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Spaces of the same clone type

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1. Introduction.

Two spaces X and Y are said to have the same n-type, if the n-stage Postnikov towers $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent. When two spaces have the same n-type for all n, we call them SNT-equivalent spaces. SNT-equivalence has been studied by many topologists, e.g. J.F. Adams, B.I. Gray, J.R. Harper, C.A. McGibbon, J.M. Møller, J. Roitberg, Y. Shitanda and C. Wilkerson, etc.. Two spaces are said to have the same genus, if the p-localizations are homotopy equivalent for all prime p. C.A. McGibbon and J.M. Møller [**10**] constructed two interesting spaces which are SNT-equivalent and have the same genus. In this case, spaces are said to have the same clone type. We set $N^m(I, J)=M^m(I, J)\times M^m(J, I)$ where $M^m(I, J)$ is a pull-back of $S_I^{2m+1}\to K(Q, 2m+1)\leftarrow K(Z_J, 2m+1)$. C.A. McGibbon [**7**] showed that the family of spaces $\{N^1(I, J)\mid \{I, J\}$ partition of all primes$\}$ has the same clone type and classified the homotopy type by using the ordinary cohomology operations. His method seems not to discriminate between $N^1(I, J)$ and Example A of [**10**]. In this paper, we calculate $\mathrm{End}(\Omega^k N^m(I, J))$ and classify the homotopy type by using End(−). By using End(−), we can also discriminate between $N^1(I, J)$ and Example A of [**10**]. In section 2, for $2m>k>0$, we show that fiber spaces $E(f)$ induced by $f: K(Z, 2m-k+1)\to\Omega^{k-1}S^{2m+1}$ are not homotopy equivalent to $\Omega^k N^m(I, J)$ for non-trivial partition $\{I, J\}$. In [**16, 17**], the author classified the homotopy type of $\{\Omega^k C(f)\mid f:\Sigma^k CP^\infty\to S^{k+3}\}$ for $k=0, 1, 2, \cdots, \infty$. If we take maps $\{f\}$ in the kernel of local expansion $\mathrm{Ph}(\Sigma^k CP^\infty, S^{k+3})\to \mathrm{Ph}(\Sigma^k CP^\infty, \prod_p (S^{k+3})_{(p)})$ (cf. [**6**]), we get uncountably many infinite loop spaces of the same clone type. In section 3, we calculate $\mathrm{Ph}(\Omega^k M^m(I, J), \Omega^h S^t)$ which is a generalization of the result of [**14**].

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1. Endomorphisms of $\Omega^k M^m(I, J)$ and $\Omega^k N^m(I, J)$.

The family of CW-complexes $\{M^m(I, J)\}$ are defined by the following pull-back diagram for partitions $\{I, J\}$ of all primes. The spaces were studied by

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W. Meier [11], C.A. McGibbon-J.M. Møller [10] and J. Roitberg [14].

$$\begin{array}{ccc} M^m(I, J) & \longrightarrow & S_I^{2m+1} \\ \downarrow & & \downarrow \\ K(Z_J, 2m+1) & \longrightarrow & K(Q, 2m+1). \end{array} \tag{1-1}$$

We define $N^m(I, J)$ by the product space $M^m(I, J)\times M^m(J, I)$. By the construction, we get $(N^m(I, J))_I = S_I^{2m+1}\times K(Z_I, 2m+1)$, $(N^m(I, J))_J = S_J^{2m+1}\times K(Z_J, 2m+1)$. $N^m(I, J)$ is the same genus as $(S^{2m+1})\times K(Z, 2m+1)$. In this section, we calculate the monoid of endomorphisms of loop spaces of $M^m(I, J)$ and $N^m(I, J)$, and classify the homotopy type of $\Omega^k N^m(I, J)$. At first, we remark the following elementary property.

The left pull-back diagram produces the right pull-back diagram where B^A is a (based) function space from A to B with compact-open topology.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array} \qquad \begin{array}{ccc} X^A & \longrightarrow & Y^A \\ \downarrow & & \downarrow \\ Z^A & \longrightarrow & W^A. \end{array}$$

LEMMA 1.1. *If CW-complex X of finite type is $(n-1)$-connected, every connected component of* $\mathrm{Map}_*(X, K(G, n))$ *is contractible where G is an abelian group.*

PROOF. Every connected component of $\mathrm{Map}_*(X^m, K(G, n))$ is contractible for finite skeleton X^m, $m>n$. Since $\mathrm{Map}_*(X, K(G, n))$ is the inverse limit of $\mathrm{Map}_*(X^m, K(G, n))$, each component is contractible.

