}_G^*(C_f^G)$, x and x^2 to be elements of $\tilde{H}_G^*(C_f)$.

Let $i^*(x)=aty$ where $a \in \mathbf{Z}/2\mathbf{Z}$. Then $i^*(x^2)=a^2t^2y$. This contradicts to localization theorem. Thus $(q, q') \neq (1, 0)$.

Next we assume that there is $\sigma^{1,q} \in \mathcal{S}^{1,q}$, $\sigma^{1,q'} \in \mathcal{S}^{1,q'}$ and a continuous G map $f: \sigma^{1,q} \rightarrow \sigma^{1,q'}$ such that

$$\tilde{H}^*(C_f) = 0.$$

a') We suppose $(q, q')=(0, 1)$. Just as we have seen in a),

$$\tilde{H}^*(C_f^G) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\tilde{H}_G^*(C_f^G) \neq 0$. This contradicts to localization theorem.

b') Next we suppose $(q, q')=(1, 0)$. Just as b),

$$\tilde{H}^*(C_f^G) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\tilde{H}_G^*(C_f^G) \neq 0$. This contradicts to localization theorem.

Actually we construct $f: \sigma^{1,q} \rightarrow \sigma^{1,q'}$ for $(q, q') = (1, 1), (0, 0)$ as follows. Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ and τ denote the involution of S^1 by conjugation. Here $(S^1, \text{id}_{S^1}) \in \mathcal{S}^{1,1}$ and $(S^1, \tau) \in \mathcal{S}^{1,0}$. Let f_0, f_1 be maps from S^1 to S^1 defined by

$$f_0(z) = z, \quad f_1(z) = z^2.$$

Both of f_0, f_1 are equivariant whether S^1 takes the trivial involution or the involution τ .

It follows easily

$$C_{f_0} \cong D^2, \quad C_{f_1} \cong \mathbf{R}P^2.$$

Thus $H^*(C_{f_0}) \cong \mathbf{Z}/2\mathbf{Z}$, $H^*(C_{f_1}) \cong \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$ where $|x| = 1$.

THEOREM 5.12. *Let $d=1, 2$ or 4 . There exists $\sigma^{2d-1,p} \in \mathcal{S}^{2d-1,p}$ which has a structure of Hopf space with involution, if and only if $p=d-1, 2d-1$.*

Actually the unit sphere of $\mathbf{R}^{0,2d}, \mathbf{R}^{d,d}$ have structures of Hopf space with involution.

PROOF. First we show that the unit spheres of $\mathbf{R}^{0,2d}$ and $\mathbf{R}^{d,d}$ are Hopf space with involution.

Let $S^{n,m}$ denote the unit sphere of $\mathbf{R}^{n,m}$. It is trivial that $S^{0,2d}$ is a Hopf space with involution. Hence we consider $S^{d,d}$.

$\mathbf{R}^{d,d}$ is identified with $L \cong K \oplus K\omega$ where $\omega^2 = -1$, $L = \mathbf{C}, \mathbf{H}$, Cayley numbers \mathbf{O} , $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$ for $d=1, 2, 4$ respectively. And also $\tau|_K = \text{id}$, $\tau|_{K\omega} = -1$. With this involution the natural product μ of \mathbf{C}, \mathbf{H} , Cayley number \mathbf{O} becomes a equivariant map. And $S^{d,d}$ with this product is a Hopf space with involution. See Iriye [4].

Consider the Hopf constructions with involution of $S^{0,2d}, S^{d,d}$, and we obtain the existence of $\sigma^{4d-1,q} \in \mathcal{S}^{4d-1,q}$, $\sigma^{2d,q'} \in \mathcal{S}^{2d,q'}$ and a continuous G map $f: \sigma^{4d-1,q} \rightarrow \sigma^{2d,q'}$ such that

$$H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \quad \text{where } |x| = 2d,$$

for $(q, q') = (4d-1, 2d)$ and $(2d-1, d)$.

For the former part of the proposition consider the Hopf construction with involution and apply Theorem 5.10.

Assume that $\sigma^{2d-1,p} \in \mathcal{S}^{2d-1,p}$ is a Hopf space with involution. Let f be the Hopf construction of the Hopf structure μ .

$$f: \sigma^{2d-1,p} * \sigma^{2d-1,p} \longrightarrow \Sigma^{0,1} \sigma^{2d-1,p}.$$

Remark that

$$\begin{aligned} \sigma^{2d-1,p} * \sigma^{2d-1,p} &\in \mathcal{S}^{4d-1,2p+1} \\ \Sigma^{0,1} \sigma^{2d-1,p} &\in \mathcal{S}^{2d,p+1}. \end{aligned}$$

Hence Theorem 5.10 says

$$(2p+1, p+1) = (4d-1, 2d) \text{ or } (2d-1, d) \text{ or } (2d-1, 0).$$

The possibility lies in cases $p=d-1, 2d-1$.

