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Topological Anosov maps of infra-nil-manifolds

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§ 0. Introduction.

We shall discuss a part of a problem of whether the universal model of Anosov diffeomorphisms exists. Concerning with this problem Manning [**Ma2**] proved that every Anosov diffeomorphism of an infra-nil-manifold is topologically conjugate to a hyperbolic infra-nil-automorphism. From the remarkable proof of his result and the work of Franks [**Fr**], Aoki and Hiraide has been studied the dynamics of covering maps of a torus ([**Ao-Hi**]).

We shall show in this paper that some of the results stated in [**Ao-Hi**] become realistic for infra-nil-manifolds as follows.

THEOREM 1. *Let* $f: N/\Gamma \to N/\Gamma$ *be a covering map of an infra-nil-manifold and denote as* $A: N/\Gamma \to N/\Gamma$ *the infra-nil-endomorphism homotopic to* f.

If f *is a TA-map, then* A *is hyperbolic and the inverse limit system of* $(N/\Gamma, f)$ *is topologically conjugate to the inverse limit system of* $(N/\Gamma, A)$.

THEOREM 2. *Let* f *and* A *be as in Theorem* 1. *Then the following statements hold*:

(1) *if* f *is a TA-homeomorphism, then* A *is a hyperbolic infra-nil-automorphism and* f *is topologically conjugate to* A,

(2) *if* f *is a topological expanding map, then* A *is an expanding infra-nil-endomorphism and* f *is topologically conjugate to* A.

In the statement of Theorem 2 it notices that (1) is a generalization of Manning [**Ma2**].

First we shall explain here the definitions and notations used above. Let X and Y be compact metric spaces and let $f: X \to X$ and $g: Y \to Y$ be continuous surjections. Then f is said to be *topologically conjugate* to g if there exists a homeomorphism $\varphi: Y \to X$ such that $f \circ \varphi = \varphi \circ g$.

Let X be a compact metric space with metric d. For $f: X \to X$ a continuous surjection, we let

$$X_f = \{(x_i) : x_i \in X \text{ and } f(x_i) = x_{i+1},\ i \in \mathbf{Z}\},$$

$$\sigma_f((x_i)) = (f(x_i)).$$

The map $\sigma_f: X_f \to X_f$ is called the *shift map* determined by f. We say that (X_f, σ_f) is the *inverse limit* of (X, f). A continuous surjection $f: X \to X$ is called *c-expansive* if there is a constant $e>0$ (called an *expansive constant*) such that for $(x_i), (y_i) \in X_f$ if $d(x_i, y_i) \leqq e$ for all $i \in \boldsymbol{Z}$ then $(x_i)=(y_i)$. In particular, if there is a constant $e>0$ such that for $x, y \in X$ if $d(f^n(x), f^n(y)) \leqq e$ for all $i \in \mathbf{N}$ then $x=y$, we say that f is *positively expansive*. A sequence of points $\{x_i: a<i<b\}$ of X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1})<\delta$ for $i \in (a, b-1)$. Given $\varepsilon>0$ a δ-pseudo orbit of $\{x_i\}$ is called to be *ε-traced* by a point $x \in X$ if $d(f^i(x), x_i)<\varepsilon$ for every $i \in (a, b-1)$. Here the symbols a and b are taken as $-\infty \leqq a<b \leqq \infty$ if f is bijective and as $-1 \leqq a<b \leqq \infty$ if f is not bijective. f has the *pseudo orbit tracing property* (abbrev. POTP) if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo orbit of f can be ε-traced by some point of X. We say that a continuous surjection $f: X \to X$ is a *topological Anosov map* (abbrev. TA-map) if f is c-expansive and has POTP, and say that f is a *topological expanding map* if f is positively expansive and open. We can check that every topological expanding map is a TA-map (see [**Ao-Hi**] Remark 2.3.10).

Let N be a simply connected nilpotent Lie group. Let C be a compact group of automorphisms of N and let $G=N \cdot C$ be the Lie group obtained by considering N as acting on itself by left translation and taking the semi-direct product of N and C in Diff(N). Let Γ be a torsion free uniform discrete subgroup of G. The space N/Γ (the quotient space of N under the action of Γ) is called an *infra-nil-manifold*. Let $\bar{A}: N \to N$ be an automorphism of N such that by conjugating Γ by $\bar{A}$ in Diff(N), $\bar{A} \circ \Gamma \circ \bar{A}^{-1} \subset \Gamma$. Then $\bar{A}$ projects to a covering map A of N/Γ. The map A is called an *infra-nil-endomorphism*. If the derivative $d\bar{A}_e$ at the identity e of N has no eigenvalues of modulas 1, we say A is hyperbolic. If A is hyperbolic, then A is a TA-covering map.

REMARK 0.1. A converse statement of Theorem 1 also holds: Let $f: N/\Gamma \to N/\Gamma$ be a covering map of an infra-nil-manifold and denote as $A: N/\Gamma \to N/\Gamma$ the infra-nil-endomorphism homotopic to f.

If A is hyperbolic and the inverse limit system of $(N/\Gamma, f)$ is topologically conjugate to the inverse limit system of $(N/\Gamma, A)$, then f is an TA-map.

See [**Ao-Hi**] Theorems 2.2.29 and 2.3.9 for details.

Let M be a closed smooth manifold and let $C^1(M, M)$ be the set of all C^1 maps of M endowed with the C^1 topology. A map $f \in C^1(M, M)$ is called an *Anosov differentiable map* if f is a C^1 regular map and if there exist $C>0$ and $0<\lambda<1$ such that for every $\boldsymbol{x}=(x_i) \in M_f=\{(x_i): x_i \in M \text{ and } f(x_i)=x_{i+1}, i \in \mathbf{Z}\}$ there is a splitting

$$T_x M=\bigcup_i T_{x_i} M=\bigcup_i (E^s_{x_i} \oplus E^u_{x_i})$$

so that for all $i\in\mathbf{Z}$

(1) $D_{x_i}f(E^\sigma_{x_i})=E^\sigma_{x_{i+1}}$ where $\sigma=s, u$,

(2) for all $n\geqq 0$

$$\|D_{x_i}f^n(v)\| \leqq C\lambda^n\|v\| \qquad \text{if } v\in E^s_{x_i},$$

$$\|D_{x_i}f^n(v)\| \geqq C^{-1}\lambda^{-n}\|v\| \qquad \text{if } v\in E^u_{x_i}.$$

If, in particular, $T_xM=\bigcup_i E^u_{x_i}$ for all $x=(x_i)\in M_f$, then f is said to be *expanding*, and if an Anosov differentiable map f is injective then f is called an *Anosov diffeomorphism*. We can check that every Anosov differentiable map is a TA-map, and that every expanding differentiable map is a topological expanding map (see [**Ao-Hi**] Theorem 1.2.1).

A map $f\in C^1(M, M)$ is said to be *C^1-structurally stable* if there is an open neighborhood $N(f)$ of f in $C^1(M, M)$ such that $g\in N(f)$ implies that f and g are topologically conjugate. Anosov [**An**] proved that every Anosov diffeomorphism is C^1-structurally stable, and Shub [**Sh**] showed the same result for expanding differentiable maps. However, Anosov differentiable maps which are not diffeomorphisms nor expanding do not be C^1-structurally stable ([**Ma-Pu**], [**Pr**]). Then we have the following.

REMARK 0.2. Under the assumption of Theorem 1 it is not true in general that f is topologically conjugate to A.

A map $f\in C^1(M, M)$ is said to be *C^1-inverse limit stable* if there is an open neighborhood $N(f)$ of f in $C^1(M, M)$ such that $g\in N(f)$ implies that the inverse limit (M_f, σ_f) of (M, f) and the inverse limit (M_g, σ_g) of (M, g) are topologically conjugate. Mañé and Pugh [**Ma-Pu**] proved that every Anosov differentiable map is C^1-inverse limit stable. If the manifold M is an infra-nil-manifold, then this fact is a corollary of Theorem 1.

REMARK 0.3 ([**Su**]). Let $f:\mathbf{T}^n\to\mathbf{T}^n$ be a covering map of an n-torus and denote $A:\mathbf{T}^n\to\mathbf{T}^n$ the toral endomorphism homotopic to f.

If f is a special TA-map, then A is a hyperbolic toral endomorphism and f is topologically conjugate to A.

We define special TA-maps as follows. Let $f:X\to X$ be a continuous surjection of a compact metric space. Define the stable and unstable sets

$$W^s(x)=\{y\in X:\lim_{n\to\infty} d(f^n(x), f^n(y))=0\},$$

$$W^u((x_i))=\{y_0\in X:\exists (y_i)\in X_f\, s.t. \lim_{i\to\infty} d(x_{-i}, y_{-i})=0\}$$

for $x\in X$ and $(x_i)\in X_f$. A TA-map $f:X\to X$ is *special* if f satisfies the property that $W^u((x_i))=W^u((y_i))$ for every (x_i), $(y_i)\in X_f$ with $x_0=y_0$. Every hyperbolic infra-nil-endomorphism is a special TA-covering map (Remark 3.13).

In [**Gr**] Gromov showed that every expanding map of an arbitrary closed smooth manifold is topologically conjugate to an expanding infra-nil-endomorphism. After this Hiraide proved in [**Hi1**] a wider result for topological expanding maps as follows.

REMARK 0.4 ([**Hi1**]). If a continuous map of a compact connected locally connected semilocally 1-connected metric space is a topological expanding map, then the space must be homeomorphic to an infra-nil-manifold and the map is topologically conjugate to an expanding infra-nil-endomorphism.

A topological space X is said to be *semilocally* 1-*connected* if for $x \in X$ there is a neighborhood U of x such that every loop contained in U with a base point x (i.e., continuous map $u:[0, 1] \to U$ satisfying $u(0)=u(1)=x$) can be deformed continuously in X to one point.

A key point in the proof of the main theorem is in the properties of the inverse limit systems of self covering maps investigated in § 3.

The outline of the proof of the main theorem can be stated as follows. If $f: N/\Gamma \to N/\Gamma$ is a TA-covering map, it is shown (see § 1) that the infra-nil-endomorphism $A: N/\Gamma \to N/\Gamma$ homotopic to f is hyperbolic. Then we shall prove in § 2 that there exists a semi-conjugacy map $\bar{h}: N \to N$ such that $\bar{h} \circ \bar{f} = \bar{A} \circ \bar{h}$ and $\bar{h}$ is continuous and surjective. Here we denote as $\bar{A}$ the automorphism of N which is a lift of A by π, and denote as $\bar{f}$ a suitable lift map of f by π. We find in § 3 a homeomorphism $\tilde{f}: (N/\Gamma)_A \to (N/\Gamma)_A$ which is topologically conjugate to the inverse limit system of $(N/\Gamma, f)$ and in § 4 a semi-conjugacy map $\tilde{h}$ between the systems $((N/\Gamma)_A, \tilde{f})$ and $((N/\Gamma)_A, \sigma_A)$. In § 5 we shall show $\Omega(f)=N/\Gamma$. By this fact $\tilde{h}$ is injective (see § 7), from which Theorem 1 will be concluded. The proof of Theorem 2(2) will be given in § 6 and Theorem 2(1) will be proved in § 7.

§ 1. Infra-nil-endomorphisms homotopic to TA-covering maps.

The aim of this section is to prepare two lemmas (Lemmas 1.3 and 1.5) that are used for the proof of Theorem 1.

Let N be a simply connected nilpotent Lie group. Let C be a compact group of automorphisms of N and let $G=N \cdot C$ be the Lie group defined as above. If Γ is a torsion free uniform discrete subgroup of G, then N/Γ is an infra-nil-manifold. If in particular Γ is a uniform discrete subgroup of N, then N/Γ is called a nil-manifold (see [**Sm**]).

Let $\bar{D}$ be a left invariant Riemannian distance for N and ρ be the restriction to Γ of the natural homomorphism mapping $G=N \cdot C$ to C. Recall that $\rho(\Gamma)$ is a finite group of automorphisms on N (see [**Au**] Theorem 1). We define a metric D for N by

$$D(x, y) = \sum_{c \in \rho(\Gamma)} \bar{D}(c(x), c(y))$$

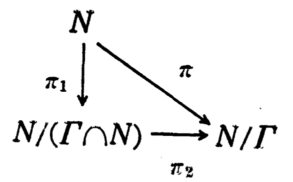
for $x, y \in N$. Then D is a left-invariant, Γ-invariant Riemannian distance. Let $\mathcal{L}(N)$ be the Lie algebra of N, and let $\exp : \mathcal{L}(N) \to N$ denote the exponential map. Since N is simply connected and nilpotent, the exponential map is a diffeomorphism (see [**Va**] Theorem 3.6.2). We claim that for any $L>0$ and $x \in N$, the closed ball $B_L(x) = \{y \in N : D(x, y) \leqq L\}$ of x with radius L is compact. Indeed, since the exponential map is a diffeomorphism, there exists $r>0$ such that $B_r(e)$ is compact. Here e denotes the identity of N. Then $B_{2r}(e) = B_r(e) \cdot B_r(e)$ is compact and thus $B_{nr}(e)$ is compact for $n \in \mathbf{N}$, from which the claim is concluded.

Let $\pi : N \to N/\Gamma$ be the natural projection and define a metric for N/Γ by

$$d(\pi(x), \pi(y)) = \inf\{D(\alpha(x), \beta(y)) : \alpha, \beta \in \Gamma\}.$$

Then d is compatible with the quotient topology on N/Γ. We can check that there exists $\lambda>0$ such that $\pi : U_\lambda(x) \to U_\lambda(\pi(x))$ is an isometry for $x \in N$ where $U_\lambda(x) = \{y \in N : D(x, y) < \lambda\}$ and $U_\lambda(\pi(x)) = \{y \in N/\Gamma : d(y, \pi(x)) < \lambda\}$. Indeed, since Γ is discrete, for $x \in N$ there is $\mu = \mu(x) > 0$ such that the subset $\{\alpha \in \Gamma : \alpha(U_\mu(x)) \cap U_\mu(x) \neq \emptyset\}$ of Γ is finite ([**Wo**] Lemma 3.1.1). Then we can take $\delta = \delta(x) > 0$ such that $\alpha(U_\delta(x)) \cap U_\delta(x) = \emptyset$ for $\alpha \in \Gamma \setminus \{id_N\}$, because Γ acts freely on N. Thus, $\pi : U_{\delta(x)/2}(x) \to U_{\delta(x)/2}(\pi(x))$ is an isometry. Since $\mathcal{U} = \{U_{\delta(x)/2}(\pi(x)) : x \in N\}$ is an open cover of N/Γ, let $\lambda>0$ be Lebesgue number of $\mathcal{U}$. Then, $\alpha(U_{\lambda/2}(x)) \cap U_{\lambda/2}(x) = \emptyset$ for $\alpha \in \Gamma \setminus \{id_N\}$ and therefore $\pi : U_{\lambda/4}(x) \to U_{\lambda/4}(\pi(x))$ is an isometry.

By a result of L. Auslander [**Au**], $\Gamma \cap N$ is a uniform discrete subgroup of N and $\Gamma \cap N$ has finite index in Γ. Then $N/(\Gamma \cap N)$ is compact and orientable ($N/(\Gamma \cap N)$ is a nil-manifold), and N/Γ is finitely covered by $N/(\Gamma \cap N)$. Denote as $\pi_1 : N \to N/(\Gamma \cap N)$ and $\pi_2 : N/(\Gamma \cap N) \to N/\Gamma$ the natural projections. Then we have



commutes.

Let $f : N/\Gamma \to N/\Gamma$ be a self-covering map and $A : N/\Gamma \to N/\Gamma$ be the infra nil-endomorphism homotopic to f. We take a homotopy $H : N/\Gamma \times [0, 1] \to N/\Gamma$ from A to f. Let $\bar{H} : N \times [0, 1] \to N$ be the lift of H by π such that $\bar{A}(x) = \bar{H}(x, 0)$ for $x \in N$, where $\bar{A} : N \to N$ is the automorphism which is a lift of A by π. Define the lift map $\bar{f} : N \to N$ of f by π by $\bar{f}(x) = \bar{H}(x, 1)$ $(x \in N)$. Let $\bar{f}_*, \bar{A}_* : \Gamma \to \Gamma$ be homomorphisms induced by $\bar{f}$, $\bar{A}$ respectively (cf. [**Ao-Hi**] §6.3 (6.1)).

LEMMA 1.1. *Let $\bar{H}$ be as above. Then there exists a homomorphism $\bar{H}_*:\Gamma\to\Gamma$ such that*

$$\bar{H}(\alpha(x), t) = \bar{H}_*(\alpha)\circ\bar{H}(x, t)$$

for $x\in N$, $t\in[0, 1]$ and $\alpha\in\Gamma$.

PROOF. For $t\in[0, 1]$ there exists a homomorphism $(\bar{H}_t)_*:\Gamma\to\Gamma$ such that

$$\bar{H}(\alpha(x), t) = (\bar{H}_t)_*(\alpha)\circ\bar{H}(x, t)$$

for $x\in N$ and $\alpha\in\Gamma$ (see [**Ao-Hi**] Lemma 6.3.10). To conclude the lemma, for $\alpha\in\Gamma$ it suffices to see that $(\bar{H}_t)_*(\alpha)$ is independent of $t\in[0, 1]$. For $\beta\in\Gamma$, the set $J_\beta=\{t\in[0, 1]:(\bar{H}_t)_*(\alpha)=\beta\}$ is open. Indeed, by the above claim there exists $\lambda>0$ such that $\gamma(U_\lambda(x))\cap U_\lambda(x)=\emptyset$ for $x\in N$ and $\gamma\in\Gamma\setminus\{id_N\}$. For $t\in J_\beta$ take a neighborhood V_t of t in $[0, 1]$ such that $\bar{H}(e, s)\in U_\lambda(\bar{H}(e, t))$, and $\bar{H}(\alpha(e), s)\in U_\lambda(\bar{H}(\alpha(e), t))$ for $s\in V_t$. Here e denotes the identity of N. Then we have that

$$\bar{H}(\alpha(e), s) = (\bar{H}_s)_*(\alpha)\circ\bar{H}(e, s) \in (\bar{H}_s)_*(\alpha)(U_\lambda(\bar{H}(e, t)))$$

and

$$\bar{H}(\alpha(e), s) \in U_\lambda(\bar{H}(\alpha(e), t)) = \beta(U_\lambda(\bar{H}(e, t)))\,.$$

Thus, $(\bar{H}_s)_*(\alpha)(U_\lambda(\bar{H}(e, t)))\cap\beta(U_\lambda(\bar{H}(e, t)))\neq\emptyset$ and then $(\bar{H}_s)_*(\alpha)=\beta$. Therefore $t\in V_t\subset J_\beta$. Since $[0, 1]$ is connected, we have $J_\beta=[0, 1]$ for some $\beta\in\Gamma$. □

Since $\Gamma\cap N$ is the maximal normal nilpotent subgroup of Γ ([**Au**] Proposition 2), we have that $\bar{f}_*(\Gamma\cap N)\subset\Gamma\cap N$. Then we can take the lift map $\hat{f}:N/(\Gamma\cap N)\to N/(\Gamma\cap N)$ of f by π_2 satisfying $\hat{f}\circ\pi_1=\pi_1\circ\bar{f}$. Since $\bar{f}_*=\bar{A}_*:\Gamma\to\Gamma$ by Lemma 1.1, we can define the lift map $\hat{A}:N/(\Gamma\cap N)\to N/(\Gamma\cap N)$ of A by π_2 satisfying $\hat{A}\circ\pi_1=\pi_1\circ\bar{A}$. Thus $\hat{A}$ is the nil-endomorphism homotopic to $\hat{f}$.

LEMMA 1.2 ([**Ma 1**]). *Let N/Γ be a nil-manifold and $A:N/\Gamma\to N/\Gamma$ be a nil-endomorphism induced by an automorphism $\bar{A}:N\to N$, then $L(A)=\prod_{i=1}^n(1-\lambda_i)$, where λ_i's are the eigenvalues of $(d\bar{A})_e$, is the Lefschetz number of A.*

The following lemma will play an important role to show our Theorem 1.

LEMMA 1.3. *Let $f:N/\Gamma\to N/\Gamma$ be a self-covering map of an infra-nil-manifold and $A:N/\Gamma\to N/\Gamma$ denote the infra-nil-endomorphism homotopic to f. If f is a TA-covering map, then A is hyperbolic.*

PROOF. For the case when N/Γ is a nil-manifold, we shall show the lemma. We know that there is $l>0$ such that for each $m\geqq l$ all fixed points of f^m have the same fixed point index 1 or -1 ([**Ao-Hi**] Proposition 10.7.2, Theorem 10.8.1 and Theorem 10.9.1).

Choose a positive integer m_0 with $m_0\geqq l$ such that f^{m_0} is topologically mixing on each elementary set, and write $g=f^{m_0}$. Obviously $g:N/\Gamma\to N/\Gamma$ is a

TA-covering map and g is homotopic to A^{m_0}. It is enough to show that A^{m_0} is hyperbolic. We have for $m\geqq 0$

$$N(g^m)=|\sum_{x\in \mathrm{Fix}(g^m)} I(g^m, x)|=|I(g^m)|$$

where $N(g^m)$ is the number of fixed points of g^m, $I(g^m, x)$ is the fixed point index of g^m at x; and $I(g^m)$ is the fixed point index of g^m. Let λ_i $(1\leqq i\leqq n)$ denote the eigenvalues of $(d\bar{A}^{m_0})_e$. Then by Lemma 1.2 it follows that

$$N(g^m)=\prod_{i=1}^{n}|1-\lambda_i^m|.$$

Since g is expansive, we have that there is $k>0$ such that $N(g^m)\leqq N(g^{m+k})$ for $m\geqq 1$. Indeed, if η is an expansive constant for g, then there is $\varepsilon>0$ such that any ε-pseudo orbit of g, (x_i), is $\eta/3$-traced by some point in $(N/\Gamma)_g$. Since g is topologically mixing on an elementary set B, there is $k>0$ such that $g^k(K)\cap K'\neq\emptyset$ for any K, K' of a finite cover consisting of $\varepsilon/2$-balls in B. Let $x\in B$ be a fixed point of g^m and choose $y\in B$ such that $d(x, y)<\varepsilon$ and $d(x, g^k(y))<\varepsilon$. Then we construct a one side $(m+k)$-periodic ε-pseudo orbit

$$(x, g(x), \cdots, g^{m-1}(x), y, g(y), \cdots, g^{k-1}(y), x, g(x), \cdots)$$

which coincides with the one sided sequence $(z_i)_0^\infty$ of a two side $(m+k)$-periodic ε-pseudo orbit (z_i) in $(N/\Gamma)^{\mathbf{Z}}$. Hence there is $(y_i)\in(N/\Gamma)_g$ such that $d(y_i, z_i)<\eta/3$ for all $i\in\mathbf{Z}$. By c-expansivity we have $g^{m+k}(y_0)=y_0$.

