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# An example of self homotopy equivalences

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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## 1. Introduction.

Let us consider the principal fibre bundle

(1.1) $$(P, q, B, G)$$

with structure group $G$. $G$ acts on $P$ freely.

Then one can consider the space of (unbased) $G$-equivariant self-homotopy equivalences of $P$, which we denote

(1.2) $$\mathrm{aut}_G(P).$$

We define

(1.3) $$\mathcal{E}_G(P) = \pi_0(\mathrm{aut}_G(P)).$$

We call this group the group of $G$-equivariant self-equivalences of the principal fibre bundle (1.1) (cf. [**4, 5**]).

Also one can consider the space of (unbased) self-homotopy-equivalences of $P$, which we denote

(1.4) $$\mathrm{aut}(P).$$

We define

(1.5) $$\mathcal{E}(P) = \pi_0(\mathrm{aut}(P)).$$

We call this group the group of self-equivalences of the space $P$.

We have a natural homomorphism from (1.3) to (1.5) forgetting the $G$-action

(1.6) $$\mathcal{E}_G(P) = \pi_0(\mathrm{aut}_G(P)) \longrightarrow \pi_0(\mathrm{aut}(P)) = \mathcal{E}(P),$$

induced by the inclusion $\mathrm{aut}_G(P) \to \mathrm{aut}(P)$.

In [**3**, Problem 13, p. 206] the author has raised the following problem in 1988: when is the homomorphism (1.6) a monomorphism.

At this point no examples are known, where this homomorphism is not a monomorphism.

In this note we give an example where this forgetting homomorphism (1.6) is not a monomorphism.

## 2. Example.

Let $G$ be a compact connected Lie group which is not a torus, and let $T$ be a maximal torus of $G$. We have the following principal fibre bundle with structure group $G$:

$$G \longrightarrow G/T \longrightarrow BT. \tag{2.1}$$

We have the following homotopy commutative diagram

$$\begin{array}{ccc} G/T & \longrightarrow & EG \\ \downarrow & Bi & \downarrow \\ BT & \longrightarrow & BG, \end{array} \tag{2.2}$$

where $i: T \subset G$ is an inclusion.

We show that for the principal fibre bundle (2.1), the natural homomorphism

$$\mathcal{F}_G(G/T) \longrightarrow \mathcal{F}(G/T) \tag{2.3}$$

cannot be a monomorphism.

EXAMPLE 1. The natural map $\mathcal{F}_G(G/T) \to \mathcal{F}(G/T)$ is not a monomorphism.

PROOF. First we show that

$$\pi_1(\mathrm{map}(BT, BG), Bi)$$

is an uncountable group.

By D. Notbohm [**1**, pp. 156-157, 163],

$$\pi_1(\mathrm{map}(BT, BG), Bi) \cong \prod_{n \geq 1} H^n(BT, \pi_{n+1}(G) \otimes Z\hat{}/Z), \tag{2.4}$$

where $Z\hat{}$ is the completion of the integer $Z$.

Since $Z\hat{}/Z$ is a rational vector space of uncountable dimension, $\pi_1(\mathrm{map}(BT, BG), Bi)$ is an uncountable group.

We consider the Serre fibration induced by the principal fibre bundle (2.1)

$$\mathrm{aut}_G(G/T) \longrightarrow \mathrm{aut}(BT).$$

Since $\pi_1(\mathrm{aut}BT, 1)=0$, we have the following exact sequence by [**4**, Theorem 1.5], which is induced by the homotopy exact sequence of this fibration

$$0 \longrightarrow \pi_1(\mathrm{map}(BT, BG), Bi) \longrightarrow \mathcal{F}_G(G/T) \longrightarrow \mathcal{F}(BT). \tag{2.5}$$

By (2.4) $\mathcal{F}_G(G/T)$ is uncountable.

Next we consider the group $\mathcal{E}(G/T)$. By S. Papadima [2, Proposition], $\mathcal{E}(G/T)$ is a finite group.

Therefore, the map $\mathcal{E}_G(G/T)\rightarrow\mathcal{E}(G/T)$ cannot be a monomorphism.

REMARK. One may think that the map (2.3) may be a surjection. Consider the following principal fibre bundle of the form (2.1)

$$S^3 \longrightarrow S^3/S^1 \longrightarrow BS^1.$$

$\mathcal{E}_{S^3}(S^3/S^1)$ is isomorphic to the group of based $G$-equivariant self equivalences $\mathcal{E}^*_{S^3}(S^3/S^1)$, since $\pi_1(S^3/S^1)=\pi_1(S^2)=0$. Hence any $S^3$-equivariant self-equivalences on the total space $S^3/S^1=S^2$ induces an identity map on the fibre $S^3$.

Therefore

$$\mathcal{E}_{S^3}(S^2) \longrightarrow \mathcal{E}(S^2)$$

is not surjective.

## Bibliography

[1] D. Notbohm, Maps between classifying spaces, Math. Z., **207** (1991), 153-168.

[2] S. Papadima, Rigidity properties of compact Lie groups modulo maximal tori, Math. Ann., **275** (1986), 637-652.

[3] R. Piccinini, Groups of self-equivalences and related topics, Springer-Verlag, **1425**, 1990.

[4] K. Tsukiyama, Equivariant self equivalences of principal fibre bundles, Math. Proc. Cambridge Philos. Soc., **98** (1985), 87-92.

[5] H. Ōshima and K. Tsukiyama, On the group of equivariant self-equivalences of free actions, Publ. Res. Inst. Math. Sci., Kyoto Univ., **22** (1986), 905-923.

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