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Uniqueness of the solution of non-linear singular partial differential equations

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Introduction.

The existence and the uniqueness of the solution of non-linear singular partial differential equations of the form

$$\text{(E)} \qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha|\le m \\ j<m}}\right)$$

were discussed in Gérard-Tahara [1], [2]; though, the uniqueness in [2] can be applied only to the solution with

$$\text{(0.1)} \qquad \left(t\frac{\partial}{\partial t}\right)^j u(t, x) = O(t^s) \quad (\text{as } t \to 0 \text{ uniformly in } x) \quad \text{for } j = 0, 1, \cdots, m-1$$

for some $s>0$.

In this paper, the author will prove the uniqueness of the solution of (E) under the following weaker assumption:

$$\text{(0.2)} \qquad \left(t\frac{\partial}{\partial t}\right)^j u(t, x) = O\left(\frac{1}{(-\log t)^s}\right) \quad (\text{as } t \to 0 \text{ uniformly in } x) \quad \text{for } j = 0, 1, \cdots, m-1$$

for some $s>0$.

The motivation for such an improvement will be illustrated in the following example.

EXAMPLE. Let us consider

$$\text{(0.3)} \qquad t\frac{\partial u}{\partial t} = \lambda u + u\frac{\partial u}{\partial x},$$

where $(t, x) \in \boldsymbol{C} \times \boldsymbol{C}$ and $\lambda \in \boldsymbol{C}$. Then:

(1) $u \equiv 0$ is a solution of (0.3).

(2) By the method of the separation of variables we can see that (0.3) has solutions of the form

$$u(t, x)=\begin{cases} t^{\lambda}\dfrac{ax+b}{-(a/\lambda)t^{\lambda}+c}, & \text{when } \lambda\neq 0,\\[2mm] \dfrac{ax+b}{a(-\log t)+c}, & \text{when } \lambda=0,\end{cases}$$

where $a, b, c\in \boldsymbol{C}$ are arbitrary constants.

(3) The condition (0.1) corresponds to the case $\lambda\neq 0$; while the condition (0.2) corresponds to the case $\lambda=0$.

Compare this with the following assertions on the uniqueness of the solution of (0.3):

(S_1) If $\mathrm{Re}\,\lambda\leqq 0$ and if $u(t, x)$ is a solution of (0.3) satisfying (0.1) for some $s>0$, we have $u(t, x)\equiv 0$.

(S_2) If $\mathrm{Re}\,\lambda<0$ and if $u(t, x)$ is a solution of (0.3) satisfying (0.2) for some $s>0$, we have $u(t, x)\equiv 0$.

The assertion (S_1) is a consequence of the result in [**1**], [**2**]. By (2) of Example we see that in the case $\mathrm{Re}\,\lambda>0$ the uniqueness of type (S_1) does not hold. Also, we can see that in the case $\lambda=0$ the uniqueness of type (S_2) does not hold.

Thus, in this paper we will discuss the case $\mathrm{Re}\,\lambda<0$ and obtain the assertion (S_2) as a consequence of the main theorem in § 1.

The paper is organized as follows. In § 1 we state our uniqueness theorem (Theorem 1) for (E). In § 2 we present some preparatory discussions and in § 3 we give a proof of our uniqueness theorem. The result is applied in § 4 to the problem of removable singularities of solutions of (E).

§ 1. Formulation and result.

Let $m\in \boldsymbol{N}^*$ $(=\{1, 2, \cdots\})$ and put:

$$I_m=\{(j, \alpha)\in \boldsymbol{N}\times\boldsymbol{N}^n\,;\ j+|\alpha|\leqq m \text{ and } j<m\},$$

$$d(m)=\text{the cardinal of } I_m,$$

where $n\in\boldsymbol{N}^*$, $\boldsymbol{N}=\{0, 1, 2, \cdots\}$, $\alpha=(\alpha_1, \cdots, \alpha_n)\in\boldsymbol{N}^n$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$. Denote:

$$t\in\boldsymbol{R},$$

$$x=(x_1, \cdots, x_n)\in\boldsymbol{C}^n,$$

$$Z=\{Z_{j,\alpha}\}_{(j,\alpha)\in I_m}\in\boldsymbol{C}^{d(m)},$$

and let $F(t, x, Z)$ be a function defined on $\{(t, x, Z)\in\boldsymbol{R}\times\boldsymbol{C}^n\times\boldsymbol{C}^{d(m)}\,;\ 0\leqq t\leqq T, |x|\leqq r \text{ and } |Z|\leqq R\}$ for some $T>0$, $r>0$ and $R>0$.

In this paper, we will consider the following non-linear singular partial differential equation:

$$(\mathrm{E}_1) \qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in I_m}\right)$$

with $u=u(t, x)$ as an unknown function.

Let $\mu(t)$ be a function on $(0, T)$ satisfying the following conditions $\mu_1)\sim\mu_4)$:

$\mu_1)$ $\mu(t) \in C^1((0, T))$,

$\mu_2)$ $\mu(t) > 0$ on $(0, T)$ and $\mu(t)$ is increasing in t,

$\mu_3)$ $\displaystyle\int_0^T \frac{\mu(s)}{s} ds < \infty$,

$\mu_4)$ $\displaystyle t\frac{d\mu}{dt}(t) = O(\mu(t))$ (as $t \to 0$).

Note that the condition $\mu(t)\to 0$ (as $t\to 0$) follows from $\mu_2)$ and $\mu_3)$. The following functions are typical examples:

$$\mu(t) = t^a, \quad \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $a>0$, $b>1$, $c>1$.

The main assumptions on the equation (E_1) are as follows.

