J. Math. Soc. Japan
Vol. 48, No. 3, 1996

On a unitary version of Suzuki's exponential product formula

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(Received Jan. 19, 1994)
(Revised Oct. 3, 1994)

1. Let $A_1, \cdots, A_q$ be bounded skew-adjoint linear operators on a separable Hilbert space $\mathcal{H}$ over $\boldsymbol{C}$. We put the sum $A=A_1+\cdots+A_q$. For an arbitrary $x\in\boldsymbol{R}$, their exponentials $e^{xA_1}, \cdots, e^{xA_q}$ and e^{xA} are unitary. For a sequence of numbers $k_1, \cdots, k_r$ for $1\leqq k_\nu\leqq q$, we consider the product

$$Q = e^{xB_1}\cdot e^{xB_2}\cdots e^{xB_r} \tag{1.1}$$

where B_ν denotes $p_\nu A_{k_\nu}$ for some $p_\nu\in\boldsymbol{R}$. The operators Q and e^{xA} have Taylor expansions as bounded operators

$$Q = \sum_{\nu_1,\cdots,\nu_r\geqq 0}\frac{B_1^{\nu_1}\cdots B_r^{\nu_r}}{\nu_1!\cdots\nu_r!}x^{\nu_1+\cdots+\nu_r} \tag{1.2}$$

$$e^{xA} = \sum_{\nu\geqq 0}\frac{A^\nu}{\nu!}x^\nu. \tag{1.3}$$

Suppose that we can choose $p_1, \cdots, p_r$ with $\sum_{\nu=1}^r p_\nu=1$ such that

$$\|Q(x)-e^{xA}\| = O|x|^{s+1} \quad \text{for } |x|<\rho \ (\rho \text{ a positive number}) \tag{1.4}$$

for some $s\in\boldsymbol{Z}_{>0}$ ($\|\ \|$ denotes the norm of vectors in $\mathcal{H}$ or bounded operators on $\mathcal{H}$).

(1.4) is equivalent to the equality

$$\sum_{\substack{0\leqq\nu_1,\cdots,\nu_r \\ \nu_1+\cdots+\nu_r\leqq s}} x^{\nu_1+\cdots+\nu_r}\frac{B_1^{\nu_1}\cdots B_r^{\nu_r}}{\nu_1!\cdots\nu_r!} = \sum_{\nu=0}^{s}\frac{x^\nu}{\nu!}A^\nu. \tag{1.5}$$

We say then $Q(x)$ is an *s-th order approximation of* e^{xA}.

We fix the above $p_1, \cdots, p_r$ and $k_1, \cdots, k_r$. Suppose now that for each $m\in\boldsymbol{Z}_{>0}$, there exist N real numbers $p_{m,1}, \cdots, p_{m,N}$ with $\sum_{j=1}^N p_{m,j}=1$ ($N=N(m)$ depends on m) such that the ordered product

$$Q^{(m)}(x) = Q(p_{m,1}x)\cdot Q(p_{m,2}x)\cdots Q(p_{m,N}x) \tag{1.6}$$

is an m-th order approximation of e^{xA}, i.e., (1.7) $$\|Q^{(m)}(x)-e^{xA}\| = O|x|^{m+1}.$$

On the background of general approximation scheme of physical systems, M. Suzuki has recently proved the following theorem as an extension of Trotter's formula (see [7]).

THEOREM OF SUZUKI. *In addition to the condition* (1.4), *suppose the following conditions are satisfied.*

(C1) $|\sum_{\nu=j}^{N} p_{m,\nu}|$ *are bounded for all* m, j, *i.e., there exists a positive number* K *such that* $|\sum_{\nu=j}^{N} p_{m,\nu}|\leqq K$.

(C2) $\lim_{m\to\infty}\sum_{j=1}^{N}|p_{m,j}|^{s+1}\to 0$.

Then for each $x\in \boldsymbol{C}$, *we have*

(1.8) $$\lim_{m\to\infty}\|Q^{(m)}(x)-e^{xA}\| = 0.$$

The convergence is uniform with respect to x *in a compact region in* $\boldsymbol{C}$.