LEMMA 1.2. *For $2m>k\geqq 0$, the homotopy group $[\Omega^k S^{2m+1}, \Omega^k S^{2m+1}]$ is $Z\times T(k, m)$ where $T(k, m)$ is 0 for $k=0$, a countable product of finite groups for $k=1$ and an inverse limit of finite abelian groups for $k>1$.*

PROOF. For $k=0$, the statement is clear. The set $\mathrm{Ph}(\Omega^k S^{2m+1}, \Omega^k S^{2m+1})$ of phantom maps is 0 by Theorem B of [23]. $[\Omega^k S^{2m+1}, \Omega^k S^{2m+1}]$ is the inverse limit of $[\Omega^k S^{2m+1}, \Omega^k (S^{2m+1})^{(s)}]$ of finitely generated abelian groups which contains Z as the free part. Since the free generator is evaluated by the degree of $\Omega^k S^{2m+1}\to\Omega^k S^{2m+1}$ at $2m-k+1$-dimension, we get the result. By the splitting $\sum\Omega S^{2m+1}=\bigvee_{j=1}^{\infty} S^{2mj+1}$, we get the result for $k=1$.

Hereafter, we assume $2m>k\geqq 0$. Hence $\Omega^k M^m(K, L)$ is simply connected. Set $\mathrm{MAP}(k, m; I)$ the connected component of $\mathrm{Map}_*(\Omega^k S_I^{2m+1}, \Omega^k S_I^{2m+1})$ which contains the constant map. Since $[\Omega^k S_I^{2m+1}, \Omega^k S_I^{2m+1}]$ is $Z_I\times T(k, m, I)$ where $T(k, m; I)$ is the inverse limit of the I-torsion subgroups, $\mathrm{Map}_*(\Omega^k S_I^{2m+1}, \Omega^k S_I^{2+1})$ is homotopy equivalent to $\mathrm{MAP}(k, m; I)\times Z_I\times T(k, m; I)$. We define a monoid structure on $[\Omega^k S_I^{2m+1}, \Omega^k, S_I^{2m+1}]$ given by the composition of maps which is denoted by $\mathrm{Mon}(k, m; I)$.

LEMMA 1.3. *A mapping space* $\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k S_K^{2m+1})$ *is weakly homotopy equivalent to* $\mathrm{MAP}(k, m; K\cap I)\times T(k, m; K\cap I)$ *or* $\mathrm{MAP}(k, m; K)\times Z_K\times T(k, m; K)$ *according to* $K\cap J\neq\emptyset$ *or* $K\cap J=\emptyset$ *respectively.*

PROOF. We get the next pull-back diagram by applying the mapping space functor $\mathrm{Map}_*(\Omega^k M^m(I, J), -)$

$$\begin{array}{ccc}
\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k S_K^{2m+1}) & \longrightarrow & \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k S_{K\cap I}^{2m+1}) \\
\downarrow & & \downarrow \\
\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k S_{K\cap J}^{2m+1}) & \longrightarrow & \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k K(Q, 2m+1)).
\end{array}$$

$\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k S_K^{2m+1})$ is weakly homotopy equivalent to a point or homotopy equivalent to the discrete set Q of rational numbers according to $K\cap J\neq\emptyset$ or $K\cap J=\emptyset$ respectively by Theorem D of [**23**]. Since $\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k K(Q, 2m+1))$ is homotopy equivalent to Q, we get the result by the property of the pull-back.

LEMMA 1.4. *Let* $\{I, J\}$ *and* $\{K, L\}$ *be partitions of all primes. Then the homotopy set* $[\Omega^k M^m(I, J), \Omega^k M^m(K, L)]$ *is* $T(k, m; K\cap I)$ *or* $Z\times T(k, m; K)$ *according to* $K\cap J\neq\emptyset$, *or* $K\cap J=\emptyset$ *respectively. In particular, it holds* $[\Omega^k M^m(I, J), \Omega^k M^m(J, I)]=0$ *for* $J\neq\emptyset$. *Each component of* $\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L))$ *is weakly homotopy equivalent to* $\mathrm{MAP}(k, m; K\cap I)$.

