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Extreme points and linear isometries of the domain of a closed *-derivation in $C(K)$

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§1. Introduction.

Unbounded derivations in non-commutative C^*-algebras have been studied in detail by many authors, which are closely related to mathematical physics and especially are one of the natural frameworks for quantum dynamics in operator algebra context ([**3, 4, 5, 14, 17, 18, 21, 22**]).

In commutative C^*-algebras, unbounded derivations, which were studied systematically by Sakai in [**21**], are also very important and interesting object to study, because it plays a role of certain differential structure of underlying space ([**1, 2, 9, 11, 12, 24**]). Indeed, known examples are given by (partial) differentiation on spaces with some differential structure.

Since the differentiation d/dt on the space $C^{(1)}([0, 1])$ of continuously differentiable functions on $[0, 1]$ is a typical example of closed derivations, for any closed derivation δ in a commutative unital C^*-algebra $C(K)$ (K: a compact Hausdorff space) we may regard the domain $\mathfrak{D}(\delta)$ of δ as a generalization of the Banach space $C^{(1)}([0, 1])$. Moreover, if $\mathfrak{D}(\delta)=C(K)$, δ is bounded and hence $\delta\equiv 0$. Thus, we wish to look for unified approach to deal with $C(K)$, $C^{(1)}([0, 1])$ and several other spaces of differentiable functions together.

Properties of the domains of closed derivations (for example, functional calculus) have been investigated by several authors. $\mathfrak{D}(\delta)$ becomes Banach algebras under several graph norms. Moreover, it was shown that $\mathfrak{D}(\delta)$ with a closed *-derivation δ is a Šilov algebra by Sakai ([**22**]), and other interesting properties of $\mathfrak{D}(\delta)$ as a Banach algebra have been studied by Batty, Goodman and Tomiyama ([**1, 2, 9, 24**]). We are also interested in some interplays between properties of $\mathfrak{D}(\delta)$ as a Banach space (or a Banach algebra) and the structure of δ.

On the other hand, a well-known Banach-Stone theorem [**8**] states that surjective linear isometries of $C(K)$ are induced by homeomorphisms of K and this theorem was extended to more general case by Novinger, Okayasu and Takagaki ([**15, 16**]). Moreover, the structure of surjective linear isometries of

$C^{(1)}([0, 1])$ with various graph norms have been studied by Cambern [**6**], Cambern and Pathak [**7**], Rao and Roy [**19**]. Recently, Jarosz and Pathak [**10**] gave a general scheme to study surjective linear isometries between many classical well known spaces.

The purpose of this paper is to discuss the Banach algebra $\mathfrak{D}(\delta)$ with Σ-norm and the Banach space $\mathfrak{D}(\delta)$ with M-norm as generalizations of spaces of differentiable functions. In section 2, we present basic notations and summarize several properties of closed *-derivations which will be used later. In section 3, we determine the form of extreme points of the closed unit ball of the conjugate space $\mathfrak{D}(\delta)^*$ of $\mathfrak{D}(\delta)$ with Σ-norm. In section 4, by using the result in section 3 we study the structure of surjective linear isometries of $\mathfrak{D}(\delta)$ with Σ-norm. In section 5 and 6, we get the form of extreme points of the closed unit ball of $\mathfrak{D}(\delta)^*$ and the structure of surjective linear isometries of $\mathfrak{D}(\delta)$ with M-norm. In section 7, characterizations of extreme points of the closed unit ball of $\mathfrak{D}(\delta)$ with M-norm will be given.

To investigate further detailed properties of $\mathfrak{D}(\delta)$ as a function space, it seems necessary to analyze δ itself in detail.

§ 2. Preliminaries.

Let K be a compact Hausdorff space and $C(K)$ denotes the space of all complex valued continuous functions on K with the supremum norm $\|\cdot\|_\infty$. A derivation δ in $C(K)$ is a linear mapping in $C(K)$ satisfying the following condition:

(1) The domain $\mathfrak{D}(\delta)$ of δ is a norm dense subalgebra of $C(K)$.

(2) $\delta(fg)=\delta(f)g+f\delta(g)$ $(f, g\in\mathfrak{D}(\delta))$.

δ is said to be a *-derivation if it also satisfies:

(3) $f\in\mathfrak{D}(\delta)$ implies $f^*\in\mathfrak{D}(\delta)$ and $\delta(f^*)=\delta(f)^*$ where f^* means the complex conjugate of f.

δ is said to be closed if $f_n\in\mathfrak{D}(\delta)$, $f_n\to f$ and $\delta(f_n)\to g$ implies $f\in\mathfrak{D}(\delta)$ and $\delta(f)=g$, that is, δ is a closed linear operator.

Now, we give three examples of closed *-derivations.

(1) Let $K=I$ or $I\cup J$ where I and J are finite closed intervals of the real line with $I\cap J=\emptyset$ and let δ be the differentiation d/dt on $C^{(1)}(K)$. Then δ is a closed *-derivation in $C(K)$. If $K=I$, then the kernel of δ is $\boldsymbol{C}1$ and if $K=I\cup J$, then $\boldsymbol{C}1$ is a proper subset of the kernel of δ.

(2) Let K be a compact Hausdorff space and let δ be the partial derivative on $C([0, 1]\times K)$, that is, for $f\in C([0, 1]\times K)$ and $(s, x)\in[0, 1]\times K$

$$\delta(f)(s, x):=\lim_{t\to s}(f(t, x)-f(s, x))/(t-s),$$

whenever the limit exists. Let

$$\mathfrak{D}(\delta) := \{f \in C([0, 1] \times K) : \delta(f) \text{ is defined and continuous on } [0, 1] \times K\}.$$

Then δ is a closed *-derivation in $C([0, 1] \times K)$ and the kernel of δ and $C(K)$ are isomorphic ([**9**]).

(3) Let Φ be a non-constant generalized Cantor function and let $C^*(1, \Phi)$ be the C^*-subalgebra of $C([0, 1])$ generated by Φ and 1. Define $\delta(f+g) := (d/dt)f$ ($f \in C^{(1)}([0, 1])$ and $g \in C^*(1, \Phi)$). Then δ is a closed *-derivation in $C([0, 1])$ which is a proper extension of d/dt and the kernel of δ is the C^*-subalgebra $C^*(1, \Phi)$. The converse statement also holds ([**22**]).

Now we state several facts on closed *-derivations. $\mathfrak{D}(\delta)$ necessarily contains the constant function 1 (and hence $\boldsymbol{C}1$) and $C^{(1)}$-functional calculus is possible in $\mathfrak{D}(\delta)$ (see Proposition A below). $\mathfrak{D}(\delta)$ becomes a Banach space under the norm.

(M-norm) $$\|f\|_M := \max(\|f\|_\infty, \|\delta(f)\|_\infty) \qquad (f \in \mathfrak{D}(\delta)).$$

Moreover, $\mathfrak{D}(\delta)$ is a Banach algebra for each of the following three norms, respectively.

(Σ-norm) $$\|f\|_\Sigma := \|f\|_\infty + \|\delta(f)\|_\infty \qquad (f \in \mathfrak{D}(\delta)),$$

(c-norm) $$\|f\|_c := \sup\{|f(x)| + |\delta(f)(x)| : x \in K\} \qquad (f \in \mathfrak{D}(\delta))$$

and

(δ-norm) $$\|f\|_\delta := \sup_{t \in K} \left\| \begin{pmatrix} f(t) & \delta(f)(t) \\ 0 & f(t) \end{pmatrix} \right\| \qquad (f \in \mathfrak{D}(\delta)).$$

We summarize three important results in [**22**] which will be used later frequently. In the following three propositions, let K be a compact Hausdorff space and let δ be a closed *-derivation in $C(K)$.

PROPOSITION A ([**22**]). *For $f(=f^*) \in \mathfrak{D}(\delta)$ and $h \in C^{(1)}([-\|f\|_\infty, \|f\|_\infty])$, $h(f)$ $(=h \circ f) \in \mathfrak{D}(\delta)$ and $\delta(h(f)) = h'(f)\delta(f)$ where h' means the derivative of h.*

PROPOSITION B ([**22**]). *If $f \in \mathfrak{D}(\delta)$ is a constant in a neighborhood of $x \in K$, then $\delta(f)(x) = 0$.*

PROPOSITION C ([**22**]). *Let J_1 and J_2 be disjoint closed subsets of K. Then there is an element $f \in \mathfrak{D}(\delta)$ such that*

$$f = 0 \ \text{ on } J_1, \quad f = 1 \ \text{ on } J_2 \ \text{ and } \ 0 \leqq f \leqq 1.$$

For other properties of unbounded derivations in C^*-algebras, we refer to [**22**].

Finally, we shall state notations. For a Banach space B, B^* denotes the dual space of B. B_1 and B_1^* denote the closed unit balls of B and B^*, respectively. Moreover, $extB_1$ and $extB_1^*$ denote the sets of extreme points of B_1 and B_1^*, respectively. Let $Ker(\delta)$ be the kernel of a closed $*$-derivation δ and $\mathfrak{R}(\delta)$ the range of δ. $\boldsymbol{T}$ denotes the unit circle $\{z\in\boldsymbol{C} : |z|=1\}$ in the complex plane. For fixed point $x\in K$ we define two linear functionals η_x, $\eta_x\circ\delta$ on $\mathfrak{D}(\delta)$ by

$$\eta_x(f) := f(x) \qquad (f\in\mathfrak{D}(\delta)),$$

$$\eta_x\circ\delta(f) := \delta(f)(x) \qquad (f\in\mathfrak{D}(\delta)).$$

Let $K(\delta)$ be the set of $x\in K$ such that $\eta_x\circ\delta\neq 0$, that is,

$$\begin{aligned} K(\delta) &= \{x\in K : \eta_x\circ\delta\neq 0\} \\ &= \{x\in K : \exists f\in\mathfrak{D}(\delta) \text{ such that } \delta(f)(x)\neq 0\}. \end{aligned}$$

Then $K(\delta)$ is an open set in K.

§ 3. Extreme points of the closed unit ball of the conjugate space $\mathfrak{D}(\delta)^*$ of $\mathfrak{D}(\delta)$ with Σ-norm.

Throughout this section, let K be a compact Hausdorff space and let δ be a closed $*$-derivation in $C(K)$; the norm of $f\in\mathfrak{D}(\delta)$ is

$$\|f\|_\Sigma = \|f\|_\infty + \|\delta(f)\|_\infty .$$

We use the following lemma later frequently.

LEMMA 3.1. *For $x_0\in K(\delta)$ and a closed subset $J(\not\ni x_0)$ of K, there exists an element $f(=f^*)\in\mathfrak{D}(\delta)$ such that*

$$\delta(f)(x_0) = 1 \quad \textit{and} \quad \delta(f) = 0 \quad \textit{on } J.$$

PROOF. As $\eta_{x_0}\circ\delta\neq 0$, there exists a function $f_0(=f_0^*)\in\mathfrak{D}(\delta)$ such that $\delta(f_0)(x_0)=1$. Since K is a compact Hausdorff space, there exist an open neighborhood U_1 of x_0 and an open neighborhood U_2 of J such that $\bar{U}_1\cap\bar{U}_2=\emptyset$. Then $g_0(=g_0^*)\in\mathfrak{D}(\delta)$ such that

$$g_0 = 1 \quad \text{on } U_1 \quad \text{and} \quad g_0 = 0 \quad \text{on } U_2.$$

Then $f_0g_0\in\mathfrak{D}(\delta)$, $\delta(f_0g_0)(x_0)=\delta(f_0)(x_0)g_0(x_0)+f_0(x_0)\delta(g_0)(x_0)=1$ and $\delta(f_0g_0)(x)=0$ for $x\in J$. This completes the proof of Lemma 3.1.

