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On the essential spectrum of the Laplacian on complete manifolds

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1. Introduction.

The Laplace-Beltrami operator Δ on a noncompact complete Riemannian manifold M is essentially self-adjoint on $C_0^\infty(M)$, and the spectrum of its self-adjoint extension to $L^2(M)$ has been studied by several authors from various points of view. For instance, Donnelly [**6**] proved that the essential spectrum $\sigma_{ess}(-\Delta)$ of $-\Delta$ is equal to $[(n-1)^2k^2/4, \infty)$ if M is an n-dimensional Hadamard manifold whose sectional curvatures approach a constant $-k^2$ at infinity. On the other hand, Escobar and Freire [**5**] consider the case M has nonnegative sectional curvatures and showed that $\sigma_{ess}(-\Delta)=[0, \infty)$, if M possesses a soul S such that the normal exponential map $\exp_S^\perp : NS \to M$ induces a diffeomorphism, and further if either $\dim M=2$, or $\dim M\geqq 3$ and $\int_1^\infty (1/v(t))\left[\int_{S_t}\mathrm{Ric}(\nabla r)\right]dt<\infty$, where NS is the normal bundle to S, $r(x)=\mathrm{dist}(x, S)$, $S_t=\{x\in M \mid \mathrm{dist}(x, S)=t\}$, and $v(t)$ denotes the volume of S_t. Recently Li [**10**] proved that if M has nonnegative Ricci curvatures and possesses a pole (i.e., a point $x\in M$ where the exponential map $\exp_x : T_xM\to M$ induces a diffeomorphism), then $\sigma_{ess}(-\Delta)$ is equal to $[0, \infty)$.

In this paper, we shall show the following:

THEOREM 1.1. *Let M be a noncompact complete Riemannian manifold of dimension n. Suppose there exists an open subset U of M with compact smooth boundary ∂U such that the outward-pointing normal exponential map* $\exp_{\partial U}^\perp$: *$N^+(\partial U)\to M-\bar{U}$ induces a diffeomorphism. Let $\varphi : [0, \infty)\to[0, \infty)$ be a continuous function satisfying*

$$\mathrm{Ric}_M(\gamma_x'(t), \gamma_x'(t)) \geqq -(n-1)\varphi(t),$$

for all $t\geqq 0$ and $x\in\partial U$, where $\gamma_x(t)=\exp_x(t\cdot\vec{n}(x))$ and $\vec{n}$ stands for the outward unit normal vector field on ∂U. Then the spectrum $\sigma(-\Delta)$ of $-\Delta$ is equal to $[0, \infty)$, provided that $\varphi(t)$ converges to zero as $t\to\infty$.

This is a generalization of the results by Escobar and Freire, and also Li mentioned above, and it will be deduced from a comparison argument and the following

THEOREM 1.2. *Let M and U be as in Theorem 1.1. We assume that U is bounded. If there exists a constant $c\in(-\infty, \infty)$ such that*

$$\sup\{|(\Delta r)(x)-c|\ ;\ \mathrm{dist}(x, U)\geqq s,\ x\in M\}\to 0 \quad as\ s\to\infty,$$

then $\sigma_{ess}(-\Delta)=[c^2/4, \infty)$, where $r(x)=\mathrm{dist}(U, x)$.

The result of Donnelly [**6**] mentioned above is also derived from this theorem, since an Hadamard manifold M satisfies all of the condition in Theorem 1.2, if the sectional curvatures approach a constant $-k^2$ at infinity, where $c=(n-1)k$ (cf. Corollary 2.1).

We shall now explain our method of proving Theorem 1.2. For any $\lambda\in(c^2/4, \infty)$ and any $\varepsilon>0$, we shall construct a sequence of functions $\{g_k\}\subset C_0^2(M)$ such that $\mathrm{supp}\, g_j\cap\mathrm{supp}\, g_k=\emptyset$ $(k\neq j)$ and

$$(a) \qquad \|(\Delta+\lambda)g_k\|_{L^2(M)}\leqq\varepsilon\|g_k\|_{L^2(M)}, \quad k=1, 2, \cdots.$$

This implies that $\sigma_{ess}(-\Delta)\supset[c^2/4, \infty)$. To construct such functions $\{g_k\}$, we shall adopt the transplantation method used in Donnelly [**6**]. We cut off a solution of $h''+ch'+\lambda h=0$ and transplant them from the real line $\boldsymbol{R}$ to M as functions depending only on the distance r from ∂U. To show that those transplanted functions satisfy such estimates as (a), we shall construct infinitely many warped product manifolds of finite intervals and level hypersurfaces of the distance function r with a warping function $e^{ct/(n-1)}$, and compare the Riemannian measures on those warped product manifolds with the Riemannian measures of M on supports of transplanted functions (cf. Remarks 2.1 and 2.2). On the other hand, the fact that $\sigma_{ess}(-\Delta)$ is bounded from below by the constant $c^2/4$ is a consequence of the decomposition principle in Donnelly and Li [8] and the Cheeger's inequality.

Finally, we remark that Theorem 1.2 certainly holds for an operator on $L^2(M, \omega dV_M)$ given by $\Delta_\omega=\Delta+\omega^{-1}\,\mathrm{grad}\,\omega$, where ω is a positive smooth function on M and dV_M is the Riemannian measure on M (cf. Theorem 2.1). Theorem 1.1 also holds for Δ_ω if we replace the Ricci tensor Ric_M with a symmetric tensor $\mathrm{Ric}_M-\omega^{-1}\mathrm{Hess}\,\omega$.

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2. Main lemma.