Note that each λ_i is not a root of unity. Indeed, this follows from the fact that $\mathrm{Per}(g)\neq\emptyset$ and $N(g^m)=\prod_{i=1}^n|1-\lambda_i^m|$. To see $|\lambda_i|\neq 1$ for $1\leqq i\leqq n$, suppose $|\lambda|=1$ $(1\leqq i\leqq s)$, $|\lambda_i|<1$ $(s+1\leqq i\leqq t)$ and $|\lambda_i|>1$ $(t+1\leqq i\leqq n)$. Since $N(g^m)\leqq N(g^{m+k})$ for $m\geqq 1$, we have

$$\frac{\prod_{s+1}^t|1-\lambda_i^m|\cdot\prod_{t+1}^n|\lambda_i^{-m-k}-\lambda_i^{-k}|}{\prod_{s+1}^t|1-\lambda_i^{m+k}|\cdot\prod_{t+1}^n|\lambda_i^{-m-k}-1|}\leqq\frac{\prod_1^s|1-\lambda_i^{m+k}|}{\prod_1^s|1-\lambda_i^m|}. \tag{1.1}$$

Then the left hand side of (1.1) tends to $\prod_{t+1}^n|\lambda_i^{-k}|$ as $m\to\infty$. Since $|\lambda_i|=1$ and λ_i is not a root of unity $(1\leqq i\leqq s)$, we can find a subsequence $\{m_j\}$ such that $\lambda_i^{m_j}\to\lambda_i^{-k}$ as $j\to\infty$. Therefore the right hand side of (1.1) tends to 0, thus contradicting.

For the case when N/Γ is an infra-nil-manifold, let $\hat{f}$, $\hat{A}$ be as above. If f is a TA-covering map, then so is $\hat{f}$. Hence we have that $\hat{A}$ is hyperbolic and therefore so is A. □

LEMMA 1.4. *Let $\bar{A}: N\to N$ be an automorphism and take a continuous map $\phi: N\to N$ by $\phi(x)=x^{-1}\cdot\bar{A}(x)$ for $x\in N$, If $\bar{A}$ is hyperbolic, then ϕ is a homeomorphism.*

PROOF. Making use of the method of Franks [**Fr**] we have that ϕ is a homeomorphism on N. Indeed, by the Baker-Campbell-Hausdorff formula (see [**Va**] Theorem 2.15.4),

$$\begin{aligned} d\phi_e(v) &= \lim_{t\to 0}\frac{1}{t}\exp^{-1}\{\exp(tv)^{-1}\cdot \bar{A}(\exp(tv))\} \\ &= \lim_{t\to 0}\frac{1}{t}\exp^{-1}\{\exp(-tv)\cdot(\exp(td\bar{A}_e v))\} \\ &= \lim_{t\to 0}\frac{1}{t}\{-tv+td\bar{A}_e v+t^2(\text{higher order terms})\} \\ &= (-I+d\bar{A}_e)v \end{aligned}$$

for $v\in\mathcal{L}(N)$. Since A is hyperbolic, by the inverse function theorem we have that ϕ is a local homeomorphism at $0\in\mathcal{L}(N)$.

Let $N=N^l\supset N^{l-1}\supset\cdots\supset N^0=\{e\}$ be the lower central series of N. Since each N^i is connected, a neighborhood of the identity e of N^i generates N^i. Assume that $N^i\subset \mathrm{Im}(\phi)=\{\phi(x)\,|\,x\in N\}$ for $i\geqq 0$ and take $\phi(x)$ and $\phi(y)\in N^{i+1}\cap \mathrm{Im}(\phi)$. Then

$$\begin{aligned} \phi(x)\cdot\phi(y) &= \phi(x)\cdot y^{-1}\cdot\bar{A}(y) \\ &= y^{-1}\cdot\phi(x)\cdot[\phi(x),\, y^{-1}]\cdot\bar{A}(y) \\ &= y^{-1}\cdot x^{-1}\cdot\bar{A}(x)\cdot[\phi(x),\, y^{-1}]\cdot\bar{A}(y). \end{aligned}$$

Since $[\phi(x),\, y^{-1}]\in N^i$ and N^i is normal in N, there exists $w\in N^i$ such that $\bar{A}(x)\cdot[\phi(x),\, y^{-1}]=w\cdot\bar{A}(x)$. Hence we can take $z\in N$ such that $\phi(z)=w$, because of $w\in N^i\subset\mathrm{Im}(\phi)$, and then

$$\begin{aligned} \phi(x)\cdot\phi(y) &= y^{-1}\cdot x^{-1}\cdot w\cdot\bar{A}(x)\cdot\bar{A}(y) \\ &= y^{-1}\cdot x^{-1}\cdot z^{-1}\cdot\bar{A}(z)\cdot\bar{A}(x)\cdot\bar{A}(y) \\ &= \phi(z\cdot x\cdot y) \\ &\in \mathrm{Im}(\phi), \end{aligned}$$

from which $N^{i+1}\subset\mathrm{Im}(\phi)$ and $N=\mathrm{Im}(\phi)$ by induction.

If $\phi(x)=\phi(y)$ $(x,\, y\in N)$, then $\bar{A}(x\cdot y^{-1})=x\cdot y^{-1}$, and then

$$(d\bar{A})_e(\exp^{-1}(x\cdot y^{-1})) = \exp^{-1}(\bar{A}(x\cdot y^{-1})) = \exp^{-1}(x\cdot y^{-1}).$$

Since A is hyperbolic, we have $\exp^{-1}(x\cdot y^{-1})=0$ from which $x\cdot y^{-1}=e$. Therefore ϕ is injective. Brouwer Theorem ensures that ϕ is a homeomorphism. □

LEMMA 1.5. *Let $f:N/\Gamma\to N/\Gamma$ be a self-covering map and let $\bar{g}:N\to N$ be a lift of f by the natural projection $\pi:N\to N/\Gamma$. If f is a TA-covering map, then $\bar{g}$ has exactly one fixed point.*

PROOF. For the proof we use that there exists $l>0$ such that for $m \geqq l$ each fixed point of $f^m : N/\Gamma \to N/\Gamma$ has the same fixed point index 1 or -1. Let $\bar{f}$, $\bar{A}$, and $\bar{H}$ be as above. Then we can find $\bar{\alpha} \in \Gamma$ such that $\bar{g} = \bar{\alpha} \circ \bar{f}$, and then $\bar{\alpha} \circ \bar{H} : N \times [0, 1] \to N$ is a homotopy from $\bar{\alpha} \circ \bar{A}$ to $\bar{g} = \bar{\alpha} \circ \bar{f}$.

Let ρ be the restriction to Γ of the natural projection mapping $G = N \cdot C$ to C. Denote as ϕ the automorphism on C defined by $\phi(c) = \bar{A} \circ c \circ \bar{A}^{-1}$ for $c \in C$. Then the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\bar{A}_*} & \Gamma \\ \rho \downarrow & & \downarrow \rho \\ \rho(\Gamma) & \xrightarrow[\phi]{} & \rho(\Gamma) \end{array}$$

Let $\bar{\alpha} = (\bar{z}, \bar{c}) \in N \cdot C$. Then we have

$$\begin{aligned} \bar{g}^l &= (\bar{\alpha} \circ \bar{f})^l \\ &= \bar{\alpha} \circ \bar{f}_*(\bar{\alpha}) \circ \cdots \circ \bar{f}_*^{l-1}(\bar{\alpha}) \circ \bar{f}^l \\ &= \bar{\alpha} \circ \bar{A}_*(\bar{\alpha}) \circ \cdots \circ \bar{A}_*^{l-1}(\bar{\alpha}) \circ \bar{f}^l \\ &= (\bar{z}, \bar{c}) \circ (\bar{A}(\bar{z}), \phi(\bar{c})) \circ \cdots \circ (\bar{A}^{l-1}(\bar{z}), \phi^{l-1}(\bar{c})) \circ \bar{f}^l, \end{aligned}$$

$$\rho(\bar{\alpha} \circ \bar{A}_*(\bar{\alpha}) \circ \cdots \circ \bar{A}_*^{l-1}(\bar{\alpha})) = \bar{c} \circ \phi(\bar{c}) \circ \cdots \circ \phi^{l-1}(\bar{c}).$$

Since $\rho(\Gamma)$ is a finite group and ϕ is a permutation of $\rho(\Gamma)$, we have

$$\rho(\bar{\alpha} \circ \bar{A}_*(\bar{\alpha}) \circ \cdots \circ \bar{A}_*^{l-1}(\bar{\alpha})) = id_N$$

for some $l \in \mathbf{N}$. Hence there exists $l \in \mathbf{N}$ such that $\bar{g}^l = \gamma \cdot \bar{f}^l$ for some $\gamma \in \Gamma \cap N$.

We assume without loss of generality that $\bar{g}^m = \gamma \cdot \bar{f}^m$. Define a continuous map $\psi : N \to N$ by $\psi(x) = x^{-1} \cdot \bar{A}^m(x)$ for $x \in N$. Since $\bar{A}^m : N \to N$ is hyperbolic by Lemma 1.3, ψ is a homeomorphism (Lemma 1.4), and there is $\bar{\gamma} \in N$ such that $\psi(\bar{\gamma}) = \gamma$. Since $\bar{\alpha} \circ \bar{A}^m(x) = \gamma \cdot \bar{A}^m(x) = \bar{\gamma}^{-1} \cdot \bar{A}^m(\bar{\gamma} \cdot x)$ $(x \in N)$, $\bar{\alpha} \circ \bar{A}^m$ is hyperbolic. Thus $\bar{\alpha} \circ \bar{A}^m$ has the single fixed point $\bar{\gamma}^{-1}$ and the fixed point index, $I(\bar{\alpha} \circ \bar{A}^m, \bar{\gamma}^{-1})$, equals to ± 1.

For $\dot{\gamma} \in \Gamma \cap N$, we have that for $x \in N$

$$\begin{aligned} (\bar{f}^m(\dot{\gamma} \cdot x))^{-1} \cdot \bar{A}^m(\dot{\gamma} \cdot x) &= (\bar{f}^m(x))^{-1} \cdot (\bar{A}^m(\dot{\gamma}))^{-1} \cdot \bar{A}^m(\dot{\gamma}) \cdot \bar{A}^m(x) \\ &= (\bar{f}^m(x))^{-1} \cdot \bar{A}^m(x) \\ &\in (\bar{f}^m(\mathcal{D}))^{-1} \cdot \bar{A}^m(\mathcal{D}) \end{aligned}$$

where $\mathcal{D}$ is a compact covering domain for the natural projection $\pi_1 : N \to N/(\Gamma \cap N)$. Let $x \in \mathrm{Fix}(\bar{g}^m)$. Since

$$\begin{aligned}\psi(\bar{\gamma}\cdot x) &= (\bar{\gamma}\cdot\bar{g}^m(x))^{-1}\cdot\bar{A}^m(\bar{\gamma}\cdot x)\\ &= (\bar{\gamma}\cdot\gamma\cdot\bar{f}^m(x))^{-1}\cdot\bar{A}^m(\bar{\gamma}\cdot x)\\ &= (\bar{\gamma}\cdot\bar{\gamma}^{-1}\cdot\bar{A}^m(\bar{\gamma})\cdot\bar{f}^m(x))^{-1}\cdot\bar{A}^m(\bar{\gamma})\cdot\bar{A}^m(x)\\ &= (\bar{f}^m(x))^{-1}\cdot(\bar{A}^m(\bar{\gamma}))^{-1}\cdot\bar{A}^m(\bar{\gamma})\cdot\bar{A}^m(x)\\ &\in (\bar{f}^m(\mathcal{D}))^{-1}\cdot\bar{A}^m(\mathcal{D}),\end{aligned}$$

we have $\mathrm{Fix}(\bar{g}^m)\subset\bar{\gamma}^{-1}\cdot\{\psi^{-1}((\bar{f}^m(\mathcal{D}))^{-1}\cdot\bar{A}^m(\mathcal{D}))\}$ and therefore $\mathrm{Fix}(\bar{g}^m)$ is compact. Since $\bar{g}^m$ is expansive, the fixed points must be isolated, and then we have that $\mathrm{Fix}(\bar{g}^m)$ is finite.

In the same fashion as above we can show that $\bigcup_{t\in[0,1]}\mathrm{Fix}(\bar{H}^m(\cdot, t))$ is compact. Therefore,

$$I(\bar{\alpha}\circ\bar{A}^m) = I(\bar{g}^m) = \sum_{x\in\mathrm{Fix}(\bar{g}^m)} I(\bar{g}^m, x).$$

By the fact that $f^m\cdot\pi=\pi\circ\bar{g}^m$, we have $I(\bar{g}^m, x)=I(f^m, \pi(x))$ $(x\in\mathrm{Fix}(\bar{g}^m))$, from which each $x\in\mathrm{Fix}(\bar{g}^m)$ has the same index. Hence

$$\sharp\mathrm{Fix}(\bar{g}^m) = \Big|\sum_{x\in\mathrm{Fix}(\bar{g}^m)} I(\bar{g}^m, x)\Big| = |I(\bar{g}^m)| = |I(\bar{\alpha}\circ\bar{A}^m)| = 1.$$

Therefore, $\bar{g}^m: N\to N$ has exactly one fixed point and so does $\bar{g}$. □

§ 2. Construction of semi-conjugacy maps on the universal covering spaces.

The aim of this section is to show Lemma 2.3. As before let N/Γ be an infra-nil-manifold and let $\pi: N\to N/\Gamma$ be the natural projection. For continuous maps f and g of N we define

$$D(f, g) = \sup\{D(f(x), g(x)) : x\in N\}$$

where D denotes a left invariant, Γ-invariant Riemannian distance for N. Notice that $D(f, g)$ is not necessary finite.

Throughout this section we suppose that $f: N/\Gamma\to N/\Gamma$ is a TA-covering map. Let $A: N/\Gamma\to N/\Gamma$ be the infra-nil-endomorphism homotopic to f, and let $\bar{A}: N\to N$ be the automorphism which is a lift of A by π. Since $d\bar{A}_e$ is hyperbolic by Lemma 1.3, the Lie algebra $\mathcal{L}(N)$ of N splits into the direct sum $\mathcal{L}(N)=E_e^s\oplus E_e^u$ of subspaces E_e^s and E_e^u such that $d\bar{A}_e(E_e^s)=E_e^s$, $d\bar{A}_e(E_e^u)=E_e^u$ and there are $c>1$, $0<\lambda<1$ so that for all $n\geqq 0$

(2.1)
$$\begin{aligned}&\|d\bar{A}_e^n(v)\|\leqq c\lambda^n\|v\| \qquad (v\in E_e^s),\\ &\|d\bar{A}_e^{-n}(v)\|\leqq c\lambda^n\|v\| \qquad (v\in E_e^u)\end{aligned}$$

where $\|\cdot\|$ is the Riemannian metric. Let $\bar{L}^{\sigma}(e)=\exp(E_e^{\sigma})$ $(\sigma=s, u)$ and let $\bar{L}^{\sigma}(x)=x\cdot\bar{L}^{\sigma}(e)$ $(\sigma=s, u)$ for $x\in N$. Since left translations are isometries under the metric D, it follows that for all $x\in N$

$$\bar{L}^{s}(x)=\{y\in N: D(\bar{A}^{i}(x), \bar{A}^{i}(y))\to 0\ (i\to\infty)\},$$

$$\bar{L}^{u}(x)=\{y\in N: D(\bar{A}^{i}(x), \bar{A}^{i}(y))\to 0\ (i\to-\infty)\}.$$

LEMMA 2.1 ([**Hi2**]). *For $x, y\in N$, $\bar{L}^{s}(x)\cap\bar{L}^{u}(y)$ consists of exactly one point.*

PROOF. The proof is similar to that in [**Hi2**] Lemma 3.2. For completeness we give here the proof.

Since $\bar{L}^{s}(e)$ and $\bar{L}^{u}(e)$ intersect transversally, we can find $\delta>0$ such that if x, y belong to a δ-neighborhood $U_{\delta}(e)$ then $\bar{L}^{s}(x)$ intersects $\bar{L}^{u}(y)$. Let x belong to the δ-neighborhood $U_{\delta}(\bar{L}^{u}(e))$ of $\bar{L}^{u}(e)$ then $x\in a\cdot U_{\delta}(e)$ for some $a\in\bar{L}^{u}(e)$, and so $\bar{L}^{s}(x)$ intersects $\bar{L}^{u}(e)$. In the same way, $\bar{L}^{s}(x)\cap\bar{L}^{u}(e)\neq\emptyset$ for $x\in U_{\delta}(U_{\delta}(\bar{L}^{u}(e)))=U_{2\delta}(\bar{L}^{u}(e))$. By induction, we have the same result for $x\in U_{n\delta}(\bar{L}^{u}(e))$ and $n>0$. Since $\bigcup_{n\geq 0}U_{n\delta}(\bar{L}^{u}(e))=N$, it follows that $\bar{L}^{s}(x)\cap\bar{L}^{u}(e)\neq\emptyset$ for all $x\in N$, from which $\bar{L}^{s}(x)\cap\bar{L}^{u}(y)\neq\emptyset$ for all $x, y\in N$. □

For $x, y\in N$ denote as $\beta(x, y)$ the point in $\bar{L}^{s}(x)\cap\bar{L}^{u}(y)$.

LEMMA 2.2 ([**Hi2**]). (1) *For $L>0$ and $\varepsilon>0$ there exists $J>0$ such that for $x, y\in N$ if $D(\bar{A}^{i}(x), \bar{A}^{i}(y))\leq L$ for all i with $|i|\leq J$, then $D(x, y)\leq\varepsilon$.*

(2) *For given $L>0$, if $D(\bar{A}^{i}(x), \bar{A}^{i}(y))\leq L$ for all $i\in\mathbf{Z}$, then $x=y$ $(x, y\in N)$.*

PROOF. This is given in [**Hi2**] Lemma 3.2 as follows.

For $L>0$ there is $\delta_L>0$ such that $\operatorname{diam}\{x, y, \beta(x, y)\}<\delta_L$ if $D(x, y)<L$, and by (2.1) there exists $c_L>0$ satisfying

$$D(\bar{A}^{i}(x), \bar{A}^{i}(y))\leq c_L\lambda^{i}D(x, y)\qquad\text{for } y\in\bar{L}^{s}(x)\cap B_{\delta_L}(x),$$

$$D(\bar{A}^{-i}(x), \bar{A}^{-i}(y))\leq c_L\lambda^{i}D(x, y)\quad\text{for } y\in\bar{L}^{u}(x)\cap B_{\delta_L}(x).$$

For given $\varepsilon>0$ choose $J>0$ such that $\delta_L c_L\lambda^{J}<\varepsilon$. Suppose $D(\bar{A}^{i}(x), \bar{A}^{i}(y))\leq L$ for $-J\leq i\leq J$ and let $z_i=\beta(\bar{A}^{i}(x), \bar{A}^{i}(y))$. Then $D(z_J, \bar{A}^{J}(y))<\delta_L$. Since $z_J\in\bar{L}^{u}(\bar{A}^{J}(y))$, we have $D(z_0, y)=D(\bar{A}^{-J}(z_J), \bar{A}^{-J}\circ\bar{A}^{J}(y))\leq\delta_L c_L\lambda^{J}<\varepsilon$. Similarly, $D(z_0, x)<\varepsilon$. Therefore $D(x, y)<2\varepsilon$. Since ε is arbitrary, (2) holds. □

If $\bar{f}$ denote the lift of f by π satisfying $\bar{f}_*=\bar{A}_*:\Gamma\to\Gamma$, then it is checked that $D(\bar{f}, \bar{A})$ is finite. Since there exists $\bar{f}(b_0)=b_0$ for some $b_0\in N$ by Lemma 1.5, we can take a homeomorphism $\bar{\phi}:N\to N$ such that $\bar{\phi}(\alpha(x))=\alpha\circ\bar{\phi}(x)$ for $x\in N$ and $\alpha\in\Gamma$, $\bar{\phi}(b_0)=e$. Thus, $\bar{\phi}\circ\bar{f}\circ\bar{\phi}^{-1}(e)=e$, from which we may assume that $\bar{f}(e)=e$.

LEMMA 2.3. *Under the assumptions and notations as above, there is a unique map* $\bar{h}: N \to N$ *such that*

(1) $\bar{A} \circ \bar{h} = \bar{h} \circ \bar{f}$,

(2) $D(\bar{h}, id_N)$ *is finite,*

where $id_N: N \to N$ *is the identity map of* N.

Furthermore $\bar{h}$ *is surjective, uniformly continuous under* D.

PROOF. For the proof we need the technique of Theorem 2.2 of Franks ([**Fr**]).

Let $Q = \{h \in C^0(N): D(h, \boldsymbol{e}) < \infty,\ h(e) = e\}$, where $C^0(N)$ is the space of continuous maps of N and $\boldsymbol{e}: N \to N$ is the map defined by $\boldsymbol{e}(x) = e$ for any $x \in N$. We define a multiplication in Q by $h_1 h_2(x) = h_1(x) \cdot h_2(x)$. Note that

$$\begin{aligned} D(h_1(x) \cdot h_2(x), e) &\leqq D(h_1(x) \cdot h_2(x), h_1(x)) + D(h_1(x), e) \\ &= D(h_2(x), e) + D(h_1(x), e), \end{aligned}$$

$$D((h(x))^{-1}, e) = D(e, h(x)) \quad (x \in N).$$

Then we can easily check that Q is a nilpotent group. Define a homomorphism $F_0: Q \to Q$ by $F_0(h) = \bar{A}^{-1} \circ h \circ \bar{f}$. This map is a homeomorphism because $\bar{A}$ is D-biuniformly continuous. Let $T: Q \to Q$ be a map defined by $T(h) = F_0(h)(h)^{-1}$.