The Krein-Milman theorem asserts that the closed unit ball of the conjugate space of any Banach space has sufficiently many extreme points in the sense that the unit ball is the w^*-closed convex hull of its extreme points. In this section, we get concrete expressions of extreme points of $\mathfrak{D}(\delta)_1^*$.

Let W be the compact Hausdorff space $K\times K\times\boldsymbol{T}$ with the product topology.

For $f\in\mathfrak{D}(\delta)$, we define $\tilde{f}\in C(W)$ by

$$\tilde{f}(x, x', z):=f(x)+z\delta(f)(x') \qquad ((x, x', z)\in W).$$

Then we may embed $\mathfrak{D}(\delta)$ as a closed subspace of $C(W)$.

LEMMA 3.2. *The mapping $\varphi: f\rightarrow\tilde{f}$ establishes a linear and norm-preserving correspondence between $\mathfrak{D}(\delta)$ and the closed subspace $S:=\{\tilde{f}: f\in\mathfrak{D}(\delta)\}$ of $C(W)$.*

$\eta_x+z\eta_{x'}\circ\delta$, $\eta_x\in\mathfrak{D}(\delta)^*$ have clearly norm one. The following lemma shows that $\eta_x\circ\delta(\neq 0)\in\mathfrak{D}(\delta)^*$ has also norm one.

LEMMA 3.3. *If $x_0\in K(\delta)$, the norm of $\eta_{x_0}\circ\delta$ is one.*

PROOF. For $x_0\in K(\delta)$, set $G:=\eta_{x_0}\circ\delta(\neq 0)$. Let Ψ be any norm-preserving extension of $(\varphi^{-1})^*(G)$ to $C(W)$. Then there exist a complex regular Borel measure μ on W such that $\|\mu\|=\|\Psi\|=\|\eta_{x_0}\circ\delta\|\leqq 1$ and

$$\Psi(g)=\int_W g\,d\mu \qquad (g\in C(W)).$$

Hence we have

$$\delta(f)(x_0)=\int_W\tilde{f}(x, x', z)d\mu=\int_W(f(x)+z\delta(f)(x'))d\mu$$

for all $f\in\mathfrak{D}(\delta)$. For any open neighborhood $U(\subset K)$ of x_0 we choose an open neighborhood V of x_0 such that $\bar{V}\subset U$. Then we take $g_1\in\mathfrak{D}(\delta)$ such that

$$g_1(x_0)=1, \quad g_1=0 \quad \text{on } K\setminus V \quad \text{and} \quad 0\leqq g_1\leqq 1,$$

then $g_1=\delta(g_1)=0$ on $K\setminus U$. For arbitrary $\varepsilon>0$, we take $f_\varepsilon(=f_\varepsilon^*)\in\mathfrak{D}(\delta)$ such that

$$\|\eta_{x_0}\circ\delta\|-\varepsilon\leqq|\delta(f_\varepsilon)(x_0)| \quad \text{and} \quad \|f_\varepsilon\|_\Sigma<1\,.$$

Put $c_\varepsilon:=\min((1-\|\delta(f_\varepsilon)\|_\infty)/(\|\delta(g_1)\|_\infty+1), \varepsilon)$. Then we take a function $h_\varepsilon\in C^{(1)}([-\|f_\varepsilon\|_\infty, \|f_\varepsilon\|_\infty])$ such that

$$\|h_\varepsilon\|_\infty\leqq c_\varepsilon, \quad h_\varepsilon(f_\varepsilon(x_0))=0, \quad h_\varepsilon'(f_\varepsilon(x_0))=1, \quad \text{and} \quad \|h_\varepsilon'\|_\infty=1.$$

Put $g_\varepsilon:=h_\varepsilon(f_\varepsilon)$. Then we have $\delta(g_1g_\varepsilon)=0$ on $K\setminus U$,

$$\delta(g_1g_\varepsilon)(x_0)=\delta(g_1)(x_0)g_\varepsilon(x_0)+g_1(x_0)\delta(g_\varepsilon)(x_0)=\delta(g_\varepsilon)(x_0)=\delta(f_\varepsilon)(x_0),$$

$$\|g_1g_\varepsilon\|_\Sigma=\|g_1g_\varepsilon\|_\infty+\|\delta(g_1g_\varepsilon)\|_\infty\leqq(1+\|\delta(g_1)\|_\infty)c_\varepsilon+\|\delta(f_\varepsilon)\|_\infty\leqq 1.$$

Then we have

$$\|\eta_{x_0}\circ\delta\|-\varepsilon \leqq |\delta(f_\varepsilon)(x_0)| = |\delta(g_1g_\varepsilon)(x_0)| = \left|\int_W((g_1g_\varepsilon)(x)+z\delta(g_1g_\varepsilon)(x'))d\mu\right|$$
$$\leqq \int_{K\times U\times T}(|(g_1g_\varepsilon)(x)|+|\delta(g_1g_\varepsilon)(x')|)d|\mu|+\int_{K\times(K\setminus U)\times T}|(g_1g_\varepsilon)(x)|d|\mu|$$
$$\leqq \|g_1g_\varepsilon\|_\Sigma\int_{K\times U\times T}d|\mu|+c_\varepsilon\int_{K\times(K\setminus U)\times T}d|\mu|$$
$$\leqq |\mu|(K\times U\times T)+\varepsilon|\mu|(K\times K\setminus U\times T)$$
$$\leqq \|\mu\|+\varepsilon = \|\eta_{x_0}\circ\delta\|+\varepsilon\,.$$

As $\varepsilon\to 0$, we have $|\mu|(K\times U\times T)=\|\mu\|$ and hence $supp|\mu|\subset K\times\{x_0\}\times T$. For $f\in\mathfrak{D}(\delta)$,

$$\delta(f)(x_0) = \int_{K\times\{x_0\}\times T}(f(x)+z\delta(f)(x'))d\mu$$
$$= \int_{K\times\{x_0\}\times T}f(x)d\mu+\delta(f)(x_0)\int_{K\times\{x_0\}\times T}zd\mu.$$

As $\eta_{x_0}\circ\delta\neq 0$, there exists $f_0(=f_0^*)\in\mathfrak{D}(\delta)$ such that $\delta(f_0)(x_0)=1$. For arbitrary $\varepsilon'>0$, we take a function $h_{\varepsilon'}\in C^{(1)}([-\|f_0\|_\infty, \|f_0\|_\infty])$ such that $\|h_{\varepsilon'}\|_\infty\leqq\varepsilon'$ and $h'_{\varepsilon'}(f_0(x_0))=1$. Then, $h_{\varepsilon'}(f_0)\in\mathfrak{D}(\delta)$ and

$$1 = |\delta(h_{\varepsilon'}(f_0))(x_0)|$$
$$\leqq \int_{K\times\{x_0\}\times T}|h_{\varepsilon'}(f_0)(x)|d|\mu|+|\delta(h_{\varepsilon'}(f_0))(x_0)|\int_{K\times\{x_0\}\times T}|z|d|\mu|$$
$$\leqq \varepsilon'\|\mu\|+\|\mu\| \leqq \varepsilon'+1.$$

As $\varepsilon'\to 0$, we conclude that $\|\eta_{x_0}\circ\delta\|=\|\Psi\|=\|\mu\|=1$. This completes the proof.

Now we state the main result of this section. If $\delta=0$, an element $G\in\mathfrak{D}(\delta)^*$ is an extreme point of $\mathfrak{D}(\delta)_1^*$ if and only if $G=\alpha\eta_{x_0}$ for some $x_0\in K$ and $\alpha\in T$. Hence we deal with the case that $\delta\neq 0$.

Now, we show that for $(x, x', z)\in K\times K(\delta)\times T$, the expression of $\alpha(\eta_x+z\eta_{x'}\circ\delta)$ is unique. Suppose that

$$\alpha_1(\eta_{x_1}+z_1\eta_{x_2}\circ\delta) = \alpha_2(\eta_{y_1}+z_2\eta_{y_2}\circ\delta) \quad \cdots\cdots (*)$$

for $\alpha_1, \alpha_2\in T$ and (x_1, x_2, z_1), $(y_1, y_2, z_2)\in K\times K(\delta)\times T$. As $1\in\mathfrak{D}(\delta)$, we have $\alpha_1=\alpha_2$. Suppose that $x_2\neq y_2$. Since $x_2\in K(\delta)$, there exists $f_1(=f_1^*)\in\mathfrak{D}(\delta)$ such that $\delta(f_1)(x_2)=1$ and $\delta(f_1)(y_2)=0$ from Lemma 3.1. Then we take $h_1\in C^{(1)}([-\|f_1\|_\infty, \|f_1\|_\infty])$ such that

$$h_1(f_1(x_1)) = h_1(f_1(y_1)) = 0 \quad\text{and}\quad h_1'(f_1(x_2)) = 1.$$

Put $g_1 := h_1(f_1)$; then

$$g_1(x_1) = g_1(y_1) = 0, \quad \delta(g_1)(x_2) = 1 \quad \text{and} \quad \delta(g_1)(y_2) = 0.$$

This contradicts with (*) and hence $x_2 = y_2$.

Next, suppose that $x_1 \neq y_1$; there exists $f_2(=f_2^*) \in \mathfrak{D}(\delta)$ such that $f_2(x_1)=1$ and $f_2(y_1)=0$. Then we take $h_2 \in C^{(1)}([-\|f_2\|_\infty, \|f_2\|_\infty])$ such that

$$h_2(0) = 0, \quad h_2(1) = 1, \quad \text{and} \quad h_2'(f_2(x_2)) = 0.$$

Set $g_2 := h_2(f_2) \in \mathfrak{D}(\delta)$; we have

$$g_2(x_1) = 1, \quad g_2(y_1) = 0 \quad \text{and} \quad \delta(g_2)(x_2) = 0.$$

This contradicts with (*) and hence $x_1 = y_1$. Hence the expression of $\alpha(\eta_x + z\eta_{x'} \circ \delta)$ $((x, x', z) \in K \times K(\delta) \times \boldsymbol{T})$ is unique.

THEOREM 3.4. *Let K be a compact Hausdorff space and let $\delta(\neq 0)$ be a closed *-derivation in $C(K)$. Then an element $G \in \mathfrak{D}(\delta)^*$ is an extreme point of $\mathfrak{D}(\delta)_1^*$ if and only if*

$$G = \alpha(\eta_{x_1} + z\eta_{x_2} \circ \delta) \quad \textit{for some } (x_1, x_2, z) \in K \times K(\delta) \times \boldsymbol{T} \textit{ and } \alpha \in \boldsymbol{T}.$$

PROOF. At first, we show 'only if part'. Let L be an extreme point of S_1^*, where S is as in Lemma 3.2. Then we can extend L to the extreme point of $C(W)_1^*$. We recall that any extreme point of $C(W)_1^*$ is a point evaluation multiplied by α $(\alpha \in \boldsymbol{T})$. Hence if $G \in \mathfrak{D}(\delta)^*$ is an extreme point of $\mathfrak{D}(\delta)_1^*$, then

$$G = \alpha(\eta_{x_1} + z\eta_{x_2} \circ \delta) \quad \text{for some } (x_1, x_2, z) \in W \text{ and } \alpha \in \boldsymbol{T}.$$

As $\delta \neq 0$, there exists $x_3 \in K(\delta)$. For $(x_1, x_2, z) \in K \times K \setminus K(\delta) \times \boldsymbol{T}$ and $\alpha \in \boldsymbol{T}$, we have

$$\begin{aligned} G_0 &:= \alpha(\eta_{x_1} + z\eta_{x_2} \circ \delta) \\ &= \alpha\eta_{x_1} = (\alpha/2)\{(\eta_{x_1} + \eta_{x_3} \circ \delta) + (\eta_{x_1} - \eta_{x_3} \circ \delta)\}, \end{aligned}$$

which implies that G_0 is not an extreme point.

