

## $L^2$ harmonic forms and stability of minimal hypersurfaces

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### 0. Introduction.

The Bernstein conjecture states that any complete minimal graph in  $E^{m+1}$  is a hyperplane. This was proved to be true for  $m \leq 7$  by Bernstein [2] ( $m=2$ ), De Giorgi [5] ( $m=3$ ), Almgren [1] ( $m=4$ ) and Simons [12] ( $m \leq 7$ ); and false for  $m \geq 8$  by Bombieri, De Giorgi and Giusti [3].

On the other hand, do Carmo and Peng [6] and Fischer-Colbrie and Schoen [8] showed that complete orientable and stable minimal surfaces in  $E^3$  are planes. Palmer [10] studied a topological restriction of a complete minimal hypersurface  $M$  in  $E^{m+1}$  which implies instability of  $M$ . This topological restriction is related to the existence of nonzero  $L^2$  harmonic 1-forms by Dodziuk's result [7].

We denote the space of all  $L^2$  harmonic  $p$ -forms on a complete orientable Riemannian manifold  $M$  by  $\mathcal{H}^p(M)$ .  $\mathcal{H}^p(M)$  consists of  $p$ -forms which are closed and coclosed by a theorem of Andreotti and Vesentini. It is well known that  $\mathcal{H}^p(E^m) = \{0\}$  for all  $p$ ;  $0 \leq p \leq m$ .

A complete minimal graph in  $E^{m+1}$  is minimizing, and any minimizing minimal hypersurface is stable. Therefore, concerning the Bernstein problem we can pose the following problem: *For a complete orientable and stable minimal hypersurface  $M$  in  $E^{m+1}$ , does  $\mathcal{H}^p(M) = \{0\}$  hold for all  $p$ ;  $0 \leq p \leq m$ .*

The case where  $m=2$  is trivial by the result of do Carmo and Peng, Fischer-Colbrie and Schoen as stated in the above. A catenoid shows that the assumption of stability in our problem is essential. Here we have the following:

**THEOREM A.** *Let  $M \subset E^{m+1}$  be a complete orientable and stable minimal hypersurface. If  $m \leq 4$ , then  $\mathcal{H}^p(M) = \{0\}$  holds for all  $p$ ;  $0 \leq p \leq m$ .*

Palmer [10] used the norm of an  $L^2$  harmonic 1-form on  $M$  to define a variation vector field and proved that a complete orientable minimal hypersurface  $M \subset E^{m+1}$  admitting a nontrivial  $L^2$  harmonic 1-form is unstable. So, Theorem A for the case where  $m=3$  is due to Palmer [10]. To prove Theorem A it suffices to show the following: Let  $M \subset E^5$  be a complete, orientable minimal hypersurface;

If  $M$  admits a nontrivial  $L^2$  harmonic 2-form, then  $M$  is unstable (Theorem 2.1).

In §3 we give definition of  $P_2$  positivity of sectional curvature, as a condition weaker than positivity of sectional curvature. In §4 (Theorem 4.1) we prove the following:

**THEOREM B.** *Let  $M$  be a complete noncompact orientable minimal hypersurface of a Riemannian manifold  $M^*$ . We assume that the sectional curvature of  $M^*$  is  $P_2$  nonnegative. If  $M$  is stable, then there are no nontrivial  $L^2$  harmonic 1-forms on  $M$ .*

This is a generalization of Palmer's result [10] (where  $M^*=E^{m+1}$ ) and Miyaoka's result [9] (where  $M^*$  is of nonnegative sectional curvature).

For the case where  $m=2$ , complete stable minimal surfaces of  $M^*$  with nonnegative scalar curvature were classified by Fischer-Colbrie and Schoen [8], Schoen and Yau [11].

The author is very grateful to the referee whose comments on the infiniteness of the volume of  $M$  simplified the proof of Theorem 2.1.