PROOF. Consider the next pull-back diagram

$$\begin{array}{ccc}
\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L)) & \longrightarrow & \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k S_K^{2m+1}) \\
\downarrow & & \downarrow \\
\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k K(Z_L, 2m+1)) & \longrightarrow & \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k K(Q, 2m+1)).
\end{array}$$

$\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k K(Z_L, 2m+1))$ and $\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k K(Q, 2m+1))$ are homotopy equivalent to Z_L and Q respectively. By using the fact and Lemma 1.3, we get the result.

LEMMA 1.5. *The homotopy set* $[\Omega^k N^m(I, J), \Omega^k M^m(K, L)]$ *is* $Z\times T(k, m; K)$, $T(k, m; K)$ *and* $Z+Z$ *according to* $I\supseteqq K\neq\emptyset$ *or* $J\supseteqq K\neq\emptyset$, $K\cap I\neq\emptyset$ *and* $K\cap J\neq\emptyset$, *or* $K=\emptyset$ *respectively.*

PROOF. The homotopy set $[\Omega^k N^m(I, J), \Omega^k M^m(K, L)]$ is equal to the free homotopy set $[\Omega^k N^m(I, J), \Omega^k M^m(K, L)]_{\mathrm{free}}$ by the simplicity of $\Omega^k M^m(K, L)$. $[\Omega^k N^m(I, J), \Omega^k M^m(K, L)]_{\mathrm{free}}$ is equal to $[\Omega^k M^m(J, I), \mathrm{Map}(\Omega^k M^m(I, J), \Omega^k M^m(K, L))]_{\mathrm{free}}$.

Consider the following fibration where $a\in[\Omega^k M^m(I, J), \Omega^k M^m(K, L)]$:

$$\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L); a) \longrightarrow$$
$$\mathrm{Map}(\Omega^k M^m(I, J), \Omega^k M^m(K, L); a) \longrightarrow \Omega^k M^m(K, L).$$

At first, we prove the case $k>0$. Note that the based homotopy set is equal to the free homotopy set. We have a homotopy equivalence $\mathrm{Map}(\Omega^k M^m(I, J), \Omega^k M^m(K, L)) \sim \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L)) \times \Omega^k M^m(K, L)$. If $K \subseteqq I$ and $K \neq \emptyset$, $[\Omega^k M^m(J, I), \Omega^k M^m(K, L)]$ is equal to 0 by Lemma 1.4, and $[\Omega^k M^m(J, I), \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L); a)]$ is equal to $[\Omega^k M^m(J, I), \mathrm{Map}_*(\Omega^k S_K^{2m+1}, \Omega^k S_K^{2m+1} : 0)] = [K(Z_K, 2m+1), \mathrm{Map}_*(\Omega^k S_K^{2m+1}, \Omega^k S_K^{2m+1} : 0)] = 0$ by Theorem D of A. Zabrodsky. We get the first part. It is similar for the case $J \supseteqq K \neq \emptyset$. If $K \cap I \neq \emptyset$ and $K \cap J \neq \emptyset$, $\mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L))$ and $\mathrm{Map}_*(\Omega^k M^m(J, I), \Omega^k M^m(K, L))$ are weakly homotopy equivalent to $\mathrm{MAP}(k, m ; K \cap I) \times T(k, m ; K \cap I)$ and $\mathrm{MAP}(k, m ; K \cap J) \times T(k, m ; K \cap J)$ respectively. Since $[\Omega^k M^m(J, I), \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L))]$ is equal to $T(k, m ; K \cap I)$, we get the second part by Lemma 1.4. Since $\Omega^k M^m(K, L)$ is $K(Z, 2m-k+1)$ for $K=\emptyset$, we get the result. In the case for $k=0$, $I \supseteqq K \neq \emptyset$, we get the result by $[M^m(J, I), M^m(K, L)] = 0$ and $[\Omega^k M^m(J, I), \mathrm{Map}_*(\Omega^k M^m(I, J), \Omega^k M^m(K, L) : a)] = 0$. The other cases are similarly proved.