Let ω be a positive smooth function on a complete Riemannian manifold M. We shall consider the differential operator $\Delta_\omega:=\Delta+\omega^{-1}\,\mathrm{grad}\,\omega$ with its domain $C_0^\infty(M)$, where $\mathrm{grad}\,\omega$ denotes the gradient of a function ω. We remark that Δ_ω is associated with the following quadratic form q:

$$q(f, h) = \int_M \langle \nabla f, \nabla h \rangle \omega dV_M,$$

where $f, h \in C_0^\infty(M)$, $\nabla f = \text{grad}\, f$, $\langle , \rangle$ is the Riemannian metric on M and dV_M is the Riemannian measure on M. Then we can verify that Δ_ω is an essentially self-adjoint operator on the Hilbert space $L^2(M, \omega dV_M)$, and hence Δ_ω has a unique self-adjoint extension $\overline{\Delta_\omega}$. (Indeed, we can show that if f is a complex-valued function in $L^2(M, \omega dV_M) \cap C^\infty(M)$ and if $\Delta_\omega f \in L^2(M, \omega dV_M)$, then $|\nabla f| \in L^2(M, \omega dV_M)$ and $-\int_M (\Delta_\omega f)\bar{f} \omega dV_M = \int_M |\nabla f|^2 \omega dV_M$. Hence spectral theory says that Δ_ω on $L^2(M, \omega dV_M)$ is essentially self-adjoint. See Karp [**9**] for the case that $\omega \equiv 1$). The domain of $\overline{\Delta_\omega}$ is given by $\text{Dom}(\overline{\Delta_\omega}) = \{u \in L^2(M, \omega dV_M) \mid \Delta_\omega u$ (distribution sense)$\in L^2(M, \omega dV_M)\}$ and $\overline{\Delta_\omega} u = \Delta_\omega u$ (distribution sense) if $u \in \text{Dom}(\overline{\Delta_\omega})$. In what follows, we shall denote this extended operator $\overline{\Delta_\omega}$ by the same letter Δ_ω.

LEMMA 2.1 (main lemma). *Let M be an n-dimensional noncompact complete Riemannian manifold and ω a positive smooth function on M. Let f be a smooth function on M satisfying the following three conditions* $(\alpha)\sim(\gamma)$:

(α) *For some constant b, $S_b = \{x \in M \mid f(x) = b\}$ is a compact hypersurface of M and $M = U_1 \cup S_b \cup U_2$ (disjoint union), where U_1 is a union of bounded domains and U_2 is a union of unbounded domains. Moreover, $|\nabla f| > \delta > 0$ on $S_b \cup U_2$ for some constant δ and* $\sup\{||\nabla f|(x) - 1|\,;\, \text{dist}(x, U_1) \geqq r,\ x \in M\} \to 0$ *as* $r \to \infty$;

(β) $\sup\{|\Delta_\omega f(x) - c|\,;\, \text{dist}(x, U_1) \geqq r,\ x \in M\} \to 0$ *as* $r \to \infty$, *for some constant* $c \in \boldsymbol{R} := (-\infty, \infty)$;

(γ) $\sup\{|(\nabla f)|\nabla f||(x)\,;\, \text{dist}(x, U_1) \geqq r,\ x \in M\} \to 0$ *as* $r \to \infty$, *where* $(\nabla f)|\nabla f| = |\nabla f|^{-1} \cdot \text{Hess}\, f(\nabla f, \nabla f)$ *is the derivative of a function $|\nabla f|$ in the direction of ∇f.*

Then $\sigma_{ess}(-\Delta_\omega) = [c^2/4, \infty)$.

PROOF. To prove $(c^2/4, \infty) \subset \sigma_{ess}(-\Delta_\omega)$, we first recall the following fact from functional analysis:

LEMMA 2.2. *Let A be a self-adjoint operator on a Hilbert space. Then, for $\lambda \in \boldsymbol{R}$, the following two conditions* (α) *and* (β) *are equivalent*:

(α) $\lambda \in \sigma_{ess}(A)$;

(β) *For any $\varepsilon > 0$, there exists an infinite dimensional subspace G_ε in the domain of A such that*

$$\|(A - \lambda) f\| \leqq \varepsilon \|f\| \quad \text{for all } f \in G_\varepsilon.$$

For any $\lambda\in(c^2/4, \infty)$ and $\varepsilon>0$, we shall construct a sequence $\{g_k\}_{k\in N}$ of functions in $C_0^2(M)$ so that $\operatorname{supp} g_j\cap\operatorname{supp} g_k=\emptyset$ if $j\neq k$ and the linear hull of $\{g_k \mid k\in N\}$, denoted by G_ε, satisfies the condition (β) in Lemma 2.2. But, in order to prove $(c^2/4, \infty)\subset\sigma_{ess}(-\Delta_\omega)$, it suffices to construct such functions g_k on one end of M, and hence in the following we shall assume that U_2 consists of only one unbounded domain. Moreover we suppose that ∇f restricted on S_b points into U_2, that is, $U_2=\{x\in M \mid f(x)>b\}$. (The case where $(\nabla f)|_{S_b}$ points into U_1 can be treated in exactly the same way).

We shall construct a chart outside a compact subset, using the function f. For any $u\in S_b$, let $\varphi_u(t): [b, \infty)\to S_b\cup U_2$ be the integral curve of the vector field $(\nabla f)/|\nabla f|^2$ such that $\varphi_u(b)=u\in S_b$ and define $\varphi: [b, \infty)\times S_b\to S_b\cup U_2$ by $\varphi(t, u)=\varphi_u(t)$ for $(t, u)\in[b, \infty)\times S_b$. It is easy to see that $\varphi: [b, \infty)\times S_b\to S_b\cup U_2$ is a diffeomorphism and satisfies $f(\varphi(t, u))=t$ for all $(t, u)\in[b, \infty)\times S_b$. In the following, using this diffeomorphism φ, we shall simply write $g(t, u):=g\circ\varphi(t, u)$ for any function g defined on U_2. In this notation, a direct computation shows that if a function $g\in C^2(U_2)$ depends only on the first component $t\in[b, \infty)$, then

$$(1)\qquad \Delta g = |\nabla f|^{-2}\langle D_{\nabla f}(\nabla f), \nabla f\rangle(\partial_t g)+|\nabla f|^2(\partial_t^2 g)+(\operatorname{tr} A_t)|\nabla f|(\partial_t g) \quad \text{at } (t, u),$$

where A_t, $\operatorname{tr} A_t$, and D are, respectively, the shape operator of the hypersurface $S_t:=\{x\in U_2 \mid f(x)=t\}$, the trace of A and the covariant derivative of the Levi-Civita connection of M. In particular,

$$(2)\qquad \Delta f = |\nabla f|^{-2}\langle D_{\nabla f}(\nabla f), \nabla f\rangle+(\operatorname{tr} A_t)|\nabla f| = |\nabla f|^{-1}\cdot(\nabla f)|\nabla f|+(\operatorname{tr} A_t)|\nabla f|$$

at (t, u).