### 1. $L^2$ harmonic 2-forms.

Let  $w$  be an  $L^2$  harmonic 2-form on a complete orientable Riemannian manifold  $M=(M, g)$ . It is known that  $w$  is closed and coclosed. The Riemannian curvature tensor, the Ricci curvature tensor and the Riemannian connection are denoted by  $R=(R_{jki}^i)$ ,  $\rho=(R_{ji})$  and  $\nabla$ . The expression of  $\Delta w$  is given by

$$\Delta w_{ij} = \nabla^r \nabla_r w_{ij} - R_i^r w_{rj} - R_j^r w_{ir} + R^{rs}{}_{ij} w_{rs} = 0.$$

Putting  $\|w\|^2 = w_{rs} w^{rs}$  and  $\|\nabla w\|^2 = \nabla_r w_{ij} \nabla^r w^{ij}$ , we obtain

$$(1.1) \quad \Delta \|w\|^2 = 4(\rho; w, w) - 2\langle [R]w, w \rangle + 2\|\nabla w\|^2,$$

where  $[R]$  denotes the curvature operator and

$$(\rho; w, w) = R_{rs} w^r{}_i w^{si}, \quad \langle [R]w, w \rangle = R^{rs}{}_{ij} w_{rs} w^{ij}.$$

On the other hand, we obtain

$$(1.2) \quad \begin{aligned} \Delta \|w\|^2 &= 2\|w\|\Delta\|w\| + 2\|\nabla\|w\|\|^2 \\ &= 2\|w\|\Delta\|w\| + 2\|\nabla w\|^2 - 2F(w), \end{aligned}$$

where we have put  $F(w) = \|\nabla w\|^2 - \|\nabla\|w\|\|^2$ .  $F(w) \geq 0$  is Kato's inequality. The equality  $F(w) = 0$  holds, if and only if

$$(1.3) \quad 2\|w\|^2 \nabla_k w_{ij} = \nabla_k \|w\|^2 \cdot w_{ij},$$

i.e.,  $w/\|w\|$  is parallel, if  $\|w\| \neq 0$ . By (1.1) and (1.2) we get

$$(1.4) \quad -\|w\|\Delta\|w\| = -2(\rho; w, w) + \langle [R]w, w \rangle - F(w).$$

Next we prove the following Lemma for later use in the next section :

LEMMA 1. *Let  $A, B$  be  $m \times m$  real matrices such that*

- (i)  *$A$  is symmetric and trace free,*
- (ii)  *$B$  is skew-symmetric.*

*If  $2 \leq m \leq 4$ , then  $\|A\|^2\|B\|^2 + 2\text{Tr}(AB)^2 + 2\text{Tr} A^2 B^2 \geq 0$ .*

PROOF. First we diagonalize  $A$  to the form  $(a_i \delta_{ij})$  by an orthogonal transformation. Let  $B = (b_{ij})$ . Then we have the following :

$$\begin{aligned} \|A\|^2\|B\|^2 &= (\sum a_i^2)(\sum_{i \neq j} b_{ij}^2), \\ \text{Tr}(AB)^2 &= -\sum_{i \neq j} a_i a_j b_{ij}^2, \\ \text{Tr} A^2 B^2 &= -\sum_{i \neq j} a_i^2 b_{ij}^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\|A\|^2\|B\|^2 + 2\text{Tr}(AB)^2 + 2\text{Tr} A^2 B^2 \\ &= 2b_{12}^2[a_3^2 + a_4^2 + \dots + a_m^2 - 2a_1 a_2] + 2b_{13}^2[\dots] + \dots + 2b_{m-1m}^2[\dots]. \end{aligned}$$

By assumption  $\text{Tr} A = 0$  we have

$$a_3^2 + a_4^2 + \dots + a_m^2 - 2a_1 a_2 = a_3^2 + a_4^2 + \dots + a_m^2 + 2a_2^2 + 2a_2 a_3 + \dots + 2a_2 a_m.$$

Thus, for  $m=2, 3, 4$  we obtain  $-2a_1 a_2 = 2a_2^2 \geq 0$ , and

$$\begin{aligned} a_3^2 - 2a_1 a_2 &= a_2^2 + (a_2 + a_3)^2 \geq 0, \\ a_3^2 + a_4^2 - 2a_1 a_2 &= (a_2 + a_3)^2 + (a_2 + a_4)^2 \geq 0, \end{aligned}$$

etc.. So, the proof is completed.