Let $\Delta = \{k \in C^0(N, \mathcal{L}(N)): \|k\| < \infty,\ k(e) = 0\}$, where $C^0(N, \mathcal{L}(N))$ is the space of continuous maps from N into the Lie algebra $\mathcal{L}(N)$ of N. Since the exponential map is a diffeomorphism, we can define a homeomorphism $\mathrm{Log}: Q \to \Delta$ by $\mathrm{Log}(k) = \exp^{-1} \circ k$. We write $\mathrm{Exp} = \mathrm{Log}^{-1}$. Define $F: \Delta \to \Delta$ by $F = \mathrm{Log} \circ F_0 \circ \mathrm{Log}^{-1}$, then since $\exp \circ d\bar{A}_e = \bar{A} \circ \exp$, it follows that $F(k) = d\bar{A}_e^{-1} \circ k \circ \bar{f}$. Hence F is a linear map. Let $T': \Delta \to \Delta$ be a map defined by $T' = \mathrm{Log} \circ T \circ \mathrm{Log}^{-1}$.

CLAIM 1. We have that T' is a C^∞-map and that T' is a local homeomorphism at the constant map $0: N \to \mathcal{L}(N)$ by $0(x) = 0\ (x \in N)$.

Indeed, since

$$\begin{aligned} T'(k) &= \mathrm{Log} \circ T \circ \mathrm{Log}^{-1}(k) \\ &= \mathrm{Log}(F_0(\exp \circ k)(\exp \circ k)^{-1}) \\ &= \mathrm{Log}((\bar{A}^{-1} \circ \exp \circ k \circ \bar{f})(\exp \circ (-k))) \\ &= \mathrm{Log}((\exp \circ F(k))(\exp \circ (-k))) \\ &= \mathrm{Log}(\mathrm{Exp}(F(k))\mathrm{Exp}(-k)), \end{aligned}$$

T' is a C^∞-map. We now compute the derivative of T' at 0. For $k \in \Delta$ we have

$$\lim_{t \to 0} \frac{1}{t} T'(tk) = \lim_{t \to 0} \frac{1}{t} \mathrm{Log}(\mathrm{Exp}(F(tk))\mathrm{Exp}(-tk))$$

$$= \lim_{t\to 0}\frac{1}{t}\{F(tk)-tk+t^2(\text{higher order terms})\}$$
$$= F(k)-k.$$

Thus the derivative $dT_0'=F-I$ where $I:\Delta\to\Delta$ is the identity.

We now show that $F-I$ is an isomorphism. Let $\Delta^s=\{k\in\Delta : k(N)\subset E_e^u\}$ and define Δ^u similarly. Clearly Δ^σ $(\sigma=s, u)$ are invariant under F. It is easily seen that $\Delta=\Delta^u\oplus\Delta^s$ and that $\|F^i(k)\|\leqq c\lambda^i\|k\|$ for $i\geqq 0$ and $k\in\Delta^s$. Moreover F restricted to Δ^u is invertible and $\|F^{-i}(k)\|\leqq c\lambda^i\|k\|$ for $i\geqq 0$ and $k\in\Delta^u$. On Δ^s we have $(F-I)^{-1}=-\sum_{i=0}^\infty F^i$. The right side converges because $\|F^i\|\leqq c\lambda^i$ for $i\geqq 0$. Similarly in Δ^u we have $(I-F^{-1})^{-1}=\sum_{i=0}^\infty F^{-i}$, so $(F-I)^{-1}=F^{-1}(I-F^{-1})^{-1}$ exists. Hence $F-I$ is an isomorphism of Δ. From this it follows by the inverse function theorem that T' is a local homeomorphism at 0. By Claim 1, T is a local homeomorphism at $\boldsymbol{e}$.

CLAIM 2. We can show that $T: Q\to Q$ is a surjection.

Indeed, let $Q=Q^l\supset Q^{l-1}\supset\cdots\supset Q^0=\{\boldsymbol{e}\}$ be the lower central series of Q. Since $\exp(t\exp^{-1}\circ h)$ $(t\in[0,1])$ is a path between $h\in Q$ and $\boldsymbol{e}$, Q is (path) connected. Then $Q^{l-1}=[Q, Q^l]$ is connected. Inductively so is Q^i $(0\leqq i\leqq l)$, and therefore a neighborhood of the identity $\boldsymbol{e}$ of Q^i generates Q^i. Assume that $Q^i\subset\text{Im}(T)$ for $i\geqq 0$ and take $T(h_1)$ and $T(h_2)\in Q^{i+1}\cap\text{Im}(T)$. Then

$$\begin{aligned}T(h_1)T(h_2) &= F_0(h_1)h_1^{-1}T(h_2)\\ &= F_0(h_1)[h_1, T(h_2)^{-1}]T(h_2)h_1^{-1}\\ &= F_0(h_1)[h_1, T(h_2)^{-1}]F_0(h_2)h_2^{-1}h_1^{-1}.\end{aligned}$$

Since $[h_1, T(h_2)^{-1}]\in Q^i$ and Q^i is normal in Q, there exists $h'\in Q^i$ such that $[h_1, T(h_2)^{-1}]F_0(h_2)=F_0(h_2)h'$. Hence we can take $h_3\in Q$ such that $T(h_3)=h'$, because of $h'\in Q^i\subset\text{Im}(T)$, and then

$$\begin{aligned}T(h_1)T(h_2) &= F_0(h_1)F_0(h_2)h'h_2^{-1}h_1^{-1}\\ &= F_0(h_1)F_0(h_2)F_0(h_3)h_3^{-1}h_2^{-1}h_1^{-1}\\ &= T(h_1h_2h_3)\end{aligned}$$

from which $Q^{i+1}\subset\text{Im}(T)$ because we have that $\boldsymbol{e}\in\text{int}_{Q^{i+1}}\{\text{Im}(T)\}$ by Claim 1, and $Q=\text{Im}(T)$ by induction.

CLAIM 3. We claim that T is a bijection.

Since $F-I$ is an isomorphism, F fixes only $0\in\Delta$ and hence F_0 has only the fixed point $\boldsymbol{e}\in Q$. Thus if $T(h_1)=T(h_2)$ $(h_1, h_2\in Q)$, then $T(h_1h_2^{-1})=\boldsymbol{e}$ so $h_1=h_2$. Therefore T is bijective from Claim 2.

Let $\tilde{h}=F_0((id_N)^{-1})(id_N)$. By the definition of $\tilde{h}$, we have

$$\sup\{D(\tilde{h}(x), e): x\in N\} = \sup\{D((\bar{A}^{-1}(\bar{f}(x))^{-1})\cdot x, e): x \in N\}$$
$$= \sup\{D(x, \bar{A}^{-1}\circ\bar{f}(x)): x \in N\}.$$

Since $\bar{A}^{-1}\circ\bar{f}\circ\alpha(x)=\alpha\circ\bar{A}^{-1}\circ\bar{f}(x)$ for any $\alpha\in\Gamma$, we have $D(\tilde{h}, e)<\infty$. Therefore $h\in Q$.

Let $\hat{h}=T^{-1}(\tilde{h})$ and define $\bar{h}=id_N\hat{h}$. Thus we have

$$T(\bar{h}) = F_0(id_N\hat{h})\hat{h}^{-1}(id_N)^{-1} = F_0(id_N)\tilde{h}(id_N)^{-1}$$
$$= F_0(id_N)F_0((id_N)^{-1})(id_N)(id_N)^{-1} = e,$$

and so $\bar{A}^{-1}\circ\bar{h}\circ\bar{f}=\bar{h}$, from which (1) is obtained.

Since $\hat{h}\in Q$ and $\bar{h}=id_N\hat{h}$, we have $D(\bar{h}, id_N)=D(\hat{h}, e)<\infty$. Hence (2) holds.

The uniqueness of $\bar{h}$ is easily checked as follows. If a map $\bar{k}: N\to N$ satisfies (1) and (2), then for $x\in N$ and $i\in\mathbf{Z}$

$$D(\bar{A}^i\circ\bar{h}(x), \bar{A}^i\circ\bar{k}(x)) \leqq \sup\{D(\bar{A}^i\circ\bar{h}(x), \bar{A}^i\circ\bar{k}(x)): x \in N\}$$
$$= \sup\{D(\bar{h}\circ\bar{A}^i(x), \bar{k}\circ\bar{A}^i(x)): x \in N\}$$
$$= \sup\{D(\bar{h}(x), \bar{k}(x)): x \in N\} < \infty.$$

Thus $\bar{h}(x)=\bar{k}(x)$ by Lemma 2.2(2).

By (2) the map $\phi=\exp^{-1}\circ\bar{h}\circ\exp$ is extended to a continuous map $\tilde{\phi}$ on $S^n=\mathbf{R}^n\cup\{\infty\}$ by $\tilde{\phi}(v)=\phi(v)$ for $v\in\mathbf{R}^n$ and $\tilde{\phi}(\infty)=\infty$, and a homotopy h_t between $\tilde{\phi}$ and the identity map is defined by

$$h_t(v) = t\phi(v)+(1-t)v \ (v \in \mathbf{R}^n) \quad \text{and} \quad h_t(\infty) = \infty.$$

Hence $\tilde{\phi}: S^n\to S^n$ is surjective and so $\bar{h}: N\to N$ is surjective.

To show uniform continuity of $\bar{h}$, we take $K>0$ such that $D(\bar{h}, id_N)\leqq K$. For given $\varepsilon>0$, by Lemma 2.2(1) there is $L>0$ such that if $D(\bar{A}^i(x), \bar{A}^i(y))<3K$ for i with $|i|\leqq L$, then $D(x, y)<\varepsilon$. Since $\bar{A}$ is uniformly continuous, we can take $\gamma>0$ satisfying the property that $D(\bar{A}^i(x), \bar{A}^i(y))<K$ $(-L\leqq i\leqq L)$ whenever $D(x, y)<\gamma$. If $D(x, y)<\gamma$, then we have for i with $|i|\leqq L$

$$D(\bar{A}^i\circ\bar{h}(x), \bar{A}^i\circ\bar{h}(y)) = D(\bar{h}\circ\bar{A}^i(x), \bar{h}\circ\bar{A}^i(y))$$
$$< D(\bar{h}\circ\bar{A}^i(x), \bar{A}^i(x))+D(\bar{A}^i(x), \bar{A}^i(y))$$
$$+D(\bar{A}^i(y), h\circ\bar{A}^i(y))$$
$$< K+K+K = 3K,$$

which implies $D(\bar{h}(x), \bar{h}(y))<\varepsilon$. □

Hereafter, let $\bar{h}$: $N\to N$ be the semi-conjugacy map obtained in Lemma 2.3. In the remainder of this section we mention some properties of $\bar{h}$ that suffice for our needs.

LEMMA 2.4. (1) *There exists* $K>0$ *such that* $D(\bar{h}\circ\alpha(x),\ \alpha\circ\bar{h}(x))<K$ *for* $x\in N$ *and* $\alpha\in\Gamma$.

(2) *For any* $\lambda>0$, *there exists* $L\in\mathbf{N}$ *such that* $D(\bar{h}\circ\alpha(x),\ \alpha\circ\bar{h}(x))<\lambda$ *for* $x\in N$ *and* $\alpha\in\bar{A}_*^L(\Gamma)$.

(3) *For* $x\in N$ *and* $\alpha\in\bigcap_{i=0}^{\infty}\bar{A}_*^i(\Gamma)$, *we have* $\bar{h}\circ\alpha(x)=\alpha\circ\bar{h}(x)$.

(4) *For* $x\in N$ *and* $\alpha\in\Gamma$, *we have* $\bar{h}\circ\alpha(x)\in\bar{L}^s(\alpha\circ\bar{h}(x))$.

PROOF. (1): By Lemma 2.3(2), there is $K'>0$ such that $D(\bar{h}(x),\ x)<K'$ for $x\in N$. Then

$$
\begin{aligned}
D(\bar{h}\circ\alpha(x),\ \alpha\circ\bar{h}(x)) &\leqq D(\bar{h}\circ\alpha(x),\ \alpha(x))+D(\alpha(x),\ \alpha\circ\bar{h}(x))\\
&\leqq 2K'
\end{aligned}
$$

for $\alpha\in\Gamma$.

(2): Let $K'>0$ be as above. For given $\lambda>0$, by Lemma 2.2(1) we can find $L>0$ such that for $x,\ y\in N$

$$
D(\bar{A}^j(x),\ \bar{A}^j(y))\leqq 2K'\ \ (|j|\leqq L)\Rightarrow D(x,\ y)<\lambda. \tag{2.2}
$$

For $x\in N$ and $\alpha\in\bar{A}_*^L(\Gamma)$, we have

$$
\begin{aligned}
D(\bar{A}^i\circ\bar{h}\circ\alpha(x),\ \bar{A}^i\circ\alpha\circ\bar{h}(x)) &= D(\bar{h}\circ\bar{f}^i\circ\alpha(x),\ \bar{A}_*^i(\alpha)\circ\bar{A}^i\circ\bar{h}(x))\\
&\leqq D(\bar{h}\circ\bar{A}_*^i(\alpha)\circ\bar{f}^i(x),\ \bar{A}_*^i(\alpha)\circ\bar{f}^i(x))\\
&\quad +D(\bar{A}_*^i(\alpha)\circ\bar{f}^i(x),\ \bar{A}_*^i(\alpha)\circ\bar{h}\circ\bar{f}^i(x))\\
&\leqq 2K'
\end{aligned}
$$

for $|j|\leqq L$, and hence $D(\bar{h}\circ\alpha(x),\ \alpha\circ\bar{h}(x))<\lambda$ by (2.2). (2) was proved.

(3): Noticing that λ is arbitrary, (3) is concluded.

(4): By (2), we have

$$
\begin{aligned}
D(\bar{A}^i\circ\bar{h}\circ\alpha(x),\ \bar{A}^i\circ\alpha\circ\bar{h}(x)) &= D(\bar{h}\circ\bar{f}^i\circ\alpha(x),\ \bar{A}_*^i(\alpha)\circ\bar{A}^i\circ\bar{h}(x))\\
&= D(\bar{h}\circ\bar{A}_*^i(\alpha)\circ\bar{f}^i(x),\ \bar{A}_*^i(\alpha)\circ\bar{h}\circ\bar{f}^i(x))\\
&\to 0\quad \text{as } i\to\infty.
\end{aligned}
$$

Therefore $\bar{h}\circ\alpha(x)\in\bar{L}^s(\alpha\circ\bar{h}(x))$. □

§ 3. Inverse limit system of self-covering maps on infra-nil-manifolds.

In this section we prepare Lemma 3.10 that needs for the proof of Theorem 1.

Let N be a simply connected nilpotent Lie group with left invariant, Γ-invariant Riemannian distance D and let N/Γ be an infra-nil-manifold with metric d induced by D. Remark that the natural projection $\pi: N\to N/\Gamma$ is a local isometry.

Let $f: N/\Gamma \to N/\Gamma$ be a continuous surjection of an infra-nil-manifold and $\sigma: (N/\Gamma)_f \to (N/\Gamma)_f$ be the inverse limit system constructed by $(N/\Gamma, f)$. We denote as $p_0: (N/\Gamma)_f \to N/\Gamma$ the natural projection to the zero-th coordinate. Define a metric $\bar{d}_f$ for $(N/\Gamma)_f$ by

$$\bar{d}_f((x_i), (y_i)) = \sum_{i \in \mathbf{Z}} \frac{1}{2^{|i|}} d(x_i, y_i)$$

for $(x_i), (y_i) \in (N/\Gamma)_f$. For simplicity we write $\bar{d}_f = \bar{d}$ in subsequent.

LEMMA 3.1 ([**Ao-Hi**] Theorem 6.5.1). *If $f: N/\Gamma \to N/\Gamma$ is a self-covering map of an infra-nil-manifold and the covering degree is greater than one, then $((N/\Gamma)_f, N/\Gamma, C, p_0)$ is a fiber bundle where C denotes the Cantor set.*

Let $f: N/\Gamma \to N/\Gamma$ be a self-covering map of an infra-nil-manifold. We denote as $\Theta(f)$ the family of all lift of f by π.

LEMMA 3.2 ([**Ao-Hi**] Lemma 6.5.4). *For $\varepsilon > 0$ there is $\delta > 0$ such that for all $\bar{g} \in \Theta(f)$ and for all $x, y \in N$ with $D(x, y) < \delta$*

$$\max\{D(\bar{g}(x), \bar{g}(y)), D(\bar{g}^{-1}(x), \bar{g}^{-1}(y))\} < \varepsilon .$$

Define a product set $N^{\mathbf{Z}} = \{(u_i): u_i \in N, i \in \mathbf{Z}\}$ and a shift map $\bar{\sigma}: N^{\mathbf{Z}} \to N^{\mathbf{Z}}$ as usual by $\bar{\sigma}((u_i)) = (u_{i+1})$. Then it is clear that $\bar{\sigma}(N_f) = N_f$ where $N_f = \{(x_i) \in N^{\mathbf{Z}}: f(\pi(x_i)) = \pi(x_{i+1}), i \in \mathbf{Z}\}$. Let $\boldsymbol{u} = (u_i) \in N_f$. For each $i \in \mathbf{Z}$ denote as $\bar{f}_{u_i, u_{i+1}}$ the element $\bar{f}$ in $\Theta(f)$ such that $\bar{f}(u_i) = u_{i+1}$ and define

$$\bar{f}_{\boldsymbol{u}}^i = \begin{cases} \bar{f}_{u_{i-1}, u_i} \circ \cdots \circ \bar{f}_{u_0, u_1} & \text{if } i > 0 \\ (\bar{f}_{u_i, u_{i+1}})^{-1} \circ \cdots \circ (\bar{f}_{u_{-1}, u_0})^{-1} & \text{if } i < 0 \\ id_N & \text{if } i = 0 . \end{cases}$$

We define a map $\tau_{\boldsymbol{u}}^f: N \to (N/\Gamma)_f$ by

$$\tau_{\boldsymbol{u}}^f(x) = (\pi \circ \bar{f}_{\boldsymbol{u}}^i(x))_{i=-\infty}^{\infty} \quad (x \in N).$$

For simplicity we write $\tau_{\boldsymbol{u}} = \tau_{\boldsymbol{u}}^f$ in subsequent.

LEMMA 3.3 ([**Ao-Hi**] Lemma 6.5.5). *For $\boldsymbol{u} = (u_i) \in N_f$ the following hold:*
(1) *$\tau_{\boldsymbol{u}}: N \to (N/\Gamma)_f$ is continuous,*
(2) *$\tau_{\boldsymbol{u}}(N)$ is dense in $(N/\Gamma)_f$,*
(3) *$\tau_{\boldsymbol{u}}(N)$ is the path connected component of $\tau_{\boldsymbol{u}}(u_0)$ in $(N/\Gamma)_f$.*

LEMMA 3.4. *For $x \in (N/\Gamma)_f$ there is $\boldsymbol{u} \in N_f$ such that $x \in \tau_{\boldsymbol{u}}(N)$.*

PROOF. Since $x \in (N/\Gamma)_f$, we choose $u_i \in N$ $(i \in \mathbf{Z})$ such that $x = (\pi(u_i))_{i \in \mathbf{Z}} \in (N/\Gamma)_f$. Clearly $f(\pi(u_i)) = \pi(u_{i+1})$. By the definition of N_f, we have that $\boldsymbol{u} = (\cdots, u_{-1}, u_0, u_1, \cdots) \in N_f$, and by the definition of $\bar{f}_{\boldsymbol{u}}$

$$x=(\pi(u_i))_{i\in\mathbf{Z}}=(\pi\circ\bar{f}_u^i(u_0))_{i\in\mathbf{Z}}=\tau_u(u_0). \qquad \square$$

Suppose that the covering degree of f is greater than one. From Lemma 3.1 it follows that $((N/\Gamma)_f, N/\Gamma, C, p_0)$ is a fiber bundle where C is the Cantor set. We note that a coordinate function $\varphi: U\times C\to p_0^{-1}(U)$ for $((N/\Gamma)_f, N/\Gamma, C, p_0)$ exists whenever U is a connected open set of N with small diameter.

Let $\boldsymbol{u}\in N_f$. We define a family $\mathcal{T}_u$ of subsets of $\tau_u(N)$ as follows: $V\in\mathcal{T}_u$ if and only if there is a connected open set U of X such that V is expressed as $V=\varphi(U\times\{a\})$ by a coordinate function $\varphi: U\times C\to p_0^{-1}(U)$ for $((N/\Gamma)_f, N/\Gamma, C, p_0)$, where a is a point in C. It is easily checked that

(1) any point in $\tau_u(N)$ belongs to some $V\in\mathcal{T}_u$,

(2) if $V_1, V_2\in\mathcal{T}_u$ and $x\in V_1\cap V_2$, then there is $V_3\in\mathcal{T}_u$ such that $x\in V_3\subset V_1\cap V_2$.

Hence the family $\mathcal{T}_u$ generates a topology of $\tau_u(N)$, which is called the *intrinsic topology* of $\tau_u(N)$. If $f: N/\Gamma\to N/\Gamma$ is a homeomorphism, then we have $\tau_u(N)=(N/\Gamma)_A$ for $\boldsymbol{u}\in N_A$. For this case define the intrinsic topology of $\tau_u(N)$ by the topology of $(N/\Gamma)_A$.