Now, we shall study the behavior of the measure ωdV_M. We denote by dV_{S_b} the Riemannian measure on S_b with respect to the induced metric on S_b by the inclusion $S_b\subset M$. Using diffeomorphism φ, for $(t, u)\in[b, \infty)\times S_b$, we define the positive function $a(t, u)$ by $dV_M=a(t, u)dtdV_{S_b}$, where dV_M is the Riemannian measure of M. A direct computation shows that $a(u, t)$ satisfies

$$(3)\qquad \frac{\partial_t a}{a} = -\frac{\langle D_{\nabla f}\nabla f, \nabla f\rangle}{|\nabla f|^4}+\frac{\operatorname{tr} A_t}{|\nabla f|}.$$

We put $\tilde{a}(t, u):=a(t, u)\omega(t, u)$ for $(t, u)\in[b, \infty)\times S_b$. (3) implies

$$\begin{aligned}(\partial_t\tilde{a})/\tilde{a} &= (\partial_t a)/a+(\partial_t\omega)/\omega\\ &= -|\nabla f|^{-3}[(\nabla f)|\nabla f|]+|\nabla f|^{-1}\operatorname{tr} A_t+\omega^{-1}|\nabla f|^{-2}\langle\nabla f, \nabla\omega\rangle.\end{aligned}$$

Therefore, by using (2), we obtain

$$(4)\qquad \begin{aligned}(\partial_t\tilde{a})/\tilde{a} &= -2|\nabla f|^{-3}\cdot(\nabla f)|\nabla f|+|\nabla f|^{-2}\{\Delta f+\omega^{-1}\langle\nabla\omega, \nabla f\rangle\}\\ &= -2|\nabla f|^{-3}\cdot(\nabla f)|\nabla f|+|\nabla f|^{-2}\Delta_\omega f.\end{aligned}$$

On the other hand, we note that

$$f(x) \leqq K\cdot \mathrm{dist}(S_b, x)+b \quad \text{for all } x \in U_2, \tag{5}$$

where $K=\sup_{U_2}|\nabla f|>0$. Indeed, for $x\in U_2$, let $c:[b, l]\to \bar{U}_2$ be a geodesic such that $c(b)\in S_b$, $c(l)=x$ and $l-b=\mathrm{dist}(S_b, x)$. Then, $f(x)=\int_b^l d(f(c(t)))/dt\cdot dt+b\leqq\int_b^l|\nabla f|dt+b\leqq K(l-b)+b$. Hence, (5) holds. The assumptions of the main lemma together with (4) and (5) imply that

$$\sup\left\{\left|\frac{\partial_t\tilde{a}(t, u)}{\tilde{a}(t, u)}-c\right|;\ u\in S_b\right\}\to 0 \quad \text{as } t\to\infty. \tag{6}$$

For a monotone increasing sequence $\{t_k\}\subset(b, \infty)$, which will be determined later, we shall consider the following sequence of manifolds: let $W(t_k)=[t_k, \infty)\times S_{t_k}$ be a warped product manifold with a Riemannian metric $dt^2+\{e^{c(t-t_k)}\}^{2/(n-1)}\cdot\{\omega(t_k, u)\}^{2/(n-1)}ds_{t_k}^2$ for $(t, u)\in[t_k, \infty)\times S_b$, where S_{t_k} is the hypersurface $\{x\in U_2|f(x)=t_k\}$ with a Riemannian metric $\{\omega(t_k, u)\}^{2/(n-1)}ds_{t_k}^2$ and $ds_{t_k}^2$ is the induced metric on S_{t_k} by the inclusion $S_{t_k}\subset M$. In the above definition, we think that $\omega(t_k, u)$ $(u\in S_b)$ is the restriction of ω on S_{t_k} and that S_{t_k} is parametrized by $u\in S_b$ through the diffeomorphism $\varphi|_{\{t_k\}\times S_b}:\{t_k\}\times S_b\to S_{t_k}$.

We take cut-off function $H\in C_0^\infty(R)$ such that $\mathrm{supp}\,H\subset(0, 3)$, $H(t)=1$ if $t\in[1, 2]$, and $0\leqq H(t)\leqq 1$ for all $t\in(-\infty, \infty)$. We set $E=\sup\{|H'(t)|+|H''(t)|;\ t\in(-\infty, \infty)\}$.

Let $\varepsilon\in(0, 1)$ be an any given constant and λ an arbitrary constant greater than $c^2/4$. We define a function f_k by $f_k(t)=\exp(-c(t-t_k)/2)\cdot\sin(\lambda_c^{-1}(t-t_k))$, where $\lambda_c^{-1}=(\lambda-c^2 4^{-1})^{1/2}$. The function f_k satisfies the equation $-f_k''-cf_k'=\lambda f_k$. f_k can be considered as a function on $W(t_k)$ depending only on the first component $t\in[t_k, \infty)$. When we denote by Δ_{t_k} the Laplace operator on $W(t_k)$, f_k satisfies $-\Delta_{t_k}f_k=-\partial_t^2 f_k-c\partial_t f_k=\lambda f_k$. We note that f_k has the infinite number of zeros.