Actually his statement is more general. He does not assume that the operators A_j are skew-adjoint.

2. The aim of this note is to prove the same result, in case where $A_1, \cdots, A_q$ are not necessarily bounded, but with additional conditions.

We denote by $\mathcal{D}(S)$ the domain of an operator S on $\mathcal{H}$.

Let $A_1, \cdots, A_q$ be skew-adjoint operators on $\mathcal{H}$ such that the intersection $\tilde{\mathcal{D}}$ of all the domains $\mathcal{D}(A_{j_1}A_{j_2}\cdots A_{j_n})$, $1\leqq j_\nu \leqq q$, is dense in $\mathcal{H}$: $\tilde{\mathcal{D}}=\bigcap_{n=1}^{\infty}\bigcap_{1\leq j_1,\cdots,j_n\leq q}\mathcal{D}(A_{j_1}A_{j_2}\cdots A_{j_n})$. Let the sum $A_1+\cdots+A_q$ be an *essentially skew adjoint operator*. We denote by A its closure which is a skew adjoint operator. By Stone's Theorem, each e^{xA_j} and e^{xA} define one parameter unitary group of operators on $\mathcal{H}$.

In the sequel we make full use of Nelson's results in [2]. First we introduce the absolute values $|A_j|$, $|A|$ of the operators A_j and A and their norms respectively as follows. For $\rho>0$, $\rho_1>0, \cdots, \rho_q>0$, we put

(2.1) $$\|e^{\rho(1+|A|)}\phi\| = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\rho^{\mu+\nu}\|A^\nu\phi\|}{\mu!\,\nu!} \quad \text{for } \phi\in\tilde{\mathcal{D}}(A)=\bigcap_{\nu=1}^{\infty}\mathcal{D}(A^\nu)$$

(2.2) $$\|e^{\rho_1|A_1|+\cdots+\rho_q|A_q|}\phi\| = \sum_{n=0}^{\infty}\sum_{1\leq j_\nu\leq q}\frac{\|A_{j_1}A_{j_2}\cdots A_{j_n}\phi\|}{n!}\rho_{j_1}\rho_{j_2}\cdots\rho_{j_n},$$

for $\phi\in\tilde{\mathcal{D}}$ respectively.

We denote by $\mathcal{H}_\rho(A)$ the set of all $\phi\in\tilde{\mathcal{D}}(A)$ such that $\|e^{\rho(1+|A|)}\phi\|<\infty$, and by $\mathcal{H}_{\rho_1,\cdots,\rho_q}$ the set of all $\phi\in\tilde{\mathcal{D}}$ such that $\|e^{\rho_1|A_1|+\cdots+\rho_q|A_q|}\phi\|<\infty$. We simply write $\mathcal{H}_{\rho_1,\cdots,\rho_q}$ by $\mathcal{H}_\rho$ if ρ_j are all equal to ρ. We put further $\mathcal{H}_0(A)=$

$\bigcup_{\rho>0}\mathcal{H}_\rho(A)$ and $\mathcal{H}_0=\bigcup_{\rho>0}\mathcal{H}_\rho=\bigcup_{\rho_1,\cdots,\rho_q>0}\mathcal{H}_{\rho_1,\cdots,\rho_q}$. ϕ in $\mathcal{H}_0(A)$ or $\mathcal{H}_0$ is called an *analytic vector* for A or the system $\{A_1,\cdots,A_q\}$. Then $\mathcal{H}_\rho\subset\mathcal{H}_\rho(A)$ and $\|e^{\rho|A|}\phi\|\leqq\|e^{\rho A}\phi\|$ for $A=|A_1|+\cdots+|A_q|$. Since A is skew adjoint, $\mathcal{H}_0(A)$ is dense in $\mathcal{H}$ (Corollary to Theorem 1 in [**2**]).