Next, we prove the converse statement. For $(x_1, x_2, z_0) \in K \times K(\delta) \times \boldsymbol{T}$, we set $G := \eta_{x_1} + z_0\eta_{x_2} \circ \delta$. Let Ψ be any norm-preserving extension of $(\varphi^{-1})^*(G)$ to $C(W)$. There exists a regular Borel measure μ on W such that $\Psi(g) = \int_W g\,d\mu$ $(g \in C(W))$ and $\|\mu\| = \|\Psi\| = 1$. Hence we have

$$f(x_1) + z_0\delta(f)(x_2) = \int_W \tilde{f}(x, x', z)\,d\mu = \int_W (f(x) + z\delta(f)(x'))\,d\mu$$

for all $f \in \mathfrak{D}(\delta)$. Moreover, since $\Psi(\tilde{1}) = (\varphi^{-1})^*(G)(\tilde{1}) = G(1) = 1$, μ is non-negative. For any open neighborhood $U(\subset K)$ of x_2 we choose an open neighborhood V of x_2 such that $\bar{V} \subset U$. Then we take $g_1 \in \mathfrak{D}(\delta)$ such that

$$g_1(x_2) = 1, \quad g_1 = 0 \quad \text{on } K \setminus V \quad \text{and} \quad 0 \leqq g_1 \leqq 1,$$

then $g_1=\delta(g_1)=0$ on $K\setminus U$. For arbitrary $\varepsilon>0$, there exists $f_\varepsilon(=f_\varepsilon^*)\in\mathfrak{D}(\delta)$ such that

$$1-\varepsilon=\|\eta_{x_2}\circ\delta\|-\varepsilon<|\delta(f_\varepsilon)(x_2)| \quad\text{and}\quad \|f_\varepsilon\|_\Sigma<1.$$

Put $c_\varepsilon:=\min((1-\|\delta(f_\varepsilon)\|_\infty)/(\|\delta(g_1)\|_\infty+1),\ \varepsilon)$. Then we take a function $h_\varepsilon\in C^{(1)}([-\|f_\varepsilon\|_\infty,\ \|f_\varepsilon\|_\infty])$ such that

$$\|h_\varepsilon\|_\infty\leqq c_\varepsilon,\quad h_\varepsilon(f_\varepsilon(x_1))=h_\varepsilon(f_\varepsilon(x_2))=0,\quad h_\varepsilon'(f_\varepsilon(x_2))=1,\quad\text{and}\quad \|h_\varepsilon'\|_\infty=1.$$

Put $g_\varepsilon:=h_\varepsilon(f_\varepsilon)$. Then we have $(g_1g_\varepsilon)(x_1)=0$, $\delta(g_1g_\varepsilon)=0$ on $K\setminus U$,

$$\delta(g_1g_\varepsilon)(x_2)=\delta(g_1)(x_2)g_\varepsilon(x_2)+g_1(x_2)\delta(g_\varepsilon)(x_2)=\delta(g_\varepsilon)(x_2)=\delta(f_\varepsilon)(x_2),$$

and $\quad \|g_1g_\varepsilon\|_\Sigma=\|g_1g_\varepsilon\|_\infty+\|\delta(g_1g_\varepsilon)\|_\infty\leqq(1+\|\delta(g_1)\|_\infty)c_\varepsilon+\|\delta(f_\varepsilon)\|_\infty\leqq 1.$

Then

$$\begin{aligned}1-\varepsilon&\leqq|\delta(f_\varepsilon)(x_2)|=|(g_1g_\varepsilon)(x_1)+z\delta(g_1g_\varepsilon)(x_2)|\\&=\left|\int_W((g_1g_\varepsilon)(x)+z\delta(g_1g_\varepsilon)(x'))d\mu\right|\\&\leqq\int_{K\times U\times T}(|(g_1g_\varepsilon)(x)|+|\delta(g_1g_\varepsilon)(x')|)d\mu+\int_{K\times(K\setminus U)\times T}|(g_1g_\varepsilon)(x)|\,d\mu\\&\leqq\|g_1g_\varepsilon\|_\Sigma\int_{K\times U\times T}d\mu+c_\varepsilon\int_{K\times(K\setminus U)\times T}d\mu\\&\leqq\mu(K\times U\times \boldsymbol{T})+\varepsilon\mu(K\times K\setminus U\times\boldsymbol{T})\\&\leqq 1+\varepsilon.\end{aligned}$$

As $\varepsilon\to0$, we get $\mu(K\times U\times\boldsymbol{T})=1$. Since μ is a regular measure, $\mu(K\times\{x_2\}\times\boldsymbol{T})=1$, which means that the support of μ is concentrated on the set $K\times\{x_2\}\times\boldsymbol{T}$.

Next, We show that $\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T})=1$. For any open neighborhood U of x_1, we get a function $f_2(=f_2^*)\in\mathfrak{D}(\delta)$ such that

$$f_2(x_1)=1,\quad f_2=0\ \text{ on } K\setminus U \quad\text{and}\quad 0\leqq f_2\leqq 1.$$

There exists $h_2\in C^{(1)}([-\|f_2\|_\infty,\ \|f_2\|_\infty])$ such that

$$h_2(1)=1,\quad h_2(0)=0,\quad 0\leqq h_2\leqq 1\quad\text{and}\quad h_2'(f_2(x_2))=0.$$

Put $g_2:=h_2(f_2)\in\mathfrak{D}(\delta)$; then we have

$$\|g_2\|_\infty=1,\quad g_2(x_1)=1,\quad g_2=0\ \text{ on } K\setminus U \text{ and } \delta(g_2)(x_2)=0.$$

Since

$$\begin{aligned}1=g_2(x_1)+z_0\delta(g_2)(x_2)&=\left|\int_{K\times\{x_2\}\times T}g_2(x)d\mu\right|\\&\leqq\int_{U\times\{x_2\}\times T}|g_2(x)|\,d\mu+\int_{(K\setminus U)\times\{x_2\}\times T}|g_2(x)|\,d\mu\\&\leqq\mu(U\times\{x_2\}\times\boldsymbol{T})\leqq 1,\end{aligned}$$

we have $\mu(U\times\{x_2\}\times\boldsymbol{T})=1$. Since μ is a regular measure, we have $\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T})=1$. Hence for $f\in\mathfrak{D}(\delta)$

$$f(x_1)+z_0\delta(f)(x_2)=\int_{\{x_1\}\times\{x_2\}\times T}f(x)d\mu+\int_{\{x_1\}\times\{x_2\}\times T}z\delta(f)(x')d\mu$$

$$=f(x_1)+\delta(f)(x_2)\int_{\{x_1\}\times\{x_2\}\times T}zd\mu\,.$$

As $\eta_{x_2}\circ\delta\neq 0$, we have $z_0=\int_{\{x_1\}\times\{x_2\}\times T}zd\mu$. Put $h(x_1,\,x_2,\,z):=z$ for $(x_1,\,x_2,\,z)\in W$. Then

$$z_0=z_0\mu(\{x_1\}\times\{x_2\}\times\{z_0\})+\int_{\{x_1\}\times\{x_2\}\times T\setminus\{z_0\}}hd\mu,$$

that is,

$$z_0\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T}\setminus\{z_0\})=\int_{\{x_1\}\times\{x_2\}\times T\setminus\{z_0\}}hd\mu.$$

Therefore we have

$$\int_{\{x_1\}\times\{x_2\}\times T\setminus\{z_0\}}|h|\,d\mu=\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T}\setminus\{z_0\})=\left|\int_{\{x_1\}\times\{x_2\}\times T\setminus\{z_0\}}h\,d\mu\right|.$$

Thus there exists $a\in\boldsymbol{C}$ such that

$$h=a\,|h|=a\quad(a.e.\ \mu).$$

Hence we get

$$z_0\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T}\setminus\{z_0\})=\int_{\{x_1\}\times\{x_2\}\times T\setminus\{z_0\}}a\,d\mu=a\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T}\setminus\{z_0\})\,.$$

Suppose that $\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T}\setminus\{z_0\})>0$; then $z_0=a$, that is, $h(x_1,\,x_2,\,z)=z_0(a.e.\ \mu)$, which implies $\mu(\{x_1\}\times\{x_2\}\times\boldsymbol{T}\setminus\{z_0\})=0$. This is a contradiction and hence we get $\mu(\{x_1\}\times\{x_2\}\times\{z_0\})=1$. Since μ is the dirac measure at $(x_1,\,x_2,\,z_0)$, any norm-preserving extension of $(\varphi^{-1})^*(G)$ is an extreme point of $C(W)_1^*$. Let

$$(\varphi^{-1})^*(G):=(1/2)(F_1+F_2)\qquad(F_1,\,F_2\in S_1^*)\,.$$

Let $\tilde{F}_1$ and $\tilde{F}_2$ be any norm-preserving extensions of F_1 and F_2 to $C(W)$, respectively. Since $(1/2)(\tilde{F}_1+\tilde{F}_2)$ is a norm-preserving extension of $(\varphi^{-1})^*(G)$, $\tilde{F}_1=\tilde{F}_2$. Hence we have $F_1=F_2$, that is, $(\varphi^{-1})^*(G)$ is an extreme point of S_1^*, which implies that $G=\eta_{x_1}+z_0\eta_{x_2}\circ\delta$ is an extreme point of $\mathfrak{D}(\delta)_1^*$. This completes the proof.

We remark that if $1\in\mathfrak{N}(\delta)$, then $K(\delta)=K$.

§4. Linear isometries between $\mathfrak{D}(\delta_1)$ and $\mathfrak{D}(\delta_2)$ with Σ-norm.

Rao and Roy ([19]) investigated the structure of surjective linear isometries of $C^{(1)}([0, 1])$ with Σ-norm.

In this section we use the results in section 3 to get the structure theorem of surjective linear isometries of $\mathfrak{D}(\delta)$ with the Σ-norm as one of generalizations of the result by Rao and Roy.

Let K_i be a compact Hausdorff space and let δ_i be a closed *-derivation in $C(K_i)$ $(i=1, 2)$. Let T be a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$. Then we have the following two lemmas.

LEMMA 4.1. *For all* $y \in K_2$

$$|T1(y)| = 1 \quad \textit{and} \quad \delta_2(T1)(y) = 0\,.$$

PROOF. If $\delta_i=0$ $(i=1, 2)$, it is clear. Suppose that $\delta_i \neq 0$ $(i=1, 2)$. Since T^* carries $ext\mathfrak{D}(\delta_2)_1^*$ onto $ext\mathfrak{D}(\delta_1)_1^*$, for $y_1 \in K_2$, $y_2 \in K_2(\delta_2)$ and $z \in \boldsymbol{T}$ there exist $x_1 \in K_1$, $x_2 \in K_1(\delta_1)$, $z' \in \boldsymbol{T}$ and $\alpha \in \boldsymbol{T}$ such that

$$T^*(\eta_{y_1} + z\eta_{y_2} \circ \delta_2) = \alpha(\eta_{x_1} + z'\eta_{x_2} \circ \delta_1),$$

which implies

$$|T1(y_1) + z\delta_2(T1)(y_2)| = 1\,.$$

Hence for arbitrary $y_1 \in K_2$ and $y_2 \in K_2(\delta_2)$,

$$|T1(y_1)| + |\delta_2(T1)(y_2)| = 1 \quad \text{or} \quad ||T1(y_1)| - |\delta_2(T1)(y_2)|| = 1,$$

which implies that $|T1(y_1)|=1$ or $T1(y_1)=0$. Suppose $T1 \equiv 0$; then $\delta_2(T1) \equiv 0$, which is a contradiction. Hence there exists $y \in K_2$ such that $|T1(y)|=1$, that is, $\|T1\|_\infty=1$. Since

$$1 = \|1\|_\Sigma = \|T1\|_\Sigma = \|T1\|_\infty + \|\delta_2(T1)\|_\infty = 1 + \|\delta_2(T1)\|_\infty\,,$$

we see $\delta_2(T1) \equiv 0$. Hence $|T1(y)|=1$ and $\delta_2(T1)(y)=0$ for all $y \in K_2$. Similarly, we observe that it holds in the case that $\delta_1=0$ and $\delta_2 \neq 0$. Considering of T^{-1}, we can conclude that it holds in the case that $\delta_1=0$ and $\delta_2 \neq 0$. This completes the proof.