LEMMA 3.5 ([**Ao-Hi**] Lemma 6.5.6). *For $\boldsymbol{u}\in N_f$ the map $\tau_u(N): N\to\tau_u(N)$ and the restriction $p_0:\tau_u(N)\to N/\Gamma$ are both covering maps under the intrinsic topology of $\tau_u(N)$, and the following diagram commutes:*

$$\begin{array}{ccc} N & \xrightarrow{\tau_u} & \tau_u(N) \\ & p\searrow \quad \swarrow p_0 & \\ & N/\Gamma & \end{array} .$$

LEMMA 3.6 ([**Ao-Hi**] Lemma 6.5.9). *For $\boldsymbol{u}\in N_f$, $\sigma(\tau_u(N))=\tau_{\bar{\sigma}(u)}(N)$ and the restriction $\sigma:\tau_u(N)\to\tau_{\bar{\sigma}(u)}(N)$ is a homeomorphism under the intrinsic topologies.*

Furthermore the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{\bar{f}_u} & N \\ \tau_u\downarrow & & \downarrow\tau_{\bar{\sigma}(u)} \\ \tau_u(N) & \xrightarrow{\sigma} & \tau_{\bar{\sigma}(u)}(N) \\ p_0\downarrow & & \downarrow p_0 \\ N/\Gamma & \xrightarrow{f} & N/\Gamma \end{array} .$$

LEMMA 3.7. (1) *For $\varepsilon>0$ and $L\in\mathbf{N}$, there exists $\delta=\delta(\varepsilon, L)>0$ such that if $\tilde{d}(\tau_u(x), \tau_u(y))<\delta$ $(x, y\in N)$, then $y\in\alpha(U_\varepsilon(x))$ for some $\alpha\in(\bar{f}_{\bar{\sigma}^{-L}(u)}^L)_*(\Gamma)$.*

(2) *Conversely, for $\varepsilon>0$ there are $L=L(\varepsilon)\in\mathbf{N}$ and $\delta=\delta(\varepsilon)>0$ such that if $y\in\alpha(U_\delta(x))$ $(x, y\in N)$ for some $\alpha\in(\bar{f}_{\bar{\sigma}^{-L}(u)}^L)_*(\Gamma)$, then $\tilde{d}(\tau_u(x), \tau_u(y))<\varepsilon$.*

PROOF. As we saw in § 1, there is $\lambda>0$ such that $\alpha(U_\lambda(x))\cap U_\lambda(x)=\emptyset$ for $\alpha\in\Gamma\setminus\{id_N\}$ and $x\in N$. For given $\varepsilon>0$ and $L\in\mathbf{N}$, by Lemma 3.2 we can find $0<\mu<\min\{\lambda, \varepsilon\}$ such that

$$D(x, y)<\mu \ (x, y\in N) \Longrightarrow D(\bar{f}^L_{\sigma^{-L}(u)}(x), \bar{f}^L_{\sigma^{-L}(u)}(y))<\lambda. \tag{3.1}$$

Let $\delta<\mu/2^L$. If $\tilde{d}(\tau_u(x), \tau_u(y))<\delta$, by the definition of τ_u

$$\tilde{d}(\tau_u(x), \tau_u(y))=\sum_{i\in\mathbf{Z}}\frac{1}{2^{|i|}}d(\pi\circ\bar{f}^i_u(x), \pi\circ\bar{f}^i_u(y))<\delta,$$

from which $d(\pi\circ\bar{f}^i_u(x), \pi\circ\bar{f}^i_u(y))<\mu$ for $|i|\leq L$.

By the definition of d, we have $\bar{f}^i_u(y)\in\alpha_i(U_\mu(\bar{f}^i_u(x)))$ for some $\alpha_i\in\Gamma$ ($|i|\leqq L$), and in particular

$$\begin{aligned} y &\in \alpha_0(U_\mu(x)) \ (\subset \alpha_0(U_\varepsilon(x))), \\ y &\in (\bar{f}^{-L}_u)^{-1}\circ\alpha_{-L}(U_\mu(\bar{f}^{-L}_u(x))) \\ &= (\bar{f}^L_{\sigma^{-L}(u)})_*(\alpha_{-L})\circ(\bar{f}^L_{\sigma^{-L}(u)})(U_\mu(\bar{f}^{-L}_u(x))) \\ &\subset (\bar{f}^L_{\sigma^{-L}(u)})_*(\alpha_{-L})(U_\lambda(n_1))\quad \text{(by (3.1))}. \end{aligned}$$

Remark that $(\bar{f}^{-L}_u)^{-1}=\bar{f}^L_{\sigma^{-L}(u)}$. Then

$$\alpha_0(U_\lambda(x))\cap(\bar{f}^L_{\sigma^{-L}(u)})_*(\alpha_{-L})(U_\lambda(x))\neq\emptyset,$$

and $\alpha_0=(\bar{f}^L_{\sigma^{-L}(u)})_*(\alpha_{-L})\in(\bar{f}^L_{\sigma^{-L}(u)})_*(\Gamma)$. Therefore the proof of (1) is completed.

For $\varepsilon>0$, we choose $L\in\mathbf{N}$ such that

$$\sum_{|i|\geq L+1}\frac{1}{2^{|i|}}d(\pi\circ\bar{f}^i_u(x), \pi\circ\bar{f}^i_u(y))<\frac{\varepsilon}{2}\quad (x, y\in N).$$

Let $\lambda>0$ be as above and let $\mu=\min\{\lambda, \varepsilon/6\}$. By Lemma 3.2 there is $\delta>0$ such that $\sup_{|i|\leq L}\{D(\bar{f}^i_{\sigma^{-i}(u)}(x), \bar{f}^i_{\sigma^{-i}(u)}(y))\}<\mu$ whenever $D(x, y)<\delta$ for $x, y\in N$. If $y\in\alpha(U_\delta(x))$ ($\alpha\in(\bar{f}^L_{\sigma^{-L}(u)})_*(\Gamma)$), then we have

$$\begin{aligned} \bar{f}^i_u(y) &\in \bar{f}^i_u(\alpha(U_\delta(x))) \\ &= (\bar{f}^i_u)_*(\alpha)\circ\bar{f}^i_u(U_\delta(x)) \\ &\subset (\bar{f}^i_u)_*(\alpha)(U_\mu(\bar{f}^i_u(x))) \end{aligned}$$

for $|i|\leqq L$. Since $\pi: U_\lambda(\bar{f}^i_u(x))\to U_\lambda(\pi\circ\bar{f}^i_u(x))$ is an isometry, we have

$$d(\pi\circ\bar{f}^i_u(x), \pi\circ\bar{f}^i_u(y))<\frac{\varepsilon}{6}\quad\text{for } |i|\leqq L,$$

and so

$$
\begin{aligned}
\tilde{d}(\tau_u(x), \tau_u(y)) &= \sum_{|i|\le L} \frac{1}{2^{|i|}} d(\pi\circ \bar{f}_u^i(x), \pi\circ \bar{f}_u^i(y)) \\
&\quad + \sum_{|i|\ge L+1} \frac{1}{2^{|i|}} d(\pi\circ \bar{f}_u^i(x), \pi\circ \bar{f}_u^i(y)) \\
&< \sum_{|i|\le L} \frac{1}{2^{|i|}}\cdot\frac{\varepsilon}{6}+\frac{\varepsilon}{2} \\
&< \varepsilon .
\end{aligned}
$$

(2) was proved. □

The following result is easily checked by Lemma 3.7.

LEMMA 3.8. *For* $x, y\in N$, $y=\alpha(x)$ *for some* $\alpha\in\bigcap_{i=0}^{\infty}(\bar{f}^{i}_{\sigma^{-i}(u)})_*(\Gamma)$ *if and only if* $\tau_u(x)=\tau_u(y)$.

Let $f: N/\Gamma\to N/\Gamma$ be a self-covering map of an infra-nil-manifold, and let $A: N/\Gamma\to N/\Gamma$ be an infra-nil-endomorphism homotopic to f. Let $\bar{A}: N\to N$ be the automorphism which is a lift of A by π. Choose a lift map $\bar{f}: N\to N$ of f by π satisfying $\bar{f}_*=\bar{A}_*: \Gamma\to\Gamma$.

For $\boldsymbol{e}=(\cdots, e, e, e, \cdots)\in N_A$, we have that $\bar{A}_{\boldsymbol{e}}^i=\bar{A}^i$ ($i\in\mathbf{Z}$) and

$$\tau_{\boldsymbol{e}}^A(x)=(\pi\circ\bar{A}^i(x))_{i=-\infty}^{\infty} \quad (x\in N).$$

If $\tau_{\boldsymbol{e}}^A(x)=\tau_{\boldsymbol{e}}^A(y)$, by Lemma 3.8 we have that $x=\alpha(y)$ for some $\alpha\in\bigcap_{i=0}^{\infty}\bar{A}_*^i(\Gamma)$. Since $\bar{f}(x)=\bar{f}_*(\alpha)\circ\bar{f}(y)=\bar{A}_*(\alpha)\circ\bar{f}(y)$ ($\bar{A}_*(\alpha)\in\bigcap_{i=0}^{\infty}\bar{A}_*^i(\Gamma)$), it follows that $\tau_{\boldsymbol{e}}^A(\bar{f}(x))=\tau_{\boldsymbol{e}}^A(\bar{f}(y))$. Therefore we can define a map $\tilde{f}: \tau_{\boldsymbol{e}}^A(N)\to\tau_{\boldsymbol{e}}^A(N)$ by

$$\tilde{f}(\tau_{\boldsymbol{e}}^A(x))=\tau_{\boldsymbol{e}}^A(\bar{f}(x))$$

for $x\in N$.

LEMMA 3.9. $\tilde{f}$ *is* $\tilde{d}$*-biuniformly continuous.*

PROOF. For given $\varepsilon>0$, by Lemma 3.7(2) there exist $L\in\mathbf{N}$ and $\lambda>0$ such that if $y\in\alpha(U_\lambda(x))$ for some $\alpha\in\bar{A}_*^L(\Gamma)$, then $\tilde{d}(\tau_{\boldsymbol{e}}^A(x), \tau_{\boldsymbol{e}}^A(y))<\varepsilon$. Let $\mu>0$ be a number at satisfying

$$D(x, y)<\mu \ (x, y\in N) \Longrightarrow D(\bar{f}(x), \bar{f}(y))<\lambda .$$

Lemma 3.7(1) ensures the existence of $\delta>0$ satisfying

$$\tilde{d}(\tau_{\boldsymbol{e}}^A(x), \tau_{\boldsymbol{e}}^A(y))<\delta \Longrightarrow y\in\alpha(U_\mu(x)) \quad \text{for some } \alpha\in\bar{f}_*^L(\Gamma).$$

Since $\bar{f}(y)\in\bar{f}(\alpha(U_\mu(x)))=\bar{f}_*(\alpha)\bar{f}(U_\mu(x))\subset\bar{f}_*(\alpha)U_\lambda(\bar{f}(x))$ and $\bar{f}_*(\alpha)\in\bar{f}_*^L(\Gamma)$, we have $\tilde{d}(\tilde{f}(\tau_{\boldsymbol{e}}^A(x)), \tilde{f}(\tau_{\boldsymbol{e}}^A(y)))=\tilde{d}(\tau_{\boldsymbol{e}}^A(\bar{f}(x)), \tau_{\boldsymbol{e}}^A(\bar{f}(y)))<\varepsilon$.

Similarly the $\tilde{d}$-uniform continuity of $\tilde{f}^{-1}$ is proved. □

Since $\tau_{\boldsymbol{e}}^A(N)$ is dense in $(N/\Gamma)_A$ by Lemma 3.3(2), it follows from Lemma 3.9 that $\tilde{f}$ is extended to a homeomorphism of $(N/\Gamma)_A$, which is denoted as the same symbol. It is checked that $f\circ p_0=p_0\circ\tilde{f}$ on $(N/\Gamma)_A$. Indeed, by the definition of $\tilde{f}$ we have that

$$f\circ p_0\circ\tau_e=f\circ\pi=\pi\circ\bar{f}=p_0\circ\tau_e\circ\bar{f}=p_0\circ\tilde{f}\circ\tau_e.$$

Since $\tau_e(N)$ is dense in $(N/\Gamma)_A$, we obtain the assertion. Let $\sigma_f:(N/\Gamma)_f\to(N/\Gamma)_f$ be a shift map constructed by f.

LEMMA 3.10. *$((N/\Gamma)_A,\tilde{f})$ is topologically conjugate to $((N/\Gamma)_f,\sigma_f)$.*

PROOF. For $u=(\cdots,\bar{f}^{-1}(e),e,\bar{f}(e),\cdots)\in N_f$, we have $\bar{f}_u^i=\bar{f}^i$ $(i\in\mathbf{Z})$ and $\tau_u^f(x)=(\pi\circ\bar{f}^i(x))_{i=-\infty}^{\infty}$ $(x\in N)$. Lemma 3.8 ensures that

$$y=\alpha(x)\text{ for some }\alpha\in\bigcap_{i=0}^{\infty}\bar{A}_*^i(\Gamma)\text{ if and only if }\tau_e^A(x)=\tau_e^A(y),$$

$$y=\alpha(x)\text{ for some }\alpha\in\bigcap_{i=0}^{\infty}\bar{f}_*^i(\Gamma)\text{ if and only if }\tau_u^f(x)=\tau_u^f(y).$$

Thus we have that $\tau_e^A(x)=\tau_e^A(y)$ if and only if $\tau_u^f(x)=\tau_u^f(y)$. Therefore a bijection $\varphi:\tau_e^A(N)\to\tau_u^f(N)$ is defined by

$$\varphi(\tau_e^A(x))=\tau_u^f(x)\quad(x\in N),$$

and we have that $\sigma_f\circ\varphi=\varphi\circ\tilde{f}$ on $\tau_e^A(N)$. This is easily checked as follows. By the definition of $\tilde{f}$ and Lemma 3.6, $\tilde{f}\circ\tau_e^A=\tau_e^A\circ\bar{f}$ and $\sigma_f\circ\tau_u^f=\tau_{\sigma(u)}^f\circ\bar{f}=\tau_u^f\circ\bar{f}$ on N, from which

$$\sigma_f\circ\varphi\circ\tau_e^A=\sigma_f\circ\tau_u^f=\tau_u^f\circ\bar{f}=\varphi\circ\tau_e^A\circ\bar{f}=\varphi\circ\tilde{f}\circ\tau_e^A.$$

It is checked that φ is $\tilde{d}$-biuniformly continuous. Indeed, for given $\varepsilon>0$, by Lemma 3.7(2) there are $L=L(\varepsilon)\in\mathbf{N}$ and $\lambda=\lambda(\varepsilon)>0$ such that if $y\in\alpha(U_\lambda(x))$ $(x,y\in N)$ for some $\alpha\in\bar{f}_*^L(\Gamma)$, then $\tilde{d}_f(\tau_u^f(x),\tau_u^f(y))<\varepsilon$. By Lemma 3.7(1) we can take $\delta>0$ satisfying $y\in\alpha(U_\lambda(x))$ for some $\alpha\in\bar{A}_*^L(\Gamma)=\bar{f}_*^L(\Gamma)$ whenever $\tilde{d}_A(\tau_e^A(x),\tau_e^A(y))<\delta$. This implies that φ is $\tilde{d}$-uniformly continuous. Analogously we can prove that φ^{-1} is $\tilde{d}$-uniformly continuous.

Since $\tau_e^A(N)$ and $\tau_u^f(N)$ are dense in $(N/\Gamma)_A$ and $(N/\Gamma)_f$ respectively by Lemma 3.3(2), φ is extended to a homeomorphism between $(N/\Gamma)_A$ and $(N/\Gamma)_f$, which is denoted as the same symbol. Therefore, $\sigma_f\circ\varphi=\varphi\circ\tilde{f}$ on $(N/\Gamma)_A$. □

REMARK 3.11. *Suppose that N/Γ is a torus. Then for any covering transformation $\alpha\in\Gamma$, there exists a homeomorphism $\tilde{\alpha}:(N/\Gamma)_A\to(N/\Gamma)_A$ satisfying $\tilde{\alpha}\circ\tau_e^A=\tau_e^A\circ\alpha$ (see* [**Ao-Hi**] *Theorem* 6.5.3). *However if Γ is not abelian, then it is not true in general.*

Indeed, we can find the following counter-example ([**Sh**]). Let N be the simply connected nilpotent Lie group defined by

$$N=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} : x,\ y,\ z\in\mathbf{R}\right\},$$

and let Γ be the discrete uniform subgroup of N obtained by

$$\Gamma=\left\{\begin{pmatrix}1 & \alpha & \gamma\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix} : \alpha,\ \beta,\ \gamma\in\mathbf{Z}\right\}.$$

Then N/Γ is a nil-manifold. Define the nil-endomorphism $A: N/\Gamma\to N/\Gamma$ induced by the automorphism $\bar{A}: N\to N$ represented as

$$\bar{A}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2x & 6z\\ 0 & 1 & 3y\\ 0 & 0 & 1\end{pmatrix}.$$

Let

$$\alpha=\begin{pmatrix}1 & 5 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad \beta_i=\begin{pmatrix}1 & 2^i & 6^i\\ 0 & 1 & 3^i\\ 0 & 0 & 1\end{pmatrix}\quad (i\geqq 0)\in\Gamma.$$

Then

$$\bar{A}^i(\Gamma)=\left\{\begin{pmatrix}1 & 2^i\alpha & 6^i\gamma\\ 0 & 1 & 3^i\beta\\ 0 & 0 & 1\end{pmatrix} : \alpha,\ \beta,\ \gamma\in\mathbf{Z}\right\}.$$

We can take the map $\tilde{\alpha}$ of $\tau_e^A(N)$ satisfying $\tilde{\alpha}\circ\tau_e^A=\tau_e^A\circ\alpha$ because τ_e^A is bijective. Then $\tilde{d}(\tau_e^A(e),\ \tau_e^A(\beta_i(e)))\to 0$ $(i\to\infty)$ by Lemma 3.7. On the other hand, we have

$$\tilde{d}(\tilde{\alpha}\circ\tau_e^A(e),\ \tilde{\alpha}\circ\tau_e^A(\beta_i(e)))=\tilde{d}(\tau_e^A\circ\alpha(e),\ \tau_e^A\circ\alpha\circ\beta_i(e))$$

does not converge to 0 as $i\to\infty$. Indeed, for $i\geqq 0$

$$\alpha\circ\beta_i(e)=\alpha\circ\beta_i\circ\alpha^{-1}(\alpha(e)),\ \alpha\circ\beta_i\circ\alpha^{-1}\in\Gamma\setminus\bar{A}(\Gamma).$$

By Lemma 3.7, $\tilde{\alpha}$ is not continuous under the metric $\tilde{d}$ for $(N/\Gamma)_A$.

REMARK 3.12. *Furthermore in the case when N/Γ is a torus, we can show that the inverse limit space $(N/\Gamma)_A$ has a structure of compact connected finite dimensional abelian group, which is called the solenoidal group.*

See [**Ao-Hi**] Theorem 7.2.4 for the proof.

REMARK 3.13. *Every hyperbolic infra-nil-endomorphism $A: N/\Gamma\to N/\Gamma$ is a special TA-covering map.*

PROOF. Since A is an Anosov differentiable map, we have A is a TA-covering map ([**Ao-Hi**] Theorem 1.2.1).

Let $\boldsymbol{u}=(\cdots, u_{-1}, u_0, u_1, \cdots)\in N_A$. By the definition of N_A, we can take $\alpha_i\in\Gamma$ such that $\bar{A}_{u_{-i},u_{-i+1}}=\alpha_i\circ\bar{A}$ $(i\geqq 0)$. Let $\alpha_i^{-1}=(z_i, c_i)\in N\cdot C$, and let ρ, ϕ be as in Lemma 1.5. Then we have that

$$\begin{aligned}\bar{A}_{\boldsymbol{u}}^{-i} &= (\bar{A}_{u_{-i},u_{-i+1}})^{-1}\circ(\bar{A}_{u_{-i+1},u_{-i+2}})^{-1}\circ\cdots\circ(\bar{A}_{u_{-1},u_0})^{-1}\\ &= (\alpha_i\circ\bar{A})^{-1}\circ(\alpha_{i-1}\circ\bar{A})^{-1}\circ\cdots\circ(\alpha_1\circ\bar{A})^{-1}\\ &= \bar{A}^{-1}\circ(z_i, c_i)\circ\bar{A}^{-1}\circ(z_{i-1}, c_{i-1})\circ\cdots\circ\bar{A}^{-1}\circ(z_1, c_1)\\ &= (\bar{A}^{-1}(z_i), \phi^{-1}(c_i))\circ(\bar{A}^{-2}(z_{i-1}), \phi^{-2}(c_{i-1}))\circ\cdots\circ(\bar{A}^{-i}(z_1), \phi^{-i}(c_1))\circ\bar{A}^{-i}.\end{aligned}$$

Since D is left invariant and Γ-invariant, for x, $y\in N$ and $i\geqq 0$,

$$D(\bar{A}_{\boldsymbol{u}}^{-i}(x), \bar{A}_{\boldsymbol{u}}^{-i}(y)) = D(\bar{A}^{-i}(x), \bar{A}^{-i}(y)).$$

Therefore $\bar{L}_{\boldsymbol{u}}^{u}(x)=\bar{L}^{u}(x)$ where $\bar{L}_{\boldsymbol{u}}^{u}(x)=\{y\in N: D(\bar{A}_{\boldsymbol{u}}^{-i}(x), \bar{A}_{\boldsymbol{u}}^{-i}(y))\to 0\ (i\to\infty)\}$. For $\boldsymbol{u}\in N_A$, we have

$$L^{u}(\tau_{\boldsymbol{u}}(x)) = \pi(\bar{L}_{\boldsymbol{u}}^{u}(x)) = \pi(\bar{L}^{u}(x))$$

where $L^{u}(\tau_{\boldsymbol{u}}(x))=\{y_0\in N/\Gamma: \exists(y_i)\in(N/\Gamma)_A\ s.t.\ \lim_{i\to\infty} d(\pi\circ\bar{A}_{\boldsymbol{u}}^{-i}(x), y_{-i})\to 0\}$ (**[Ao-Hi]** Lemma 6.6.8), and then A is a special TA-covering map. □

§4. Construction of semi-conjugacy maps on the inverse limit systems.