(C_1) $F(t, x, Z)$ is continuous in $t\in[0, T]$ and holomorphic in (x, Z);

(C_2) $\displaystyle\max_{|x|\le r}|F(t, x, 0)| = O(\mu(t)^m)$ (as $t\to 0$);

(C_3) $\displaystyle\max_{|x|\le r}\left|\frac{\partial F}{\partial Z_{j,\alpha}}(t, x, 0)\right| = O(\mu(t)^{|\alpha|})$ (as $t\to 0$) for any $(j, \alpha)\in I_m$.

Under (C_1), (C_2), (C_3) we denote by $\lambda_1(0), \cdots, \lambda_m(0)$ the roots of

$$\lambda^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0, 0, 0)\lambda^j = 0$$

and call them the characteristic exponents of (E_1) at $x=0$.

DEFINITION. Let $\varepsilon>0$ and $\delta>0$. We denote by $\mathcal{S}(\varepsilon, \delta; \mu(t))$ the set of all functions $u(t, x)$ satisfying the following (i), (ii) and (iii):

(i) $u(t, x)$ is a function on $\{(t, x)\in \boldsymbol{R}\times\boldsymbol{C}^n;\ 0<t<\varepsilon$ and $|x|\le\delta\}$;

(ii) $u(t, x)$ is of C^m class in $t\in(0, \varepsilon)$ and holomorphic in x;

(iii) There is an $s>0$ such that for $j=0, 1, \cdots, m-1$ we have

$$\max_{|x|\leq\delta}\left|\left(t\frac{\partial}{\partial t}\right)^j u(t, x)\right| = O(\mu(t)^s) \quad (\text{as } t\to 0).$$

Note that by (iii) and the Cauchy's inequality in x we easily see for any $0<\delta_2<\delta$

$$\max_{|x|\leq\delta_2}\left|\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u(t, x)\right| = O(\mu(t)^s) \quad (\text{as } t\to 0)$$
$$\text{for any } (j, \alpha)\in I_m.$$

We will use $\mathcal{S}(\varepsilon, \delta; \mu(t))$ as a framework of our solutions of (E_1). Since $\mu(t)\to 0$ (as $t\to 0$) holds, $(t(\partial/\partial t))^j u(t, x)$ $(j=0, 1, \cdots, m-1)$ can be continuous in $t\in[0, \varepsilon)$ (including $t=0$). Then:

THEOREM 1. *Let $\mu(t)$ be a function on $(0, T)$ satisfying the conditions $\mu_1)\sim\mu_4)$. Assume (C_1), (C_2), (C_3) and the condition*

(1.1) $$\operatorname{Re}\lambda_i(0) < 0 \quad \textit{for } i=1, \cdots, m.$$

Then, if $u_1(t, x)$ and $u_2(t, x)$ are two solutions of (E_1) belonging to $\mathcal{S}(\varepsilon, \delta; \mu(t))$, we have $u_1(t, x)=u_2(t, x)$ on $\{(t, x)\in \boldsymbol{R}\times\boldsymbol{C}^n; 0<t<\varepsilon_1$ and $|x|\leqq\delta_1\}$ for some $\varepsilon_1>0$ and $\delta_1>0$.

REMARK 1. (1) In the case $\mu(t)=t^c$, $c>0$, the above result is obtained by the discussion in [2].

(2) In §4 we will use this theorem in the case

$$\mu(t) = \frac{1}{(-\log t)^c}, \quad c>1.$$

Note that in this case the discussion in [2] does not work in general.

REMARK 2. The author believes that Theorem 1 should be improved in the following form: if (C_1), (C_2), (C_3) and (1.1) are satisfied, the uniqueness of the solution of (E_1) is valid under the condition

$$\max_{|x|\leq\delta}\left|\left(t\frac{\partial}{\partial t}\right)^j u(t, x)\right| = o(1) \quad (\text{as } t\to 0) \text{ for } j=0, 1, \cdots, m-1.$$

Though, at present he has no idea to prove this conjecture.

In the proof of Theorem 1 (in §3) we will use the following norm: for a convergent power series $f(t, x)$ in x with coefficients in $C^0((0, T))$ of the form

$$f(t, x) = \sum_{\alpha\in\boldsymbol{N}^n} f_\alpha(t)x^\alpha, \quad f_\alpha(t)\in C^0((0, T))$$

we write

$$\|f(t)\|_\rho = \sum_{\alpha\in N^n} |f_\alpha(t)| \frac{\alpha!}{|\alpha|!}\rho^{|\alpha|} \tag{1.2}$$

(which is a convergent power series in ρ with coefficients in $C^0((0, T))$).

The following lemma holds:

LEMMA 1. *For $f(t, x)$ and $g(t, x)$ we have*:

(1) $\|(fg)(t)\|_\rho \leqq \|f(t)\|_\rho \|g(t)\|_\rho$.

(2) $\left\|\left(\frac{\partial}{\partial x_i}\right)f(t)\right\|_\rho \leqq \frac{\partial}{\partial \rho}\|f(t)\|_\rho$ *for* $i=1, \cdots, n$.

§ 2. Preparatory discussion.

Before the proof of Theorem 1, we will prove here the following proposition.