If $\delta_i=0$ $(i=1, 2)$, for $y_0 \in K_2$ there exist $x_0 \in K_1$ and $\alpha_0 \in \boldsymbol{T}$ such that $T^*(\eta_{y_0}) = \alpha_0 \eta_{x_0}$. From Theorem 3.4, if $\delta_i \neq 0$ $(i=1, 2)$, for $y_1 \in K_2$ and $y_2 \in K_2(\delta_2)$ there exist $x_1 \in K_1$, $x_2 \in K_1(\delta_1)$, $z_1 \in \boldsymbol{T}$ and $\alpha_1 \in \boldsymbol{T}$ such that

$$T^*(\eta_{y_1} + \eta_{y_2} \circ \delta_2) = \alpha_1(\eta_{x_1} + z_1 \eta_{x_2} \circ \delta_1)\,.$$

LEMMA 4.2. *In the above situation,*

$$T^*(\eta_{y_1}) = \alpha_1 \eta_{x_1} \quad \textit{and} \quad T^*(\eta_{y_2} \circ \delta_2) = \alpha_1(z_1 \eta_{x_2} \circ \delta_1)\,.$$

PROOF. For $y_1 \in K_2$, $y_2 \in K_2(\delta_2)$, there exist $x_1 \in K_1$, $x_2 \in K_1(\delta_1)$, $z_1 \in \boldsymbol{T}$ and $\alpha_1 \in \boldsymbol{T}$ such that

$$T^*(\eta_{y_1}+\eta_{y_2}\circ\delta_2) = \alpha_1(\eta_{x_1}+z_1\eta_{x_2}\circ\delta_1).$$

Put $F_1 := T^*(\eta_{y_1})$ and $F_2 := T^*(\eta_{y_2}\circ\delta_2)$. Since $F_1-F_2 \in ext\mathfrak{D}(\delta_1)_1^*$, there exist $x_3 \in K_1$, $x_4 \in K_1(\delta_1)$, $z_2 \in \boldsymbol{T}$ and $\alpha_2 \in \boldsymbol{T}$ such that

$$F_1-F_2 = \alpha_2(\eta_{x_3}+z_2\eta_{x_4}\circ\delta_1).$$

Since $F_2(1)=0$ from Lemma 4.1, we see $\alpha_1=\alpha_2=F_1(1)$. Thus we have

$$F_1 = (\alpha_1/2)\{(\eta_{x_1}+z_1\eta_{x_2}\circ\delta_1)+(\eta_{x_3}+z_2\eta_{x_4}\circ\delta_1)\},$$

and

$$F_2 = (\alpha_1/2)\{(\eta_{x_1}+z_1\eta_{x_2}\circ\delta_1)-(\eta_{x_3}+z_2\eta_{x_4}\circ\delta_1)\}.$$

Since $F_1+iF_2 \in ext\mathfrak{D}(\delta_1)_1^*$, there exist $x_5 \in K_1$, $x_6 \in K_1(\delta_1)$, $z_3 \in \boldsymbol{T}$ and $\alpha_3 \in \boldsymbol{T}$ such that

$$\begin{aligned} F_1+iF_2 &= \alpha_3(\eta_{x_5}+z_3\eta_{x_5}\circ\delta_1) := A \\ &= (\alpha_1/2)\{(1+i)(\eta_{x_1}+z_1\eta_{x_2}\circ\delta_1)+(1-i)(\eta_{x_3}+z_2\eta_{x_4}\circ\delta_1)\} := B. \end{aligned}$$

Now, we show that $x_6=x_2$ and $x_6=x_4$, that is, $x_2=x_4$. Suppose that $x_6 \neq x_2$ and $x_6=x_4$, then from $x_6 \in K_1(\delta_1)$ and Lemma 3.1 there exists $f_1(=f_1^*) \in \mathfrak{D}(\delta_1)$ such that $\delta_1(f_1)(x_6)=1$ and $\delta_1(f_1)(x_2)=0$. Then we take $h_1 \in C^{(1)}([-\|f_1\|_\infty, \|f_1\|_\infty])$ such that

$$h_1(f_1(x_1)) = h_1(f_1(x_3)) = h_1(f_1(x_5)) = 0 \quad \text{and} \quad h_1'(f_1(x_6)) = 1.$$

Put $g_1 := h_1(f_1)$; then

$$g_1(x_1) = g_1(x_3) = g_1(x_5) = 0, \quad \delta_1(g_1)(x_2) = 0 \quad \text{and} \quad \delta_1(g_1)(x_6) = 1.$$

Then $|A(g_1)|=1$, but $|B(g_1)|=\sqrt{2}/2$. This is a contradiction and hence this case does not occur. Similarly, the case that $x_6=x_2$ and $x_6 \neq x_4$ does not occur. Suppose that $x_6 \neq x_2$ and $x_6 \neq x_4$. Since $x_6 \in K_1(\delta_1)$, there exists $f_2(=f_2^*) \in \mathfrak{D}(\delta_1)$ such that

$$\delta_1(f_2)(x_6) = 1 \quad \text{and} \quad \delta_1(f_2)(x_2) = \delta_1(f_2)(x_4) = 0$$

from Lemma 3.1. Then we take $h_2 \in C^{(1)}([-\|f_2\|_\infty, \|f_2\|_\infty])$ such that

$$h_2(f_2(x_1)) = h_2(f_2(x_3)) = h_2(f_2(x_5)) = 0, \quad \text{and} \quad h_2'(f_2(x_6)) = 1.$$

Put $g_2 := h_2(f_2)$; then

$$g_2(x_1)=g_2(x_3)=g_2(x_5)=0, \quad \delta_1(g_2)(x_2)=\delta_1(g_2)(x_4)=0 \quad \text{and} \quad \delta_1(g_2)(x_6)=1.$$

Then $|A(g_2)|=1$, but $|B(g_2)|=0$. This is a contradiction and thus this case does not occur, too. Hence we get that $x_6=x_2$ and $x_6=x_4$, that is, $x_4=x_2$.

Next, we show that $x_5=x_1$ and $x_5=x_3$, that is, $x_1=x_3$. Suppose that $x_5 \neq x_1$

and $x_5=x_3$, there exists $f_3(=f_3^*)\in\mathfrak{D}(\delta_1)$ such that $f_3(x_5)=1$ and $f_3(x_1)=0$. Then we take $h_3\in C^{(1)}([-\|f_3\|_\infty, \|f_3\|_\infty])$ such that

$$h_3(0)=0, \quad h_3(1)=1, \quad \text{and} \quad h_3'(f_3(x_2))=0.$$

Set $g_3:=h_3(f_3)\in\mathfrak{D}(\delta_1)$; we have

$$g_3(x_1)=0, \quad g_3(x_5)=1 \quad \text{and} \quad \delta_1(g_3)(x_2)=0.$$

Then $|A(g_3)|=1$, but $|B(g_3)|=\sqrt{2}/2$. This is a contradiction and hence this case does not occur. Similarly, the case that $x_5=x_1$ and $x_5\neq x_3$ does not occur. Suppose that $x_5\neq x_1$ and $x_5\neq x_3$. Then we can take $g_4\in\mathfrak{D}(\delta_1)$ such that

$$g_4(x_1)=0, \quad g_4(x_5)=1 \quad \text{and} \quad \delta_1(g_4)(x_2)=0$$

and $g_5\in\mathfrak{D}(\delta_1)$ such that

$$g_5(x_3)=0, \quad g_5(x_5)=1 \quad \text{and} \quad \delta_1(g_5)(x_2)=0$$

as a function g_3 above. Then $g_4g_5\in\mathfrak{D}(\delta_1)$ and $|A(g_4g_5)|=1$, but $B(g_4g_5)=0$. This is a contradiction and thus this case does not occur, too. Hence we get that $x_5=x_1$ and $x_5=x_3$, that is, $x_1=x_3$. Hence we obtain

$$F_1=\alpha_1\eta_{x_1}+(\alpha_1/2)(z_1+z_2)(\eta_{x_2}\circ\delta_1)$$

$$F_2=(\alpha_1/2)(z_1-z_2)(\eta_{x_2}\circ\delta_1).$$

Since $\|\eta_y\circ\delta_2\|=1$ $(y\in K_2(\delta_2))$ from Lemma 3.2 and T^* is a linear isometry,

$$1=\|T^*(\eta_{y_2}\circ\delta_2)\|=\|F_2\|=(1/2)|z_1-z_2|\,\|\eta_{x_2}\circ\delta_1\|=(1/2)|z_1-z_2|$$

and hence $z_2=-z_1$. Consequently, we have

$$T^*(\eta_{y_1})=\alpha_1\eta_{x_1} \quad \text{and} \quad T^*(\eta_{y_2}\circ\delta_2)=\alpha_1(z_1\eta_{x_2}\circ\delta_1).$$

Thus the proof is completed.

REMARK 4.3. The same argument as in the above proof implies that if there exists a surjective linear isometry between $\mathfrak{D}(\delta_1)$ and $\mathfrak{D}(\delta_2)$, then $\delta_i=0$ $(i=1, 2)$ or $\delta_i\neq 0$ $(i=1, 2)$.

From this lemma, we get the following theorem.

THEOREM 4.4. *Let K_i be a compact Hausdorff space and let δ_i be a closed $*$-derivation in $C(K_i)$ $(i=1, 2)$.*

(1) *Let T be a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$. Then the following statements are held.*

(i) *There exist a homeomorphism τ from K_2 to K_1, $w_1\in Ker(\delta_2)$ and a continuous function w_2 on $K_2(\delta_2)$ such that $\tau(K_2(\delta_2))=K_1(\delta_1)$, $|w_1(y)|=1$ for all $y\in K_2$,*

$|w_2(y)|=1$ for all $y\in K_2(\delta_2)$,

$$(Tf)(y)=w_1(y)f(\tau(y)) \qquad \textit{for } f\in\mathfrak{D}(\delta_1) \textit{ and } y\in K_2,$$

$$\delta_2(Tf)(y)=w_2(y)\delta_1(f)(\tau(y)) \quad \textit{for } f\in\mathfrak{D}(\delta_1) \textit{ and } y\in K_2(\delta_2).$$

(ii) $T(Ker(\delta_1))=Ker(\delta_2)$.

(2) *Suppose that there exist* $w\in Ker(\delta_2)$ *and a homeomorphism* τ *from* K_2 *to* K_1 *such that* $|w(y)|=1$ *for all* $y\in K_2$, $\tau(K_2(\delta_2))=K_1(\delta_1)$, $f\circ\tau\in\mathfrak{D}(\delta_2)$ *for all* $f\in\mathfrak{D}(\delta_1)$, $g\circ\tau^{-1}\in\mathfrak{D}(\delta_1)$ *for all* $g\in\mathfrak{D}(\delta_2)$, *and* $|\delta_2(f\circ\tau)(y)|=|\delta_1(f)(\tau(y))|$ *for all* $f\in\mathfrak{D}(\delta_1)$ *and all* $y\in K_2(\delta_2)$. *Then the operator* T *from* $\mathfrak{D}(\delta_1)$ *to* $\mathfrak{D}(\delta_2)$ *defined by*

$$(Tf)(y):=w(y)f(\tau(y)) \qquad \textit{for } f\in\mathfrak{D}(\delta_1) \textit{ and } y\in K_2$$

is a surjective linear isometry.