THEOREM 5.13. *Let $d=1, 2$ or 4 . There exists $\sigma^{2d-1,p} \in S^{2d-1,p}$ which is a Hopf space with anti involution, if and only if $p=0, d$.*

Actually the unit spheres of $\mathbf{R}^{2d-1,1}, \mathbf{R}^{d-1,d+1}$ are Hopf spaces with anti involution.

PROOF. First we show that $S^{2d-1,1}, S^{d-1,d+1}$ have structures of Hopf space with anti involution.

Identify \mathbf{R}^{2d} with $L \cong K \oplus K\omega$ where $\omega^2 = -1, L = \mathbf{C}, \mathbf{H}$, Cayley numbers \mathbf{O} , $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$ for $d=1, 2, 4$ respectively. We introduce linear involutions τ_0, τ_1 of $K \oplus K\omega$ as follows

$$\begin{aligned} \tau_0(x + y\omega) &= \bar{x} + y\omega \\ \tau_1(x + y\omega) &= \bar{x} - y\omega \quad \text{for } x, y \in K. \end{aligned}$$

Here the standard product of \mathbf{C}, \mathbf{H} , Cayley numbers \mathbf{O} has the property

$$\tau_i(zw) = \tau_i(w)\tau_i(z) \quad \text{for } z, w \in L, i = 0, 1.$$

Identify $S^{d-1,d+1}, S^{2d-1,1}$ with the unit spheres of $(\mathbf{R}^{2d}, \tau_0), (\mathbf{R}^{2d}, \tau_1)$ respectively and then $S^{d-1,d+1}, S^{2d-1,1}$ become Hopf space with anti involution with the standard product of L .

Consider the Hopf constructions with anti involution of $S^{2d-1,1}, S^{d-1,d+1}$ and we obtain the existence of $\sigma^{4d-1,q} \in S^{4d-1,q}, \sigma^{2d,q'} \in S^{2d,q'}$ and a continuous G map $f: \sigma^{4d-1,q} \rightarrow \sigma^{2d,q'}$ such that

$$H^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \quad \text{where } |x| = 2d,$$

for $(q, q') = (2d-1, 0)$ and $(2d-1, d)$.

For the former part of the proposition consider the Hopf construction with anti involution and apply Theorem 5.10.

Assume that $\sigma^{2d-1,p} \in S^{2d-1,p}$ has a structure of Hopf space with anti involution. Let f be the Hopf construction of the Hopf structure μ .

$$f: \sigma^{2d-1,p} * \sigma^{2d-1,p} \rightarrow \Sigma^{1,0} \sigma^{2d-1,p}.$$

Remark that

$$\begin{aligned} (\sigma^{2d-1,p} * \sigma^{2d-1,p})^G &= \left\{ \left(x, \frac{1}{2}, \tau x \right) \in \sigma^{2d-1,p} * \sigma^{2d-1,p} \mid x \in \sigma^{2d-1,p} \right\} \\ &\in S^{2d-1,2d-1} \end{aligned}$$

$$(\Sigma^{1,0}\sigma^{2d-1,p})^G = \left\{ \left(\frac{1}{2}, x \right) \in \Sigma^{1,0}\sigma^{2d-1,p} \mid x \in (\sigma^{2d-1,p})^G \right\} \in \mathcal{S}^{p,p}$$

Thus we have that

$$\sigma^{2d-1,p} * \sigma^{2d-1,p} \in \mathcal{S}^{4d-1,2d-1}$$

$$\Sigma^{1,0}\sigma^{2d-1,p} \in \mathcal{S}^{2d,p}.$$

Hence Theorem 5.10 says

$$(2d-1, p) = (4d-1, 2d) \text{ or } (2d-1, d) \text{ or } (2d-1, 0).$$

The possibility lies in cases $p=d, 0$.

References

- [1] J.F. Adams, On the non-existence of Hopf invariant one, *Ann. of Math.*, **72** (1960), 20-104.
- [2] G.E. Bredon, Introduction to compact transformation groups, *Pure Appl. Math.*, **46**, Academic Press, New York, London, 1972.
- [3] D. Quillen, The spectrum of an equivariant cohomology ring I, II, *Ann. of Math.*, **94** (1971), 549-602.
- [4] K. Iriye, Hopf τ space and τ homotopy groups, *J. Math. Kyoto Univ.*, **22-4** (1983), 719-727.
- [5] J.D. Stasheff, *H*-spaces from a homotopy point of view, *Lecture Notes in Math.*, **161**, Springer-Verlag, 1970.
- [6] E. Thomas, On functional cup-products and the transgression operator, *Arch. Math.*, **12** (1963), 435-444.

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