PROPOSITION 1. *Let $\mu(t)$ be as before. Assume* (C_1), (C_2), (C_3) *and* (1.1). *Then, if $u(t, x)$ is a solution of* (E_1) *belonging to* $\mathcal{S}(\varepsilon, \delta; \mu(t))$ *and if $\delta>0$ is sufficiently small, we have for any* $0<\delta_2<\delta$

$$\max_{|x|\leq\delta_2}\left|\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u(t, x)\right| = O(\mu(t)^m) \quad (as\ t\to 0) \tag{2.1}$$
$$for\ any\ (j, \alpha)\in I_m.$$

To prove this, we need a result on the ordinary differential equation:

$$\left(t\frac{\partial}{\partial t}\right)^m u = \sum_{j<m} a_j(t, x)\left(t\frac{\partial}{\partial t}\right)^j u + f(t, x), \tag{2.2}$$

where

$$a_j(t, x) = \frac{\partial F}{\partial Z_{j,0}}(t, x, 0), \quad j=0, 1, \cdots, m-1.$$

For $k\in \boldsymbol{N}$, $\varepsilon>0$ and $\delta>0$ we denote by $\mathcal{T}_k(\varepsilon, \delta)$ the set of all functions $u(t, x)$ satisfying the following properties i), ii) and iii):

i) $u(t, x)$ is a function on $\{(t, x)\in \boldsymbol{R}\times \boldsymbol{C}^n; 0<t<\varepsilon$ and $|x|\leqq\delta\}$;

ii) $u(t, x)$ is of C^k class in $t\in(0, \varepsilon)$ and holomorphic in x;

iii) For any $j=0, 1, \cdots, k$ we have

$$\max_{|x|\leq\delta}\left|\left(t\frac{\partial}{\partial t}\right)^j u(t, x)\right| = O(1) \quad (\text{as } t\to 0).$$

Then, we have:

LEMMA 2. *If*

$$\mathrm{Re}\,\lambda_i(0) < 0 \quad for\ i=1, \cdots, m$$

and if $\varepsilon>0$ *and* $\delta>0$ *are sufficiently small, then*:

(1) *For any* $f(t, x)\in\mathcal{T}_0(\varepsilon, \delta)$ *there exists a unique solution* $u(t, x)\in\mathcal{T}_m(\varepsilon, \delta)$ *of* (2.2); *moreover the estimates*

$$\sum_{j<m}\sup_{0<\tau\leqq t}\left|\left(\tau\frac{\partial}{\partial\tau}\right)^j u(\tau, x)\right| \leqq C\left(\sup_{0<\tau\leqq t}|f(\tau, x)|\right)$$

$$\text{for any } 0<t<\varepsilon \text{ and } |x|\leqq\delta$$

hold for some $C>0$ *which is independent of* u *and* f.

(2) *If* $f(t, x)$ *satisfies*

$$\max_{|x|\leqq\delta}|f(t, x)| = O(\mu(t)^s) \quad (as\ t\to 0)$$

for some $s>0$, *then the unique solution* $u(t, x)$ *of* (2.2) *also satisfies for any* $0<\delta_2<\delta$

$$\max_{|x|\leqq\delta_2}\left|\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u(t, x)\right| = O(\mu(t)^s) \quad (as\ t\to 0)$$

$$\text{for any } (j, \alpha)\in I_m.$$

(1) is obtained from the discussion in Tahara [3, §2]. (2) is a corollary of (1). By using this lemma, let us give a proof of Proposition 1.

PROOF OF PROPOSITION 1. Let $u(t, x)$ be a solution of (E_1) belonging to $\mathcal{S}(\varepsilon, \delta; \mu(t))$. Then, there is an $s>0$ such that for any $0<\delta_2<\delta$

$$\max_{|x|\leqq\delta_2}\left|\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u(t, x)\right| = O(\mu(t)^s) \quad (\text{as } t\to 0) \tag{2.3}$$

$$\text{for any } (j, \alpha)\in I_m.$$

Since $u(t, x)$ is a solution of (E_1), by the Taylor expansion we get

$$\left(t\frac{\partial}{\partial t}\right)^m u = \sum_{(j,\alpha)\in I_m} b_{j,\alpha}(t, x)\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u + F(t, x, 0),$$

where

$$\begin{aligned} b_{j,\alpha}(t, x) &= \int_0^1 \frac{\partial F}{\partial Z_{j,\alpha}}\left(t, x, \theta\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in I_m}\right)d\theta \\ &= \frac{\partial F}{\partial Z_{j,\alpha}}(t, x, 0) + O(\mu(t)^s) \\ &= O(\mu(t)^{|\alpha|}) + O(\mu(t)^s) \quad (\text{as } t\to 0) \end{aligned} \tag{2.4}$$

(from (2.3) and (C_3)). Therefore, if we put

$$R[u] = F(t, x, 0) + \sum_{j<m}\left(b_{j,0}(t, x) - \frac{\partial F}{\partial Z_{j,0}}(t, x, 0)\right)\left(t\frac{\partial}{\partial t}\right)^j u$$

$$+\sum_{\substack{(j,\alpha)\in I_m\\|\alpha|>0}} b_{j,\alpha}(t,x)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u,$$

we have

$$\Big(t\frac{\partial}{\partial t}\Big)^m u=\sum_{j<m} a_j(t,x)\Big(t\frac{\partial}{\partial t}\Big)^j u+R[u] \tag{2.5}$$

and for $r=\min\{1, s\}>0$ we see

$$R[u]=O(\mu(t)^m)+\sum_{(j,\alpha)\in I_m} O(\mu(t)^r)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u \tag{2.6}$$

(from (C_2) and (2.4)).

Now, choose a sequence $s_1, s_2, \cdots, s_p$ so that the following properties are satisfied:

(a-1) $s_1=s<s_2<\cdots<s_p=m$;

(a-2) $s_{i+1}<r+s_i$ holds for $i=1, \cdots, p-1$.

Then, let us show that

$$\max_{|x|\le\delta_2}\left|\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u(t,x)\right|=O(\mu(t)^{s_k})\quad(\text{as } t\to 0) \tag{2.7$_k$}$$

for any $(j, \alpha)\in I_m$

holds for any $0<\delta_2<\delta$ and $k=1, 2, \cdots, p$.