THEOREM 1.6. *Let $\{I, J\}$ be partition of all primes.*

(1) *For the case $I \neq \emptyset$ and $J \neq \emptyset$, the monoid $\mathrm{End}(\Omega^k N^m(I, J))$ of endomorphisms of $\Omega^k N^m(I, J)$ given by the composition of maps is isomorphic to $\mathrm{Mon}(k, m ; I) + \mathrm{Mon}(k, m ; J)$ which is equal to $(Z+T(k, m ; I)) + (Z+T(k, m ; J)) = Z+Z+T(k, m)$ as sets*

(2) *For the case $I=\emptyset$ or $J=\emptyset$, the monoid $\mathrm{End}(\Omega^k N^m(I, J))$ is isomorphic to the set of the triangle matrices which is equal to $Z+Z+\mathrm{Mon}(k, m) = Z+Z+Z+T(k, m)$ as sets:*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \begin{aligned} a &\in [K(2m-k+1), K(2m-k+1)] = Z, \\ b &\in [\Omega^k S^{2m+1}, K(2m-k+1)] = Z, \\ d &\in [\Omega^k S^{2m+1}, \Omega^k S^{2m+1}] = Z+T(k, m). \end{aligned}$$

The composition law is given by the following equation:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} e & f \\ 0 & g \end{pmatrix} = \begin{pmatrix} ae & af+bg \\ 0 & dg \end{pmatrix}.$$

PROOF. If $I \neq \emptyset$ and $J \neq \emptyset$, we have $\mathrm{End}(\Omega^k N^m(I, J)) = [\Omega^k N^m(I, J), \Omega^k M^m(I, J)] \times [\Omega^k N^m(I, J), \Omega^k M^m(J, I)] = Z+T(k, m ; I) + Z+T(k, m ; J)$ by Lemma 1.5. A map from $\Omega^k N^m(I, J)$ to $\Omega^k M^m(I, J)$ is obtained by

$$[a]\,\mathrm{Proj} : \Omega^k N^m(I, J) \longrightarrow \Omega^k M^m(I, J) \longrightarrow \Omega^k M^m(I, J)$$

where $[a]: \Omega^k M^m(I, J) \to \Omega^k M^m(I, J)$ is a map of degree a at $2m-k+1$-dimension. Hence we get the result. Since we have $\Omega^k N^m(I, J) = \Omega^k S^{2m+1} \times K(Z, 2m-k+1)$ for $I=\emptyset$ or $J=\emptyset$, we can easily see the result (cf. [**18**]).

REMARK. For $k=0, 1$, a map $\Omega^k S^{2m+1} \to \Omega^k S^{2m+1}$ of degree ± 1 at $(2m-k+1)$-dimension induces a homotopy equivalence. Hence we get the next result. For $k=0, 1$ and $I \neq \emptyset$ and $J \neq \emptyset$, the group $\mathrm{Aut}(\Omega^k N^m(I, J))$ is isomorphic to the direct sum $(Z/2+T(k, m; I))+(Z/2+T(k, m; J))$. For $k=0, 1$ and $I=\emptyset$ or $J=\emptyset$, it is isomorphic to the subset of the upper triangle matrices which satisfies $a=\pm 1$, $d=\pm 1+d' \in Z+T(k, m)$.

THEOREM 1.7. *Let $\{I, J\}$ and $\{K, L\}$ be partitions of all primes. $\Omega^k N^m(I, J)$ and $\Omega^k N^m(K, L)$ are homotopy equivalent if and only if partitions $\{I, J\}$ and $\{K, L\}$ are equal.*

PROOF. $\Omega^k N^m(I, J)$ and $\Omega^k N^m(K, L)$ are not homotopy equivalent by Theorem 1.6, if I and J are not empty and K or L is empty. If I, J, K and L are not empty, and $\{I, J\}$ and $\{K, L\}$ are different partitions, since a map from $\Omega^k N^m(I, J)$ to $\Omega^k M^m(K, L)$ or $\Omega^k M^m(L, K)$ is homotopic to the map of degree 0 by Lemma 1.5, $\Omega^k N^m(I, J)$ and $\Omega^k N^m(K, L)$ are not homotopy equivalent.