Suppose $f: N/\Gamma\to N/\Gamma$ is a TA-covering map, and let $A: N/\Gamma\to N/\Gamma$ be the infra-nil-endomorphism homotopic to f. Let $\bar{A}: N\to N$ be the automorphism which is a lift of A by π. By Lemma 1.3, A is hyperbolic. Let $\bar{f}: N\to N$ be a lift of f by π satisfying $\bar{f}_*=\bar{A}_*: \Gamma\to\Gamma$. We may assume that $\bar{f}(e)=e$, and let $\bar{h}: N\to N$ denote the semi-conjugacy map obtained in Lemma 2.3.

Let $\sigma_A: (N/\Gamma)_A\to(N/\Gamma)_A$ be the inverse limit system of $(N/\Gamma, A)$, and let $\tau_e: N\to(N/\Gamma)_A$ be the continuous map defined by $\tau_e(x)=(\pi\circ\bar{A}^i(x))_{i=-\infty}^{\infty}$ for $x\in N$. As saw in §3 a homeomorphism $\tilde{f}: (N/\Gamma)_A\to(N/\Gamma)_A$ is constructed by f.

LEMMA 4.1. *Under the assumptions and notations as above, there is a continuous surjection* $\tilde{h}: (N/\Gamma)_A\to(N/\Gamma)_A$ *such that*

(1) $\tilde{h}\circ\tau_e=\tau_e\circ\bar{h}$ *on* N,

(2) $\sigma_A\circ\tilde{h}=\tilde{h}\circ\tilde{f}$ *on* $(N/\Gamma)_A$.

PROOF. By Lemma 3.8 and Lemma 2.4(3) we define a map $\tilde{h}: \tau_e(N)\to\tau_e(N)$ by

$$\tilde{h}(\tau_e(x)) = \tau_e\circ\bar{h}(x)\quad(x\in N).$$

Then $\sigma_A\circ\tilde{h}=\tilde{h}\circ\tilde{f}$ on $\tau_e(N)$. This follows from Lemma 3.6 and Lemma 2.3; i.e.,

$$\sigma_A\circ\tilde{h}\circ\tau_e = \sigma_A\circ\tau_e\circ\bar{h} = \tau_e\circ\bar{A}\circ\bar{h} = \tau_e\circ\bar{h}\circ\bar{f} = \tilde{h}\circ\tau_e\circ\bar{f} = \tilde{h}\circ\tilde{f}\circ\tau_e.$$

To show that $\tilde{h}$ is $\tilde{d}$-uniformly continuous, for $\varepsilon>0$ there are $L\in\mathbf{N}$ and $\delta>0$ such that if $y\in\alpha(U_\delta(x))$ $(x, y\in N)$ for some $\alpha\in\bar{A}^L_*(\Gamma)$, then $\tilde{d}(\tau_e(x), \tau_e(y))<\varepsilon$. Since $\bar{h}$ is D-uniformly continuous by Lemma 2.3, we can take $\lambda>0$ such that if $D(x, y)<\lambda$ $(x, y\in N)$ then $D(\bar{h}(x), \bar{h}(y))<\delta/2$. By Lemma 2.4(2), there exists $J\in\mathbf{N}$ with $J\geqq L$ such that $D(\bar{h}\circ\alpha(x), \alpha\circ\bar{h}(x))<\delta/2$ for $\alpha\in\bar{f}^J_*(\Gamma)$, and by Lemma 3.7(1) we can take $\mu>0$ satisfying $y\in\alpha(U_\lambda(x))$ for some $\alpha\in\bar{f}^J_*(\Gamma)$ whenever $\tilde{d}(\tau_e(x), \tau_e(y))<\mu$. Thus

$$\begin{aligned}\tilde{d}(\tau_e(x), \tau_e(y)) < \mu &\Longrightarrow y \in \alpha(U_\lambda(x)) \quad \text{for some } \alpha \in \bar{A}^J_*(\Gamma)\\ &\Longrightarrow \bar{h}(y) \in \bar{h}\circ\alpha(U_\lambda(x)) = \bar{h}(U_\lambda(\alpha(x)))\\ &\qquad \subset U_{\delta/2}(\bar{h}\circ\alpha(x)) \subset U_\delta(\alpha(\bar{h}(x)))\\ &\qquad = \alpha(U_\delta(\bar{h}(x))) \quad \text{for some } \alpha \in \bar{A}^L_*(\Gamma)\\ &\Longrightarrow \tilde{d}(\tilde{h}(\tau_e(x)), \tilde{h}(\tau_e(y))) = \tilde{d}(\tau_e\circ\bar{h}(x), \tau_e\circ\bar{h}(y))\\ &\qquad < \varepsilon .\end{aligned}$$

This shows that $\tilde{h}$ is $\tilde{d}$-uniformly continuous.

Since $\tau_e(N)$ is dense in $(N/\Gamma)_A$ by Lemma 3.3, it follows that $\tilde{h}$ is extended to a continuous map on $(N/\Gamma)_A$, which is denoted as the same symbol. Therefore (2) holds.

Since $\bar{h}$ is surjective, we have $\tilde{h}(\tau_e(N))=\tau_e(N)$. Hence $\tilde{h}((N/\Gamma)_A)\supset\tau_e(N)$. Since $\tau_e(N)$ is dense in $(N/\Gamma)_A$, we have $\tilde{h}((N/\Gamma)_A)=(N/\Gamma)_A$. Therefore $\tilde{h}$ is surjective. □

For $\boldsymbol{u}=(\cdots, u_{-1}, u_0, u_1, \cdots)\in N_A$ define

$$\boldsymbol{u}(j) = (\cdots, \bar{A}^{-3}(u_{-j}), \bar{A}^{-2}(u_{-j}), \bar{A}^{-1}(u_{-j}), u_{-j}, u_{-j+1}, \\ \cdots, u_{-1}, u_0, u_1, \cdots) \in N_A$$

for $j\in\mathbf{N}$. By the definition of $\bar{A}_{\boldsymbol{u}(j)}$ we have

$$\bar{A}^i_{\boldsymbol{u}(j)} = \begin{cases} \bar{A}^i_{\boldsymbol{u}} & \text{if } i \geqq -j \\ \bar{A}^{i+j}\circ\bar{A}^{-j}_{\boldsymbol{u}} & \text{if } i < -j. \end{cases} \tag{4.1}$$

For $j\in\mathbf{N}$ we have $(\bar{A}^{-j}_{\boldsymbol{u}})^{-1}=\alpha^{-1}_{\boldsymbol{u}(j)}\circ\bar{A}^j$ for some $\alpha_{\boldsymbol{u}(j)}\in\Gamma$ (see [**Ao-Hi**] Theorem 6.3.9). Then

$$\bar{A}^j\circ\bar{A}^{-j}_{\boldsymbol{u}} = \bar{A}^j\circ\bar{A}^{-j}\circ\alpha_{\boldsymbol{u}(j)} = \alpha_{\boldsymbol{u}(j)} . \tag{4.2}$$

We define $\bar{h}_{\boldsymbol{u}(j)} : N\to N$ by

$$\bar{h}_{\boldsymbol{u}(j)} = \alpha^{-1}_{\boldsymbol{u}(j)}\circ\bar{h}\circ\alpha_{\boldsymbol{u}(j)} \quad \text{on } N.$$

LEMMA 4.2. (1) $\tilde{d}(\tau_{\boldsymbol{u}}, \tau_{\boldsymbol{u}(j)})=\sup\{\tilde{d}(\tau_{\boldsymbol{u}}(x), \tau_{\boldsymbol{u}(j)}(x)) : x\in N\}\to 0$ *as* $j\to\infty$.

(2) *For* $j\in\mathbf{N}$ $\tau_{u(j)}=\tau_e\circ\alpha_{u(j)}$ *on* N.

(3) *The following diagram commutes*;

$$\begin{array}{ccc} N & \xrightarrow{\bar{h}_{u(j)}} & N \\ \tau_{u(j)}\downarrow & & \downarrow\tau_{u(j)} \\ \tau_{u(j)}(N) & \xrightarrow[\tilde{h}]{} & \tau_{u(j)}(N). \end{array}$$

PROOF. (1): Since N/Γ is compact, there exists $M>0$ such that $d(x, y)\leqq M$ $(x, y\in N/\Gamma)$. By the definition of τ_u and $\tau_{u(j)}$, we have

$$\begin{aligned} \tilde{d}(\tau_u(x), \tau_{u(j)}(x)) &= \sum_{i\in\mathbf{Z}}\frac{1}{2^{|i|}}d(\pi\circ\bar{A}^i_u(x), \pi\circ\bar{A}^i_{u(j)}(x)) \\ &= \sum_{i<-j}\frac{1}{2^{|i|}}d(\pi\circ\bar{A}^i_u(x), \pi\circ\bar{A}^i_{u(j)}(x)) \quad \text{(by (4.1))} \\ &\leqq \sum_{i<-j}\frac{M}{2^{|i|}} \leqq \frac{M}{2^j} \end{aligned}$$

for $x\in N$ and $j\in\mathbf{N}$. This shows (1).

(2): By (4.1) and (4.2), we have that if $i<-j$

$$\bar{A}^i_{u(j)} = \bar{A}^{i+j}\circ\bar{A}^{-j}_u = \bar{A}^{i+j}\circ\bar{A}^{-j}\circ\alpha_{u(j)} = \bar{A}^i\circ\alpha_{u(j)},$$

and that if $i\geqq -j$

$$\bar{A}^i_{u(j)} = \bar{A}^i_u = \bar{A}^{i+j}_{\bar{\sigma}^{-j}(u)}\circ\bar{A}^{-j}_u = \beta_i\circ\bar{A}^{i+j}\circ\bar{A}^{-j}\circ\alpha_{u(j)} = \beta_i\circ\bar{A}^i\circ\alpha_{u(j)}$$

for some $\beta_i\in\Gamma$ (see [**Ao-Hi**] Theorem 6.3.9). Hence

$$\tau_{u(j)}(x) = (\pi\circ\bar{A}^i_{u(j)}(x))_{i\in\mathbf{Z}} = (\pi\circ\bar{A}^i\circ\alpha_{u(j)}(x))_{i\in\mathbf{Z}} = \tau_e\circ\alpha_{u(j)}(x)$$

for $x\in N$ and $j\in\mathbf{N}$.

(3): By (2), we have that for $j\in\mathbf{N}$

$$\begin{aligned} \tilde{h}\circ\tau_{u(j)} &= \tilde{h}\circ\tau_e\circ\alpha_{u(j)} = \tau_e\circ\bar{h}\circ\alpha_{u(j)} \\ &= \tau_e\circ\alpha_{u(j)}\circ\alpha^{-1}_{u(j)}\circ\bar{h}\circ\alpha_{u(j)} = \tau_{u(j)}\circ\bar{h}_{u(j)} \end{aligned}$$

on N. □

LEMMA 4.3. *For* $u\in N_A$, *there exists a surjective map* $\bar{k}_u : N\to N$ *such that*

(1) $\tilde{h}\circ\tau_u=\tau_u\circ\bar{k}_u$ *on* N,

(2) $D(\bar{k}_u, id_N)<\infty$,

(3) $\bar{k}_u$ *is D-uniformly continuous*,

(4) $(\bar{k}_u)_* : \bigcap_{i=0}^{\infty}(\bar{A}^i_{\bar{\sigma}^{-i}(u)})_*(\Gamma)\to\bigcap_{i=0}^{\infty}(\bar{A}^i_{\bar{\sigma}^{-i}(u)})_*(\Gamma)$ *is the identity map.*

PROOF. Let $u(j)\in N_A$ and $\alpha_{u(j)}\in\Gamma$ be as above. By Lemma 2.4(4) we have that for $x\in N$ and $j\in\mathbf{N}$

$$\begin{aligned}\bar{h}_{u(j)}(x) &= \alpha_{u(j)}^{-1}\circ\bar{h}\circ\alpha_{u(j)}(x)\\ &\in \bar{L}^s(\alpha_{u(j)}^{-1}\circ\alpha_{u(j)}\circ\bar{h}(x)) = \bar{L}^s(\bar{h}(x)),\end{aligned}$$

and by Lemma 2.4(1) there exists $K>0$ such that for $x\in N$ and $j\in\mathbf{N}$

$$\begin{aligned}D(\bar{h}_{u(j)}(x), \bar{h}(x)) &= D(\alpha_{u(j)}^{-1}\circ\bar{h}\circ\alpha_{u(j)}(x), \bar{h}(x))\\ &= D(\bar{h}\circ\alpha_{u(j)}(x), \alpha_{u(j)}\circ\bar{h}(x))\\ &< K.\end{aligned} \tag{4.3}$$

Then we can take $\bar{x}\in\bar{L}^s(\bar{h}(x))$ and a subsequence $\{j(n)\}_{n\in\mathbf{N}}\subset\mathbf{N}$ with $j(n)\nearrow\infty$ $(n\nearrow\infty)$ such that $D(\bar{h}_{u(j(n))}(x), \bar{x})\to 0$ $(j\to\infty)$. Remark that the above sequence $\{j(n)\}_{n\in\mathbf{N}}\subset\mathbf{N}$ depends on $x\in N$. By Lemma 4.2(1), (3)

$$\begin{aligned}\tilde{d}(\tau_u(\bar{x}), \tilde{h}\circ\tau_u(x)) &\leqq \tilde{d}(\tau_u(\bar{x}), \tau_u\circ\bar{h}_{u(j(n))}(x))\\ &\quad+\tilde{d}(\tau_u\circ\bar{h}_{u(j(n))}(x), \tau_{u(j(n))}\circ\bar{h}_{u(j(n))}(x))\\ &\quad+\tilde{d}(\tau_{u(j(n))}\circ\bar{h}_{u(j(n))}(x), \tilde{h}\circ\tau_u(x))\\ &\leqq \tilde{d}(\tau_u(\bar{x}), \tau_u\circ\bar{h}_{u(j(n))}(x))+\tilde{d}(\tau_u, \tau_{u(j(n))})\\ &\quad+\tilde{d}(\tilde{h}\circ\tau_{u(j(n))}(x), \tilde{h}\circ\tau_u(x))\\ &\to 0 \quad (n\to\infty),\end{aligned}$$

from which

$$\tau_u(\bar{x}) = \tilde{h}\circ\tau_u(x). \tag{4.4}$$

We claim that $\{\bar{h}_{u(j)}(x)\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $\bar{L}^s(\bar{h}(x))$. Indeed, assume that there exist $\bar{x}^i\in\bar{L}^s(\bar{h}(x))$ $(i=1, 2)$ and subsequences $\{j^i(n)\}_{n\in\mathbf{N}}$ $(i=1, 2)$ with $j^i(n)\nearrow\infty$ $(n\nearrow\infty)$ such that $D(\bar{h}_{u(j^i(n))}(x), \bar{x}^i)\to 0$ as $n\to\infty$ $(i=1, 2)$. Then by (4.4) $\tau_u(\bar{x}^1)=\tilde{h}\circ\tau_u(x)=\tau_u(\bar{x}^2)$. Hence $\bar{x}^2=\bar{\alpha}(\bar{x}^1)$ for some $\bar{\alpha}\in\bigcap_{i=0}^{\infty}\bar{A}_*^i(\Gamma)$ (Lemma 3.8). Since $\pi:\bar{L}^s(\bar{h}(x))\to\pi(\bar{L}^s(\bar{h}(x)))$ is bijective by [**Ao-Hi**] Lemma 6.6.8(2), we have $\bar{\alpha}=id_N$. This implies the claim.

If $\bar{h}_{u(j)}(x)\to\bar{k}_u(x)$ $(j\to\infty)$ for $x\in N$, then $\bar{k}_u: N\to N$ is a map. By (4.4) we have that $\tau_u\circ\bar{k}_u=\tilde{h}\circ\tau_u$ on N. Since

$$D(\bar{h}_{u(j)}(x), x) \leqq D(\bar{h}_{u(j)}(x), \bar{h}(x))+D(\bar{h}(x), x) \quad (x\in N),$$

by (4.3) we have that $D(\bar{k}_u, id_N)<\infty$. By uniform continuity of $\bar{h}$, we can take $\delta>0$ such that $D(\bar{h}(x), \bar{h}(y))<\varepsilon$ whenever $D(x, y)<\delta$. If $D(x, y)<\delta$, then we have that for $j\in\mathbf{N}$

$$D(\bar{h}_{u(j)}(x),\ \bar{h}_{u(j)}(y)) = D(\alpha_{u(j)}^{-1}\circ\bar{h}\circ\alpha_{u(j)}(x),\ \alpha_{u(j)}^{-1}\circ\bar{h}\circ\alpha_{u(j)}(y))$$
$$= D(\bar{h}\circ\alpha_{u(j)}(x),\ \bar{h}\circ\alpha_{u(j)}(y))$$
$$< \varepsilon .$$

Hence $D(\bar{k}_u(x),\ \bar{k}_u(y))\leqq\varepsilon$ and so $\bar{k}_u$ is uniformly continuous under D. By (2) we can prove that $\bar{k}_u$ is surjective.

Since $\bar{k}_u(x)\in\bar{L}^s(\bar{h}(x))$ $(x\in N)$, we have that for $\alpha\in\bigcap_{i=0}^{\infty}(\bar{A}_{\sigma^{-i}(u)}^{i})_*(\Gamma)$

$$(\bar{k}_u)_*(\alpha)\circ\bar{k}_u(e) = \bar{k}_u(\alpha(e)) \in \bar{L}^s(\bar{h}(\alpha(e))) = \bar{L}^s(\alpha\circ\bar{h}(e)) = \alpha\circ\bar{L}^s(e).$$

Hence $\alpha^{-1}\circ(\bar{k}_u)_*(\alpha)\circ\bar{k}_u(e)\in\bar{L}^s(e)$. Since $\pi:\ \bar{L}^s(\bar{h}(x))\to\pi(\bar{L}^s(\bar{h}(x)))$ is bijective, we have that

$$\alpha^{-1}\circ(\bar{k}_u)_*(\alpha)\circ\bar{k}_u(e) = \bar{k}_u(e) \in \bar{L}^s(e).$$

Therefore $(\bar{k}_u)_*(\alpha)=\alpha$. □

LEMMA 4.4. *Each path connected component in $(N/\Gamma)_A$ is h-invariant.*

PROOF. This is clear from Lemma 3.4 and Lemma 4.3(1). □

§ 5. Nonwandering set.

The purpose of this section is to show Lemma 5.4.

LEMMA 5.1. *If $x_0\in\mathrm{Per}(\sigma_A)$, then $\tilde{h}^{-1}(x_0)$ is the set of one point.*

PROOF. Without loss of generality we may suppose that $\sigma_A(x_0)=x_0$ is satisfied. Since $\sigma_A\circ\tilde{h}=\tilde{h}\circ\tilde{f}$, we have $\tilde{f}(\tilde{h}^{-1}(x_0))=\tilde{h}^{-1}(x_0)$.

Since $\tilde{f}|_{\tilde{h}^{-1}(x_0)}$ is expansive and has POTP, $\tilde{h}^{-1}(x_0)$ contains a periodic point y_0 of $\tilde{f}$. We can check POTP of $\tilde{f}|_{\tilde{h}^{-1}(x_0)}$ as follows. Since $f: N/\Gamma\to N/\Gamma$ has POTP, for $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo orbit of $\tilde{f}$ is ε-traced by some point of $(N/\Gamma)_A$. If $\{v_i\}\subset\tilde{h}^{-1}(x_0)$ is a δ-pseudo orbit of $\tilde{f}$, then an ε-tracing point v for $\{v_i\}$ exists in $(N/\Gamma)_A$. Since $\tilde{h}(v_i)=x_0$ for all i, each of $\sigma_A\circ\tilde{h}(v)$ is near to x_0 in $(N/\Gamma)_A$, and hence $\tilde{h}(v)=x_0$ by expansivity of σ_A, i.e., $v\in\tilde{h}^{-1}(x_0)$. Therefore $\tilde{f}|_{\tilde{h}^{-1}(x_0)}$ has POTP. To avoid complication, suppose $\tilde{f}(y_0)=y_0$.

By $\sigma_A(x_0)=x_0$, there exists $u_0\in N$ such that $x_0=(\pi(u_0))_{i\in\mathbf{Z}}\in(N/\Gamma)_A$. By the definition of $(N/\Gamma)_A$ we have that $\bar{\alpha}\circ\bar{A}(u_0)=u_0$ for some $\bar{\alpha}\in\Gamma$. Let $\hat{A}=\bar{\alpha}\circ\bar{A}$ and take $\boldsymbol{u}=(\cdots,\ u_0,\ u_0,\ u_0,\ \cdots)\in N_A$. By the definition of τ_u we have $x_0=\tau_u(u_0)\in(N/\Gamma)_A$. Since $\tilde{h}$ preserves each path connected component of $(N/\Gamma)_A$ (Lemma 4.4), by Lemma 3.3 there exists $v_0\in N$ such that $y_0=\tau_u(v_0)\in\tau_u(N)$.

By Lemma 3.6, $\sigma_A\circ\tau_u=\tau_u\circ\hat{A}$ on N. Since $\tilde{f}(x_0)=x_0$, we have $\tilde{f}(\tau_u(N))=\tau_u(N)$. By Lemmas 3.6 and 3.10, $\tilde{f}|_{\tau_u(N)}:\tau_u(N)\to\tau_u(N)$ is a homeomorphism under the intrinsic topology of $\tau_u(N)$. Therefore there is the lift map $\hat{f}: N\to N$ of $\tilde{f}|_{\tau_u(N)}$ such that $\hat{f}(u_0)=u_0$ by Lemma 3.5. Since

$$\pi\circ\hat{f} = p_0\circ\tau_u\circ\hat{f} = p_0\circ\tilde{f}\circ\tau_u = f\circ p_0\circ\tau_u = f\circ\pi$$

on N, we have that $\hat{f}$ is the lift map of f by π, and then $\hat{f}$ is expansive and has POTP.

By Lemmas 3.5 and 4.3, $\tilde{h}|_{\tau_u(N)}:\tau_u(N)\to\tau_u(N)$ is continuous surjection under the intrinsic topology of $\tau_u(N)$. Take the lift map $\hat{h}:N\to N$ of $\tilde{h}|_{\tau_u(N)}$ satisfying $\hat{h}(v_0)=u_0$ by Lemma 3.5. Let $\bar{k}_u$ be the lift of $\tilde{h}|_{\tau_u(N)}$ obtained in Lemma 4.3. Then there exists $\beta\in\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$ such that

$$\hat{h} = \beta\circ\bar{k}_u \quad \text{on } N.$$

Therefore we have that $\hat{h}$ is proper (i.e., the inverse image by $\hat{h}$ of any compact subset is compact) by Lemma 4.3(2).