Since $s_1=s$, $(2.7)_1$ is clear from (2.3). Then, by (2.6) and (a-2) we see

$$\begin{aligned}\max_{|x|\le\delta_2}|R[u](t,x)| &= O(\mu(t)^m)+O(\mu(t)^{r+s_1})\\ &= O(\mu(t)^{s_2})\quad(\text{as } t\to 0);\end{aligned}$$

therefore by applying (2) of Lemma 2 to (2.5) we have $(2.7)_2$. By substituting this into (2.6) we have

$$\begin{aligned}\max_{|x|\le\delta_2}|R[u](t,x)| &= O(\mu(t)^m)+O(\mu(t)^{r+s_2})\\ &= O(\mu(t)^{s_3})\quad(\text{as } t\to 0)\end{aligned}$$

and hence by using (2) of Lemma 2 again we obtain $(2.7)_3$.

Repeating the same argument, we see that $(2.7)_k$ holds for any $k=1, 2, \cdots, p$. Since $s_p=m$, $(2.7)_p$ is the same as (2.1). Thus, Proposition 1 is proved.

§3. Proof of Theorem 1.

Let $\mu(t)$ be a function on $(0, T)$ satisfying the conditions $\mu_1)\sim\mu_4)$. Assume (C_1), (C_2), (C_3) and the condition (1.1). Then we can choose $A>0$ and $h>0$ so that

$$\left|t\frac{d\mu}{dt}(t)\right| \leqq A\mu(t), \quad 0<t<T;$$

$$\mathrm{Re}\,\lambda_i(0) < -2h < 0, \quad i=1, \cdots, m.$$

Put

$$\Theta_0 = 1,$$

$$\Theta_1 = \left(t\frac{\partial}{\partial t} - \lambda_1(0)\right),$$

$$\Theta_2 = \left(t\frac{\partial}{\partial t} - \lambda_2(0)\right)\left(t\frac{\partial}{\partial t} - \lambda_1(0)\right),$$

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$$\Theta_m = \left(t\frac{\partial}{\partial t} - \lambda_m(0)\right)\left(t\frac{\partial}{\partial t} - \lambda_{m-1}(0)\right) \cdots \left(t\frac{\partial}{\partial t} - \lambda_1(0)\right).$$

Let $u_1(t, x)$ and $u_2(t, x)$ be two solutions of (E_1) belonging to $\mathcal{S}(\varepsilon, \delta; \mu(t))$ and put

(3.1) $$w(t, x) = u_2(t, x) - u_1(t, x).$$

Then, $w(t, x)$ is a solution of

(3.2) $$\left(t\frac{\partial}{\partial t}\right)^m w = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u_1 + \left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha w\right\}_{(j,\alpha)\in I_m}\right) - F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u_1\right\}_{(j,\alpha)\in I_m}\right).$$

Moreover by Proposition 1 we have

(3.3) $$\max_{|x|\leqq\delta_2}\left|\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u_1(t, x)\right| = O(\mu(t)^m) \quad (\text{as } t\to 0),$$

(3.4) $$\max_{|x|\leqq\delta_2}\left|\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha w(t, x)\right| = O(\mu(t)^m) \quad (\text{as } t\to 0)$$

for any $0<\delta_2<\delta$ and any $(j, \alpha)\in I_m$.

Our aim is to show that $w(t, x)\equiv 0$ holds on $\{(t, x)\in \boldsymbol{R}\times\boldsymbol{C}^n;\ 0<t<\varepsilon_1$ and $|x|\leqq\delta_1\}$ for some $\varepsilon_1>0$ and $\delta_1>0$. Let us show this from now.

For $(j, k)\in\boldsymbol{N}\times\boldsymbol{N}$ satisfying $j+k\leqq m-1$ we put

(3.5) $$\phi_{j,k}(t, \rho) = \int_0^t \left(\frac{\tau}{t}\right)^{-\mathrm{Re}\,\lambda_{j+1}(0)} \mu(\tau)^k \times\left\{\sum_{|\alpha|=k}\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\right\|_\rho + kA\sum_{|\alpha|=k}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\right\|_\rho\right\}\frac{d\tau}{\tau},$$

where $\|\cdot\|_\rho$ is the norm introduced in (1.2). Then we have:

LEMMA 3. *Let* $0<\varepsilon_2<\varepsilon$ *and* $0<\delta_2<\delta$. *Then,* $\phi_{j,k}(t, \rho)$ $(j+k\leqq m-1)$ *are well-defined in* $C^0([0, \varepsilon_2]\times[0, \delta_2])$ *and satisfy the following properties on* $\{(t, \rho);\ 0<t\leqq\varepsilon_2$ *and* $0\leqq\rho\leqq\delta_2\}$:

(1) $\phi_{j,k}(t, \rho)$ *is of* C^1 *class in* $t\in(0, \varepsilon_2]$ *and analytic in* $\rho\in[0, \delta_2]$.