PROOF. At first, we prove the statement (i). We may assume that $\delta_i\neq 0$ ($i=1, 2$). From Lemma 4.2, for each point $y\in K_2$, there exist $x\in K_1$ and $\alpha\in\boldsymbol{T}$ such that

$$T^*\eta_y=\alpha\eta_x\,.$$

Defining τ and w_1 by $T^*\eta_y:=w_1(y)\eta_{\tau(y)}$, then $w_1=T1\in Ker(\delta_2)$ from Lemma 4.1 and τ is a homeomorphism from K_2 to K_1. We note that $f\circ\tau=w_1^*(Tf)\in\mathfrak{D}(\delta_2)$ for $f\in\mathfrak{D}(\delta_1)$.

From Lemma 4.2, for each point $y\in K_2(\delta_2)$ there exist $x\in K_1(\delta_1)$ and $\alpha\in\boldsymbol{T}$ such that

$$T^*(\eta_y\circ\delta_2)=\alpha(\eta_x\circ\delta_1)\neq 0\,.$$

Defining τ_0 and w_2 by $T^*(\eta_y\circ\delta_2):=w_2(y)\eta_{\tau_0(y)}\circ\delta_1$, then τ_0 is a mapping from $K_2(\delta_2)$ to $K_1(\delta_1)$.

Next we show that $\tau=\tau_0$ on $K_2(\delta_2)$. Since $w_1\in Ker(\delta_2)$,

$$\begin{aligned} w_1(y)\delta_2(f\circ\tau)(y) &= \delta_2(w_1f\circ\tau)(y)=\delta_2(Tf)(y) \\ &= w_2(y)\delta_1(f)(\tau_0(y)) \qquad \cdots\cdots (*) \end{aligned}$$

for all $f\in\mathfrak{D}(\delta_1)$ and all $y\in K_2(\delta_2)$. Suppose that there exists $y_0\in K_2(\delta_2)$ such that $\tau(y_0)\neq\tau_0(y_0)$. Then there exist $f_1\in\mathfrak{D}(\delta_1)$ such that $\delta_1(f_1)(\tau_0(y_0))=1$ and $\delta_1(f_1)(\tau(y_0))=0$ from Lemma 3.1. Since K_1 is a compact Hausdorff space, there exist an open neighborhood $U_1(\subset K_1)$ of $\tau_0(y_0)$ and an open neighborhood $U_2(\subset K_1)$ of $\tau(y_0)$ such that $\bar{U}_1\cap\bar{U}_2=\emptyset$. Then we take $g_1\in\mathfrak{D}(\delta_1)$ such that $g_1=1$ on $\bar{U}_1$ and $g_1=0$ on $\bar{U}_2$. Since τ is a homeomorphism, $\tau^{-1}(U_2)$ is an open neighborhood of y_0 and hence $(f_1g_1)\circ\tau=0$ on $\tau^{-1}(U_2)$, which implies $\delta_2((f_1g_1)\circ\tau)(y_0)=0$. But $\delta_1(f_1g_1)(\tau_0(y_0))=\delta_1(f_1)(\tau_0(y_0))g_1(\tau_0(y_0))=1$. This contradicts with $(*)$, which implies that $\tau=\tau_0$ on $K_2(\delta_2)$. Hence we have $T^*(\eta_y\circ\delta_2)=w_2(y)\eta_{\tau(y)}\circ\delta_1$ for $y\in K_2(\delta_2)$.

Finally, we show that w_2 is continuous on $K_2(\delta_2)$. Since $\tau(K_2(\delta_2))=K_1(\delta_1)$, for arbitrary $y_0 \in K_2(\delta_2)$, there is $f_0 \in \mathfrak{D}(\delta_1)$ such that $\delta_1(f_0)(\tau(y_0)) \neq 0$. We take an open neighborhood $U(\subset K_1(\delta_1))$ of $\tau(y_0)$ such that $\delta_1(f_0)(x) \neq 0$ $(x \in U)$. Then we have $w_2(y) = w_1(y)\delta_2(f \circ \tau)(y)/\delta_1(f)(\tau(y))$ for $y \in \tau^{-1}(U)$ and w_2 is continuous at y_0, that is, w_2 is continuous on $K_2(\delta_2)$. This completes the proof of (i). (ii) follows easily from (i).

Next, we prove the converse statement (2). From the assumption in (2), T is well-defined as a surjective linear operator from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$. Then we have

$$\begin{aligned} \|Tf\|_\Sigma &= \|Tf\|_\infty + \|\delta_2(Tf)\|_\infty \\ &= \|Tf\|_\infty + \|w\delta_2(f \circ \tau)\|_\infty \\ &= \|f\|_\infty + \|\delta_1(f)\|_\infty = \|f\|_\Sigma . \end{aligned}$$

Thus all the proofs of Theorem 4.4 are completed.

REMARK 4.5. If $\mathfrak{R}(\delta_1) \ni 1$ (especially, $\mathfrak{R}(\delta_1) = C(K_1)$), then $K_1(\delta_1) = K_1$ and there exists $f_0 \in \mathfrak{D}(\delta_1)$ such that $\delta_1(f_0) = 1$ and hence $w_2(y) = w_1(y)\delta_2(f_0 \circ \tau)(y)$ for all $y \in K_2$.

We note that if $\delta = d/dt$ on $C^{(1)}([0, 1])$, then $\mathfrak{R}(\delta) = C([0, 1])$ and if $\delta = d/dz$ on $C^{(1)}(\boldsymbol{T})$, then $1 \in \mathfrak{R}(\delta) \subsetneqq C(\boldsymbol{T})$.

Let δ be a closed *-derivation in $C([0, 1])$ such that δ extends d/dt, that is, $C^{(1)}([0, 1]) \subset \mathfrak{D}(\delta)$ and $\delta(f) = f'$ for $f \in C^{(1)}([0, 1])$. Applying Theorem 4.4 to δ, we can investigate further structure of T. At first, we shall state several facts for our purpose.

A real valued function Φ on $[0, 1]$ is said to be a generalized Cantor function (abbreviated GCF) if Φ is increasing on $[0, 1]$, but not strictly increasing on any subinterval of $[0, 1]$ ([**22**]). For a GCF Φ, there exists a family of non-empty disjoint open intervals $\{I_k\}$ such that $\bigcup_{k=1}^{\infty} I_k$ is dense in $[0, 1]$ and Φ is constant on each I_k.

Let δ be a closed *-derivation in $C([0, 1])$ such that δ extends d/dt. Then there is a GCF Φ such that $Ker(\delta) = C^*(1, \Phi)$ and $\mathfrak{D}(\delta) = C^{(1)}([0, 1]) + Ker(\delta)$ ([**22**]).

We denote the identity mapping of $[0, 1]$ by id.

COROLLARY 4.6. *Let δ be a closed *-derivation in $C([0, 1])$ such that δ extends d/dt. Suppose that the norm of $f \in \mathfrak{D}(\delta)$ is*

$$\|f\|_\Sigma = \|f\|_\infty + \|\delta(f)\|_\infty .$$

(1) *Let T be a surjective linear isometry of $\mathfrak{D}(\delta)$. Then there exist w, τ_0, $\rho_0 \in Ker(\delta)$ such that $|w(x)| = 1$ for all $x \in [0, 1]$, $\tau := id + \tau_0$ is a homeomorphism of $[0, 1]$, $\tau^{-1} = id + \rho_0$, $\tau_0(0) = \tau_0(1) = \rho_0(0) = \rho_0(1) = 0$ and further,*

$$(Tf)(x) = w(x)f\circ(id+\tau_0)(x) \qquad \textit{for } f\in\mathfrak{D}(\delta) \textit{ and } x\in[0, 1],$$

or

$$(Tf)(x) = w(x)f\circ(1-(id+\tau_0))(x) \qquad \textit{for } f\in\mathfrak{D}(\delta) \textit{ and } x\in[0, 1].$$

(2) *Suppose that there exist* w, $\tau_0\in Ker(\delta)$ *such that* $|w(x)|=1$ *for all* $x\in[0, 1]$, $\tau:=id+\tau_0$ *is a homeomorphism of* $[0, 1]$ *and* $f\circ\tau$, $f\circ\tau^{-1}\in\mathfrak{D}(\delta)$ *for all* $f\in Ker(\delta)$. *Then the operators* T_1 *and* T_2 *on* $\mathfrak{D}(\delta)$ *defined respectively by*

$$(T_1f)(x) := w(x)f\circ(id+\tau_0)(x) \qquad \textit{for } f\in\mathfrak{D}(\delta) \textit{ and } x\in[0, 1],$$

$$(T_2f)(x) := w(x)f\circ(1-(id+\tau_0))(x) \qquad \textit{for } f\in\mathfrak{D}(\delta) \textit{ and } x\in[0, 1]$$

are surjective linear isometries of $\mathfrak{D}(\delta)$.

PROOF. We shall prove the statement (1). From (1) of Theorem 4.4, there exist a homeomorphism τ of $[0, 1]$ and $w\in Ker(\delta)$ such that $|w(x)|=1$ for all $x\in[0, 1]$ and

$$(Tf)(x) = w(x)f(\tau(x)) \qquad \text{for } f\in\mathfrak{D}(\delta) \text{ and } x\in[0, 1].$$

Since τ is a homeomorphism, we may suppose that τ is strictly increasing on $[0, 1]$. Since 1, id and $w=T(1)\in\mathfrak{D}(\delta)$, $\tau=w^*T(id)\in\mathfrak{D}(\delta)$. Then there exist a real function $\tau\in C^{(1)}([0, 1])$ and a real function $\tau_0\in Ker(\delta)$ such that $\tau=\tau_1+\tau_0$, $\tau_1(0)=\tau_0(0)=0$ and $\tau_1(1)+\tau_0(1)=1$. Then there exists $h\in C([-\|\Phi\|_\infty, \|\Phi\|_\infty])$ such that $\tau_0=h\circ\Phi$, which implies that τ_0 is constant on each I_k. Now, since $\tau=\tau_1+h\circ\Phi$ and τ is strictly increasing on $[0, 1]$, we get τ_1 is strictly increasing on each I_k. We show that τ_1 is strictly increasing on $[0, 1]$. Suppose that there exist x_0 and y_0 in $[0, 1]$ such that $x_0<y_0$ and $\tau_1(y_0)\leqq\tau_1(x_0)$. Since τ_1 is in $C^{(1)}([0, 1])$, then there exists an open subinterval J $(\subset(x_0, y_0))$ on which τ_1 is non-increasing. But τ_1 is strictly increasing on $J\cap I_k$ for some I_k. This is a contradiction. Thus τ_1 is strictly increasing on $[0, 1]$. Hence $\tau_1'\geqq 0$. Next, we show $\tau_1'\equiv 1$. Since $\delta(w)=0$, we have

$$\begin{aligned} 2 &= \|id\|_\Sigma = \|T(id)\|_\Sigma \\ &= \|w\tau\|_\infty+\|\delta(w\tau)\|_\infty \\ &= \|\tau\|_\infty+\|\tau_1'\|_\infty. \end{aligned}$$

From this and $\|\tau\|_\infty=1$, we have $\|\tau_1'\|_\infty=1$. Suppose that there exists x_0 in $[0, 1]$ such that $\tau_1'(x_0)<1$. Then we choose a function $h\in C^{(1)}([0, 1])$ such that $\|h\|_\infty<1$, $\|h'\|_\infty=1$, $h'(\tau(x_0))=1$ and $0<h'(x)<1$ for all $x(\neq\tau(x_0))\in[0, 1]$. Since $\tau_1'(x_0)<1$ and $0<h'(x)<1$ for all $x(\neq\tau(x_0))\in[0, 1]$, there exists an open neighborhood U_{x_0} of x_0 such that

$$\sup\{\tau_1'(x) : x\in\overline{U_{x_0}}\}(:= c_1) < 1$$

and

$$\sup\{h'\circ\tau(x) : x\in([0, 1]\setminus U_{x_0})\}\,(:= c_2) < 1 .$$

Then

$$\begin{aligned} 1 &= \|h'\|_\infty = \|\delta(h)\|_\infty \\ &= \|\delta(Th)\|_\infty = \|\delta(h\circ\tau)\|_\infty \\ &= \sup\{h'\circ\tau(x)\tau_1'(x) : x\in[0, 1]\} \\ &\leqq \max(c_1, c_2) < 1 . \end{aligned}$$

This is a contradiction. Thus $\tau_1'\equiv 1$. Since $\tau_1(0)=0$, we get $\tau_1=id$. Considering T^{-1}, we get $\rho_0\in Ker(\delta)$ such that $\tau^{-1}=id+\rho_0$. If τ is strictly decreasing, we may consider $1-\tau$.