By the definition of $\hat{f}$ and $\hat{h}$, we have

$$\begin{aligned}\tau_u\circ\hat{A}\circ\hat{h} &= \sigma_A\circ\tau_u\circ\hat{h} = \sigma_A\circ\tilde{h}\circ\tau_u = \tilde{h}\circ\tilde{f}\circ\tau_u\\ &= \tilde{h}\circ\tau_u\circ\hat{f} = \tau_u\circ\hat{h}\circ\hat{f} \quad \text{on } N,\\ \hat{A}\circ\hat{h}(v_0) &= \hat{A}(u_0) = u_0 = \hat{h}(v_0) = \hat{h}\circ\hat{f}(v_0),\end{aligned}$$

and then $\hat{A}\circ\hat{h}=\hat{h}\circ\hat{f}$ on N. Therefore $\hat{f}(\hat{h}^{-1}(u_0))=\hat{h}^{-1}(u_0)$. Since $\hat{h}$ is proper, $\hat{h}^{-1}(u_0)$ is compact. It is not difficult to see that $\hat{f}:\hat{h}^{-1}(u_0)\to\hat{h}^{-1}(u_0)$ has POTP. Therefore $\hat{f}|_{\hat{h}^{-1}(u_0)}$ is TA-homeomorphism of a compact metric space.

Denote as Ω the nonwandering set of $\hat{f}|_{\hat{h}^{-1}(u_0)}$. Then the set of all periodic points of $\hat{f}|_{\hat{h}^{-1}(u_0)}$ is dense in Ω. Since $\hat{f}:N\to N$ has exactly one fixed point by Lemma 1.5, Ω consists of one point. This implies $\hat{h}^{-1}(u_0)=\Omega$. Therefore $\hat{h}^{-1}(u_0)=v_0$.

Since $(\bar{k}_u)_*:\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)\to\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$ is the identity map by Lemma 4.3(4), we have that for $\alpha\in\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$

$$\hat{h}\circ\alpha = \beta\circ\bar{k}_u\circ\alpha = \beta\circ\alpha\circ\bar{k}_u = \beta\circ\alpha\circ\beta^{-1}\circ\beta\circ\bar{k}_u = \beta\circ\alpha\circ\beta^{-1}\circ\hat{h}$$

on N. Hence $\hat{h}_*(\alpha)=\beta\circ\alpha\circ\beta^{-1}$ for $\alpha\in\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$ and $\hat{h}_*:\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)\to\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$ is bijective.

Let $z\in\tilde{h}^{-1}(x_0)$. Since $\tilde{h}$ preserves each path connected component of $(N/\Gamma)_A$ and $x_0\in\tau_u(N)$, there exists $w\in N$ such that $z=\tau_u(w)$. Hence

$$\tau_u\circ\hat{h}(w) = \tilde{h}\circ\tau_u(w) = \tilde{h}(z) = x_0 = \tau_u(u_0),$$

and then $\hat{h}(w)=\alpha(u_0)$ for some $\alpha\in\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$. Since $\hat{h}_*:\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)\to\bigcap_{i=0}^{\infty}\hat{A}_*^i(\Gamma)$ is bijective, we have $w=\hat{h}_*^{-1}(\alpha)(\hat{h}^{-1}(u_0))=\hat{h}_*^{-1}(\alpha)(v_0)$ and so $z=\tau_u(w)=\tau_u(v_0)=y_0$. Therefore $\tilde{h}^{-1}(x_0)=\{y_0\}$. □

Let $N\supset N^1\supset\cdots\supset N^k\supsetneqq N^{k+1}=e$ be the lower central series where $N^{i+1}=[N,N^i]$, $N^1=[N,N]$. N_e, N_e^i will denote the tangent spaces of N, N^i at the identity.

LEMMA 5.2 ([**Pa**]). *If $d\bar{A}_e : N_e/N_e^1 \to N_e/N_e^1$ has no root of unity as eigenvalues, then $A : N/\Gamma \to N/\Gamma$ is ergodic with respect to Haar measure.*

PROOF. See [**Pa**] Corollary 2. □

LEMMA 5.3. *$\sigma_A : (N/\Gamma)_A \to (N/\Gamma)_A$ is transitive.*

PROOF. Let $\hat{A} : N/(\Gamma \cap N) \to N/(\Gamma \cap N)$ be a nil-endomorphism induced by $A : N \to N$ (§1). By Lemma 5.2 and the hyperbolicity of A, $\hat{A}$ is ergodic with respect to Haar measure. Thus $\hat{A}$ is transitive, from which A is transitive. Therefore $\sigma_A : (N/\Gamma)_A \to (N/\Gamma)_A$ is transitive. □

LEMMA 5.4. *Let N/Γ be an infra-nil-manifold. If $f : N/\Gamma \to N/\Gamma$ is a TA-covering map, then the nonwandering set $\Omega(f)$ coincides with the entire space N/Γ.*

PROOF. By Lemma 5.3 the periodic points of σ_A are dense in $(N/\Gamma)_A$ and then we have that $\tilde{h}(\Omega(\tilde{f}))=(N/\Gamma)_A$. Indeed, if $\tilde{h}(\Omega(\tilde{f}))\neq(N/\Gamma)_A$ then $\tilde{h}(\Omega(\tilde{f}))$ is a proper compact subset of $(N/\Gamma)_A$. Hence we can find $z\in(N/\Gamma)_A$ such that $\sigma_A^r(z)=z$ for some r and $z\notin\tilde{h}(\Omega(\tilde{f}))$. Then $\bigcup_{i=1}^r \tilde{h}^{-1}(\sigma_A^i(z))$ is a non-empty compact $\tilde{f}$-invariant subset of $(N/\Gamma)_A$ that disjoints from $\Omega(\tilde{f})$, which is impossible.

Let z be a point in $(N/\Gamma)_A$ such that the orbit $\{\sigma_A^i(z) : i\in\mathbf{Z}\}$ is dense in $(N/\Gamma)_A$. By the above fact there is $x\in\Omega(\tilde{f})$ such that $\tilde{h}(x)=z$. If Ω_1 is the basic set in which x belongs, then we have that $\tilde{h}(\Omega_1)=(N/\Gamma)_A$.

By Lemma 5.1, $\tilde{h} : \tilde{h}^{-1}(\mathrm{Per}(\sigma_A))\to\mathrm{Per}(\sigma_A)$ is bijective and so $\Omega(\tilde{f})$ itself a basic set. Thus $\tilde{f}|_{\Omega(\tilde{f})}$ is topologically transitive, in which case we have $\Omega(\tilde{f}) =(N/\Gamma)_A$ because $\tilde{f}|_{\Omega(\tilde{f})}$ is a TA-homeomorphism. □

§6. Injectivity of semi-conjugacy maps 1.

The purpose of this section is to show Theorem 2(2). For the proof we need the following Lemmas.

LEMMA 6.1 ([**Re**]). *If X is a compact metric space and $f : X\to X$ is a positively expansive map, then there exist a compatible metric ρ and constants $\delta'>0$, $\lambda>1$ such that for $x, y\in X$, if $\rho(x, y)\leqq\delta'$ then $\rho(f(x), f(y))\geqq\lambda\rho(x, y)$.*

PROOF. See [**Ao-Hi**] Theorem 2.2.10. □

LEMMA 6.2. *Let X be a compact metric space with metric ρ and let $\bar{X}$ be a topological space. Let $p : \bar{X}\to X$ be a covering map. If X is locally connected, then there are a compatible metric $\bar{\rho}$ for $\bar{X}$ and a constant $\delta_0>0$ such that*

(1) *for $0<\delta\leqq\delta_0$ and $x\in\bar{X}$*

$$p : U_\delta(x) \longrightarrow U_\delta(p(x))$$

is an isometry where $U_\delta(x)=\{y\in\bar{X} : \bar{\rho}(x, y)<\delta\}$ *and* $U_\delta(p(x))=\{y\in X : \rho(p(x), y)<\delta\}$,

(2) *all covering transformations for* p *are isometries,*

(3) $\bar{X}$ *is a complete metric space with respect to* $\bar{\rho}$.

PROOF. See [**Ao-Hi**] Theorem 6.4.1. □

Let (N, D) and $(N/\Gamma, d)$ be as in § 1. Suppose $f: N/\Gamma\to N/\Gamma$ is an expanding map, and let $A: N/\Gamma\to N/\Gamma$ be the infra-nil-endomorphism homotopic to f. Then A is hyperbolic by Lemma 1.3. As before we denote as $\bar{A}: N\to N$ a lift of A by π, and as the lift map $\bar{f}: N\to N$ of f by π satisfying that $\bar{f}_*=\bar{A}_*: \Gamma\to\Gamma$.

LEMMA 6.3 ([**Co-Re**]). *If* $f: N/\Gamma\to N/\Gamma$ *is topological expanding, then there exist a constant* $\lambda>1$ *and a compatible metric* $\bar{D}$ *for* N *such that*

(1) $\bar{D}$ *is complete,*

(2) *all covering transformations for* π *are isometries under* $\bar{D}$,

(3) $\bar{D}(\bar{f}(x), \bar{f}(y))\geqq\lambda\bar{D}(x, y)$ *for* $x, y\in N$.

PROOF. Since f is positively expansive, by Lemma 6.1 there exist a compatible metric ρ for N/Γ and constants $\delta'>0$ and $\lambda>1$ such that $\rho(x, y)\leqq\delta'$ implies $\rho(f(x), f(y))\geqq\lambda\rho(x, y)$. Since $\pi: N\to N/\Gamma$ is a covering map, there exist a metric $\bar{\rho}$ for N and a constant $\delta_0>0$ satisfying the properties in Lemma 6.2. For $\delta=\min\{\delta', \delta_0\}$, Lemma 3.2 ensures the existence of $0<\delta_1<\delta$ such that $\bar{\rho}(\bar{f}(x), \bar{f}(y))<\delta_1$ implies $\bar{\rho}(x, y)<\delta$. Note that $\bar{\rho}(x, y)=\rho(\pi(x), \pi(y))$ since $\delta\leqq\delta_0$. From these facts we have that $\bar{\rho}(x, y)<\delta_1/\lambda$ if $\bar{\rho}(\bar{f}(x), \bar{f}(y))<\delta_1$.

For $x, y\in N$ let $\{x_i: 0\leqq i\leqq l+1\}$ be a δ_1-chain from x to y (i.e., $\bar{\rho}(x_i, x_{i+1})<\delta_1$ for $0\leqq i\leqq l$) and define $\bar{D}$ by

$$\bar{D}(x, y)=\inf\left\{\sum_{i=0}^{l}\bar{\rho}(x_i, x_{i+1})\right\}$$

where the infimum is taken over all finite δ_1-chains from x to y. By the triangle inequality of $\bar{\rho}$ we have $\bar{D}(x, y)\geqq\bar{\rho}(x, y)$, from $\bar{D}$ is a metric for N. Clearly $\bar{\rho}(x, y)=\bar{D}(x, y)$ if $\bar{\rho}(x, y)\leqq\delta_1$. Thus $\bar{D}$ is compatible and by Lemma 6.2(3), (1) holds. (2) is clear from the construction of $\bar{D}$ together with Lemma 6.2(2). It remains to show only (3).

Let $\{x_i: 0\leqq i\leqq l\}$ be a finite sequence from $\bar{f}(x)$ to $\bar{f}(y)$ with $\bar{\rho}(x_i, x_{i+1})<\delta_1$ for $0\leqq i\leqq l-1$. Then $\{\bar{f}^{-1}(x_0), \cdots, \bar{f}^{-1}(x_l)\}$ is a finite sequence from x to y such that

$$\bar{\rho}(\bar{f}^{-1}(x_i), \bar{f}^{-1}(x_{i+1}))<\delta_1/\lambda$$

for $0\leqq i\leqq l-1$ and thus the sequence is a δ_1-chain. Thus we have

$$\bar{\rho}(x_i, x_{i+1})=\bar{\rho}(\bar{f}\circ\bar{f}^{-1}(x_i), \bar{f}\circ\bar{f}^{-1}(x_{i+1}))\geqq\lambda\bar{\rho}(\bar{f}^{-1}(x_i), \bar{f}^{-1}(x_{i+1}))$$

and therefore $\sum\bar{\rho}(x_i, x_{i+1})\geqq\lambda\bar{D}(x, y)$, from which $\bar{D}(\bar{f}(x), \bar{f}(y))\geqq\lambda\bar{D}(x, y)$. □

LEMMA 6.4. *Under the assumptions and notations of Lemma* 6.3, *given* $K>0$ *there exists* $\delta_K>0$ *such that for any* K*-pseudo orbit* $\{x_i : i\geqq 0\}$ *of* $\bar{f}$ *there is a unique* $x\in N$ *so that* $\bar{D}(\bar{f}^i(x), x_i)\leqq\delta_K$ *for* $i\geqq 0$.

PROOF. The proof is similar to that in [**Ao-Hi**] Lemma 8.2.6. For completeness we give here the proof.

Put $x_i^0=\bar{f}^{-i}(x_i)$ for $i\geqq 0$. By Lemma 6.3(3) we have

$$\begin{aligned}\bar{D}(x_{i-1}^0, x_i^0) &= \bar{D}(\bar{f}^{-i}\circ\bar{f}(x_{i-1}), \bar{f}^{-i}(x_i))\\ &\leqq \frac{1}{\lambda^i}\bar{D}(\bar{f}(x_{i-1}), x_i) \leqq \frac{K}{\lambda^i} \quad (i\geqq 0).\end{aligned}$$

Thus $\{x_i^0\}$ is a Cauchy sequence and so there is a point x in N such that $x_i^0\to x$ as $i\to\infty$. Fix $i>0$ and let $0\leqq j<i$, then we have

$$\begin{aligned}\bar{D}(x_j, \bar{f}^j(x_i^0)) &= \bar{D}(x_j, \bar{f}^{j-i}(x_i))\\ &\leqq \bar{D}(x_j, \bar{f}^{-1}(x_{j+1}))+\cdots+\bar{D}(\bar{f}^{j-i+1}(x_{i-1}), \bar{f}^{j-i}(x_i))\\ &\leqq K(\lambda^{-1}+\cdots+\lambda^{-(i-j)})<\delta_K\end{aligned}$$

where $\delta_K=K/(\lambda-1)$. Therefore $\bar{D}(x_j, \bar{f}^j(x))<\delta_K$ for $j\geqq 0$. □

LEMMA 6.5. *Under the assumptions and notations of Lemma* 6.3, *there exists a unique continuous surjection* $\bar{k}: N\to N$ *such that*

(1) $\bar{f}\circ\bar{k}=\bar{k}\circ\bar{A}$,

(2) $\sup\{\bar{D}(\bar{k}(x), x): x\in N\}$ *is finite*,

(3) $\bar{k}$ *is uniformly continuous under* $\bar{D}$.

PROOF. Since $\bar{f}_*=\bar{A}_*$, we can take $K>0$ such that $\bar{D}(\bar{A}(x), \bar{f}(x))<K$ for all $x\in N$. Let $\delta_K>0$ be as in Lemma 6.4. For any $x\in N$ the sequence $\{\bar{A}^j(x): j\in\mathbf{Z}\}$ is a K-pseudo orbit of $\bar{f}$. Hence there is a unique $y\in N$ such that

$$\bar{D}(\bar{A}^j(x), \bar{f}^j(y))\leqq\delta_K \quad \text{for } j\in\mathbf{Z}.$$

We define a map $\bar{k}: N\to N$ by $\bar{k}(x)=y$. Since $\bar{D}(x, y)\leqq\delta_K$, obviously $\sup\{\bar{D}(\bar{k}(x), x): x\in N\}\leqq\delta_K$. Hence (2) holds. Since $\{\bar{A}^j(\bar{A}(x)): j\in\mathbf{Z}\}$ is δ_K-traced by a point $\bar{f}(y)$, we have $\bar{f}(\bar{k}(x))=\bar{f}(y)=\bar{k}(\bar{A}(x))$, from which (1) is obtained. The proof of (3) and the uniqueness of $\bar{k}$ is similar to that of Lemma 2.3. □

LEMMA 6.6. *Let* $f: N/\Gamma\to N/\Gamma$ *be topological expanding and let* $\bar{h}: N\to N$ *be the semi-conjugacy map obtained in Lemma* 2.3. *Then* $\bar{h}$ *is a homeomorphism and satisfies* $\bar{h}\circ\alpha(x)=\alpha\circ\bar{h}(x)$ *for* $x\in N$ *and* $\alpha\in\Gamma$.

PROOF. This is given in [**Ao-Hi**] Proposition 8.4.1 as follows. We already know that there exists a metric $\bar{D}$ for N such that $\bar{f}$ has the property of Lemma 6.5 and, further, a tracing property in Lemma 6.4. Let $\bar{k}: N\to N$ be a semi-conjugacy map as in Lemma 6.5. In the similar way as the proof of Lemma 2.4(3), we have $\bar{k}\circ\alpha(x)=\alpha\circ\bar{k}(x)$ for $x\in N$ and $\alpha\in\Gamma$. Thus, $\sup\{D(\bar{k}(x), x): x\in N\}$ is finite, i.e., there is $K'>0$ such that $D(\bar{k}(x), x)<K'$ for $x\in N$. From Lemma 2.3(1) and Lemma 6.5(1) it follows that

$$(\bar{h}\circ\bar{k})\circ\bar{A}=\bar{A}\circ(\bar{h}\circ\bar{k}), \quad (\bar{k}\circ\bar{h})\circ\bar{f}=\bar{f}\circ(\bar{k}\circ\bar{h}).$$

Since $D(\bar{h}(x), x)<K$ for $x\in N$, we have for all $x\in N$

$$D(\bar{h}\circ\bar{k}(x), x)<L, \quad D(\bar{k}\circ\bar{h}(x), x)<L$$

where $L=K'+K$. Lemma 6.3 implies $\bar{D}(\bar{f}^j\circ(\bar{k}\circ\bar{h})(x), \bar{f}^j(x))\to\infty$ as $j\to\infty$ when $\bar{k}\circ\bar{h}(x)\neq x$. But $D(\bar{f}^j\circ(\bar{k}\circ\bar{h})(x), \bar{f}^j(x))<L$ for $j\geqq 0$. This is impossible since D and $\bar{D}$ are uniformly equivalent. Therefore, $\bar{k}\circ\bar{h}(x)=x$, and so $\bar{k}\circ\bar{h}$ is the identity map. Similarly, $\bar{h}\circ\bar{k}$ is the identity map. □

By Lemma 6.6 $\bar{h}$ induces a homeomorphism $h: N/\Gamma\to N/\Gamma$ and $A\circ h=h\circ f$ holds on N/Γ. Therefore, Theorem 2(2) was concluded.

§7. Injectivity of semi-conjugacy maps 2.

The purpose of this section is to show Theorems 1 and 2(1).

Let (N, D) and $(N/\Gamma, d)$ be as in §1. Suppose $f: N/\Gamma\to N/\Gamma$ is a TA-covering map, and let $A: N/\Gamma\to N/\Gamma$ be the infra-nil-endomorphism homotopic to f. Then A is hyperbolic by Lemma 1.3. As before we denote as $\bar{A}: N\to N$ a lift of A by π, and as the lift map $\bar{f}: N\to N$ of f by π satisfying that $\bar{f}_*=\bar{A}_*: \Gamma\to\Gamma$. We may assume that $\bar{f}(e)=e$. Let $\bar{h}: N\to N$ the semi-conjugacy map obtained in Lemma 2.3.

Fix $\boldsymbol{u}\in N_f=\{(x_i)\in N^{\mathbf{Z}}: f(\pi(x_i))=\pi(x_{i+1}), i\in\mathbf{Z}\}$. For $x\in N$ we define a local stable and a local unstable sets by

$$\bar{W}^s_\varepsilon(x;\boldsymbol{u})=\{y\in N: D(\bar{f}^i_{\boldsymbol{u}}(x), \bar{f}^i_{\boldsymbol{u}}(y))\leqq\varepsilon,\ i\geqq 0\},$$

$$\bar{W}^u_\varepsilon(x;\boldsymbol{u})=\{y\in N: D(\bar{f}^i_{\boldsymbol{u}}(x), \bar{f}^i_{\boldsymbol{u}}(y))\leqq\varepsilon,\ i\leqq 0\}.$$

Hence $D(\bar{f}^i_{\boldsymbol{u}}(x), \bar{f}^i_{\boldsymbol{u}}(y))=D(\bar{f}^i(x), \bar{f}^i(y))$ for $i\geqq 0$, from which the local stable set $\bar{W}^s_\varepsilon(x;\boldsymbol{u})$ does not depend on the choice of $\boldsymbol{u}$. For simplicity we write

$$\bar{W}^s_\varepsilon(x)=\bar{W}^s_\varepsilon(x;\boldsymbol{u}) \quad (x\in N \text{ and } \boldsymbol{u}\in N_f).$$

For $x\in N$ define a stable and unstable sets as follows:

$$\bar{W}^s(x;\boldsymbol{u})=\{y\in N: D(\bar{f}^i_{\boldsymbol{u}}(x), \bar{f}^i_{\boldsymbol{u}}(y))\to 0\ (i\to\infty)\},$$

$$\bar{W}^u(x;\boldsymbol{u})=\{y\in N: D(\bar{f}^i_{\boldsymbol{u}}(x), \bar{f}^i_{\boldsymbol{u}}(y))\to 0\ (i\to-\infty)\}.$$

Since $\bar{W}^s(x;\boldsymbol{u})$ is independent of $\boldsymbol{u}$, we write $\bar{W}^s(x)=\bar{W}^s(x;\boldsymbol{u})$.