(2) *For any* (j, k) *we have*

$$\mu(t)^k \sum_{|\alpha|=k} \left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(t)\right\|_\rho \leqq \phi_{j,k}(t, \rho).$$

(3) *When* $k>0$, *we have*

$$\begin{aligned}&\left(t\frac{\partial}{\partial t}+2h\right)\phi_{j,k}(t, \rho)\\ &\leqq n\mu(t)\frac{\partial}{\partial\rho}\phi_{j+1,k-1}(t, \rho)+nkA\mu(t)\frac{\partial}{\partial\rho}\phi_{j,k-1}(t, \rho).\end{aligned}$$

(4) *When* $k=0$ *and* $j=0, 1, \cdots, m-2$, *we have*

$$\left(t\frac{\partial}{\partial t}+2h\right)\phi_{j,0}(t, \rho) \leqq \phi_{j+1,0}(t, \rho).$$

(5) *When* $k=0$ *and* $j=m-1$, *we have*

$$\begin{aligned}&\left(t\frac{\partial}{\partial t}+2h\right)\phi_{m-1,0}(t, \rho)\\ &\leqq \gamma(t, \rho)\sum_{j<m}\phi_{j,0}(t, \rho)+B\mu(t)\frac{\partial}{\partial\rho}\sum_{j+k\leqq m-1}\phi_{j,k}(t, \rho)\end{aligned}$$

for some $B>0$ *and some* $\gamma(t, \rho)\in C^0([0, \varepsilon_2]\times[0, \delta_2])$ *satisfying* $\gamma(0, 0)=0$.

PROOF. (1) is clear from the definition. For (j, k) and $|\alpha|=k$ we have

$$\begin{aligned}&\left(t\frac{\partial}{\partial t}-\lambda_{j+1}(0)\right)\left(\mu(t)^k\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right)\\ &=\mu(t)^k\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha w+k\mu(t)^{k-1}t\frac{d\mu(t)}{dt}\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w.\end{aligned}$$

By integrating this we get

$$\begin{aligned}\mu(t)^k\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(t)=\int_0^t\left(\frac{\tau}{t}\right)^{-\lambda_{j+1}(0)}&\Big\{\mu(\tau)^k\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\\ &+k\mu(\tau)^{k-1}\tau\frac{d\mu(\tau)}{d\tau}\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\Big\}\frac{d\tau}{\tau}\end{aligned}$$

and hence by taking the norm we see

$$\mu(t)^k\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(t)\right\|_\rho$$

$$\leqq \int_0^t\left(\frac{\tau}{t}\right)^{-\mathrm{Re}\,\lambda_{j+1}(0)}\left\{\mu(\tau)^k\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\right\|_\rho\right.$$

$$\left.+k\mu(\tau)^{k-1}\left|\tau\frac{d\mu(\tau)}{d\tau}\right|\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\right\|_\rho\right\}\frac{d\tau}{\tau}$$

$$\leqq \int_0^t\left(\frac{\tau}{t}\right)^{-\mathrm{Re}\,\lambda_{j+1}(0)}\mu(\tau)^k$$

$$\times\left\{\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\right\|_\rho+kA\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(\tau)\right\|_\rho\right\}\frac{d\tau}{\tau}$$

which implies (2).

Let us show (3). Since $0<2h<-\mathrm{Re}\,\lambda_{j+1}(0)$ and $\phi_{j,k}(t,\rho)\geqq 0$ hold, we have

$$\left(t\frac{\partial}{\partial t}+2h\right)\phi_{j,k}(t,\rho) \tag{3.6}$$

$$\leqq \left(t\frac{\partial}{\partial t}-\mathrm{Re}\,\lambda_{j+1}(0)\right)\phi_{j,k}(t,\rho)$$

$$=\mu(t)^k\left\{\sum_{|\alpha|=k}\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha w(t)\right\|_\rho+kA\sum_{|\alpha|=k}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w(t)\right\|_\rho\right\}.$$

Since $|\alpha|=k>0$, we can decompose α into

$$\alpha=\alpha'+e_i,\quad |\alpha'|=k-1 \tag{3.7}$$

for some i $(1\leqq i\leqq n)$, where $e_1=(1,0,\cdots,0),\cdots,e_n=(0,\cdots,0,1)\in\boldsymbol{N}^n$. Therefore by (3.6), (2) of Lemma 1 and (2) of this lemma (which is proved already) we obtain

$$\left(t\frac{\partial}{\partial t}+2h\right)\phi_{j,k}(t,\rho)$$

$$\leqq \mu(t)^k\left\{\sum_{|\alpha|=k}\frac{\partial}{\partial\rho}\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha'} w(t)\right\|_\rho+kA\sum_{|\alpha|=k}\frac{\partial}{\partial\rho}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha'} w(t)\right\|_\rho\right\}$$

$$\leqq \mu(t)^k\left\{n\sum_{|\alpha'|=k-1}\frac{\partial}{\partial\rho}\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha'} w(t)\right\|_\rho\right.$$

$$\left.+kAn\sum_{|\alpha'|=k-1}\frac{\partial}{\partial\rho}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha'} w(t)\right\|_\rho\right\}$$

$$=n\mu(t)\frac{\partial}{\partial\rho}\left\{\mu(t)^{k-1}\sum_{|\alpha'|=k-1}\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha'} w(t)\right\|_\rho\right.$$

$$\left.+kA\mu(t)^{k-1}\sum_{|\alpha'|=k-1}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha'} w(t)\right\|_\rho\right\}$$

$$\leqq n\mu(t)\frac{\partial}{\partial\rho}\{\phi_{j+1,k-1}(t,\rho)+kA\phi_{j,k-1}(t,\rho)\}.$$

This implies (3).

When $k=0$ and $j=0, 1, \cdots, m-2$, by the same argument as in (3.6) and by (2) of this lemma we get

$$\Big(t\frac{\partial}{\partial t}+2h\Big)\phi_{j,0}(t, \rho) \leqq \Big(t\frac{\partial}{\partial t}-\mathrm{Re}\,\lambda_{j+1}(0)\Big)\phi_{j,0}(t, \rho) = \|\Theta_{j+1}w(t)\|_{\rho}$$
$$\leqq \phi_{j+1,0}(t, \rho)$$

which implies (4).