We prove the converse statement (2). Since $\tau_0\circ\tau^{-1}\in\mathfrak{D}(\delta)$ from assumption in (2), $\tau^{-1}=id-\tau_0\circ\tau^{-1}\in\mathfrak{D}(\delta)$. Hence we have $f\circ\tau$ and $f\circ\tau^{-1}\in\mathfrak{D}(\delta)$ for all $f\in\mathfrak{D}(\delta)$. Thus T_1 defined in (2) is a surjective linear operator on $\mathfrak{D}(\delta)$. Next, we shall show that $|\delta(f\circ\tau)(x)|=|\delta(f)(\tau(x))|$ for all $f\in\mathfrak{D}(\delta)$ and all $x\in[0, 1]$. For each $f\in\mathfrak{D}(\delta)$, there exist $f_1\in C^{(1)}([0, 1])$ and $f_0\in Ker(\delta)$ such that $f=f_1+f_0$. Since

$$\begin{aligned} |\delta(f\circ\tau)(x)| &= |\delta(f_1\circ\tau+f_0\circ\tau)(x)| \\ &= |(f_1'\circ\tau\delta(\tau)+\delta(f_0\circ\tau))(x)| \\ &= |(f_1'\circ\tau+\delta(f_0\circ\tau))(x)|, \end{aligned}$$

it is sufficient to show $f_0\circ\tau\in Ker(\delta)$. From our assumption, $g:=f_0\circ\tau\in\mathfrak{D}(\delta)$. Therefore there exist $g_1\in C^{(1)}([0, 1])$ and $g_0\in Ker(\delta)$ such that $f_0\circ\tau=g_1+g_0$. For each I_m, $\bigcup_{k=1}^\infty\{\tau^{-1}(I_m)\cap I_k\}$ is dense in $\tau^{-1}(I_m)$. Since $g=g_1+g_0$ and g and g_0 are constant on each non-empty open interval $\tau^{-1}(I_m)\cap I_k$, then g_1 is constant and hence $g_1'\equiv 0$ on each such interval $\tau^{-1}(I_m)\cap I_k$. Since g_1 is in $C^{(1)}([0, 1])$, then $g_1'\equiv 0$ on each $\tau^{-1}(I_m)$. Since $\bigcup_{k=1}^\infty\tau^{-1}(I_m)$ is dense in $[0, 1]$, $g_1'\equiv 0$ on $[0, 1]$, which implies that g_1 is constant and hence $f_0\circ\tau\in Ker(\delta)$. Thus, from (2) of Theorem 4.4, we get $\|T_1 f\|_\Sigma=\|f\|_\Sigma$ for all $f\in\mathfrak{D}(\delta)$. Similarly, we can prove that T_2 is a surjective isometry of $\mathfrak{D}(\delta)$. This completes the proof.

Considering the case that Φ is constant and $\delta=d/dt$ in Corollary 4.6, we get the following corollary.

COROLLARY 4.7 (Rao and Roy). *Suppose that the norm of $f\in C^{(1)}([0, 1])$ is*

$$\|f\|_\Sigma = \|f\|_\infty+\|\delta(f)\|_\infty .$$

Let T be a surjective linear isometry of $C^{(1)}([0, 1])$. Then T has the following expression

$$(Tf)(x) = \alpha f(\tau(x)) \qquad \text{for all } f \in C^{(1)}([0, 1]) \text{ and } x \in [0, 1]$$

with $T(1)=\alpha$ $(\alpha \in \boldsymbol{T})$ *is a constant and* τ *is one of the two functions id or* $1-id$.

§ 5. Extreme points of the closed unit ball of the conjugate space $\mathfrak{D}(\delta)^*$ of $\mathfrak{D}(\delta)$ with M-norm.

Throughout this section, let K be a compact Hausdorff space and let δ be a closed *-derivation in $C(K)$; the norm of $f \in \mathfrak{D}(\delta)$ is

$$\|f\|_M = \max(\|f\|_\infty, \|\delta(f)\|_\infty).$$

In this section, we get concrete expressions of extreme points of $\mathfrak{D}(\delta)_1^*$ with M-norm.

Let W be the compact Hausdorff space $X \cup Y$ (topological sum with $X=Y=K$). We define $\tilde{f}$ on $C(W)$ $(f \in \mathfrak{D}(\delta))$ by

$$\tilde{f}(w) := \begin{cases} f(w) & \text{if } w \in X \\ \delta(f)(w) & \text{if } w \in Y\,. \end{cases}$$

Then we may embed $\mathfrak{D}(\delta)$ as a closed subspace of $C(W)$.

LEMMA 5.1. *The mapping* $\phi: f \to \tilde{f}$ *establishes a linear and norm-preserving correspondence between* $\mathfrak{D}(\delta)$ *and the closed subspace* $S := \{\tilde{f} : f \in \mathfrak{D}(\delta)\}$ *of* $C(W)$.

We recall that $C(W) = C(K) \oplus_{l^\infty} C(K)$ and $C(W)^* = C(K)^* \oplus_{l^1} C(K)^*$ by the canonical correspondence. Let Ψ be an extension of $(\phi^{-1})^*(F)$ $(F \in \mathfrak{D}(\delta)^*)$ to $C(W)$, where this extension is not necessary unique; then $\Psi \in C(W)^*$ has the form

$$\Psi(g) = \int_K g\,d\mu + \int_K \delta(g)\,d\nu \qquad (\forall g \in C(W))$$

for some complex regular Borel measures μ and ν on K. Hence $F(\in \mathfrak{D}(\delta)^*)$ has the form

$$F(f) = \Psi(\tilde{f}) = \int_K f\,d\mu + \int_K \delta(f)\,d\nu \qquad (\forall f \in \mathfrak{D}(\delta)).$$

If Ψ is a norm-preserving extension, $\|F\| = \|\Psi\| = \|\mu\| + \|\nu\|$.

Now we state the main result of this section. We note that the expressions of η_x and $\eta_x \circ \delta \neq 0$ are unique.

THEOREM 5.2. *Let* K *be a compact Hausdorff space and let* δ *be a closed *-derivation in* $C(K)$. *Then an element* $G \in \mathfrak{D}(\delta)^*$ *is an extreme point of* $\mathfrak{D}(\delta)_1^*$ *if and only if*

$$G = \alpha\eta_x \ (\alpha \in \boldsymbol{T},\ x \in K) \quad \text{or} \quad \alpha(\eta_x \circ \delta)\ (\alpha \in \boldsymbol{T},\ x \in K(\delta)).$$

PROOF. At first, we prove 'only if part'. Suppose that L is an extreme point of S_1^*, where S is as in Lemma 5.1; then we can extend L to the extreme point of $C(W)^*$. We recall that the form of an extreme point of $C(W)_1^*$ is a point evaluation multiplied by α ($\alpha\in \boldsymbol{T}$). Hence, if an element G of $\mathfrak{D}(\delta)^*$ is an extreme point of $\mathfrak{D}(\delta)_1^*$, then

$$G=\alpha\eta_x\ (\alpha\in \boldsymbol{T},\ x\in K) \quad \text{or} \quad \alpha(\eta_x\circ\delta)\ (\alpha\in \boldsymbol{T},\ x\in K(\delta)).$$

Next, we show 'if part'. For $x_0\in X(=K)$, we set $G_1:=\eta_{x_0}\in\mathfrak{D}(\delta)^*$. Let Ψ_1 be any norm-preserving extension of $(\phi^{-1})^*(G_1)$ to $C(W)$. Then there exist complex regular Borel measures $\mu\in C(X)^*$ and $\nu\in C(Y)^*$ such that $\|\Psi_1\|=\|\mu\|+\|\nu\|$ and

$$f(x_0)=\Psi_1(\tilde{f})=\int_K f d\mu+\int_K \delta(f)d\nu$$

for all $f\in\mathfrak{D}(\delta)$. Since

$$1=\Psi_1(\tilde{1})=\int_K d\mu=\left|\int_K d\mu\right|\leqq\int_K d|\mu|=\|\mu\|\leqq\|\Psi_1\|=1\,,$$

we have $\|\mu\|=\mu(1)=1$ and hence μ is a positive measure. Since $1=\|\Psi_1\|=\|\mu\|+\|\nu\|$, we have $\nu=0$. Hence for all $f\in\mathfrak{D}(\delta)$, $f(x_0)=\Psi_1(\tilde{f})=\int_K f d\mu$. Since $\mathfrak{D}(\delta)$ is dense in $C(K)$, μ is the dirac measure at x_0. Thus Ψ_1 (arbitrary norm-preserving extension of $(\phi^{-1})^*(G_1)$ to $C(W)$) is an extreme point of $C(W)_1^*$ and hence we conclude that $(\phi^{-1})^*(G_1)$ is an extreme point of S_1^*, which implies that $G_1=\eta_{x_0}$ is an extreme point of $\mathfrak{D}(\delta)_1^*$.