LEMMA 7.1. *Let* $\varepsilon>0$ *be an enough small number and let* $\bar{\sigma}: N_f\to N_f$ *be a shift map defined by* $\bar{\sigma}((x_i))=(x_{i+1})$. *Then the following hold.*

(1) *For* $\gamma>0$ *there exists* $n_\gamma>0$ *such that for* $\boldsymbol{u}\in N_f$ *and* $x\in N$

$$\bar{f}_{\boldsymbol{u}}^n(\overline{W}_\varepsilon^s(x))\subset \overline{W}_\gamma^s(\bar{f}_{\boldsymbol{u}}^n(x)),$$

$$\bar{f}_{\boldsymbol{u}}^{-n}(\overline{W}_\varepsilon^u(x\,;\boldsymbol{u}))\subset \overline{W}_\gamma^u(\bar{f}_{\boldsymbol{u}}^n(x)\,;\bar{\sigma}^{-n}(\boldsymbol{u}))$$

for all $n\geqq n_\gamma$.

(2)

$$\overline{W}^s(x)=\bigcup_{i\geqq 0}\bar{f}_{\bar{\sigma}^i(\boldsymbol{u})}^{-i}(\overline{W}_\varepsilon^s(\bar{f}_{\boldsymbol{u}}^i(x))),$$

$$\overline{W}^u(x\,;\boldsymbol{u})=\bigcup_{i\geqq 0}\bar{f}_{\bar{\sigma}^{-i}(\boldsymbol{u})}^{i}(\overline{W}_\varepsilon^u(\bar{f}_{\boldsymbol{u}}^{-i}(x)\,;\bar{\sigma}^{-i}(\boldsymbol{u}))).$$

PROOF. See [**Ao-Hi**] Lemma 6.6.3 and 6.6.4. □

Let $f: N/\Gamma\to N/\Gamma$ be a TA-covering map which is not a topological expanding map, we have the following lemma.

LEMMA 7.2 (Lifting of local product structure). *Let* $\boldsymbol{u}\in N_f$, $\varepsilon>0$ *be an enough small number and* $x\in N$. *Then there are a connected open neighborhood* $\overline{N}(x\,;\boldsymbol{u})$ *of* x *in* N *and a continuous map* $\bar{\alpha}_{\boldsymbol{u}}:\overline{N}(x\,;\boldsymbol{u})\times\overline{N}(x\,;\boldsymbol{u})\to\overline{N}(x\,;\boldsymbol{u})$ *such that*

(1) $\{\bar{\alpha}_{\boldsymbol{u}}(y,z)\}=\overline{W}_\varepsilon^u(y\,;\boldsymbol{u})\cap\overline{W}_\varepsilon^s(z)$ *for* $y, z\in\overline{N}(x\,;\boldsymbol{u})$,

(2) *for* $y, z, w\in\overline{N}(x\,;\boldsymbol{u})$

$$\bar{\alpha}_{\boldsymbol{u}}(y,y)=y,$$

$$\bar{\alpha}_{\boldsymbol{u}}(y,\bar{\alpha}_{\boldsymbol{u}}(z,w))=\bar{\alpha}_{\boldsymbol{u}}(y,w)=\bar{\alpha}_{\boldsymbol{u}}(\bar{\alpha}_{\boldsymbol{u}}(y,z),w),$$

(3) *the restriction* $\bar{\alpha}_{\boldsymbol{u}}:\bar{D}^s(x)\times\bar{D}^u(x\,;\boldsymbol{u})\to\overline{N}(x\,;\boldsymbol{u})$ *is a homeomorphism where* $\bar{D}^s(x)=\overline{W}_\varepsilon^s(x)\cap\overline{N}(x\,;\boldsymbol{u})$ *and* $\bar{D}^u(x\,;\boldsymbol{u})=\overline{W}_\varepsilon^u(x\,;\boldsymbol{u})\times\overline{N}(x\,;\boldsymbol{u})$,

(4) *there is a constant* $\rho>0$ *independent of* $x\in N$ *and* $\boldsymbol{u}\in N_f$ *such that* $\overline{N}(x\,;\boldsymbol{u})\supset\bar{B}_\rho(x)$ *where* $\bar{B}_\rho(x)=\{y\in N: D(x,y)\leqq\rho\}$,

(5) $\bar{f}_{\boldsymbol{u}}(\bar{D}^s(x))\subset\bar{D}^s(\bar{f}_{\boldsymbol{u}}(x))$ *and* $\bar{f}_{\boldsymbol{u}}(\bar{D}^u(x\,;\boldsymbol{u}))\supset\bar{D}^u(\bar{f}_{\boldsymbol{u}}(x)\,;\bar{\sigma}(\boldsymbol{u}))$,

(6) $\bar{D}^s(x)\supsetneqq\{x\}$ *and* $\bar{D}^u(x\,;\boldsymbol{u})\supsetneqq\{x\}$.

PROOF. See [**Ao-Hi**] Theorem 6.6.5. □

Let M be a connected topological manifold without boundary and let $\mathcal{F}$ be a family of subsets of M. We say that $\mathcal{F}$ is a *generalized foliation* on M if the following holds;

(1) $\mathcal{F}$ is a decomposition of M,

(2) each $L\in\mathcal{F}$, called a *leaf*, is path connected,

(3) if $x\in M$ then there exist non-trivial connected subsets D_x, K_x with $\{x\}=D_x\cap K_x$, a connected open neighborhood N_x of x, and a homeomorphism

$\varphi_x : D_x \times K_x \to N_x$, called a *local coordinate at* x, such that

(a) $\varphi_x(x, x) = x$,

(b) $\varphi_x(y, x) = y$ $(y \in D_x)$ and $\varphi_x(x, z) = z$ $(z \in K_x)$,

(c) for each $L \in \mathcal{F}$ there is an at most countable set $B \subset K_x$ such that $N_x \cap L = \varphi_x(D_x \times B)$.

LEMMA 7.3. *Let $\mathcal{F}$ be a generalized foliation on M and let U be an open subset of M. Denote by $L(x)$ the leaf through x of $\mathcal{F}$ and put*

$$V = \{x \in M : L(x) \cap U \neq \emptyset\}.$$

Then V is open in M.

PROOF. See [**Ao-Hi**] Remark 6.7.2. □

Let $\mathcal{F}$ and $\mathcal{F}'$ be generalized foliations on M. We say that $\mathcal{F}$ is *transverse* to $\mathcal{F}'$ if, for $x \in M$, there exist non-trivial connected subsets D_x, D'_x with $\{x\} = D_x \cap D'_x$, a connected open neighborhood N_x of x in M (such a neighborhood N_x is called a *coordinate domain at* x), and a homeomorphism $\phi_x : D_x \times D'_x \to N_x$ (in particular called a *canonical coordinate at* x) such that

(a) $\phi_x(x, x) = x$,

(b) $\phi_x(y, x) = y$ $(y \in D_x)$ and $\phi_x(x, z) = z$ $(z \in D'_x)$,

(c) for any $L \in \mathcal{F}$ there is an at most countable set $B' \subset D'_x$ such that $N_x \cap L = \phi_x(D_x \times B')$,

(d) for any $L' \in \mathcal{F}'$ there is an at most countable set $B \subset D_x$ such that $N_x \cap L' = \phi_x(B \times D'_x)$.

It is clear that if $\mathcal{F}$ is transverse to $\mathcal{F}'$ then $\mathcal{F}'$ is transverse to $\mathcal{F}$.

LEMMA 7.4. *Let f be as above. For $\boldsymbol{u} \in N_f$ the families $\bar{\mathcal{F}}^s = \{\bar{W}^s(x) : x \in N\}$ and $\bar{\mathcal{F}}^u_{\boldsymbol{u}} = \{\bar{W}^u(x ; \boldsymbol{u}) : x \in N\}$ are transverse generalized foliations on N.*

PROOF. See [**Ao-Hi**] Theorem 6.7.4. □

For $\boldsymbol{e} = (\cdots, e, e, e, \cdots) \in N_f$ we write

$$\bar{W}^u_\varepsilon(x) = \bar{W}^u_\varepsilon(x ; \boldsymbol{e}) \quad \text{and} \quad \bar{W}^u(x) = \bar{W}^u(x ; \boldsymbol{e}).$$

Since $\bar{A} \circ \bar{h} = \bar{h} \circ \bar{f}$ holds and $\bar{h} : N \to N$ is a D-uniformly continuous surjection, we have

$$\bar{h}(\bar{W}^s(x)) \subset \bar{L}^s(\bar{h}(x)), \quad \bar{h}(\bar{W}^u(x)) \subset \bar{L}^u(\bar{h}(x)) \quad \text{for all } x \in N. \tag{7.1}$$

Under the above assumption we can prove the following lemma.

LEMMA 7.5. *Let f and $\bar{W}^\sigma(x)$ $(x \in N,\ \sigma = s, u)$ be as above. Then $\bar{W}^u(x) \cap \bar{W}^s(y)$ is at most one point for $x, y \in N$.*

PROOF. The proof is similar to that in [**Ao-Hi**] Lemma 8.4.4. However, for completeness we give here the proof.

Let a, $b\in\overline{W}^s(x)\cap\overline{W}^u(y)$ and suppose $a\neq b$. Then there is $m>0$ such that $D(\bar{f}^{-m}(a), \bar{f}^{-m}(b))<\rho$ where ρ is as in Lemma 7.2(4). Put $a'=\bar{f}^{-m}(a)$ and $b'=\bar{f}^{-m}(b)$, and let $\varepsilon>0$ be as in Lemma 7.2. For sufficiently large m we have

$$b'\notin\overline{W}_\varepsilon^s(a'). \tag{7.2}$$

It is clear that $b'\in\overline{W}^s(a')$ since a, $b\in\overline{W}^s(x)$, and that $b'\in\bar{B}_\rho(a')\subset\bar{N}(a'; \boldsymbol{e})$ since $D(a', b')<\rho$. Hence there is $(b_1, b_2)\in\bar{D}^s(a')\times\bar{D}^u(a'; \boldsymbol{e})$ such that $b'=\bar{\alpha}_{\boldsymbol{e}}(b_1, b_2)\in\overline{W}_\varepsilon^s(b_2)$. Then we obtain $b_2\neq a'$. For, if $b_2=a'$ then $\overline{W}_\varepsilon^s(a')=\overline{W}_\varepsilon^s(b_2)\ni b'$, which is inconsistent with (7.2).

Let $U_{a'}$ and U_{b_2} be open neighborhoods of a' and b_2 in $\bar{D}^u(a'; \boldsymbol{e})$, respectively, such that $U_{a'}\cap U_{b_2}=\emptyset$, and put

$$N_{a'}=\bar{\alpha}_{\boldsymbol{e}}(\bar{D}^s(a')\times U_{a'}),\quad N_{b'}=\bar{\alpha}_{\boldsymbol{e}}(\bar{D}^s(a')\times U_{b_2}).$$

Obviously $N_{a'}$ and N_{b_2} are open neighborhoods of a' and b' in N respectively. Since $N_{a'}\cap N_{b'}=\emptyset$, we have

$$\overline{W}_\varepsilon^s(v)\cap\overline{W}_\varepsilon^s(w)=\emptyset\quad\text{for } v\in N_{a'} \text{ and } w\in N_{b'}. \tag{7.3}$$

If $V_s=\{z\in N: \overline{W}^s(z)\cap N_{b'}\neq\emptyset\}$, then V_s is open in N since $\mathcal{F}^s$ is a generalized foliation on N, and $a'\in V_s$ since $b'\in\overline{W}^s(a')$. Since Per(f) is dense in N/Γ by Lemma 5.4, there is $p\in V_s\cap N_{a'}$ such that $\pi(p)\in$Per(f). Let k be a period of $\pi(p)$ and let $\boldsymbol{u}=(u_i)\in N_f$ be a k-periodic sequence with $p=u_0$. Write $\bar{g}=\bar{f}_u^k$ for simplicity. Then $\bar{g}(p)=p$. Since $p\in V_s$, we can choose $w\in\overline{W}^s(p)\cap N_{b'}$. Since p, $w\in\bar{N}(a'; \boldsymbol{e})$, we have $\overline{W}_\varepsilon^u(p; \boldsymbol{u})\cap\overline{W}_\varepsilon^s(w)=\{q\}$ for some $q\in\bar{N}(a'; \boldsymbol{u})$. Hence $\lim_{i\to\infty}\bar{g}^i(q)=p$ since $\overline{W}^s(w)=\overline{W}^s(p)$, and $\lim_{i\to-\infty}\bar{g}^i(q)=p$. Using (7.3), we have $p\neq q$ because $p\in N_{a'}$, $q\in\overline{W}_\varepsilon^s(w)$ and $w\in N_{b'}$. Let $\mu=\min\{D(p, q), \varepsilon'\}/4$ where ε' is an expansive constant for $\bar{g}$. Then there is $0<\delta<2\mu$ such that every δ-pseudo orbit of $\bar{g}$ is μ-traced by some point of N. Choose $l>0$ such that $D(\bar{g}^{l+1}(q), p)<\delta/2$ and $D(\bar{g}^{-l}(q), p)<\delta/2$. Then the sequence

$$\{\cdots, \bar{g}^{-l}(q), \cdots, \bar{g}^{-1}(q), q, \bar{g}(q), \cdots, \bar{g}^{l}(q), \cdots\}$$

is a $(2l+1)$-periodic δ-pseudo orbit of $\bar{g}$. By using POTP and expansivity we can find $q_0\in N$ such that $\bar{g}^{2l+1}(q_0)=q_0$ and $D(q, q_0)<\mu$. It is checked that $\bar{g}^{l+1}(q_0)\neq q_0$. Indeed, if $\bar{g}^{l+1}(q_0)=q_0$ then

$$D(p, \bar{g}^{l+1}(q_0))\leqq D(p, \bar{g}^{l+1}(q))+D(\bar{g}^{l+1}(q), \bar{g}^{l+1}(q_0))<\frac{\delta}{2}+\mu<2\mu.$$

Thus we have $D(p, q)<3\mu$ which is impossible since $4\mu\leqq D(p, q)$. Therefore $\bar{g}^{2l+1}$ has at least two distinct fixed points, which contradicts Lemma 1.5. □

LEMMA 7.6 ([**Fr**]). *Let f and $\overline{W}^{\sigma}(x)$ $(x\in N,\ \sigma=s,\ u)$ be as in Lemma 7.5. Then $\overline{W}^{u}(x)\cap\overline{W}^{s}(y)$ is the set of one point for $x,\ y\in N$.*

PROOF. The proof is described in [**Ao-Hi**] Lemma 8.4.5. But we give here the proof for completeness.

Let $y_0\in N$ and put $s=\overline{W}^{s}(y_0)$. It is enough to show that $\overline{W}^{u}(x)\cap s\neq\emptyset$ for all $x\in N$. Let us put

$$Q=\{x\in N:\overline{W}^{u}(x)\cap s\neq\emptyset\},$$

then we have

$$Q=\{x\in N:\overline{W}^{u}(x)\cap U(s)\neq\emptyset\}$$

where $U(s)=\bigcup_{z\in s}\overline{N}(z\,;\boldsymbol{e})$. Indeed, choose x from the right hand set of the above equality. Then $z\in\overline{W}^{u}(x)\cap U(s)$ and hence $z\in\overline{W}^{u}(x)\cap\overline{N}(z'\,;\boldsymbol{e})$ for some $z'\in s$. Since $\overline{N}(z'\,;\boldsymbol{e})=\bar{\alpha}_{\boldsymbol{e}}(\bar{D}^{s}(z')\times\bar{D}^{u}(z'\,;\boldsymbol{e}))$, there is $(y_1,\ y_2)\in\bar{D}^{s}(z')\times\bar{D}^{u}(z'\,;\boldsymbol{e})$ such that $z=\bar{\alpha}_{\boldsymbol{e}}(y_1,\ y_2)\in\overline{W}^{u}_{\varepsilon}(y_1\,;\boldsymbol{e})$. Hence $y_1\in\overline{W}^{u}_{\varepsilon}(z\,;\boldsymbol{e})\subset\overline{W}^{u}(z\,;\boldsymbol{e})$ and on the other hand, $y_1\in\bar{D}^{s}(z')\subset s$. Therefore, $\overline{W}^{u}_{\varepsilon}(z\,;\boldsymbol{e})\cap s\neq\emptyset$ which implies $x\in Q$.

Hence Q is open in N. If $Q=N$ then the Lemma holds. Thus we suppose $Q\subsetneqq N$ and then derive a contradiction. Let $w\in Q$. If $\overline{N}(w\,;\boldsymbol{e})\not\subset Q$, then Q does not contain $\bar{D}^{s}(w)$. For $x\in\overline{N}(w\,;\boldsymbol{e})$, then there is

$$(x',\ x'')\in\bar{D}^{s}(w)\times\bar{D}^{u}(w\,;\boldsymbol{e})$$

such that $x=\bar{\alpha}_{\boldsymbol{e}}(x',\ x'')\in\overline{W}^{u}_{\boldsymbol{e}}(x'\,;\boldsymbol{e})$. If $\bar{D}^{s}(w)\subset Q$ then $\overline{W}^{u}(x'\,;\boldsymbol{e})\cap s\neq\emptyset$ since $x'\in\bar{D}^{s}(w)\subset Q$. Since $\overline{W}^{u}(x\,;\boldsymbol{e})=\overline{W}^{u}(x'\,;\boldsymbol{e})$, we have $\overline{W}^{u}(x\,;\boldsymbol{e})\cap s\neq\emptyset$ and therefore $x\in Q$, i.e., $\overline{N}(w\,;\boldsymbol{e})\subset Q$, thus contradicting.

Choose and fix $a\in\bar{D}^{s}(w)\setminus Q$. Let $\gamma:[0,\ 1]\to\bar{D}^{s}(w)$ be a path such that $\gamma(0)=w$ and $\gamma(1)=a$, and $\rho:[0,\ 1]\to\overline{W}^{u}(w\,;\boldsymbol{e})$ be a path such that $\rho(0)=w$ and $\rho(1)\in\overline{W}^{u}(w\,;\boldsymbol{e})\cap s$. We set

$$R=\{(r,\ t)\in[0,\ 1]\times[0,\ 1]:\overline{W}^{u}(\gamma(r)\,;\boldsymbol{e})\cap\overline{W}^{s}(\rho(t))\neq\emptyset\},$$

then R is not empty since $([0,\ 1]\times\{0\})\cup(\{0\}\times[0,\ 1])\subset R$ and by transversality of $\mathfrak{F}^{s}$ and $\mathfrak{F}^{u}_{\boldsymbol{e}}$, R is open in $[0,\ 1]\times[0,\ 1]$. Note that $R\subsetneqq[0,\ 1]\times[0,\ 1]$. Since $\overline{W}^{u}(\gamma(r)\,;\boldsymbol{e})\cap\overline{W}^{s}(\rho(t))$ is a single point for $(r,\ t)\in R$ (Lemma 7.5), we can define a map $\theta:R\to N$ by

$$\theta(r,\ t)=\overline{W}^{u}(\gamma(r)\,;\boldsymbol{e})\cap\overline{W}^{s}(\rho(t))\quad((r,\ t)\in R).$$

Then θ is continuous. By (7.1) we have

$$\bar{h}(\overline{W}^{u}(\gamma(r)\,;\boldsymbol{e}))\subset\bar{L}^{u}(\bar{h}\circ\gamma(r))\quad\text{and}\quad\bar{h}(\overline{W}^{s}(\rho(t)))\subset\bar{L}^{s}(\bar{h}\circ\rho(t)).$$

Then it follows that

$$\begin{aligned}\bar{h}(\theta(R)) &= \bar{h}\{\overline{W}^u(\gamma(r);\boldsymbol{e})\cap\overline{W}^s(\rho(t)):(r,t)\in R\}\\ &\subset \{\bar{h}(\overline{W}^u(\gamma(r);\boldsymbol{e}))\cap\bar{h}(\overline{W}^s(\rho(t))):(r,t)\in R\}\\ &\subset \{\bar{L}^u(\bar{h}\circ\gamma(r))\cap\bar{L}^s(\bar{h}\circ\rho(t)):(r,t)\in R\}\\ &\subset \{\bar{L}^u(\bar{h}\circ\gamma(r))\cap\bar{L}^s(\bar{h}\circ\rho(t)):(r,t)\in[0,1]\times[0,1]\}.\end{aligned}$$

Notice that the last part of the above relation is compact. Since $\bar{h}$ is proper by Lemma 2.3, we obtain that $\theta(R)$ is bounded.