We define a positive function $b_k(t, u)$ for $(t, u)\in[t_k, \infty)\times S_b$ by the identity $dV_{W(t_k)}=b_k(t, u)dtdV_{S_b}$, where $dV_{W(t_k)}$ is the Riemannian measure on $W(t_k)$. Then $b_k(t, u)$ satisfies $\partial_t(\log b_k(t, u))\equiv c$ for all $(t, u)\in[t_k, \infty)\times S_b$. On the other hand, as is showed above, $\tilde{a}(t, u)$ satisfies $\sup\{|\partial_t(\log\tilde{a}(t, u))-c|;\ u\in S_b\}\to 0$ as $t\to\infty$. Hence, noting that $\tilde{a}(t, u)=\tilde{a}(t_k, u)\cdot\exp\int_{t_k}^t\partial_t(\log\tilde{a})(\tau, u)d\tau$ and that $b_k(t, u)=b_k(t_k, u)\cdot\exp\int_{t_k}^t c\,d\tau$ for $t\in[t_k, \infty)$, we see that there exist constants $\delta\in(1, \infty)$, $t_0\in[1, \infty)$ and a positive integer q such that the following six conditions hold:

(i) $\sin(\lambda_c^{-1}\delta)=0$;

(ii) $E/\delta<\varepsilon$;

(iii) If we set t_k by $t_k=t_0+2\lambda_c\pi qk$ for $k=0, 1, 2, \cdots$, then $[t_k, t_k+3\delta]\cap[t_j, t_j+3\delta]=\emptyset$ if $k\neq j$;

(iv) $2^{-1}\leqq|\tilde{a}(t, u)/b_k(t, u)|^{1/2}\leqq 2$, for all $t\in I_k:=[t_k, t_k+3\delta]$ and all $u\in S_b$ and all $k=0, 1, 2, \cdots$;

(v) $1-\varepsilon\leqq|\nabla f|(t, u)\leqq 1+\varepsilon$ for all $(t, u)\in[t_0, \infty)\times S_b$;

(vi) $|\Delta_\omega f-c|(t, u)<\varepsilon$ for all $(t, u)\in[t_0, \infty)\times S_b$.

Indeed, we shall take δ so that (i) and (ii) hold, and then for this δ, we can choose t_0 so that (iv), (v) and (vi) are satisfied. We note that $b_k(t_k, u)=\tilde{a}(t_k, u)$ by the construction of the warped product manifolds $W(t_k)$ and that $f_k(t_k)=f_k(t_k+3\delta)=0$.

Now, we define $g_k\in C_0^2(R)$ by $g_k(t)=h_k(t)f_k(t)$, where we put $h_k(t):=H(\delta^{-1}(t-t_k))$. We transplant g_k onto U_2 as a function depending only on the first component t with respect to the coordinates system $(t, u)\in[b, \infty)\times S_b$ on U_2. Then, since (1) and (2) imply $\Delta_\omega g=|\nabla f|^2\partial_t^2 g+(\Delta_\omega f)\partial_t g$ for $g\in C^2(U_2)$ which depends only on the first parameter t, we have

$$\begin{aligned}-\Delta_\omega g_k &= -\Delta g_k-\omega^{-1}\langle\nabla\omega, \nabla g_k\rangle\\ &= -(\Delta_\omega h_k)f_k-h_k\Delta_\omega f_k-2\langle\nabla h_k, \nabla f_k\rangle\\ &= -\{|\nabla f|^2 h_k''+(\Delta_\omega f)h_k'\}f_k-\{|\nabla f|^2 f_k''+(\Delta_\omega f)f_k'\}h_k-2|\nabla f|^2 f_k' h_k'\\ &= -\{|\nabla f|^2 H''\delta^{-2}+(\Delta_\omega f)H'\delta^{-1}\}f_k+\lambda f_k h_k+h_k\{1-|\nabla f|^2\}f_k''\\ &\quad +h_k\{c-\Delta_\omega f\}f_k'-2|\nabla f|^2 H'\delta^{-1}f_k',\end{aligned}$$

where we have used the fact that $f_k''+cf_k'=-\lambda f_k$. Hence,

$$\begin{aligned}&\|(-\Delta_\omega-\lambda)g_k\|_{L^2(M, \omega dV_M)}\\ &\leqq(1+\varepsilon)^2 E\delta^{-2}\|\chi_k f_k\|_{L^2(M, \omega dV_M)}+E\delta^{-1}(|c|+\varepsilon)\|\chi_k f_k\|_{L^2(M, \omega dV_M)}\\ &\quad+3\varepsilon\|\chi_k\partial_t^2 f_k\|_{L^2(M, \omega dV_M)}+\varepsilon\|\chi_k\partial_t f_k\|_{L^2(M, \omega dV_M)}+2(1+\varepsilon)^2 E\delta^{-1}\|\chi_k\partial_t f_k\|_{L^2(M, \omega dV_M)},\end{aligned}$$

where χ_k is the characteristic function of I_k. Also, by the condition (v), we have

$$\begin{aligned}\|\chi_k\partial_t f_k\|_{L^2(M, \omega dV_M)} &\leqq \sup_{(t, u)\in I_k\times S_b}\left|\frac{\tilde{a}(t, u)}{b_k(t, u)}\right|^{1/2}\cdot\|\chi_k\partial_t f_k\|_{L^2(W(t_k))}\\ &\leqq 2\|\chi_k\partial_t f_k\|_{L^2(W(t))}.\end{aligned}\tag{7}$$

On the other hand, Green's formula on $W(t_k)$ implies

$$\|\chi_k\partial_t f_k\|_{L^2(W(t_k))}=\sqrt{\lambda}\,\|\chi_k f_k\|_{L^2(W(t_k))}.\tag{8}$$

LEMMA 2.3. *Following inequality holds:*

$$\|\chi_k f_k\|_{L^2(W(t_k))}\leqq\sqrt{3}\,\|g_k\|_{L^2(W(t_k))}.$$

PROOF.