THEOREM 1.8. *Let $\{I, J\}$ be any partitions of all primes. $\Omega^k N^m(I, J)$ is the same clone as $\Omega^k(S^{2m+1} \times K(Z, 2m+1))$.*

PROOF. It is clear that $\Omega^k N^m(I, J)$ has the same genus as $\Omega^k(S^{2m+1} \times K(Z, 2m+1))$. Set $X = \Omega^k(S^{2m+1} \times K(Z, 2m+1))$. To show that $\Omega^k N^m(I, J)$ has the same n-type for all n as X, we use a theorem of A. Zabrodsky [**23**] (cf. [**10**]). The proof is the same as [**10**], [**7**]. We have the exact sequence:

$$\varepsilon_t(X^{(n)}) \xrightarrow{d} (Z/tZ)^*/\pm 1 \xrightarrow{g} G(X^{(n)}) \longrightarrow 0$$

which is defined as follows. $\varepsilon_t(-)$ denotes a monoid of self maps which are local equivalence at prime divisors of t. $(Z/tZ)^*$ is the group of units of rings of integers modulo t. $G(X^{(n)})$ denotes the set of spaces of the same genus as $X^{(n)}$. A map d is given by the determinant of $H^*(f; Z)/\mathrm{Tor}$ for $f: X^{(n)} \to X^{(n)}$. Since d is onto-map by an elementary observation, we have $G(X^{(n)}) = *$. Hence $\Omega^k(S^{2m+1} \times K(Z, 2m+1))$ and $N^m(I, J)$ have the same n-type for all n. We get the result.

2. Various spaces of the same n-type for all n.

In this section, we study spaces in $\mathrm{SNT}(K(Z, 2m-k+1) \times \Omega^k S^{2m+1})$. Let $E(f)$ be a fiber space over $K(Z, 2m)$ with a fiber ΩS^{2m+1} induced by $f: K(Z, 2m)$

$\to S^{2m+1}$. Clearly $\Omega^{k-1}E(f)$ is SNT-equivalent to $\Omega^k N^m(I, J)$.

THEOREM 2.1. *$\Omega^{k-1}E(f)$ is not homotopy equivalent to $\Omega^k N^m(I, J)$ for any $f: K(Z, 2m)\to S^{2m+1}$ and any non-trivial partition $\{I, J\}$.*

PROOF. Let $\Phi: \Omega^k N^m(I, J)\to\Omega^{k-1}E(f)$ be a homotopy equivalence. We observe the homomorphism of $(2m-k+1)$-dimension homotopy group $\pi_{2m-k+1}(\Phi)$. Since $\pi_{2m-k+1}(\Omega^k N^m(I, J))$ is isomorphic to the direct sum $\pi_{2m-k+1}(\Omega^k M^m(I, J))$ and $\pi_{2m-k+1}(\Omega^k M^m(J, I))$, and $\pi_{2m-k+1}(\Omega^{k-1}E(f))$ is canonically isomorphic to the direct sum of $\pi_{2m-k+1}(K(Z, 2m-k+1))$ and $\pi_{2m-k+1}(\Omega^k S^{2m+1})$, we set

$$\pi_{2m-k+1}(\Phi)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a: \pi_{2m-k+1}(\Omega^k M^m(I, J))\to\pi_{2m-k+1}(K(Z, 2m-k+1))$, $b: \pi_{2m-k+1}(\Omega^k M^m(I, J))\to\pi_{2m-k+1}(\Omega^k S^{2m+1})$, $c: \pi_{2m-k+1}(\Omega^k M^m(J, I))\to\pi_{2m-k+1}(K(Z, 2m-k+1))$, $d: \pi_{2m-k+1}(\Omega^k M^m(J, I))\to\pi_{2m-k+1}(\Omega^k S^{2m+1})$.

Here the determinant of $\pi_{2m-k+1}(\Phi)$ is ± 1. We calculate the completion at I. Since $E(f)\hat{}_I$ is the product space $K(Z\hat{}_I, 2m-k+1)\times(\Omega^k S^{2m+1})\hat{}_I$, the (2, 2)-component of $\pi_{2m-k+1}(\Phi\hat{}_I)$ is equal to $d\hat{}_I=d\otimes Z\hat{}_I=0$ by Theorem C, D of [23]. Hence we get $d=0$. Similarly we get $b=0$. This contradicts the assumption.