REMARK. Note that, if  $m=5$ , for example :

$$a_3^2 + a_4^2 + a_5^2 - 2a_1 a_2 = -3 < 0$$

holds for  $a_1=3, a_2=2, a_3=a_4=-2, a_5=-1$ . Thus, considering a matrix  $B$  with components  $(b_{12}=-b_{21} \neq 0, \text{ otherwise } =0)$ , we see that the condition  $m \leq 4$  in Lemma 1 is essential.

## 2. Minimal hypersurfaces of a Riemannian manifold.

Let  $M$  be an orientable minimal hypersurface of an  $(m+1)$ -dimensional orientable Riemannian manifold  $M^* = (M^*, \langle, \rangle)$ . Let  $n$  be a unit normal vector field on  $M$  and let  $A$  denote the shape operator with respect to  $n$ . We denote

the Riemannian curvature tensor and the Ricci curvature tensor of  $M^*$  by  $R^*$  and  $\rho^*$ .

We assume that  $M$  admits a nontrivial  $L^2$  harmonic 2-form  $w$ . After Palmer [10] we use the following cut off function  $h$ . Let  $p$  be a point of  $M$ . By  $B_r(p)$  we denote the geodesic  $r$ -ball centered at  $p$  ( $r$ -neighborhood of  $p$  in  $M$ ).  $h$  is a smooth function such that  $0 \leq h \leq 1$  and

- (i)  $h=1$  on  $B_{r/2}(p)$  and  $h=0$  outside  $B_r(p)$ ,
- (ii)  $\|\nabla h\|^2 \leq c/r^2$ , where  $c$  is a constant.

We consider  $V=h\|w\|n$  as a variation vector field of  $M$ . Then the second variation formula is given by

$$\begin{aligned} a''(0) &= \int_M (\|\nabla^\perp V\|^2 - \rho^*(V, V) - \|A\|^2 \|V\|^2) \\ &= - \int_M h^2 (\|w\| \Delta \|w\| + \rho^*(n, n) \|w\|^2 + \|A\|^2 \|w\|^2) + \int_M \|\nabla h\|^2 \|w\|^2. \end{aligned}$$

By (1.4) we obtain

$$\begin{aligned} a''(0) &= - \int_M h^2 [2(\rho; w, w) - \langle [R]w, w \rangle + F(w) + \rho^*(n, n) \|w\|^2 + \|A\|^2 \|w\|^2] \\ &\quad + \int_M \|\nabla h\|^2 \|w\|^2. \end{aligned}$$

Let  $\{e_j, e_0=n; 1 \leq j \leq m\}$  be a local orthonormal frame along  $M$ . Then

$$(2.1) \quad R_{jkl}^i = R^{*j}_{kl} + A_k^i A_{jl} - A_l^i A_{jk},$$

$$(2.2) \quad R_{jl} = R^{*j}_l - R^{*0}_{j0l} - A_j^r A_{rl}.$$

Therefore we obtain

$$\begin{aligned} (\rho; w, w) &= (\rho^*; w, w) - (R_0^*; w, w) - \langle Aw, Aw \rangle, \\ \langle [R]w, w \rangle &= \langle [R^*]w, w \rangle - 2A^{ik} w_{kl} A^{lj} w_{ji}, \end{aligned}$$

where  $(\rho^*; w, w) = \rho^*_{jl} w^j w^l$ ,  $(R_0^*; w, w) = R^{*0}_{j0l} w^j w^l$ , and

$$\langle Aw, Aw \rangle = A_r^i w^{rj} A_{is} w^s, \quad \langle [R^*]w, w \rangle = R^*_{ijkl} w^{ij} w^{kl}.$$