$$
\begin{aligned}
&\|\chi_k f_k\|^2_{L^2(W(t_k))}\\
&=\int_{S_b} dV_{S_b}(u)\int_{t_k}^{t_k+3\delta} f_k^2(t) b_k(t, u)dt\\
&=\int_{S_b} dV_{S_b}(u)\int_{t_k}^{t_k+3\delta} \exp(-c(t-t_k))\cdot\sin^2(\lambda_c^{-1}(t-t_k))\cdot\tilde{a}(t_k, u)\exp(c(t-t_k))dt\\
&=\int_{S_b} dV_{S_b}(u)\int_0^{3\delta}\tilde{a}(t_k, u)\sin^2(\lambda_c^{-1}\tau)d\tau\\
&=3\int_{S_b}\tilde{a}(t_k, u)dV_{S_b}(u)\int_\delta^{2\delta}\sin^2(\lambda_c^{-1}\tau)d\tau\\
&\leqq 3\int_{S_b}\tilde{a}(t_k, u)dV_{S_b}(u)\int_0^{3\delta}H^2(\tau/\delta)\sin^2(\lambda_c^{-1}\tau)d\tau\\
&\leqq 3\int_{S_b}\tilde{a}(t_k, u)dV_{S_b}(u)\int_{t_k}^{t_k+3\delta}H^2\Big(\frac{t-t_k}{\delta}\Big)f_k^2(t)\exp(c(t-t_k))dt\\
&=3\|g_k\|^2_{L^2(W(t_k))}.
\end{aligned}
$$

In the last equality, we have used the fact that $b_k(t, u)=\tilde{a}(t_k, u)\exp(c(t-t_k))$. □

By virtue of Lemma 2.3 together with (7) and (8), we get

$$
\begin{aligned}
(9)\qquad &\|\chi_k\partial_t f_k\|_{L^2(M,\omega dV_M)}\leqq 2\sqrt{\lambda}\|\chi_k f_k\|_{L^2(W(t_k))}\leqq 2\sqrt{3\lambda}\|g_k\|_{L^2(W(t_k))}\\
&\leqq 2\sqrt{3\lambda}\sup_{(t,u)\in I_k\times S_b}\left|\frac{b_k(t, u)}{\tilde{a}(t, u)}\right|^{1/2}\|g_k\|_{L^2(M,\omega dV_M)}\leqq 4\sqrt{3\lambda}\|g_k\|_{L^2(M,\omega dV_M)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(10)\qquad &\|\chi_k f_k\|_{L^2(M,\omega dV_M)}\leqq\sup_{(t,u)\in I_k\times S_b}\left|\frac{\tilde{a}(t, u)}{b_k(t, u)}\right|^{1/2}\cdot\|\chi_k f_k\|_{L^2(W(t_k))}\\
&\leqq 2\|\chi_k f_k\|_{L^2(W(t_k))}\leqq 2\sqrt{3}\|g_k\|_{L^2(W(t_k))}\leqq 4\sqrt{3}\|g_k\|_{L^2(M,\omega dV_M)}.
\end{aligned}
$$

The inequality (9), (10) and the fact that $f_k''+cf_k'+\lambda f_k=0$ imply

$$
\begin{aligned}
\|\chi_k\partial_t^2 f_k\|_{L^2(M,\omega dV_M)}&\leqq\lambda\|\chi_k f_k\|_{L^2(M,\omega dV_M)}+|c|\cdot\|\chi_k\partial_t f_k\|_{L^2(M,\omega dV_M)}\\
&\leqq 4\lambda\sqrt{3}\|g_k\|_{L^2(M,\omega dV_M)}+|c|\cdot 4\sqrt{3\lambda}\|g_k\|_{L^2(M,\omega dV_M)}.
\end{aligned}
$$

Therefore, by combining the above inequalities and by using (v), (vi) and $\varepsilon\in(0, 1)$, we obtain

$$
\begin{aligned}
&\|(-\Delta_\omega-\lambda)g_k\|_{L^2(M,\omega dV_M)}\\
&\leqq\{4\sqrt{3}(1+\varepsilon)^2\varepsilon+4\sqrt{3}\varepsilon(|c|+\varepsilon)+12\sqrt{3}\lambda\varepsilon+12\sqrt{3\lambda}\varepsilon|c|+4\sqrt{3\lambda}\varepsilon+8\sqrt{3\lambda}\varepsilon(1+\varepsilon)^2\}\\
&\quad\times\|g_k\|_{L^2(M,\omega dV_M)}\\
&\leqq 4\sqrt{3}\varepsilon\{5+|c|(1+3\sqrt{\lambda})+3\lambda+9\sqrt{\lambda}\}\|g_k\|_{L^2(M,\omega dV_M)}.
\end{aligned}
$$

We note that $g_k \in C_0^2(M)$ and that supp $g_k \cap$ supp $g_j = \emptyset$ if $k \neq j$ (see (iii)). Thus, by Lemma 2.2, we get $\lambda \in \sigma_{ess}(-\Delta_\omega)$. Since $\lambda \in (c^2/4, \infty)$ is arbitrary, we have

$$[c^2/4, \infty) \subset \sigma_{ess}(-\Delta_\omega). \tag{11}$$

When $c=0$, we have thus shown $\sigma_{ess}(-\Delta_\omega)=[0, \infty)$, as claimed. When $c>0$, we shall need the following decomposition principle in Donnelly and Li [8] and Cheeger's inequality:

LEMMA 2.4 (decomposition principle). *Let M be a noncompact complete Riemannian manifold and ω a positive smooth function on M. Let $\Delta_\omega = \Delta + \omega^{-1}$ grad ω be a self-adjoint operator on $L^2(M, \omega dV_M)$, as defined above. We assume that $N \subset M$ is a compact submanifold with boundary, of the same dimension as M. Let Δ'_ω be the self-adjoint extension of $\Delta_\omega|_{C_0^\infty(M-N)}$ to $L^2(M-N, \omega dV_M)$ which satisfies Dirichlet boundary condition. Then, Δ_ω and Δ'_ω have the same essential spectrum.*