Let $E'(g)$ be a fiber space over $K(Z, 3)$ with a fiber S^3 induced by $g: K(Z, 3)\to HP^\infty$. We can get the next theorem by the similar method.

THEOREM 2.2. *$E'(g)$ is not homotopy equivalent to $N^1(I, J)$ for any $g: K(Z, 3)\to HP^\infty$ and any non-trivial partition $\{I, J\}$.*

Since S_P^{2m+1} can be delooped for a set $P=\{\text{prime } p \mid m|(p-1)\}$ by the result of D. Sullivan [19], $E(f)_P$ can be delooped for some map f. Let $E'(g)_P$ be a fiber space over $K(Z_P, 2m+1)$ with a fiber S_P^{2m+1} induced by $g: K(Z_P, 2m+1)\to BS_P^{2m+1}$. We get the analogous result.

THEOREM 2.3. *$E'(g)_P$ is not homotopy equivalent to $N^m(I, J)_P$ for any $g: K(Z_P, 2m+1)\to BS_P^{2m+1}$ and non-trivial partition $\{I, J\}$.*

McGibbon and Møller [10] defined an interesting space by using the construction of C. Wilkerson [21]. Define a space X_B by the following pull-back diagram where $B: \bar{X}_0\to\bar{X}_0$:

(2.1)
$$\begin{array}{ccccc} X_B & & \longrightarrow & & X^\wedge \\ \downarrow & & & & \downarrow \\ X_Q & \longrightarrow & \bar{X}_0 & \xrightarrow[B]{} & \bar{X}_0 \end{array}$$

They calculated $\mathrm{Aut}(X_B)=Z/2Z$ for $X=K(Z, 3)\times S^3$, and proved that it is the same clone as $K(Z, 3)\times S^3$. Clearly the same results hold for $K(Z, 2m+1)\times S^{2m+1}$. Up to date, we have three types of spaces in $\mathrm{SNT}(K(Z, 3)\times S^3)$, that is, the spaces $\{E'(g)|g: K(Z, 3)\to HP^\infty\}$, $\{(K(Z, 3)\times S^3)_B\}$ and $\{N^1(I, J)|\{I, J\}$ partition of all primes$\}$. By Theorem 2.2, the first type and the third type are different. By Theorem 1.6, the second type and the third type are different. This is easily generalized to $\mathrm{SNT}(K(Z, 2m+1)\times S^{2m+1})$ and $\mathrm{SNT}(K(Z, 2m)\times \Omega S^{2m+1})$. As in [**10**], we define $B=\{B_P|p \text{ prime}\}\in\prod GL(Q_p^\wedge, 2)\subseteq GL(Q^\wedge, 2)$ as follows:

$$B_P=\begin{pmatrix}1 & 0\\ c_p & 1\end{pmatrix}\qquad \begin{array}{l} c_2=1\in Z_2^\wedge,\ c_3=-1\in Z_3^\wedge,\\ c_p=0\in Z_p^\wedge\quad (p>3).\end{array}$$

By the similar calculation, we get $\mathrm{Aut}((K(Z, 2m)\times\Omega S^{2m+1})_B)=Z/2Z\times T(1, m)$. By Theorem 1.6 and its remark, we have $\mathrm{Aut}(\Omega N^m(I, J))=Z/2Z\times Z/2Z\times T(1, m)$ for any non-trivial partition $\{I, J\}$. Hence we get the next theorem.

THEOREM 2.4. *The space $\Omega^k(K(Z, 2m+1)\times S^{2m+1})_B$ is not homotopy equivalent to $\{\Omega^k N^m(I, J)|\{I, J\}$ non-trivial partition$\}$ for $k=0$, 1 and B defined above.*

REMARK 1. The author does not know whether a map $f: \Omega^k S^{2m+1}\to\Omega^k S^{2m+1}$ of degree ±1 at $(2m-k+1)$-dimension induces a homotopy equivalence for $k>1$. If it is true, Theorem 2.4 holds for $\Omega^k(K(Z, 2m+1)\times S^{2m+1})_B$ for $k>1$.