Lastly let us show (5). Since $w(t, x)$ is a solution of (3.2), by the Taylor expansion we get

$$\Big(t\frac{\partial}{\partial t}\Big)^m w = \sum_{(j,\alpha)\in I_m} a_{j,\alpha}(t, x)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w, \tag{3.8}$$

where

$$a_{j,\alpha}(t, x) = \int_0^1 \frac{\partial F}{\partial Z_{j,\alpha}}\Big(t, x, \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_1+\theta\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w\Big\}_{(j,\alpha)\in I_m}\Big)d\theta\,.$$

Moreover, it is easy to see

$$a_{j,\alpha}(0, x) = \frac{\partial F}{\partial Z_{j,\alpha}}(0, x, 0) \tag{3.9}$$

and by (3.3), (3.4) and (C_3) we have

$$\begin{aligned}\max_{|x|\leqq\delta_2}|a_{j,\alpha}(t, x)| &= \frac{\partial F}{\partial Z_{j,\alpha}}(t, x, 0)+O(\mu(t)^m)\\ &= O(\mu(t)^{|\alpha|})+O(\mu(t)^m)\\ &= O(\mu(t)^{|\alpha|}) \quad (\text{as } t\to 0).\end{aligned} \tag{3.10}$$

Therefore, (3.8) can be rewritten into the form

$$\Big(\Big(t\frac{\partial}{\partial t}\Big)^m - \sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0, 0, 0)\Big(t\frac{\partial}{\partial t}\Big)^j\Big)w$$
$$= \sum_{j<m}(a_{j,0}(t, x)-a_{j,0}(0, 0))\Big(t\frac{\partial}{\partial t}\Big)^j w + \sum_{\substack{(j,\alpha)\in I_m\\ |\alpha|>0}} a_{j,\alpha}(t, x)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w$$

and hence

$$\Theta_m w = \sum_{j<m}\gamma_j(t, x)\Theta_j w + \sum_{\substack{(j,\alpha)\in I_m\\ |\alpha|>0}} c_{j,\alpha}(t, x)\Theta_j\Big(\frac{\partial}{\partial x}\Big)^\alpha w \tag{3.11}$$

for some $\gamma_j(t, x)$ and some $c_{j,\alpha}(t, x)$ satisfying the following:

(b-1) $\gamma_j(0, 0) = 0$;

(b-2) $\max_{|x|\leqq\delta_2}|c_{j,\alpha}(t, x)| = O(\mu(t)^{|\alpha|})$ (as $t\to 0$).

By (3.5), (3.11) and (b-2) it is easy to see:

(3.12)
$$\begin{aligned}&\left(t\frac{\partial}{\partial t}+2h\right)\phi_{m-1,0}(t,\rho)\\ &\leqq\left(t\frac{\partial}{\partial t}-\operatorname{Re}\lambda_m(0)\right)\phi_{m-1,0}(t,\rho)=\|\Theta_m w(t)\|_\rho\\ &\leqq\sum_{j<m}\|\gamma_j(t)\|_\rho\|\Theta_j w\|_\rho+\sum_{\substack{(j,\alpha)\in I_m\\ |\alpha|>0}}\|c_{j,\alpha}(t)\|_\rho\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right\|_\rho\\ &=\sum_{j<m}\|\gamma_j(t)\|_\rho\|\Theta_j w\|_\rho+\sum_{\substack{(j,\alpha)\in I_m\\ |\alpha|>0}}O(\mu(t)^{|\alpha|})\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right\|_\rho.\end{aligned}$$

Using the decomposition

$$\alpha=\alpha'+e_i$$

in (3.7), we see

(b-3) $\mu(t)^{|\alpha|}=\mu(t)\mu(t)^{|\alpha'|}$

(b-4) $\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right\|_\rho\leqq\frac{\partial}{\partial\rho}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha'} w\right\|_\rho.$

Thus, by substituting (b-3) and (b-4) into (3.12) we obtain

$$\begin{aligned}&\left(t\frac{\partial}{\partial t}+2h\right)\phi_{m-1,0}(t,\rho)\\ &\leqq\sum_{j<m}\|\gamma_j(t)\|_\rho\|\Theta_j w\|_\rho+O(1)\mu(t)\frac{\partial}{\partial\rho}\left(\sum_{j+|\alpha'|\leqq m-1}\mu(t)^{|\alpha'|}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha'} w\right\|_\rho\right)\\ &\leqq\sum_{j<m}\|\gamma_j(t)\|_\rho\phi_{j,0}(t,\rho)+O(1)\mu(t)\frac{\partial}{\partial\rho}\sum_{j+k\leqq m-1}\phi_{j,k}(t,\rho)\end{aligned}$$

which implies (5). Thus, all the parts of Lemma 3 are proved.

Next, we first choose $\sigma_j>0$ ($j=0, 1, \cdots, m-1$) so that

(3.13)
$$\frac{\sigma_j}{\sigma_{j+1}}<\frac{h}{2},\quad j=0,1,\cdots,m-2$$

and then choose $\varepsilon_3>0$, $\delta_3>0$ sufficiently small so that

(3.14)
$$\gamma(t,\rho)\frac{\sigma_{m-1}}{\sigma_j}<\frac{h}{2},\quad j=0,1,\cdots,m-1$$

hold on $\{(t,\rho);\ 0\leqq t\leqq\varepsilon_3 \text{ and } 0\leqq\rho\leqq\delta_3\}$, where $\gamma(t,\rho)$ is the one in (5) of Lemma 3.