LEMMA 2.5 (Cheeger's inequality). *Let M be a compact Riemannian manifold with nonempty boundary ∂M and ω a positive smooth function on M. We set*

$$\mathfrak{H}_\omega(M) := \inf_{\Omega} A_\omega(\partial\Omega)/V_\omega(\Omega),$$

where Ω ranges over all open submanifold of M, with $\partial\Omega \cap \partial M = \emptyset$, and with smooth boundary $\partial\Omega$, and we set $A_\omega(\partial\Omega) = \int_{\partial\Omega} \omega dV_{\partial\Omega}$, $V_\omega(\Omega) = \int_\Omega \omega dV_M$, where $dV_{\partial\Omega}$ is the Riemannian measure on $\partial\Omega$ with respect to the induced metric. Then we have

$$\lambda_\omega \geqq \mathfrak{H}_\omega(M)^2/4,$$

where λ_ω is the first Dirichlet eigenvalue of $-\Delta_\omega$.

We can prove Lemma 2.4 and 2.5 in exactly the same way as in the case that $\omega \equiv 1$. Hence, we shall omit the proof. (As to the Cheeger's inequality, see Chavel [3]).

Now, in case $c>0$, making use of Lemma 2.4 and 2.5, we shall show that $\sigma_{ess}(-\Delta_\omega)$ is bounded from below by the constant $c^2/4$. Let $\delta \in (0, c)$ be any given constant. For all bounded domain Ω sufficiently apart from U_1, we have by the assumption of the main lemma

$$1-\delta < |\nabla f| < 1+\delta \quad \text{on } \Omega$$

and

$$|\Delta_\omega f - c| < \delta \qquad \text{on } \Omega.$$

Hence, we get

$$(1+\delta)A_\omega(\partial\Omega) \geqq \int_{\partial\Omega} |\nabla f| \omega dV_{\partial\Omega}$$

$$\geqq \int_{\partial\Omega} \langle \nabla f, \vec{n} \rangle \omega dV_{\partial\Omega}$$

$$= \int_{\Omega} \operatorname{div}(\omega \nabla f) dV_M$$

$$= \int_{\Omega} (\Delta_\omega f) \omega dV_M$$

$$\geqq (c-\delta) V_\omega(\Omega),$$

where $\vec{n}$ is the outward unit normal vector field on $\partial\Omega$. So, we have $A_\omega(\partial\Omega)/V_\omega(\Omega) \geqq (c-\delta)/(1+\delta)$. Therefore, for all $\phi \in C_0^\infty(U_2)$ with its support sufficiently apart from U_1, Lemma 2.5 implies

$$\frac{\int_M |\nabla\phi|^2 \omega dV_M}{\int_M \phi^2 \omega dV_M} \geqq \frac{1}{4}\left(\frac{c-\delta}{1+\delta}\right)^2.$$

Hence, by using Lemma 2.4, we obtain

$$\inf \sigma_{ess}(-\Delta_\omega) \geqq \frac{1}{4}\left(\frac{c-\delta}{1+\delta}\right)^2 \quad \text{for all } \delta \in (0, c).$$

Thus, we have shown that $\sigma_{ess}(-\Delta_\omega)$ is bounded from below by the constant $c^2/4$. From this fact and (11), in case $c>0$, we obtain $\sigma_{ess}(-\Delta_\omega)=[c^2/4, \infty)$.

In case $c<0$, it suffices to substitute $-f$ for f in the above argument. We have thus proved the main lemma.

REMARK 2.1. In the proof above, we have used a solution of

$$-h''-ch' = \lambda h$$

(defined on the real line), which satisfies the equation $-\Delta_\omega h=\lambda h$ on warped product manifolds $W(t_k)$ with warping function $e^{ct/(n-1)}$, if we transplant it to $W(t_k)$. On the other hand, under the assumption that the sectional curvatures of an Hadamard manifold tend to $-b^2$ at infinity, Donnelly [**6**] employed a solution of different equations:

$$(12) \qquad -f''(r)-c\coth(cr/(n-1))\cdot f'(r) = \lambda f(r)$$

or

$$(13) \qquad -f''(r)-(n-1)r^{-1}f'(r) = \lambda f(r),$$

which satisfies $-\Delta f=\lambda f$ on the hyperbolic space $\boldsymbol{H}^n(-b^2)$ with constant curvatures $-b^2$ or Euclidean space $\boldsymbol{R}^n$, if we transplant it to $\boldsymbol{H}^n(-b^2)$ or $\boldsymbol{R}^n$ as a function depending only on the distance from a single point, where $b=c/(n-1)>0$. Our choice for transplanted functions makes it easier to obtain such estimates as $\|(\Delta_\omega+\lambda)g_k\|_{L^2(M,\omega dV_M)}\leqq\varepsilon\|g_k\|_{L^2(M,\omega dV_M)}$. Compare Lemma 2.3 with [**6**, Lemma 4.6 and Lemma 4.10]).

The following is an immediate consequence of Lemma 2.1.

THEOREM 2.1. *Let M be a noncompact complete Riemannian manifold, and ω a positive smooth function on M. Suppose there exists a bounded open subset U of M with smooth boundary ∂U such that the outward-pointing normal exponential map $\exp_{\partial U}^{\perp}: N^+(\partial U)\to M-\bar{U}$ induces a diffeomorphism. If there exists a constant $c\in(-\infty, \infty)$ such that*

$$\sup\{|(\Delta_\omega r)(x)-c|\ ;\ \mathrm{dist}(x, U)\geqq s,\ x\in M\}\to 0 \quad as\ s\to\infty,$$

then $\sigma_{ess}(-\Delta_\omega)=[c^2/4, \infty)$, where $r(x)=\mathrm{dist}(U, x)$.