Let us put

$$t_0=\sup\{\hat{t}:\overline{W}^u(\gamma(r);\boldsymbol{e})\cap\overline{W}^s(\rho(t))\neq\emptyset,\ 0\leqq r\leqq 1 \text{ and } 0\leqq t\leqq\hat{t}\},$$

$$r_0=\sup\{\hat{r}:\overline{W}^u(\gamma(r);\boldsymbol{e})\cap\overline{W}^s(\rho(t_0))\neq\emptyset,\ 0\leqq r\leqq\hat{r}\}.$$

Then $(r_0, t_0)\notin R$. Since $\theta(R)$ is bounded, we can choose a sequence

$$\{(r_n, t_n): r_n<r_{n+1},\ t_n<t_{n+1}\}\subset R$$

converging to (r_0, t_0), such that $\theta(r_n, t_n)$ converges in N. Let $\lim\theta(r_n, t_n)=v$. Take a compact neighborhoods C^s and C^u of v in $\bar{D}^s(v)$ and $\bar{D}^u(v;\boldsymbol{e})$ respectively, and let $C=\bar{\alpha}_{\boldsymbol{e}}(C^s\times C^u)$. Then C is a compact neighborhood of v in N. Since $\lim_{i\to\infty}\theta(r_n, t_n)=v$, we may assume $\theta(r_n, t_n)\in C$ for $n\geqq 1$. Then for $n\geqq 1$ there is $(u_n, v_n)\in C^s\times C^u$ such that

$$\theta(r_n, t_n)=\bar{\alpha}_{\boldsymbol{e}}(u_n, v_n)$$

and hence

$$\begin{aligned}\overline{W}^u(\gamma(r_n);\boldsymbol{e})\cap\overline{W}^s(\rho(t_n)) = \{\theta(r_n, t_n)\} &= \{\bar{\alpha}_{\boldsymbol{e}}(u_n, v_n)\}\\ &\subset \overline{W}^u(u_n;\boldsymbol{e})\cap\overline{W}^s(v_n),\end{aligned}$$

from which

$$\overline{W}^u(\gamma(r_n);\boldsymbol{e})=\overline{W}^u(u_n;\boldsymbol{e}),\quad \overline{W}^s(\rho(t_n))=\overline{W}^s(v_n).$$

Thus we have

$$\begin{aligned}\{\theta(r_1, t_n)\} &= \overline{W}^u(\gamma(r_1);\boldsymbol{e})\cap\overline{W}^s(\rho(t_n))\\ &= \overline{W}^u(u_1;\boldsymbol{e})\cap\overline{W}^s(v_n)\\ &\ni \bar{\alpha}_{\boldsymbol{e}}(u_1, v_n),\end{aligned}$$

and so $\theta(r_1, t_n)=\bar{\alpha}_{\boldsymbol{e}}(u_1, v_n)\in C$. In the similar way, $\theta(r_n, t_1)=\bar{\alpha}_{\boldsymbol{e}}(u_n, v_1)\in C$. Since θ is continuous on R and (r_1, t_n), $(r_0, t_1)\in R$, we have $\theta(r_1, t_n)\to\theta(r_1, t_0)$ and $\theta(r_n, t_1)\to\theta(r_0, t_1)$ $(n\to\infty)$. Thus $\theta(r_1, t_0)$, $\theta(r_0, t_1)\in C$, from which there are (w, z), $(\acute{w}, \acute{z})\in C^s\times C^u$ such that

$$\theta(r_1, t_0)=\bar{\alpha}_{\boldsymbol{e}}(w, z),\ \theta(r_0, t_1)=\bar{\alpha}_{\boldsymbol{e}}(\acute{w}, \acute{z}).$$

In the same fashion we have

$$\overline{W}^s(\rho(t_0))=\overline{W}^s(z),\quad \overline{W}^u(\gamma(r_0)\,;\boldsymbol{e})=\overline{W}^u(\dot{w}\,;\boldsymbol{e})$$

and hence

$$\overline{W}^u(\gamma(r_0)\,;\boldsymbol{e})\cap\overline{W}^s(\rho(t_0))=\overline{W}^u(\dot{w}\,;\boldsymbol{e})\cap\overline{W}^s(z)\ni\bar{\alpha}_e(\dot{w},\ z).$$

Therefore $(r_0,\ t_0)\in R$, thus contradicting. □

By using Lemma 7.6, define $\bar{i}: N\times N\to N$ by

$$\{\bar{i}(x,\ y)\}=\overline{W}^u(x)\cap\overline{W}^s(y)\quad\text{for }(x,\ y)\in N\times N,$$

then $\bar{i}$ satisfies the following properties; for $x,\ y,\ z\in N$

$$\begin{aligned}&\bar{i}(x,\ x)=x\,,\\ (7.4)\qquad &\bar{i}(x,\ \bar{i}(y,\ z))=\bar{i}(x,\ z)\,,\\ &\bar{i}(\bar{i}(x,\ y),\ z)=\bar{i}(x,\ z)\,.\end{aligned}$$

Define for $y\in\bar{h}^{-1}(x)$

$$I^s_{x,y}=\bar{h}^{-1}(x)\cap\overline{W}^s(y),\quad I^u_{x,y}=\bar{h}^{-1}(x)\cap\overline{W}^u(y)\,.$$

LEMMA 7.7. $\bar{i}(I^s_{x,y}\times I^u_{x,y})=\bar{h}^{-1}(x)$.

PROOF. For $v,\ w\in\bar{h}^{-1}(x)$

$$\begin{aligned}\bar{h}\circ\bar{i}(v,\ w)&=\bar{h}(\overline{W}^u(v\,;\boldsymbol{e})\cap\overline{W}^s(w))\\ &\subset\bar{L}^u(\bar{h}(v))\cap\bar{L}^s(\bar{h}(w))\quad\text{(by (7.1))}\\ &=\{x\}\end{aligned}$$

and so $\bar{i}(v,\ w)\in\bar{h}^{-1}(x)$. Since $I^\sigma_{x,y}\subset\bar{h}^{-1}(x)$ for $\sigma=s,\ u$, we have $\bar{i}(I^s_{x,y}\times I^u_{x,y})\subset\bar{h}^{-1}(x)$. Conversely, let $y\in\bar{h}^{-1}(x)$. Then for any $z\in\bar{h}^{-1}(x)$

$$\bar{i}(z,\ y)\in\bar{h}^{-1}(x)\,,\quad \bar{i}(z,\ y)\in\overline{W}^s(y)$$

from which $\bar{i}(z,\ y)\in I^s_{x,y}$. Similarly $\bar{i}(y,\ z)\in I^u_{x,y}$. Therefore

$$z=\bar{i}(\bar{i}(z,\ y),\ \bar{i}(y,\ z))\in\bar{i}(I^s_{x,y}\times I^u_{x,y})\,.$$ □

By Lemma 2.3, we have $D(\bar{h}(x),\ x)<K$ $(x\in N)$ for some $K>0$ and so $\operatorname{diam}(\bar{h}^{-1}(x))\leqq 2K$, i.e., $\bar{h}^{-1}(x)\subset\bar{B}_{2K}(y)$ for $y\in\bar{h}^{-1}(x)$ where $\bar{B}_K(y)=\{z\in N: D(z,\ y)\leqq K\}$.

LEMMA 7.8. $I^s_{x,y}\subset\bar{i}(\bar{B}_{2K}(y),\ y)$, $I^u_{x,y}\subset\bar{i}(y,\ \bar{B}_{2K}(y))$.

PROOF. By Lemma 7.7 we have

$$\begin{aligned}I^s_{x,y}&=\bar{i}(I^s_{x,y},\ y)=\bar{i}(\bar{i}(I^s_{x,y}\times I^u_{x,y}),\ y)\\ &=\bar{i}(\bar{h}^{-1}(x),\ y)\subset\bar{i}(\bar{B}_{2K}(y),\ y)\,.\end{aligned}$$

Also we obtain the same result for $\sigma=u$. □

Let us put $R_L(x)=\bar{i}(\bar{B}_L(x)\times\bar{B}_L(x))$ for $x\in N$ and $L>0$.

LEMMA 7.9. *For $L>0$ there is $L_0>0$ such that $R_L(x)\subset\bar{B}_{L_0}(x)$ for all $x\in N$.*

PROOF. Since $\overline{W}^s(x)\subset\bar{h}^{-1}(\bar{L}^s(\bar{h}(x)))$, and $\overline{W}^u(x)\subset\bar{h}^{-1}(\bar{L}^u(\bar{h}(x)))$ by (7.1), we have

$$\begin{aligned} R_L(x) &= \bar{i}(\bar{B}_L(x)\times\bar{B}_L(x)) \\ &= \bigcup_{v,w\in\bar{B}_L(x)}\overline{W}^u(v)\cap\overline{W}^s(w) \\ &\subset \bigcup_{v,w\in\bar{B}_L(x)}\bar{h}^{-1}(\bar{L}^u(\bar{h}(v)))\cap\bar{h}^{-1}(\bar{L}^s(\bar{h}(w))) \\ &= \bar{h}^{-1}\{\bigcup_{v,w\in\bar{B}_L(x)}\bar{L}^u(\bar{h}(v))\cap\bar{L}^s(\bar{h}(w))\} \\ &\subset \bar{h}^{-1}\{\bigcup_{v,w\in\bar{B}_{L+K}(x)}\bar{L}^u(v)\cap\bar{L}^s(w)\} \\ &= \bar{h}^{-1}\{\bigcup_{v,w\in\bar{B}_{L+K}(e)}\bar{L}^u(x\cdot v)\cap\bar{L}^s(x\cdot w)\} \\ &= \bar{h}^{-1}\{x\cdot(\bigcup_{v,w\in\bar{B}_{L+K}(e)}\bar{L}^u(v)\cap\bar{L}^s(w))\}. \end{aligned}$$

Since $\bigcup_{v,w\in\bar{B}_{L+K}(e)}\bar{L}^u(v)\cap\bar{L}^s(w)$ is compact, there exists $L'>0$ such that

$$R_L(x)\subset\bar{h}^{-1}(x\cdot\bar{B}_{L'}(e))=\bar{h}^{-1}(B_{L'}(x))\subset\bar{B}_{L'+K}(x)$$

Therefore $L_0=L'+K$ satisfies the above condition. □

Let $\varepsilon>0$ be an enough small number and let $x\in N$. We define for $y\in\overline{W}^s(x)$

$$D(x,\ y\ ;\ \overline{W}^s(x))=\min\{m\geqq 0: \bar{f}^m(y)\in\overline{W}^s_\varepsilon(\bar{f}^m(x))\},$$

and for $y\in\overline{W}^u(x)$

$$D(x,\ y\ ;\ \overline{W}^u(x))=\min\{m\geqq 0: \bar{f}^{-m}(y)\in\overline{W}^u_\varepsilon(\bar{f}^{-m}(x))\}.$$

Note that these are well defined by Lemma 7.1(2).

LEMMA 7.10. *For $L>0$ there exists $K_0\in\mathbf{N}$ such that for $x\in N$*
(1) *if $v\in R_L(x)$ and $w\in R_L(w)\cap\overline{W}^s(v)$, then $D(v,\ w\ ;\ \overline{W}^s(v))\leqq K_0$,*
(2) *if $v\in R_L(x)$ and $w\in R_L(x)\cap\overline{W}^u(v)$, then $D(v,\ w\ ;\ \overline{W}^u(v))\leqq K_0$.*

PROOF. The proof is given by the technique described in §8.4 Claim 4 of [**Ao-Hi**].

Let ρ be as in Lemma 7.2(4) and L_0 be as in Lemma 7.9. Then there are $l>0$ and a sequence $\{x_1, \cdots, x_l\}\subset N$ such that $\bar{B}_{L_0}(x)\subset\bigcup_1^l\bar{B}_\rho(x_i)$. Hence $R_L(x)\subset\bigcup_1^l\bar{N}(x_i\,;\,\boldsymbol{e})$ by Lemma 7.9. Let $v\in R_L(x)$ and define

$$D=R_L(x)\cap\overline{W}^s(v).$$

Then we have $D=\bar{i}(\bar{B}_L(x),\ v)$ and hence D is connected. Indeed, if

$$z \in \bar{i}(\bar{B}_L(x), v) \subset \overline{W}^s(v)$$

then $z=\bar{i}(x_1, v)$ for some $x_1 \in \bar{B}_L(x)$. Since $v \in R_L(x)$, there is $(v_1, v_2) \in \bar{B}_L(x) \times \bar{B}_L(x)$ such that $v=\bar{i}(v_1, v_2)$. Hence

$$z = \bar{i}(x_1, \bar{i}(v_1, v_2)) = \bar{i}(x_1, v_2) \in \bar{i}(\bar{B}_L(x) \times \bar{B}_L(x)) = R_L(x)$$

and so $z \in D$. Conversely, let $z \in D$. Then $z=\bar{i}(w_1, w_2)$ for some $(w_1, w_2) \in \bar{B}_L(x) \times \bar{B}_L(x)$. Since $z=\bar{i}(z, v)$, we have

$$z = \bar{i}(\bar{i}(w_1, w_2), v) \in \bar{i}(\bar{B}_L(x), v)$$

and therefore $D \subset \bar{i}(\bar{B}_L(x), v)$.

Since $R_L(x) \subset \bigcup_1^l \bar{N}(x_i; \boldsymbol{e})$, we have $D=\bigcup_1^l \bar{N}(x_i; \boldsymbol{e}) \cap D$. To avoid complication, we may suppose that each $\bar{N}(x_i; \boldsymbol{e}) \cap D$ is non-empty. Choose $y_i \in D \cap \bar{N}(x_i; \boldsymbol{e})$ for $1 \leqq i \leqq l$.

$$D \cap \bar{N}(x_i; \boldsymbol{e}) \subset \overline{W}^s_{2\varepsilon}(y_i) \quad (1 \leqq i \leqq l).$$

This is checked as follows. Since $y_i \in \bar{N}(x_i; \boldsymbol{e})$, there is $z_i \in \bar{D}^u(x_i; \boldsymbol{e})$ such that $y_i \in \overline{W}^s(z_i)$. If $y' \in D \cap \bar{N}(x_i; \boldsymbol{e})$ then we have also $y' \in \overline{W}^s_\varepsilon(z)$ for some $z \in \bar{D}^u(x_i; \boldsymbol{e})$. Since $y_i, y' \in D \subset \overline{W}^s(v)$, clearly $z_i, z \in \overline{W}^s(v)$ and so

$$z_i, z \in \bar{D}^u(x_i; \boldsymbol{e}) \cap \overline{W}^s(v) \subset \overline{W}^u(x_i; \boldsymbol{e}) \cap \overline{W}^s(v)$$

which shows $z=z_i$. Therefore $y' \in \overline{W}^s_{2\varepsilon}(y_i)$, from which

$$D \subset \bigcup_1^l \overline{W}^s_{2\varepsilon}(y_i).$$

By Lemma 7.1 there is $K_0>0$ such that

$$\bar{f}^{K_0}(\overline{W}^s_{2\varepsilon}(z)) \subset \overline{W}^s_{\varepsilon/4l}(\bar{f}^{K_0}(z))$$

for $z \in N$. Hence we have

$$\bar{f}^{K_0}(D) \subset \bigcup_1^l \bar{f}^{K_0}(\overline{W}^s_{2\varepsilon}(y_i)) \subset \bigcup_1^l \overline{W}^s_{\varepsilon/4l}(\bar{f}^{K_0}(y_i)).$$

Since D is connected, for i_1, i_2 with $1 \leqq i_1, i_2 \leqq l$ we can find a sequence $j_1 = i_1, j_2, \cdots, j_m = i_2$ such that

$$\overline{W}^s_{2\varepsilon}(y_{j_i}) \cap \overline{W}^s_{2\varepsilon}(y_{j_{i+1}}) \neq \emptyset \quad (1 \leqq i \leqq m-1).$$

By using this fact we have

$$\bar{f}^{K_0}(D) \subset \overline{W}^s_{\varepsilon/2}(\bar{f}^{K_0}(y_i))$$

and therefore $D(v, w; \overline{W}^s(v)) \leqq K_0$ for any $w \in D$. The analogous result holds for $\overline{W}^u(v; \boldsymbol{e})$. □

LEMMA 7.11. *Let $\bar{h}: N \to N$ be as above. Then $\bar{h}$ is bijective.*

PROOF. Let $v, w \in I^s_{x,y}$. If $v \neq w$ then there is $n_0 > 0$ such that $\bar{f}^{-n}(v) \notin \overline{W}^s_\varepsilon(\bar{f}^{-n}(w))$ for $n \geqq n_0$ (since $\bar{f}$ is expansive) and hence

$$D(\bar{f}^{-n}(v),\ \bar{f}^{-n}(w)\,;\ \overline{W}^s(\bar{f}^{-n}(v))) \geqq n - n_0 .$$

Let K_0 be as in Lemma 7.10 for $L = 2K$, and write

$$n_1 = n_0 + K_0 + 1, \quad v' = \bar{f}^{-n_1}(v) \quad \text{and} \quad w' = \bar{f}^{-n_1}(w).$$

Then we have $D(v', w'\,; \overline{W}^s(v')) \geqq K_0 + 1$. Since $\bar{A} \circ \bar{h} = \bar{h} \circ \bar{f}$ on N, it follows that

$$\begin{aligned} \bar{f}^{-n_1}(I^s_{x,y}) &= \bar{f}^{-n_1}(\bar{h}^{-1}(x) \cap \overline{W}^s(y)) \\ &= \bar{h}^{-1} \circ \bar{A}^{-n_1}(x) \cap \overline{W}^s(\bar{f}^{-n_1}(y)) \\ &= I^s_{x',y'} \end{aligned}$$

where $x' = \bar{A}^{-n_1}(x)$ and $y' = \bar{f}^{-n_1}(y)$. Therefore, $v', w' \in I^s_{x',y'} \subset \bar{i}(\bar{B}_{2K}(y'), y') \subset R_{2K}(y') \cap \overline{W}^s(y')$ (Lemma 7.8). Using Lemma 7.10, we have $D(v', w'\,; \overline{W}^s(v')) \leqq K_0$, thus contradicting. This shows that $I^s_{x,y}$ is a set consisting of one point. In the same fashion we have that $I^u_{x,y}$ is a single point set. Since $\bar{i}(I^s_{x,y} \times I^u_{x,y}) = \bar{h}^{-1}(x)$ by Lemma 7.7, we obtain that $\bar{h}^{-1}(x)$ is a one point set. □

LEMMA 7.12. *Let $\bar{f}$ be as above, and let $K > 0$ be the number satisfying that $D(\bar{h}, id_N) < K$. Then for $\lambda > 0$ there is $L > 0$ such that if*

$$D(\bar{f}^{-L}(x),\ \bar{f}^{-L}(y)) \leqq 3K$$

and

$$D(\bar{f}^{L}(x),\ \bar{f}^{L}(y)) \leqq 3K,$$

then $D(x, y) < \lambda$.

PROOF. By Lemma 7.10 we have that there exists $K_0 > 0$ such that for all $x \in N$

$$D(v, w\,; \overline{W}^s(v)) \leqq K_0 \ \text{ if } v \in R_{3K}(x) \text{ and } w \in R_{3K}(x) \cap \overline{W}^s(v),$$

$$D(v, w\,; \overline{W}^u(v)) \leqq K_0 \ \text{ if } v \in R_{3K}(x) \text{ and } w \in R_{3K}(x) \cap \overline{W}^u(v).$$

By Lemma 7.1(1) it follows that for $\lambda > 0$ there exists $m > 0$ such that

$$\bar{f}^m(\overline{W}^s_\varepsilon(z)) \subset \overline{W}^s_{\lambda/3}(\bar{f}^m(z)),$$

$$\bar{f}^{-m}(\overline{W}^u_\varepsilon(z)) \subset \overline{W}^u_{\lambda/3}(\bar{f}^{-m}(z)).$$

To see that $L = m + K_0$ is our requirement, suppose $D(\bar{f}^j(x), \bar{f}^j(y)) \leqq 3K$ for $j = L$ and $j = -L$. For the case $j = -L$ we have

$$\bar{i}(\bar{f}^{-L}(x),\ \bar{f}^{-L}(y)) \in R_{3K}(\bar{f}^{-L}(x)) \cap \overline{W}^s(\bar{f}^{-L}(y))$$

and thus

$$D(\bar{f}^{-L}(y),\ \bar{i}(\bar{f}^{-L}(x),\ \bar{f}^{-L}(y));\ \overline{W}^s(\bar{f}^{-L}(y))) \leqq K_0.$$

This implies that

$$\bar{i}(\bar{f}^{-m}(x),\ \bar{f}^{-m}(y)) = \bar{f}^{K_0}(\bar{i}(\bar{f}^{-L}(x),\ \bar{f}^{-L}(y))) \in \overline{W}^s_\varepsilon(\bar{f}^{-l}(y)),$$

from which

$$\bar{i}(x,\ y) = \bar{f}^m(\bar{i}(\bar{f}^{-m}(x),\ \bar{f}^{-m}(y))) \in \overline{W}^s_{\lambda/3}(x).$$

Therefore $D(\bar{i}(x, y), x)<\lambda/2$. For the case $j=L$ we have $D(\bar{i}(x, y), y)<\lambda/2$ in the same argument. □

LEMMA 7.13. *Let $\bar{h}$ be as in Lemma* 7.11. *Then $\bar{h}^{-1}$ is D-uniformly continuous.*

PROOF. For given $\lambda>0$, we have $L>0$ as in Lemma 7.12. Then by uniform continuity of $\bar{A}$, we can find $\delta>0$ such that $D(x, y)<\delta$ implies $D(\bar{A}^j(x), \bar{A}^j(y))<K$ for $j=L$ and $j=-L$. Since $\bar{h}^{-1}: N\to N$ is bijective by Lemma 7.11, using the fact that $\bar{h}^{-1}\circ\bar{A}=\bar{f}\circ\bar{h}^{-1}$ and $D(\bar{h}^{-1}(x), x)<K$ $(x\in N)$, we have that for $j=L$ and $j=-L$

$$\begin{aligned} &D(\bar{f}^j\circ\bar{h}^{-1}(x),\ \bar{f}^j\circ\bar{h}^{-1}(y)) \\ &\leqq D(\bar{h}^{-1}\circ\bar{A}^j(x),\ \bar{A}^j(x))+D(\bar{A}^j(x),\ \bar{A}^j(y))+D(\bar{A}^j(y),\ \bar{h}^{-1}\circ\bar{A}^j(y)) \\ &\leqq 3K \end{aligned}$$

and so $D(\bar{h}^{-1}(x), \bar{h}^{-1}(y))<\lambda$. Therefore $\bar{h}^{-1}$ is D-uniformly continuous. □

By Lemma 7.13, $\bar{h}^{-1}$ satisfies all condition of Lemma 2.4(1)(2)(3). Thus we can define a map $\tilde{h}^{-1}: \tau_e(N)\to\tau_e(N)$ by

$$\tilde{h}^{-1}(\tau_e(x)) = \tau_e\circ\bar{h}^{-1}(x) \quad (x\in N).$$

Then $\tilde{h}^{-1}$ is surjective (by Lemma 4.1) and it is an inverse map of $\tilde{h}$. Thus $\tilde{h}$ is a conjugacy map from $((N/\Gamma)_A, \tilde{f})$ to $((N/\Gamma)_A, \sigma_A)$. Therefore Theorem 1 is obtained by Lemma 3.10.

For the case when f is a TA-homeomorphism, we have that

$$\bar{h}(\alpha(x)) = \alpha\circ\bar{h}(x)$$

for $\alpha\in\Gamma$ by Lemma 2.4, which shows that $\bar{h}$ induces a homeomorphism $h: N/\Gamma \to N/\Gamma$. Since $\bar{A}\circ\bar{h}=\bar{h}\circ\bar{f}$ on N, we have $A\circ h=h\circ f$ on N/Γ. Theorem 2(1) was proved.

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