REMARK 2. Since the local expansion map $\mathrm{ex}: Z\to\prod Z_{(p)}$ induces the exact sequence:

$$0\longrightarrow \mathrm{Hom}(Q, \prod Z_{(p)}/Z)\longrightarrow \mathrm{Ext}(Q, Z)\longrightarrow \mathrm{Ext}(Q, \prod Z_{(p)})\longrightarrow 0$$

the local expansion map $\mathrm{Ph}(CP^\infty, S^3)\to\mathrm{Ph}(CP^\infty, \prod(S^3)_{(p)})$ has a non-trivial kernel which contains uncountably many clone maps (cf. [**6**], [**7**]). By Theorem 3.2 of [**16**], we get uncountably many loop spaces of the same clone type as $\Omega^k\Sigma^k(S^3\vee\sum CP^\infty)$. By considering stable version [**17**], we get uncountably many infinite loop spaces of the same clone type as $Q(S^3\vee\sum CP^\infty)$.

3. Set $\mathrm{Ph}(\Omega^k M^m(I, J), \Omega^h S^t)$ of phantom maps.

J. Roitberg calculated the set of phantom maps $\mathrm{Ph}(\Omega^k M^m(I, J), \Omega^h S^t)$ for some cases (cf. [**14, 15**]). In this section, we calculate it completely. Hereafter we assume $\{I, J\}$ is an non-trivial partition.

THEOREM 3.1. *$\mathrm{Ph}(\Omega^k M^m(I, J), \Omega^h S^{2t+1})$ is $Z_J^\wedge/Z=Z_J^\wedge/Z_J+\oplus Z/p^\infty$ $(p\in I)$ or 0 according to $2m-k+1=2t-h$ or otherwise respectively. $\mathrm{Ph}(\Omega^k M^m(I, J), \Omega^h S^{2t})$ is $Z_J^\wedge/Z=Z_J^\wedge/Z_J+\oplus Z/p^\infty$ $(p\in I)$ or 0 according to $2m-k+1=2t-h-1$, $4t-h-2$ or otherwise respectively.*

PROOF. For the case of odd sphere, we have the following equalities:

$$\begin{aligned}&[\Omega^k M^m(I, J), (\Omega^{h+1}S^{2t+1})^\wedge]\\&=[\Omega^k M^m(I, J), (\Omega^{h+1}S^{2t+1})^\wedge_I]\times[\Omega^k M^m(I, J), (\Omega^{h+1}S^{2t+1})^\wedge_J]\\&=[\Omega^k S^{2m+1}, (\Omega^{h+1}S^{2t+1})^\wedge_I]\times[\Omega^k K(Z_J, 2m+1), (\Omega^{h+1}S^{2t+1})^\wedge_J]\\&=[\Omega^k S^{2m+1}, (\Omega^{h+1}S^{2t+1})^\wedge_I]\end{aligned}$$

by Theorem B of [**23**].

The homotopy fiber $(\Omega^h S^{2t+1})\rho$ of the Sullivan completion $e^\wedge$: $(\Omega^h S^{2t+1})\to(\Omega^h S^{2t+1})^\wedge$ is $K(Z^\wedge/Z, 2t-h)$ by Section 3 of [**18**].

By using the fiber sequence:

$$\Omega^{h+1}(S^{2t+1})^\wedge \xrightarrow{\xi} (\Omega^h S^{2t+1})\rho \xrightarrow{\rho} \Omega^h S^{2t+1} \xrightarrow{e^\wedge} \Omega^h(S^{2t+1})^\wedge \tag{3.2}$$

we get $\mathrm{Ph}(\Omega^k M^m(I, J), \Omega^h S^{2t+1})=[\Omega^k M^m(I, J), (\Omega^h S^{2t+1})\rho]/\mathrm{Im}\,\xi^*=(Z^\wedge/Z)/(Z^\wedge_I/Z)=Z^\wedge_J/Z=Z^\wedge_J/Z_J+\oplus Z/p^\infty$ $(p\in I)$ if $2m-k+1=2t-h$, and 0 otherwise. Here $\mathrm{Im}\,\xi^*=Z^\wedge_I/Z$ is proved as follows, if $2m-k+1=2t-h$. There exist always canonical maps:

$$S^{2m+1}\longrightarrow \Omega S^{2m+2}\longrightarrow \Omega^2 S^{2m+3}\longrightarrow \Omega^3 S^{2m+4}\longrightarrow \Omega^4 S^{2m+5}. \tag{3.3}$$