Consequently, we get

$$a''(0) = - \int_M h^2 [P^*(w) + F(w) + D(w)] + \int_M \|\nabla h\|^2 \|w\|^2,$$

where we have put

$$(2.3) \quad P^*(w) = 2(\rho^*; w, w) - 2(R_0^*; w, w) - \langle [R^*]w, w \rangle + \rho^*(n, n) \|w\|^2$$

$$(2.4) \quad D(w) = \|A\|^2 \|w\|^2 - 2\langle Aw, Aw \rangle + 2A^{ik} w_{kl} A^{lj} w_{ji}.$$

Now we assume that  $M^*$  is the  $(m+1)$ -dimensional Euclidean space  $E^{m+1}$ . Then we have

$$(2.5) \quad a''(0) = -\int_M h^2[F(w)+D(w)] + \int_M \|\nabla h\|^2 \|w\|^2.$$

**THEOREM 2.1.** *Let  $M \subset E^5$  be a complete, orientable minimal hypersurface. If  $M$  admits a nontrivial  $L^2$  harmonic 2-form, then  $M$  is unstable.*

**PROOF.** We assume that a complete orientable minimal hypersurface  $M$  in  $E^5$  is stable and  $M$  admits a nontrivial  $L^2$  harmonic 2-form  $w$ . Lemma 1 implies that  $D(w) \geq 0$  holds on  $M$ . Then (2.5) implies the following:

$$0 \leq a''(0) \leq -\int_{B_{r/2}(p)} [F(w)+D(w)] + (c/r^2) \int_M \|w\|^2.$$

Letting  $r \rightarrow \infty$ , we obtain  $F(w)=D(w)=0$ . The equality  $F(w)=0$  shows (1.3). We consider (1.3) on an open set where  $w \neq 0$ .  $\delta w=0$  implies that  $w^k_j \nabla_k \|w\|^2=0$  holds.  $dw=0$  is equivalent to  $\nabla_k w_{ij} + \nabla_i w_{jk} + \nabla_j w_{ki}=0$ . By (1.3) and the last equality multiplied by  $w^{ij}$ , we get  $\nabla_k \|w\|^2=0$ , and hence  $\|w\|$  is constant. This is a contradiction, because the volume of any complete minimal hypersurface of  $E^{m+1}$  is infinite (cf. Burago and Zalgaller [4], p. 215). q.e.d.

**PROOF OF THEOREM A.** It suffices to show the case  $m=4$ . By the assumption that  $M \subset E^5$  is stable we see that  $\mathcal{H}^1(M)=\{0\}$  holds by Palmer's result [10] and  $\mathcal{H}^3(M)=\{0\}$  holds by the duality. On the other hand we have  $\mathcal{H}^0(M)=\mathcal{H}^4(M)=\{0\}$ . Combining these with Theorem 2.1, proof is completed.

**REMARK.** It is not clear to the author if there is some nice curvature condition of  $M^*$  which implies  $P^*(w) \geq 0$ . If there is some, then one may obtain some generalized version of Theorem 2.1.

### 3. $P_2$ positivity of sectional curvature.

Let  $M=(M, g)$  be an  $m$ -dimensional Riemannian manifold. Let  $p$  be a point of  $M$  and let  $\{e_1, e_2, \dots, e_m\}$  be an orthonormal basis of the tangent space  $T_p M$  at  $p$ . By  $K_{rs}$  we denote the sectional curvature;  $K(e_r, e_s)$ . The scalar curvature of  $M$  is denoted by  $S$ .

**DEFINITION.** The sectional curvature of  $M$  is said to be  $P_2$  positive ( $P_2$  nonnegative), if for any point  $p$  of  $M$  and for any orthonormal pair  $\{X, Y\}$  at  $p$

$$\rho(X, X) + \rho(Y, Y) - K(X, Y) > 0 \quad (\geq 0, \text{ resp.}).$$

It is clear that positivity of sectional curvature of  $M$  implies  $P_2$  positivity of sectional curvature of  $M$ .