LEMMA 1. *Let* $L_j=\sum_{\nu=1}^q a_{j,\nu}A_\nu$, $a_{j,\nu}\in\boldsymbol{R}$ *and* $s_\nu=\sum_{j=1}^r t_j a_{j,\nu}$. *Suppose* $t_1,\cdots,t_r>0$. *Then*

$$\|e^{t_1|L_1|+\cdots+t_r|L_r|}\phi\| \leqq \|e^{|s_1||A_1|+\cdots+|s_q||A_q|}\phi\|. \tag{2.3}$$

In fact,

$$\begin{aligned}\|e^{t_1|L_1|+\cdots+t_r|L_r|}\phi\| &= \sum_{n=0}^{\infty}\sum_{1\leqq j_1,\cdots,j_n\leqq r}\frac{\|L_{j_1}\cdots L_{j_n}\phi\|}{n!}t_{j_1}\cdots t_{j_n}\\ &= \sum_{n=0}^{\infty}\sum_{1\leqq \nu_1,\cdots,\nu_n\leqq q}\frac{\|A_{\nu_1}A_{\nu_2}\cdots A_{\nu_n}\phi\|}{n!}|s_{\nu_1}|\cdots|s_{\nu_n}|.\end{aligned}$$

In particular we have

$$\|e^{\rho|A|}\phi\| \leqq \|e^{\rho A}\phi\| \quad \text{for } \phi\in\mathcal{H}_\rho. \tag{2.4}$$

The following is an easy consequence of the definition.

LEMMA 2. *Suppose that* $\phi\in\mathcal{H}_{\rho+|t|}$ *for* $\rho>0$. *Then* $e^{tA}\phi\in\mathcal{H}_\rho$ *and*

$$\|e^{\rho A}(e^{tA}\phi)\| \leqq \|e^{(\rho+|t|)A}\phi\|. \tag{2.5}$$

LEMMA 3. $e^{tA}\phi\in\mathcal{D}(A_{j_1}\cdots A_{j_n})$ *for* $\phi\in\mathcal{H}_\rho$ *if* $|t|<\rho$. *There exists a positive constant* C_1 *depending only on* $j_1,\cdots,j_n$ *such that*

$$\|A_{j_1}\cdots A_{j_n}e^{tA}\phi\| \leqq C_1\|e^{\rho A}\phi\|. \tag{2.6}$$

PROOF. For an arbitrary positive number δ, $\mathcal{H}_\delta\subset\mathcal{D}(A_{j_1}\cdots A_{j_n})$ and

$$\|A_{j_1}\cdots A_{j_n}\varphi\| \leqq n!\frac{\|e^{\delta A}\varphi\|}{\delta^n} \quad \text{for } \varphi\in\mathcal{H}_\delta. \tag{2.7}$$

The Lemma follows from Lemma 2 if we choose δ such that $|t|+\delta<\rho$.

We now return to the operators (1.1) and (1.3). In place of (1.4), we assume that

$$\sum_{\substack{0\leqq\nu_1,\cdots,\nu_r\\ \nu_1+\cdots+\nu_r\leqq s}} x^{\nu_1+\cdots+\nu_r}\frac{B_1^{\nu_1}\cdots B_r^{\nu_r}}{\nu_1!\cdots\nu_r!}\phi = \sum_{\nu=0}^{s} x^\nu\frac{A^\nu}{\nu!}\phi \tag{2.8}$$

for $\phi\in\mathcal{H}_0$.

For simplicity we assume that x is non-negative and that $\sum_{\nu=1}^r r|p_\nu|<\rho$. Then from Lemma 1,

$$\|e^{xB_j}\cdots e^{xB_r}\phi\| \leqq \|e^{(r-j+1)|x|(|B_j|+\cdots+|B_r|)}\phi\| \leqq \|e^{r|x|(|B_1|+\cdots+|B_r|)}\phi\| \leqq \|e^{\rho A}\phi\|. \tag{2.9}$$