Put

$$\Phi(t,\rho)=\sum_{j<m}\sigma_j\phi_{j,0}(t,\rho)+\sum_{\substack{j+k\leqq m-1\\ k>0}}\phi_{j,k}(t,\rho).$$

Then we have:

LEMMA 4. *There is a $C>0$ such that*

$$\left(t\frac{\partial}{\partial t}+h\right)\Phi(t,\ \rho)\leqq C\mu(t)\frac{\partial}{\partial\rho}\Phi(t,\ \rho)$$

holds on $\{(t,\ \rho);\ 0\leqq t\leqq\varepsilon_3$ *and* $0\leqq\rho\leqq\delta_3\}$.

PROOF. By Lemma 3, (3.13) and (3.14) we have

$$(3.15)\quad \left(t\frac{\partial}{\partial t}+2h\right)\Phi$$

$$=\sum_{j<m}\sigma_j\left(t\frac{\partial}{\partial t}+2h\right)\phi_{j,0}+\sum_{\substack{j+k\leqq m-1\\ k>0}}\left(t\frac{\partial}{\partial t}+2h\right)\phi_{j,k}$$

$$\leqq\sum_{j\leqq m-2}\sigma_j\phi_{j+1,0}+\sigma_{m-1}\left\{\gamma(t,\ \rho)\sum_{j<m}\phi_{j,0}+B\mu(t)\frac{\partial}{\partial\rho}\sum_{j+k\leqq m-1}\phi_{j,k}\right\}$$

$$+\sum_{\substack{j+k\leqq m-1\\ k>0}}\left\{n\mu(t)\frac{\partial}{\partial\rho}\phi_{j+1,k-1}+nkA\mu(t)\frac{\partial}{\partial\rho}\phi_{j,k-1}\right\}$$

$$\leqq\frac{h}{2}\sum_{j\leqq m-2}\sigma_{j+1}\phi_{j+1,0}+\frac{h}{2}\sum_{j<m}\sigma_j\phi_{j,0}+(\sigma_{m-1}B+n+n(m-1)A)\mu(t)\frac{\partial}{\partial\rho}\sum_{j+k\leqq m-1}\phi_{j,k}$$

$$\leqq\frac{h}{2}\Phi+\frac{h}{2}\Phi+C\mu(t)\frac{\partial}{\partial\rho}\Phi$$

for some $C>0$. Hence, on $\{(t,\ \rho);\ 0\leqq t\leqq\varepsilon_3$ and $0\leqq\rho\leqq\delta_3\}$ we obtain

$$\left(t\frac{\partial}{\partial t}+h\right)\Phi(t,\ \rho)\leqq C\mu(t)\frac{\partial}{\partial\rho}\Phi(t,\ \rho).$$

COMPLETION OF THE PROOF OF THEOREM 1. Since

$$\|w(t)\|_\rho\leqq\phi_{0,0}(t,\ \rho)\leqq\frac{1}{\sigma_0}\Phi(t,\ \rho)$$

holds, to show Theorem 1 it is sufficient to prove that $\Phi(t,\ \rho)\equiv 0$ holds on $\{(t,\ \rho);\ 0\leqq t\leqq\varepsilon_1$ and $0\leqq\rho\leqq\delta_1\}$ for some $\varepsilon_1>0$ and $\delta_1>0$.

Let $C>0$ be as in Lemma 4. Choose $T_0>0$ so that $0<T_0<\varepsilon_3$ and

$$C\int_0^{T_0}\frac{\mu(s)}{s}ds<\delta_3.$$

Define the function $\rho_0(t)$ by

$$\rho_0(t)=C\int_t^{T_0}\frac{\mu(s)}{s}ds,\quad 0\leqq t\leqq T_0.$$

Then, $0<\rho_0(0)<\delta_3$, $\rho_0(T_0)=0$ and $\rho_0(t)$ is decreasing in t.

Put

$$\varphi(t)=\Phi(t,\ \rho_0(t)),\quad 0\leqq t\leqq T_0.$$

Then, by Lemma 4 we have

$$\begin{aligned}\Big(t\frac{d}{dt}+h\Big)\varphi(t) &= t\frac{\partial\Phi}{\partial t}(t,\ \rho_0(t))+t\frac{\partial\Phi}{\partial\rho}(t,\ \rho_0(t))\frac{d\rho_0(t)}{dt}+h\Phi(t,\ \rho_0(t))\\ &= \Big(t\frac{\partial}{\partial t}+h-C\mu(t)\frac{\partial}{\partial\rho}\Big)\Phi(t,\ \rho)\Big|_{\rho=\rho_0(t)}\\ &\leqq 0\end{aligned}$$

and therefore

$$\frac{d}{dt}(t^h\varphi(t))\leqq 0,\quad 0<t\leqq T_0.$$

By integrating this from ε to t (>0) we get

$$t^h\varphi(t)\leqq\varepsilon^h\varphi(\varepsilon),\quad 0<\varepsilon\leqq t\leqq T_0$$

and by letting $\varepsilon\to 0$ we see

$$\varphi(t)\leqq 0\quad\text{for } 0<t\leqq T_0.$$

On the other hand, $\varphi(t)\geqq 0$ is clear from the definition of $\varphi(t)$. Hence, we obtain

$$\varphi(t)=0\quad\text{for } 0\leqq t\leqq T_0$$

which implies

$$\Phi(t,\ \rho)=0\quad\text{on } \{(t,\ \rho)\,;\ 0\leqq t\leqq T_0 \text{ and } \rho=\rho_0(t)\}. \tag{3.16}$$

Since $\Phi(t,\ \rho)$ is increasing in ρ, (3.16) implies

$$\Phi(t,\ \rho)\equiv 0\quad\text{on } \{(t,\ \rho)\,;\ 0\leqq t\leqq T_0 \text{ and } 0\leqq\rho\leqq\rho_0(t)\}.$$

This completes the proof of Theorem 1.

§ 4. Application.