If  $m=2$ , then  $\rho(e_1, e_1)+\rho(e_2, e_2)-K_{12}=\rho(e_1, e_1)=S/2$ , and  $P_2$  positivity is equivalent to the fact that the scalar curvature is positive.

If  $m=3$ , then  $\rho(e_1, e_1)+\rho(e_2, e_2)-K_{12}=S/2$ , etc., and  $P_2$  positivity is equivalent to the fact that the scalar curvature is positive.

If  $m \geq 4$ , for  $i, j$  ( $1 \leq i < j \leq m$ ) we define  $P_{ij}$  by

$$P_{ij} = \sum_{r, s \neq i, j, r < s} K_{rs}.$$

Then the scalar curvature  $S$  is given by  ${}_{m-2}C_2 S = 2 \sum_{i < j} P_{ij}$ , and

$$\rho(e_i, e_i) + \rho(e_j, e_j) - K_{ij} = S/2 - P_{ij}.$$

Therefore,  $P_2$  positivity is expressed by  $S/2 - P_{ij} > 0$ .

For example, if  $m=5$ , we have  $S = 2 \sum_{i < j} K_{ij}$ , and  $P_{12} = K_{34} + K_{35} + K_{45}$ . So,

$$S - 2P_{12} = 2(K_{12} + K_{13} + K_{14} + K_{15} + K_{23} + K_{24} + K_{25}).$$

Here we give a simple example of a Riemannian manifold which is not of positive sectional curvature, but the sectional curvature is  $P_2$  positive. Let  $(S^m, g_0)$  be the odd dimensional unit sphere in  $E^{m+1}$ ,  $m=2n+1$ . It admits a standard contact structure  $\eta$ . We define a Riemannian metric  $g(t)$  by

$$g(t) = t^{-1}g_0 + t^{-1}(t^m - 1)\eta \otimes \eta,$$

where  $t$  is a real number which is determined later. For  $t \geq 1$ , the sectional curvature and the Ricci curvature are bounded as follows:

$$\begin{aligned} t(4-3t^m) &\leq K_{(t)}(X, Y) \leq t^{m+1}, \\ (m+1)t-2t^{m+1} &\leq \rho_{(t)}(X, X) \leq (m-1)t^{m+1} \end{aligned}$$

for any orthonormal pair  $\{X, Y\}$  with respect to  $g(t)$  (cf. S. Tanno [13], [14], etc.). Let  $\varepsilon$  be a small positive number, and define  $t$  by  $-\varepsilon = 4 - 3t^m$ . Then for any orthonormal pair  $\{X, Y\}$ , we obtain

$$\begin{aligned} \rho_{(t)}(X, X) + \rho_{(t)}(Y, Y) - K_{(t)}(X, Y) &\geq 2(m+1)t - 5t^{m+1} \\ &= [2m+2 - (20/3) - (5/3)\varepsilon]t. \end{aligned}$$

The above is positive by suitable choice of  $\varepsilon$ ; while some sectional curvature is negative.

#### 4. $L^2$ harmonic 1-forms.

Let  $M$  be a complete orientable minimal hypersurface of an orientable Riemannian manifold  $M^* = (M^*, \langle, \rangle)$ . We assume that  $M$  admits a nontrivial  $L^2$  harmonic 1-form  $u$ . We consider the variation of  $M$  by  $V = h\|u\|n$  as Palmer and Miyaoka did. We put

$$W(u) = \|A\|^2 \|u\|^2 - \langle Au, Au \rangle \geq 0,$$

$$E(u) = \|\nabla u\|^2 - \|\nabla \|u\|\|^2 \geq 0,$$

where  $\|u\|^2 = u_i u^i$ . Since  $\text{Tr } A = 0$ ,  $W(u) = 0$  holds, if and only if  $A$  or  $u$  vanishes.  $E(u) = 0$  holds, if and only if  $2\|u\|^2 \nabla_i u_j = \nabla_i \|u\|^2 \cdot u_j$ .