$Q(x)\phi$ ($\phi\in\mathcal{H}_\rho$) has the Taylor expansion as follows

$$Q(x)\phi - \sum_{\substack{0\leq\nu_1,\cdots,\nu_r \\ \nu_1+\cdots+\nu_r\leq s}} x^{\nu_1+\cdots+\nu_r}\frac{B_1^{\nu_1}\cdots B_r^{\nu_r}}{\nu_1!\cdots\nu_r!}\phi \tag{2.10}$$

$$= \sum_{\substack{0\leq\mu_1,\cdots,\mu_r \\ \mu_1+\cdots+\mu_r=s+1}} (s+1)!\int_{0\leq y_{s+1}\leq\cdots\leq y_1\leq x} dy_1\cdots dy_{s+1}\frac{B_1^{\mu_1}e^{y_{s+1}B_1}\cdots B_r^{\mu_r}e^{y_{s+1}B_r}}{\mu_1!\cdots\mu_r!}\phi.$$

Hence for a small positive δ and a positive constant C_2

$$\|\text{The LHS of (2.10)}\| \leqq C_2|x|^{s+1}\|e^{r(\delta+|x|)(|B_1|+\cdots+|B_r|)}\phi\| \leqq C_2|x|^{s+1}\|e^{\rho A}\phi\| \tag{2.11}$$

due to Lemma 1, Lemma 3 and the inequality

$$\|B_1^{\mu_1}e^{y_{s+1}B_1}\cdots B_r^{\mu_r}e^{y_{s+1}B_r}\phi\| \leqq \frac{(s+1)!}{r^{s+1}\delta^{s+1}}\|e^{r(\delta+|x|)(|B_1|+\cdots+|B_r|)}\phi\|. \tag{2.12}$$

(2.12) is proved as follows. The LHS is majorized by

$$\sum_{0\leq\nu_1,\cdots,\nu_r}\frac{y_{s+1}^{\nu_1+\cdots+\nu_r}}{\nu_1!\cdots\nu_r!}\|B_1^{\mu_1+\nu_1}\cdots B_r^{\mu_r+\nu_r}\phi\|$$

$$\leqq \sum_{0\leq\nu_1,\cdots,\nu_r}\frac{|x|^{\nu_1+\cdots+\nu_r}}{\nu_1!\cdots\nu_r!}\|B_1^{\mu_1+\nu_1}\cdots B_r^{\mu_r+\nu_r}\phi\|$$

$$\leqq \sum_{n=s+1}^{\infty}\sum_{\substack{0\leq\nu_1,\cdots,\nu_r \\ \nu_1+\cdots+\nu_r+s+1=n}}\frac{|x|^{\nu_1+\cdots+\nu_r}}{\nu_1!\cdots\nu_r!}\sum_{1\leq j_1,\cdots,j_n\leq r}\|B_{j_1}B_{j_2}\cdots B_{j_n}\phi\|$$

(remark that $\mu_1+\cdots+\mu_r=s+1$)

$$\leqq \frac{(s+1)!}{r^{s+1}\delta^{s+1}}\sum_{n=0}^{\infty}\frac{r^n(\delta+|x|)^n}{n!}\sum_{1\leq j_1,\cdots,j_n\leq r}\|B_{j_1}B_{j_2}\cdots B_{j_n}\phi\|$$

$$\leqq \frac{(s+1)!}{r^{s+1}\delta^{s+1}}\|e^{r(\delta+|x|)(|B_1|+\cdots+|B_r|)}\phi\|,$$

since

$$\sum_{\substack{0\leq\nu_1,\cdots,\nu_r \\ \nu_1+\cdots+\nu_r+s+1=n}}\frac{r^{s+1}\delta^{s+1}|x|^{\nu_1+\cdots+\nu_r}}{(s+1)!\,\nu_1!\cdots\nu_r!} \leqq \frac{r^n(\delta+|x|)^n}{n!}.$$

In the same manner for $0\leqq x<\rho$,

$$(2.13)\qquad \|e^{xA}\phi-\sum_{\nu=0}^{s}\frac{x^{\nu}}{\nu!}A^{\nu}\phi\| = \|\int_{0\leq y_{s+1}\leq\cdots\leq y_1\leq x} dy_1\cdots dy_{s+1}A^{s+1}e^{y_{s+1}A}\phi\|$$
$$\leq C_3|x|^{s+1}\|e^{(\delta+|x|)|A|}\phi\| \leq C_3|x|^{s+1}\|e^{\rho A}\phi\| .$$

(2.11) and (2.13) hold for a negative x too.