Finally, we shall show that $G_2:=\eta_{y_0}\circ\delta$ is an extreme point of $\mathfrak{D}(\delta)_1^*$ for $y_0\in K(\delta)(\subset Y)$. Let Ψ_2 be any norm-preserving extension of $(\phi^{-1})^*(G_2)$ to $C(W)$. Then there exist complex regular Borel measures $\mu\in C(X)^*$ and $\nu\in C(Y)^*$ such that $\|\Phi\|=\|\mu\|+\|\nu\|$ and

$$\delta(f)(y_0)=\Psi_2(\tilde{f})=\int_K f d\mu+\int_K \delta(f)d\nu$$

for all $f\in\mathfrak{D}(\delta)$. For arbitrary $\varepsilon>0$, we take $f_\varepsilon(=f_\varepsilon^*)\in\mathfrak{D}(\delta)$ such that

$$\|\eta_{y_0}\circ\delta\|-\varepsilon\leqq|\delta(f_\varepsilon)(y_0)| \quad \text{and} \quad \|f_\varepsilon\|_M<1\,.$$

For any open neighborhood $U(\subset Y)$ of y_0 we choose an open neighborhood V of y_0 such that $\bar{V}\subset U$. Then we take $g_1\in\mathfrak{D}(\delta)$ such that

$$g_1(y_0)=1,\quad g_1=0 \quad \text{on } K\setminus V \quad \text{and} \quad 0\leqq g_1\leqq 1,$$

then $g_1=\delta(g_1)=0$ on $K\setminus U$. Put $c_\varepsilon:=(1-\|\delta(f_\varepsilon)\|_\infty)/(\|\delta(g_1)\|_\infty+1)(<1)$. Then we take a function $h_\varepsilon\in C^{(1)}([-\|f_\varepsilon\|_\infty,\|f_\varepsilon\|_\infty])$ such that

$$\|h_\varepsilon\|_\infty\leqq c_\varepsilon,\quad h_\varepsilon(f_\varepsilon(y_0))=0,\quad h_\varepsilon'(f_\varepsilon(y_0))=1 \quad \text{and} \quad \|h_\varepsilon'\|_\infty=1.$$

Put $g_\varepsilon := h_\varepsilon(f_\varepsilon)$. Then we have

$$\delta(g_1 g_\varepsilon)(y_0) = \delta(g_1)(y_0)g_\varepsilon(y_0) + g_1(y_0)\delta(g_\varepsilon)(y_0) = \delta(g_\varepsilon)(y_0) = \delta(f_\varepsilon)(y_0),$$

$$\|g_1 g_\varepsilon\|_\infty \leqq c_\varepsilon, \quad \|\delta(g_1 g_\varepsilon)\|_\infty \leqq 1 \quad \text{and} \quad g_1 g_\varepsilon = \delta(g_1 g_\varepsilon) = 0 \quad \text{on } K \setminus U.$$

Since

$$\begin{aligned} \|\eta_{y_0} \circ \delta\| - \varepsilon &\leqq |\delta(f_\varepsilon)(y_0)| = |\delta(g_1 g_\varepsilon)(y_0)| \\ &= \left| \int_K g_1 g_\varepsilon d\mu + \int_K \delta(g_1 g_\varepsilon) d\nu \right| \\ &\leqq \int_K |g_1 g_\varepsilon| d|\mu| + \int_K |\delta(g_1 g_\varepsilon)| d|\nu| \\ &\leqq \int_U d|\mu| + \int_U d|\nu| \\ &\leqq \|\mu\| + \|\nu\| = \|\Psi_2\| = \|\eta_{y_0} \circ \delta\|, \end{aligned}$$

we have $\int_U d|\mu| = \|\mu\|$ and $\int_U d|\nu| = \|\nu\|$. Thus we have $supp|\mu| \subset \{y_0\}$ and $supp|\nu| \subset \{y_0\}$, which implies there exist $a, b \in \boldsymbol{C}$ such that $\mu = a\delta_{y_0}$ and $\nu = b\delta_{y_0}$ where δ_{y_0} is the dirac measure at $\{y_0\}$. Hence we have $\delta(f)(y_0) = af(y_0) + b\delta(f)(y_0)$. Since $1 \in \mathfrak{D}(\delta)$ and $y_0 \in K(\delta)$, we see $a=0$ and $b=1$. Thus ν is the dirac measure at y_0 and $\mu=0$, which implies Ψ_2 is an extreme point of $C(W)_1^*$. Then we conclude that $(\phi^{-1})^*(G_2)$ is an extreme point of S_1^*, which implies that $G_2 = \eta_{y_0} \circ \delta$ is an extreme point of $\mathfrak{D}(\delta)_1^*$. This completes the proof.

§6. Linear isometries between $\mathfrak{D}(\delta_1)$ and $\mathfrak{D}(\delta_2)$ with M-norm.

In this section we use the results in section 5 to study the structure of surjective linear isometries of $\mathfrak{D}(\delta)$ with the M-norm.

THEOREM 6.1. *Let K_i be a compact Hausdorff space and let δ_i be a closed $*$-derivation in $C(K_i)$ $(i=1, 2)$.*

(1) *Let T be a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$. Then the following statements are held.*

(i) *There exist a homeomorphism τ from K_2 to K_1, $w_1 \in Ker(\delta_2)$ and a continuous function w_2 on $K_2(\delta_2)$ such that $\tau(K_2(\delta_2)) = K_1(\delta_1)$, $|w_1(y)|=1$ for all $y \in K_2$, $|w_2(y)|=1$ for all $y \in K_2(\delta_2)$,*

$$(Tf)(y) = w_1(y) f(\tau(y)) \qquad \textit{for } f \in \mathfrak{D}(\delta_1) \textit{ and } y \in K_2,$$

$$\delta_2(Tf)(y) = w_2(y)\delta_1(f)(\tau(y)) \qquad \textit{for } f \in \mathfrak{D}(\delta_1) \textit{ and } y \in K_2(\delta_2).$$

(ii) $T(Ker(\delta_1)) = Ker(\delta_2)$.

(2) *Suppose that there exist $w \in Ker(\delta_2)$ and a homeomorphism τ from K_2 to*

K_1 such that $|w(y)|=1$ for all $y\in K_2$, $\tau(K_2(\delta_2))=K_1(\delta_1)$, $f\circ\tau\in\mathfrak{D}(\delta_2)$ for all $f\in\mathfrak{D}(\delta_1)$, $g\circ\tau^{-1}\in\mathfrak{D}(\delta_1)$ for all $g\in\mathfrak{D}(\delta_2)$, and $|\delta_2(f\circ\tau)(y)|=|\delta_1(f)(\tau(y))|$ for all $f\in\mathfrak{D}(\delta_1)$ and all $y\in K_2(\delta_2)$. Then the operator T from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$ defined by

$$(Tf)(y):=w(y)f(\tau(y)) \qquad \text{for } f\in\mathfrak{D}(\delta_1) \text{ and } y\in K_2$$

is a surjective linear isometry.

PROOF. At first, we recall that

$$ext\mathfrak{D}(\delta_i)_1^* = \{\alpha\eta_x,\ \beta(\eta_y\circ\delta_i) : \alpha,\ \beta\in\boldsymbol{T},\ x\in K_i,\ y\in K_i(\delta_i)\}$$

and its expression is unique. For each point $y\in K_2$, we have two possibilities:

1. There exist $x_1\in K_1$ and $\alpha_1\in\boldsymbol{T}$ such that

$$T^*\eta_y = \alpha_1\eta_{x_1}$$

or

2. There exist $x_2\in K_1$ and $\alpha_2\in\boldsymbol{T}$ such that

$$T^*\eta_y = \alpha_2(\eta_{x_2}\circ\delta_1).$$

Hence we have $|T1(y)|=1$ or $T1(y)=0$ for each $y\in K_2$. Put

$$\Omega := \{y\in K_2 : |T1(y)|=1\}(\neq\emptyset),$$

$$\Gamma := \{y\in K_2 : T1(y)=0\}.$$

Suppose that $\Gamma\neq\emptyset$; then Ω and Γ are non-empty open and closed sets. Put $p:=1-(T1)^*(T1)\in\mathfrak{D}(\delta_2)$; then p is a projection and hence $\delta_2(p)=0$. For $\lambda\in\boldsymbol{T}$,

$$\begin{aligned}\|T1+\lambda p\|_M &= \max(\|T1+\lambda p\|_\infty,\ \|\delta_2(T1)\|_\infty)\\ &= \max(\|T1\|_\infty,\ \|\delta_2(T1)\|_\infty)\\ &= \|T1\|_M = 1\end{aligned}$$

and $T^{-1}(T1+\lambda p)=1+\lambda T^{-1}(p)$, but there exists $\lambda_0\in\boldsymbol{T}$ such that

$$\|T^{-1}(T1+\lambda_0 p)\|_M = \|1+\lambda_0 T^{-1}(p)\|_M > 1.$$

This is a contradiction, which implies $\Gamma=\emptyset$. Hence for each point $y\in K_2$, there exist $x\in K_1$ and $\alpha\in\boldsymbol{T}$ such that

$$T^*\eta_y = \alpha\eta_x.$$

Then, defining τ and w_1 by $T^*\eta_y := w_1(y)\eta_{\tau(y)}$, we have $w_1=T1\in\mathfrak{D}(\delta_2)$ and τ is a homeomorphism from K_2 to K_1.

Since T^* carries $ext\mathfrak{D}(\delta_2)_1^*$ and $\{\alpha\eta_y : \alpha\in\boldsymbol{T},\ y\in K_2\}$ onto $ext\mathfrak{D}(\delta_1)_1^*$ and $\{\alpha\eta_x : \alpha\in\boldsymbol{T},\ x\in K_1\}$ respectively, for each $y\in K_2(\delta_2)$ there exist $x\in K_1(\delta_1)$ and $\alpha\in\boldsymbol{T}$ such that

$$T^*(\eta_y \circ \delta_2) = \alpha\eta_x \circ \delta_1 (\neq 0).$$

From this, we see $w_1 \in Ker(\delta_2)$. Defining τ_0 and w_2 by $T^*(\eta_y \circ \delta_2) := w_2(y)\eta_{\tau_0(y)} \circ \delta_1$, then τ_0 is a mapping from $K_2(\delta_2)$ to $K_1(\delta_1)$. By the same way as in the proof of Theorem 4.4, the proof of (i) is completed. (ii) follows easily from (i).

We prove the converse statement (2). From the assumption in (2), T is well-defined as a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$. Then we have

$$\|Tf\|_\infty = \|f\|_\infty \quad \text{and} \quad \|\delta_2(Tf)\|_\infty = \|\delta_1(f)\|_\infty,$$

that is, $\|Tf\|_M = \|f\|_M$ for all $f \in \mathfrak{D}(\delta_1)$. From $g \circ \tau^{-1} \in \mathfrak{D}(\delta_1)$ for all $g \in \mathfrak{D}(\delta_2)$, T is surjective. Thus, all the proofs are completed.

If $1 \in \mathfrak{R}(\delta_1)$, refer to Remark 4.5.

REMARK 6.2. Let T be a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$. From Theorem 6.1, then $\|Tf\|_\infty = \|f\|_\infty$ and $\|\delta_2(Tf)\|_\infty = \|\delta_1(f)\|_\infty$ for all $f \in \mathfrak{D}(\delta_1)$. Hence if there exist a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$, $\delta_i = 0$ ($i=$ 1, 2) or $\delta_i \neq 0$ ($i=1, 2$). If $\delta_i = 0$ ($i=1, 2$), we have a well-known Banach-Stone theorem.

In Theorem 6.1, let δ be a closed *-derivation in $C([0, 1])$ such that δ extends d/dt. By the same way as in the proof of Corollary 4.6, we get the following corollary.

COROLLARY 6.3. *Let δ be a closed *-derivation in $C([0, 1])$ such that δ extends d/dt. Suppose that the norm of $f \in \mathfrak{D}(\delta)$ is*

$$\|f\|_M = \max(\|f\|_\infty, \|\delta(f)\|_\infty).$$

(1) *Let T be a surjective linear isometry of $\mathfrak{D}(\delta)$. Then there exist $w, \tau_0, \rho_0 \in Ker(\delta)$ such that $|w(x)|=1$ for all $x \in [0, 1]$, $\tau := id + \tau_0$ is a homeomorphism of $[0, 1]$, $\tau^{-1} = id + \rho_0$, $\tau_0(0) = \tau_0(1) = \rho_0(0) = \rho_0(1) = 0$ and further,*

$$(Tf)(x) = w(x) f \circ (id + \tau_0)(x) \qquad \text{for } f \in \mathfrak{D}(\delta) \text{ and } x \in [0, 1],$$

or

$$(Tf)(x) = w(x) f \circ (1 - (id + \tau_0))(x) \qquad \text{for } f \in \mathfrak{D}(\delta) \text{ and } x \in [0, 1].$$

(2) *Suppose that there exist $w, \tau_0 \in Ker(\delta)$ such that $|w(x)|=1$ for all $x \in [0, 1]$, $\tau := id + \tau_0$ is a homeomorphism of $[0, 1]$ and $f \circ \tau$, $f \circ \tau^{-1} \in \mathfrak{D}(\delta)$ for all $f \in Ker(\delta)$. Then the operators T_1 and T_2 on $\mathfrak{D}(\delta)$ defined respectively by*

$$(T_1 f)(x) := w(x) f \circ (id + \tau_0)(x) \qquad \text{for } f \in \mathfrak{D}(\delta) \text{ and } x \in [0, 1],$$

$$(T_2 f)(x) := w(x) f \circ (1 - (id + \tau_0))(x) \qquad \text{for } f \in \mathfrak{D}(\delta) \text{ and } x \in [0, 1]$$

are surjective linear isometries of $\mathfrak{D}(\delta)$.