By the theorem of F.R. Cohen, J.C. Moore and J.A. Neisendorfer [**3**], [**4**], [**12**], there exists rational equivalences for p-localized spaces:

$$\begin{aligned}&\longrightarrow \Omega^4(S^{2m+5})_{(p)}\longrightarrow \Omega^2(S^{2m+3})_{(p)}\longrightarrow (S^{2m+1})_{(p)}\\&\longrightarrow \Omega^5(S^{2m+5})_{(p)}\longrightarrow \Omega^3(S^{2m+3})_{(p)}\longrightarrow \Omega(S^{2m+1})_{(p)}\end{aligned} \tag{3.4}$$

and the composition $(S^{2m+1})_{(p)}\to\Omega^2(S^{2m+3})_{(p)}\to(S^{2m+1})_{(p)}$ is a map of degree p. Since the former map is a map of degree 1, the latter map is a map of degree p. Consider the following diagram for even k.

$$\begin{array}{ccc}[(\Omega^k S^{2m+1})_{(p)}, (\Omega^{h+1}S^{2t+1})^\wedge_p] & \longrightarrow & [(\Omega^k S^{2m+1})_{(p)}, K(Z^\wedge_p, 2t-h)]\\ \downarrow & & \downarrow\\ [(S^{2m+1-k})_{(p)}, (\Omega^{h+1}S^{2t+1})^\wedge_p] & \longrightarrow & [(S^{2m+1-k})_{(p)}, K(Z^\wedge_p, 2t-h)].\end{array}$$

Since the left vertical map is onto by the theorem of F.R. Cohen, J.C. Moore and J.A. Neisendorfer, the upper horizontal map is onto. Since ξ can be factored as $(\Omega^{h+1}S^{2t+1})^\wedge_I\to K(Z^\wedge_I, 2t-h)\to K(Z^\wedge/Z, 2t-h)$, we get $\mathrm{Im}\,\xi^*=Z^\wedge_I/Z$. By using the decomposition of I.M. James, it is similarly proved for odd k. For the case of even sphere, we have a fibration by Section 3 of [**18**]:

$$K(Z^\wedge/Z, 4t-h-2)\longrightarrow(\Omega^h S^{2t})\rho\longrightarrow K(Z^\wedge/Z, 2t-h-1).$$

For $h>0$, we have $(\Omega^h S^{2t})_Q = K(Q, 2t-h)\times K(Q, 4t-h-1)$, $(\Omega^h S^{2t})^{\wedge}_Q = K(Q^{\wedge}, 2t-h)\times K(Q^{\wedge}, 4t-h-1)$ and $(\Omega^h S^{2t})_P = K(Z^{\wedge}/Z, 2t-h-1)\times K(Z^{\wedge}/Z, 4t-h-2)$. By the similar argument, we get the latter result for the case $h>0$. For the general case, it is sufficient to prove the case $2m-k+1=2t-h-1$, $4t-h-2$. By the similar argument, the group $[\Omega^k M^m(I, J), (\Omega^{h+1}S^{2t})^{\wedge}]$ is mapped to $Z^{\wedge}_I$ in $[\Omega^k M^m(I, J), (\Omega^h S^{2t})_P] = Z^{\wedge}/Z$. Note that we use the theorem of F.R. Cohen, J.C. Moore and J.A. Neisendorfer, and the factorization $(\Omega^{h+1}S^{2t})^{\wedge} \to (\Omega^{h+1}S^{2t})^{\wedge}_Q \to (\Omega^h S^{2t})_P$.

REMARK. The theorem of A. Borel and H. Hopf induces that the orders of k-invariants of Postnikov tower of H-space are finite. If this theorem holds for the case of uncountable basis, $(\Omega^h S^{2t})_P = K(Z^{\wedge}/Z, 2t-h-1)\times K(Z^{\wedge}/Z, 4t-h-2)$ holds for $h\geqq 0$.

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