Combining (2.8), (2.11) and (2.13) we have

$$\|Q(x)\phi-e^{xA}\phi\| \leq (C_2+C_3)|x|^{s+1}\|e^{\rho A}\phi\| \quad \text{for } \phi\in\mathcal{H}_{\rho}.$$

Hence we have proved

LEMMA 4. *Suppose that* $\sum_{\nu=j}^{r} r|p_{\nu}||x|<\rho$ *for all* j. *Then there exists a constant* C *such that*

$$(2.14)\qquad \|Q(x)\phi-e^{xA}\phi\| \leq C|x|^{s+1}\|e^{\rho A}\phi\| \quad \text{for } \phi\in\mathcal{H}_{\rho}.$$

We are now in a position to prove the following Theorem.

THEOREM 1. *In addition to the condition* (2.8), *suppose the condition* (C1), (C2) *in Theorem of Suzuki are satisfied. If* $|x|$ *is sufficiently small such that* $v|x|<\rho$ *for* $v=\sup_{\substack{1\leq m\\ 1\leq j\leq N}}|\sum_{\nu=j}^{N}p_{m,\nu}|$. *Then for all* $\phi\in\mathcal{H}_{\rho}$

$$(2.15)\qquad \lim_{m\to\infty}\|Q(p_{m,1}x)\cdots Q(p_{m,N}x)\phi-e^{xA}\phi\| = 0.$$

PROOF. The way of proof is similar to Suzuki's in [7]. But we must be careful of domains of definition of operators. First we remark $\lim_{m\to\infty}\max_{1\leq j\leq N}|p_{m,j}|=0$. Whence we may assume that there exists a $\delta>0$ such that $r\ (\sum_{\nu=1}^{r}|p_{\nu}|)|p_{m,j}||x|<\delta$ and $\delta+v|x|<\rho$. Then

$$(2.16)\qquad \|Q(p_{m,1}x)\cdots Q(p_{m,N}x)\phi-e^{xA}\phi\|$$
$$\leq \sum_{j=1}^{N}\|Q(p_{m,1}x)\cdots Q(p_{m,j-1}x)(Q(p_{m,j}x)-e^{p_{m,j}xA})e^{(p_{m,j+1}+\cdots+p_{m,N})xA}\phi\|$$
$$\leq \sum_{j=1}^{N}\|(Q(p_{m,j}x)-e^{p_{m,j}xA})e^{(p_{m,j+1}+\cdots+p_{m,N})xA}\phi\|$$
$$\leq \sum_{j=1}^{N}C|p_{m,j}x|^{s+1}\|e^{\delta A}e^{(p_{m,j+1}+\cdots+p_{m,N})xA}\phi\| \quad \text{(Lemma 4)}$$
$$\leq C\sum_{j=1}^{N}|p_{m,j}|^{s+1}|x|^{s+1}\|e^{\rho A}\phi\| \quad \text{(Lemma 2)}.$$

Hence (2.15) holds.

To obtain the limit formula (2.15) for all $x\in\boldsymbol{R}$, we use the notion of analytical dominancy due to E. Nelson.

We say that $1+|A|$ *analytically dominates* $\boldsymbol{A}=|A_1|+\cdots+|A_q|$ if the following holds.

(i) $\boldsymbol{A}\leqq 1+|A|$, i.e., $\sum_{j=1}^{q}\|A_j\phi\|\leqq c(\|\phi\|+\|A\phi\|)$, $\phi\in\mathscr{D}(A)$.