REMARK 2.2. To obtain estimates $\|(\Delta_\omega+\lambda)(h_k f)\|_{L^2(M)}\leqq\varepsilon\|h_k f\|_{L^2(M)}$ for a function f which satisfies (12) or (13) and cut-off functions h_k, Donnelly [**6**] compared the Riemannian measure on M with that on the model space $\boldsymbol{H}^n(-b^2)$ or $\boldsymbol{R}^n$. But under the assumption that $\Delta_\omega r\to c$, what we can see is only the limit of the growth rate of weighted Riemannian measures on level hypersurfaces of the function r, and we can get neither the model space as in [**6**, § 6] nor the asymptotic value of measures on level hypersurfaces of r as in [**6**, § 12]. For this reason, we have constructed infinitely many warped product manifolds $W(t_k)$ to make the difference between the measures on $\{y\in M\,|\,t_k\leqq r(y)\leqq t_k+3\delta\}$ and those on $W(t_k)$ small enough.

REMARK 2.3. Let N be an arbitrary k-dimensional compact Riemannian manifold. Let W be a warped product manifold $(M\times N,\ g_M+\omega^{2/k}g_N)$, where M is a noncompact complete Riemannian manifold, g_M (resp. g_N) is a Riemannian metric on M (resp. N) and ω is a positive smooth function on M. Then it is easy to see that under the assumption of Theorem 2.1 (or Lemma 2.1) the essential spectrum $\sigma_{ess}(\Delta_\omega)$ of Δ_ω on $L^2(M, \omega dV_M)$ is equal to that of the Laplacian of the warped product manifold W.

The following Corollary 2.1 is a generalization of the result by Donnelly [**6**, Theorem 6.3] stated in Introduction.

COROLLARY 2.1. *Let M be a complete Riemannian manifold with a pole p. We assume that*

$$K(\gamma_w'(t)\wedge v)\leqq 1/(4t^2)\quad for\ all\ (t, w)\in(0, \infty)\times U_pM\ and\ v\in T_{\gamma_w(t)}M-\boldsymbol{R}\cdot\gamma_w'(t),$$

where $\gamma_w(t)=\exp_p(tw)$, $U_pM=\{v\in T_pM\ ;\ |v|=1\}$ and $K(\gamma_w'(t)\wedge v)$ is the sectional curvature of the 2-plane spanned by $\gamma_w'(t)$ and v. If there exists a nonnegative constant c such that

$$\sup\{|K(\gamma'_w(t)\wedge v)+c| \,;\, t\geqq s,\ v\in T_{\gamma_w(t)}M-\boldsymbol{R}\cdot\gamma'_w(t),\ w\in U_pM\} \to 0 \quad as\ s\to\infty,$$

then $\sigma_{ess}(-\Delta)=[c(n-1)^2/4, \infty)$.

PROOF. Under the assumption above, the Laplacian Δr of the distance function r to p converges to $(n-1)\sqrt{c}$ uniformly with respect to $w\in U_pM$ as $r\to\infty$. (The assumption $K(\gamma'_w(t)\wedge v)\leqq 1/(4t^2)$ implies that $0<(n-1)/(2r)\leqq\Delta r$). Consequently, Corollary 2.1 follows from Theorem 2.1. □

3. Proof of Theorem 1.1.

By the assumption, $r=\mathrm{dist}(U, *)$ is a smooth function on $M-\bar{U}$. By virtue of Theorem 2.1, it suffices to prove that $\sup\{|\Delta r|(\gamma_x(t))\,;\, x\in\partial U\}\to 0$ as $t\to\infty$. Let x be a fixed point of ∂U and we denote $\Delta r(\gamma_x(t))$ by $H(t)$. $H(t)$ is the mean curvature of the hypersurface $S_t=\{x\in M-\bar{U}\,|\,\mathrm{dist}(x, U)=t\}$ at the point $y=\gamma_x(t)$, that is, $\sum_{i=1}^{i=n-1}\langle\nabla_{e_i}\mathrm{grad}\, r, e_i\rangle(y)$, where $\{e_i\}$ is an orthonormal base of the tangent space $T_y(S_t)$ to S_t at y, ∇ is the Riemannian connection of M, and $\langle\ ,\ \rangle$ is the Riemannian metric of M. We observe that $H(t)$ satisfies

$$H'(t)+\frac{1}{n-1}H^2(t)-(n-1)\varphi(t)\leqq 0 \quad \text{for all } t\geqq 0.$$

(See, for example, Chavel [**3**, p. 72]). We set $S_1=\max\{H_x(0)\,|\,x\in\partial U\}$, where $H_x(0)$ is the mean curvature of ∂U at x. Let f be the solution of the equation

$$f'(t)+\frac{1}{n-1}f^2(t)-(n-1)\varphi(t)=0, \quad f(0)=S_1.$$

By means of a usual comparison argument, we see that f is defined on all of $[0, \infty)$ and satisfies $H(t)\leqq f(t)$ for all $t\geqq 0$.

Now we shall show that $\limsup_{t\to\infty} f(t)\leqq 0$ by contradiction. We assume that there exists a positive constant C_∞ such that

$$\sup\{t\in[0, \infty)\ |\ f(t)\geqq 2(n-1)C_\infty\}=\infty. \tag{14}$$

From the hypothesis: $\lim_{t\to\infty}\varphi(t)=0$, for any $\varepsilon\in(0, C_\infty)$, there exists a constant $t_2>0$ such that $-\varphi(t)\geqq-\varepsilon^2$ for all $t\geqq t_2$. We may assume $f(t_2)\geqq 2(n-1)C_\infty$. Let $G:[t_2, \infty)\to\boldsymbol{R}$ be the solution of the equation

$$G'+\frac{1}{n-1}G^2-(n-1)\varepsilon^2=0, \quad G(t_2)=f(t_2). \tag{15}$$

A standard comparison argument shows that $f(t)\leqq G(t)$ for all $t\geqq t_2$. The solution of the equation (15) is expressed as follows:

$$G(t)=(n-1)\varepsilon\frac{Be^{2\varepsilon t}+1}{Be^{2\varepsilon t}-1},$$

where B is the constant determined by $f(t_2)+(n-1)\varepsilon=Be^{2\varepsilon t_2}\{f(t_2)-(n-1)\varepsilon\}$. Since $f(t_2)\geqq 2(n-1)C_\infty\geqq 2(n-1)\varepsilon$, we have

$$Be^{2\varepsilon t_2}=\frac{f(t_2)+(n-1)\varepsilon}{f(t_2)-(n-1)\varepsilon}>1$$

and $\lim_{t\to\infty}G(t)=(n-1)\varepsilon$. Therefore, $\limsup_{t\to\infty}f(t)\leqq(n-1)\varepsilon$. But, since $\varepsilon\in(0, C_\infty)$ is arbitrary, $\limsup_{t\to\infty}f(t)\leqq 0$. This contradicts the assumption (14). Hence, for any $C_\infty>0$, $\sup\{t\in[0, \infty)\,|\,f(t)\geqq 2(n-1)C_\infty\}<+\infty$. Thus, $\limsup_{t\to\infty}f(t)\leqq 0$.

Next, we shall show that for any $\varepsilon>0$, if $-\varphi(t)\geqq-(n-1)^{-2}\varepsilon^4$ for all $t\geqq t_0$, then $H(t)$ satisfies $H(t)\geqq-\varepsilon^2$ for all $t\geqq t_0$. Assuming that there exists a constant $t_1\geqq t_0$ such that $H(t_1)<-\varepsilon^2$, we shall obtain a contradiction. Let $F\colon[t_1, l)\to\boldsymbol{R}$ is the solution of the equation

$$F'+\frac{1}{n-1}F^2-\frac{\varepsilon^4}{n-1}=0,\quad F(t_1)=H(t_1)<-\varepsilon^2, \tag{16}$$

where $[t_1, l)$ is the maximal interval of existance. Then $H(t)\leqq F(t)$ for all $t_1\leqq t<l$. But the solution of the equation (16) is expressed as follows:

$$F(t)=\varepsilon^2\Big[A_1\cdot\exp\Big\{\frac{2\varepsilon^2 t}{n-1}\Big\}+1\Big]\Big/\Big[A_1\cdot\exp\Big\{\frac{2\varepsilon^2 t}{n-1}\Big\}-1\Big]\quad\text{for } t_1\leqq t<l,$$

where A_1 is the constant determined by $F(t_1)+\varepsilon^2=A_1\cdot\exp\{(2\varepsilon^2 t_1)/(n-1)\}\cdot\{F(t_1)-\varepsilon^2\}$. The inequality, $F(t_1)<-\varepsilon^2$ means $A_1\cdot\exp\{(2\varepsilon^2 t_1)/(n-1)\}=(-F(t_1)-\varepsilon^2)/(-F(t_1)+\varepsilon^2)\in(0, 1)$. Hence, l is the bounded number so that $A_1\cdot\exp\{(2\varepsilon^2 l)/(n-1)\}=1$ and $\lim_{t\to l-0}F(t)=-\infty$. But this means $\lim_{t\to l-0}H(t)=-\infty$, contradicting the fact that $H(t)$ is the mean curvature of S_t at $\exp_x t\vec{n}(x)$. Therefore, $H(t)\geqq-\varepsilon^2$ for all $t\geqq t_0$. Since, from the assumption of Theorem 1.1, for any $\varepsilon>0$ there exists $t_0>0$ such that $-\varphi(t)\geqq-(n-1)^{-2}\varepsilon^4$ for all $t\geqq t_0$, we see $\lim_{s\to\infty}\inf\{\Delta r(\gamma_x(t))\,|\,x\in\partial U,\ t\geqq s\}\geqq 0$. We have thus proved that $\sup\{|\Delta r|(\gamma_x(t));\ x\in\partial U\}\to 0$ as $t\to\infty$. This completes the proof of Theorem 1.1.

Lower bound of the Ricci curvatures can be relaxed. For instance, we have the following

THEOREM 3.1. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of positive numbers such that* $0<a_n<1$ *and* $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=0$. *Let* M *be a noncompact complete Riemannian manifold of dimension* n. *Suppose there exists an open subset* U *of* M *with compact smooth boundary* ∂U *such that the outward-pointing normal exponential map* $\exp_{\partial U}^{\perp}\colon N^+(\partial U)\to M-\bar{U}$ *induces a diffeomorphism and that*

$$\mathrm{Ric}_M(\gamma'_x(t),\ \gamma'_x(t)) \geqq -(n-1)\chi(t),$$

for all $t \geqq 0$ *and* $x \in \partial U$, *as in Theorem* 1.1. *Here* χ *is a function defined as follows*:

$$\chi(t) = \begin{cases} -c, & t \in [0,\ k_0] \cup \bigcup_{j=k_0}^{\infty} (j,\ j+a_j), \\ -b_j, & t \in [j+a_j,\ j+1], \quad j = k_0,\ k_0+1,\ \cdots \end{cases}$$

where c *is a positive constant and* k_0 *is a positive integer. Then the spectrum of the minus Laplacian of* M *is equal to* $[0,\ \infty)$.

The proof of Theorem 3.1 will be done along the same lines as those of Theorem 1.1 and hence we shall omit the proof of Theorem 3.1.

REMARK 3.1. As is seen from the proof above, Theorem 1.1 and Theorem 3.1 certainly hold for Δ_x if we replace $\mathrm{Ric}_M(\gamma'_x(t),\ \gamma'_x(t))$ with $\mathrm{Ric}_M(\gamma'_x(t),\ \gamma'_x(t)) - \omega^{-1} \cdot \mathrm{Hess}\,\omega(\gamma'_x(t),\ \gamma'_x(t))$.

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