Considering the case that Φ is constant and $\delta=d/dt$ in Corollary 6.3, we get the following corollary.

COROLLARY 6.4. *Suppose that the norm of* $f\in C^{(1)}([0, 1])$ *is*

$$\|f\|_M = \max(\|f\|_\infty, \|f'\|_\infty).$$

Let T *be a surjective linear isometry of* $C^{(1)}([0, 1])$. *Then* T *has the following expression*

$$(Tf)(x) = \alpha f(\tau(x)) \qquad \text{for } f\in C^{(1)}([0, 1]) \text{ and } x\in[0, 1]$$

with $T(1)=\alpha$ $(\alpha\in\boldsymbol{T})$ *is a constant and* τ *is one of the two functions* id *or* $1-id$.

§7. Extreme points of the closed unit ball of $\mathfrak{D}(\delta)$ with M-norm.

Let K be a compact Hausdorff space and let δ be a closed *-derivation in $C(K)$. Extreme points of the closed unit balls of Banach spaces have been investigated for many concrete spaces by many authors. In this section we wish to characterize all the extreme points of the closed unit ball of $\mathfrak{D}(\delta)$ with M-norm, that is,

$$\|f\|_M = \max(\|f\|_\infty, \|\delta(f)\|_\infty).$$

The first thing to note is that for f to belong to $ext\mathfrak{D}(\delta)_1$, it is necessary that $\|f\|_\infty=1$. For if $\|f\|_\infty<1$, then $\|f\pm(1-\|f\|_\infty)1\|_M\leqq 1$ and

$$f = (1/2)\{(f+(1-\|f\|_\infty)1)+(f-(1-\|f\|_\infty)1)\},$$

which implies that f is not extreme. Moreover, if f belongs $\mathfrak{D}(\delta)_1$ and $|f(x)|=1$ for all $x\in K$, then f belongs to $ext\mathfrak{D}(\delta)_1$. This is because f is already extreme in $C(K)$. In the situation mentioned above, we describe the other members of $ext\mathfrak{D}(\delta)_1$.

THEOREM 7.1. *Let* K *be a compact Hausdorff space and let* δ *be a closed *-derivation with* $Ker(\delta)=\boldsymbol{C}1$. *Let* f *be an element of* $\mathfrak{D}(\delta)$ *such that* $\|f\|_M=\|f\|_\infty=1$ *and suppose that* f *is not of modulus one everywhere. Then* $f\in ext\mathfrak{D}(\delta)_1$ *if and only if* $|\delta(f)(x)|=1$ $(x\in K\setminus M_f)$, *where* $M_f:=\{x\in K: |f(x)|=\|f\|_\infty=1\}$.

PROOF. We first prove 'only if' part. Suppose that there exists $x_0\in K\setminus M_f$ such that $|\delta(f)(x_0)|<1$, that is, there exists $a>0$ such that $\max(|f(x_0)|, |\delta(f)(x_0)|)<a<1$. Let $U_{x_0}:=\{x\in K: \max(|f(x)|, |\delta(f)(x)|)<a\}$. Then there exists $f_0\in\mathfrak{D}(\delta)$ such that

$$f_0(x_0) = 1, \quad f_0(x) = 0 \quad (x\in K\setminus U_{x_0}) \quad \text{and} \quad 0\leqq f_0\leqq 1.$$

Moreover, there exists $h\in C^{(1)}([0, 1])$ such that

$$0 < h(1) < 1 \quad \text{and} \quad h(0) = h'(0) = 0.$$

Put $g_0 := h(f_0)$. Then $g_0 \in \mathfrak{D}(\delta)$ is non-zero and $g_0 = \delta(g_0) = 0$ on $K \setminus U_{x_0}$. We choose a real λ such that

$$0 < \lambda < \min(1/\|g_0\|_\infty, 1/(1+\|\delta(g_0)\|_\infty))(1-a).$$

Then we have

$$\|f \pm \lambda g_0\|_M = \max(\|f \pm \lambda g_0\|_\infty, \|\delta(f \pm \lambda g_0)\|_\infty) \leqq 1,$$

which implies that $f \pm \lambda g_0$ belong to $\mathfrak{D}(\delta)_1$. Moreover, we have

$$f = (1/2)\{(f+\lambda g_0)+(f-\lambda g_0)\}.$$

Consequently, $f \in ext\mathfrak{D}(\delta)_1$ implies that $|\delta(f)| = 1$ on $K \setminus M_f$.

To prove the converse statement, we take $f \in \mathfrak{D}(\delta)_1$ such that $\|f\|_M = \|f\|_\infty = 1$ and f is not necessarily modulus one everywhere. Suppose further that $|\delta(f)| = 1$ on $K \setminus M_f$. Let

$$f := (1/2)(g_1 + g_2) \qquad (g_1, g_2 \in \mathfrak{D}(\delta)_1).$$

Clearly,

$$g_1 = g_2 = f \qquad \text{on } M_f.$$

Also,

$$\delta(f) = (1/2)(\delta(g_1) + \delta(g_2)).$$

Hence, from the assumption we have

$$\delta(g_1) = \delta(g_2) = \delta(f) \qquad \text{on } K \setminus M_f.$$

We show that

$$\delta(g_1) = \delta(g_2) = \delta(f) \qquad \text{on } M_f.$$

To this end, we take and fix arbitrary $x \in M_f$. Suppose that there exists an open neighborhood U_x of x such that $U_x \subset M_f$. Then

$$f - g_1 = 0 \quad \text{and} \quad f - g_2 = 0 \qquad \text{on } U_x,$$

which implies

$$\delta(f)(x) = \delta(g_1)(x) \quad \text{and} \quad \delta(f)(x) = \delta(g_2)(x).$$

Next, suppose that for each open neighborhood U_x of $x \in M_f$, there exists $y \in U_x \cap (K \setminus M_f)$. Then there exists a net $\{x_\gamma\}$ in $K \setminus M_f$ such that

$$x_\gamma \longrightarrow x, \quad \delta(f)(x_\gamma) = \delta(g_1)(x_\gamma) \quad \text{and} \quad \delta(f)(x_\gamma) = \delta(g_2)(x_\gamma).$$

Therefore

$$\delta(f)(x) = \delta(g_1)(x) = \delta(g_2)(x).$$

Consequently, we have

$$\delta(g_1) = \delta(g_2) = \delta(f) \qquad \text{on } K,$$

that is,

$$\delta(f-g_1)=0 \quad \text{and} \quad \delta(f-g_2)=0 \qquad \text{on } K.$$

From our assumption, $f-g_1$ and $f-g_2$ are constant on K. Since $f=g_1$ and $f=g_2$ on M_f, we have $f=g_1$ and $f=g_2$ on K, which implies that f belongs to $ext\mathfrak{D}(\delta)_1$. This completes the proof.

REMARK 7.2. In the proof of 'only if' part of Theorem 7.1, the condition $Ker(\delta)=\boldsymbol{C}1$ is unnecessary. However, this condition can not be deleted in 'if' part and such example is easily constructed.

As stated in the first paragraph of this section, a unitary element u (that is, $|u(x)|\equiv 1$ on K) of $\mathfrak{D}(\delta)_1$ is an extreme point of $\mathfrak{D}(\delta)_1$. For a unitary $u\in\mathfrak{D}(\delta)$ it may happen that $\|\delta(u)\|_\infty>1$, and for a non-unitary $u\in\mathfrak{D}(\delta)$, $\delta(u)$ is able to be unitary. Such examples are easy to find in $C^{(1)}([0, 1])$. In connection with this, we present some examples.

EXAMPLE 7.3. Let K be a compact Hausdorff space and let δ be a closed *-derivation in $C(K)$. For any $f(=f^*)\in\mathfrak{D}(\delta)$ such that $\|\delta(f)\|_\infty=1$, there exists $h\in C^{(1)}([-\|f\|_\infty, \|f\|_\infty])$ such that $\|h'\|_\infty\leqq 1$ and h is of modulus one everywhere. Then, $\|h(f)\|_M=1$ and $h(f)$ is unitary, hence $h(f)\in ext\mathfrak{D}(\delta)_1$.

EXAMPLE 7.4. Let K be a compact connected Hausdorff space and let δ be a closed *-derivation in $C(K)$ with $\mathfrak{R}(\delta)=C(K)$. Then there exist $f(=f^*)\in\mathfrak{D}(\delta)$ and $h\in C^{(1)}([-\|f\|_\infty, \|f\|_\infty])$ such that $h(f)$ is non-unitary with $\|h(f)\|_\infty=1$ and $\delta(h(f))$ is unitary. Thus $h(f)$ is in $\mathfrak{D}(\delta)_1$ and $h(f)$ satisfies the condition of Theorem 7.1. Thus, if the kernel of δ is one dimensional $\boldsymbol{C}1$, $h(f)$ is an extreme point of $\mathfrak{D}(\delta)_1$.

Next, we consider a special case with $Ker(\delta)\neq\boldsymbol{C}1$.

PROPOSITION 7.5. *Let $K=I\cup J$ where I and J are disjoint finite closed intervals of the real line and let δ be a closed *-derivation in $C(K)$. Suppose that the kernel of δ is the set of functions which are respectively constant on I and J. Let f be an element of $\mathfrak{D}(\delta)$ such that $\|f\|_M=\|f\|_\infty=1$ and suppose that f is not of modulus one everywhere. Then $f\in ext\mathfrak{D}(\delta)_1$ if and only if $|\delta(f)(x)|=1$ $(x\in K\setminus M_f)$, $I\cap M_f\neq\emptyset$ and $J\cap M_f\neq\emptyset$, where $M_f:=\{x\in K: |f(x)|=\|f\|_\infty=1\}$.*

PROOF. We first prove 'only if' part. From Theorem 7.1, $f\in ext\mathfrak{D}(\delta)_1$ implies that $|\delta(f)|=1$ on $K\setminus M_f$. Suppose that $I\cap M_f=\emptyset$, that is, there exists $b>0$ such that $b\leqq 1-\sup\{|f(x)| : x\in I\}$. Put $g\in\mathfrak{D}(\delta)$,

$$g=b \quad \text{on } I \quad \text{and} \quad g=0 \quad \text{on } J.$$

Then $\|f\pm g\|_M\leqq 1$ and

$$f=(1/2)\{(f+g)+(f-g)\},$$

which implies that f is not extreme. Consequently, $f \in ext\mathfrak{D}(\delta)_1$ implies that $|\delta(f)(x)|=1$ $(x \in K \setminus M_f)$, $I \cap M_f \neq \emptyset$ and $J \cap M_f \neq \emptyset$.

By the same way as in the proof of the Theorem 7.1, we get the converse statement. This completes the proof.

REMARK 7.6. Let $K=I \cup J$ where I and J are disjoint finite closed intervals of the real line and let δ be the differentiation d/dt in $C(K)$; then δ satisfies the condition of the preceding proposition.

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