(ii) $(\mathrm{ad}\,\boldsymbol{A})^n(|A|)\leqq c_n(1+|A|)$, i.e.,

$$\|\mathrm{ad}\,A_{j_1}\,\mathrm{ad}\,A_{j_2}\cdots\mathrm{ad}\,A_{j_n}(A)\phi\|\leqq c_n(\|\phi\|+\|A\phi\|),\quad \phi\in\mathscr{D}(A)$$

where c and c_n denote positive constants such that $\omega(\rho)=\sum_{n=1}^{\infty}c_n\frac{\rho^n}{n!}$ is finite for a positive constant ρ. We put $\kappa(\rho)=\int_0^\rho\frac{dt}{1-\omega(t)}$. Then

LEMMA 5 ([2]).

$$\|e^{\rho\boldsymbol{A}}\phi\|\leqq\|e^{c\kappa(\rho)(1+|A|)}\phi\|,\quad \phi\in\tilde{\mathscr{D}}(A). \tag{2.17}$$

Under this circumstance we get the following

THEOREM 2. *In addition to the condition of Theorem* 1, *suppose that* $1+|A|$ *analytically dominates* $\boldsymbol{A}$. *Then* $\mathscr{H}_0(A)$ $(\subset\tilde{\mathscr{D}}(A)\subset\tilde{\mathscr{D}})$ *is dense in* $\mathscr{H}$ *and* (2.15) *holds for all* $\phi\in\mathscr{H}$ *and all* $x\in\boldsymbol{R}$.

PROOF. e^{xA} being a one parameter unitary group, $e^{xA}\mathscr{H}_\rho(A)\subset\mathscr{H}_\rho(A)$ for any $x\in\boldsymbol{R}$ and $\rho>0$. Suppose $\phi\in\mathscr{H}_{c\kappa(\rho)}(A)$. We use the inequality (2.16). Let $x\in\boldsymbol{R}$ be arbitrary. We denote by p_j^* the sum $\sum_{\nu=j}^{N}p_{m,j}$. Then since $\lim_{m\to\infty}p_{m,j}=0$, we have

$$\begin{aligned}&\|(Q(p_{m,j}x)-e^{p_{m,j}xA})e^{p_{j+1}^*xA}\phi\|\\ &\leqq C|p_{m,j}x|^{s+1}\|e^{\rho\boldsymbol{A}}e^{p_{j+1}^*xA}\phi\|\quad\text{(Lemma 4)}\\ &\leqq C|p_{m,j}x|^{s+1}\|e^{c\kappa(\rho)(1+|A|)}e^{p_{j+1}^*xA}\phi\|\\ &=C|p_{m,j}x|^{s+1}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}(c\kappa(\rho))^{\mu+\nu}\frac{\|A^\nu e^{p_{j+1}^*xA}\phi\|}{\mu!\,\nu!}\\ &=C|p_{m,j}x|^{s+1}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}(c\kappa(\rho))^{\mu+\nu}\frac{\|A^\nu\phi\|}{\mu!\,\nu!}\\ &=C|p_{m,j}x|^{s+1}\|e^{c\kappa(\rho)(1+|A|)}\phi\|.\end{aligned} \tag{2.18}$$

Therefore

$$\begin{aligned}&\|Q(p_{m,1}x)\cdots Q(p_{m,N}x)\phi-e^{xA}\phi\|\\ &\leqq C\sum_{j=1}^{N}|p_{m,j}x|^{s+1}\|e^{c\kappa(\rho)(1+|A|)}\phi\|.\end{aligned} \tag{2.19}$$

This gives the limit formula (2.15) for $\phi\in\mathscr{H}_0(A)=\bigcup_{\rho>0}\mathscr{H}_\rho(A)$. The limit in (2.15) is uniform for x in a compact region in $\boldsymbol{R}$. Since $\|Q(p_{m,1}x)\cdots Q(p_{m,N}x)-e^{xA}\|\leqq 2$ and $\tilde{\mathscr{D}}(A)$ is dense in $\mathscr{H}$, we have (2.15) for all $\phi\in\mathscr{H}$. Theorem has thus been proved.

It seems unknown if the Theorems still hold without the assumption of analytical dominancy although the original Trotter formula was proved without it (see [**1**]).

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