On the other hand, the second variation formula is given by

$$a''(0) = -\int_M h^2 [\|u\| \Delta \|u\| + \rho^*(n, n) \|u\|^2 + \|A\|^2 \|u\|^2] + \int_M \|\nabla h\|^2 \|u\|^2.$$

Corresponding to (1.1) and (1.2), we obtain

$$\Delta \|u\|^2 = 2\rho(u^*, u^*) + 2\|\nabla u\|^2,$$

$$\Delta \|u\|^2 = 2\|u\| \Delta \|u\| + 2\|\nabla u\| - 2E(u),$$

where  $u^*$  denotes the vector field dual to  $u$  with respect to the Riemannian metric. Hence, using (2.2) we obtain

$$-\|u\| \Delta \|u\| = -\rho(u^*, u^*) + \langle R^*(n, u^*)u^*, n \rangle + \langle Au, Au \rangle - E(u).$$

Therefore, we obtain

$$a''(0) = -\int_M h^2 [Q^*(u) + E(u) + W(u)] + \int_M \|\nabla h\|^2 \|u\|^2,$$

where we have put

$$Q^*(u) = \rho^*(u^*, u^*) - \langle R^*(n, u^*)u^*, n \rangle + \rho^*(n, n) \|u\|^2.$$

**THEOREM 4.1.** *Let  $M \subset M^*$  be a complete orientable and stable minimal hypersurface. We assume that the sectional curvature of  $M^*$  is  $P_2$  nonnegative.*

(i) *If  $M$  is compact and if  $M$  admits a nontrivial harmonic 1-form  $u$ , then  $u$  is parallel,  $M$  is totally geodesic in  $M^*$  and  $\rho^*(n, n) = 0$ .*

(ii) *If  $M$  is noncompact, then there are no nontrivial  $L^2$  harmonic 1-forms on  $M$ .*

**PROOF.** We assume that  $M$  is noncompact and there is a nontrivial  $L^2$  harmonic 1-form  $u$ .  $P_2$  nonnegativity of the sectional curvature of  $M^*$  shows  $Q^*(u) \geq 0$ . So we have

$$0 \leq a''(0) \leq -\int_{B_{r/2}(p)} [Q^*(u) + E(u) + W(u)] + (c/r^2) \int_M \|u\|^2.$$

Letting  $r \rightarrow \infty$ , we have  $Q^*(u) = E(u) = W(u) = 0$ . The equality  $E(u) = 0$  implies  $2\|u\|^2 \nabla_i u_j = \nabla_i \|u\|^2 u_j$ . So  $\delta u = 0$  implies  $u^i \nabla_i \|u\|^2 = 0$ . Furthermore,  $du = 0$  implies  $\|u\|$  is constant and  $u$  is parallel.  $W(u) = 0$  implies that  $M$  is totally geodesic in  $M^*$ . We check the condition  $Q^*(u) = 0$ . (2.2) means  $Q^*(u) = \rho(u^*, u^*) + \rho^*(n, n) \|u\|^2 = 0$ . Since  $u$  is parallel,  $\rho(u^*, u^*) = 0$ , and hence  $\rho^*(n, n) = 0$ . Let

$e$  be an arbitrary unit tangent vector to  $M$ .  $P_2$  nonnegativity for an orthonormal pair  $\{e, n\}$  implies

$$\rho^*(e, e) + \rho^*(n, n) - K^*(e, n) \geq 0.$$

By (2.2) and  $\rho^*(n, n) = 0$ , the left hand side of the last inequality is equal to  $\rho(e, e)$ . Thus the Ricci curvature of  $M$  is nonnegative. Because  $M$  is complete and noncompact, the volume of  $M$  is infinite. This is a contradiction.

Next, if  $M$  is compact, it suffices to put  $h=1$  in the above discussion.

REMARK. By Dodziuk's result [7] the existence of a nontrivial  $L^2$  harmonic 1-form follows from a topological condition that there exists a cycle of codimension one in  $M$  which does not disconnect  $M$  (cf. Palmer [10]).

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