Lastly, let us apply Theorem 1 to the problem of removable singularities of solutions of

$$\Big(t\frac{\partial}{\partial t}\Big)^m u=F\Big(t,\ x,\ \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big\}_{(j,\alpha)\in I_m}\Big), \tag{E$_2$}$$

where $t\in\boldsymbol{C}$, $x=(x_1,\ \cdots,\ x_n)\in\boldsymbol{C}^n$ and I_m is the same as in § 1.

On $F(t,\ x,\ Z)$ we impose the following conditions:

(A_1) $F(t,\ x,\ Z)$ is holomorphic in $(t,\ x,\ Z)$ near $(0,\ 0,\ 0)$;

(A_2) $F(0,\ x,\ 0)\equiv 0$ near $x=0$;

(A_3) $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0,\ x,\ 0)\equiv 0$ near $x=0$, if $|\alpha|>0$.

Then (E_2) is the equation discussed by Gérard-Tahara [**2**]. We define the characteristic exponents $\lambda_1(0), \cdots, \lambda_m(0)$ of (E_2) at $x=0$ by the roots of

$$\lambda^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0, 0, 0)\lambda^j = 0.$$

Denote by:

- $\mathcal{R}(\boldsymbol{C}\setminus\{0\})$ the universal covering space of $\boldsymbol{C}\setminus\{0\}$;
- $S_\theta(\varepsilon) = \{t \in \mathcal{R}(\boldsymbol{C}\setminus\{0\})\,;\ 0<|t|<\varepsilon$ and $|\arg t|<\theta\}$;
- $D_\delta = \{x \in \boldsymbol{C}^n\,;\ |x| \leqq \delta\}$.

By Gérard-Tahara [**2**] we already know the following:

THEOREM 2 ([**2**]). *Assume* (A_1), (A_2), (A_3) *and the condition*

$$\mathrm{Re}\,\lambda_i(0) \leqq 0 \quad \textit{for } i=1, \cdots, m. \tag{4.1}$$

Then, if $u(t, x)$ *is a solution of* (E_2) *holomorphic on* $S_\theta(\varepsilon)\times D_\delta$ *for some* $\theta>0$, $\varepsilon>0$, $\delta>0$ *and satisfying*

$$\max_{|x|\leqq\delta} |u(t, x)| = O(|t|^s) \quad (\textit{as } t\to 0 \textit{ in } S_\theta(\varepsilon)) \tag{4.2}$$

for some $s>0$, $u(t, x)$ *must be holomorphic in a full neighborhood of* $(0, 0)$.

This implies that under (4.1) the singularity of the form (4.2) is removable. Since the function

$$\mu(t) = \frac{1}{(-\log t)^c}, \quad c>1$$

satisfies the conditions $\mu_1)\sim\mu_4)$ in §1, by using Theorem 1 we can treat the logarithmic singularities of solutions of (E_2).

THEOREM 3. *Assume* (A_1), (A_2), (A_3) *and the condition*

$$\mathrm{Re}\,\lambda_i(0) < 0 \quad \textit{for } i=1, \cdots, m. \tag{4.3}$$

Then, if $u(t, x)$ *is a solution of* (E_2) *holomorphic on* $S_\theta(\varepsilon)\times D_\delta$ *for some* $\theta>0$, $\varepsilon>0$, $\delta>0$ *and satisfying*

$$\max_{|x|\leqq\delta} |u(t, x)| = O\Big(\frac{1}{(-\log|t|)^s}\Big) \quad (\textit{as } t\to 0 \textit{ in } S_\theta(\varepsilon)) \tag{4.4}$$

for some $s>0$, $u(t, x)$ *must be holomorphic in a full neighborhood of* $(0, 0)$.

REMARK 3. The following example shows that we can not replace (4.3) by (4.1): the equation

$$t\frac{\partial u}{\partial t} = u\Big(\frac{\partial u}{\partial x}\Big)^k$$

(where $(t, x)\in \boldsymbol{C}\times\boldsymbol{C}$ and $k\in\boldsymbol{N}^*$) has singular solutions of the form

$$u(t, x)=\left(\frac{1}{k}\right)^{1/k}\frac{x+c}{(-\log t)^{1/k}}, \quad c\in\boldsymbol{C}.$$

PROOF OF THEOREM 3. Let $u(t, x)$ be a solution of (E_2) holomorphic on $S_\theta(\varepsilon)\times D_\delta$ and satisfying (4.4). Denote by $u_R(t, x)$ the restriction of $u(t, x)$ on $\boldsymbol{R}_+\times D_\delta$. Then, by using the Cauchy's inequality in t we see that for $j=0, 1, \cdots, m-1$

$$\max_{|x|\leq\delta}\left|\left(t\frac{\partial}{\partial t}\right)^j u_R(t, x)\right|=O\left(\frac{1}{(-\log t)^s}\right) \quad (\text{as } t\to 0 \text{ in } \boldsymbol{R}_+).$$

This implies that $u_R(t, x)$ belongs to $\mathcal{S}(\varepsilon, \delta; \mu(t))$ with

$$\mu(t)=\frac{1}{(-\log t)^c}, \quad c>1.$$

On the other hand, by [2] we know that (E_2) has a solution $u_0(t, x)$ holomorphic in a full neighborhood of (0, 0) and satisfying $u_0(0, x)\equiv 0$ near $x=0$.

Hence, by applying Theorem 1 to (E_2) we obtain $u_R(t, x)=u_0(t, x)$ on $\{(t, x)\in\boldsymbol{R}\times\boldsymbol{C}^n;\ 0<t<\varepsilon_1 \text{ and } |x|\leq\delta_1\}$ for some $\varepsilon_1>0$ and $\delta_1>0$. This leads us to the conclusion of Theorem 3.

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