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Circles on quaternionic space forms

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Introduction.

A smooth curve γ parametrized by its arc-length is called a *circle* of geodesic curvature κ ($\kappa>0$) if it satisfies the following equations with an associated unit vector field Y along γ;

$$\nabla_X X = \kappa Y, \qquad \nabla_X Y = -\kappa X,$$

where $X(t)=\dot{\gamma}(t)$. Though this definition was given by Nomizu and Yano [**8**] in 1974, the study on circles is just begun. We studied in [**3**] and [**4**] circles on complex space forms, and in [**2**] we studied them on a surface of nonpositive curvature. In this paper we study circles on a quaternion projective space and on a quaternion hyperbolic space, and show that the similar properties hold as for circles on complex space forms.

In the study of circles on complex space forms, complex torsion $\theta=\langle X, JY\rangle$, where J is the complex structure, for a circle plays an important role. On a complex projective space, every circle with $\theta=0, \pm 1$ is closed, but when $0<|\theta|<1$ we have closed circles and open circles, just like geodesics on a torus. On a complex hyperbolic space, there exist a bound κ_θ for each θ such that circles with complex torsion θ are unbounded if $\kappa\leqq\kappa_\theta$ and bounded if $\kappa>\kappa_\theta$. As a corresponding invariant for circles on a quaternion Kähler manifold M, we define the *structure torsion* Θ. For a circle γ on M with associated unit vector fields X, Y, we set Θ as the norm of the projected vector $\mathrm{Proj}(X_t)$ of X_t onto the 1-dimensional quaternion subspace $\{Y_t\cdot\varepsilon \mid \varepsilon \text{ is a quaternion}\}$ (see for detail § 2). This plays the same role as the complex torsion for circles on a Kähler manifold. By using the Hopf fibration, we take a horizontal lift of a circle on a quaternionic space form. Under the identification of the algebra of quaternions with 2-dimensional complex vector space, we find it satisfies linear differential equations. By a usual method, computing eigenvalues and eigenvectors of associated matrices, we solve them and give explicit expressions of circles. With the aid of these expressions we can show some fundamental feature of them.

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§ 1. Preliminaries.

Let $\boldsymbol{H}$ denote the division algebra of quaternions;

$$\boldsymbol{H}=\{a+bi+cj+dk \mid a, b, c, d \in \boldsymbol{R}\},$$

where $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$. We identify this algebra with $\boldsymbol{C}^2=\{\alpha+\beta j \mid \alpha, \beta \in \boldsymbol{C}\}$ in the following manner;

$$(\alpha_1+\beta_1 j)+(\alpha_2+\beta_2 j)=(\alpha_1+\alpha_2)+(\beta_1+\beta_2)j,$$

$$(\alpha_1+\beta_1 j)\cdot(\alpha_2+\beta_2 j)=(\alpha_1\alpha_2-\beta_1\bar{\beta}_2)+(\alpha_1\beta_2+\beta_1\bar{\alpha}_2)j,$$

where $\bar{\alpha}$ denote the complex conjugate of α. This means that $\alpha j=j\bar{\alpha}$ for any $\alpha\in\boldsymbol{C}$. For a quaternion $\varepsilon=a+bi+cj+dk$ we denote by $\bar{\varepsilon}$ the quaternion conjugate; $\bar{\varepsilon}=a-bi-cj-dk$. Clearly we have $\overline{\varepsilon\delta}=\bar{\delta}\cdot\bar{\varepsilon}$. Under the expression $\varepsilon=\alpha+\beta j$ we find $\bar{\varepsilon}=\bar{\alpha}-\beta j$.

Let $\boldsymbol{H}^n$ be a right quaternion vector space, which means that the action of $\boldsymbol{H}$ is given as follows;

$$\xi\cdot\varepsilon=(\xi_1\varepsilon, \cdots, \xi_n\varepsilon) \qquad \text{for } \xi=(\xi_1, \cdots, \xi_n)\in\boldsymbol{H}^n \text{ and } \varepsilon\in\boldsymbol{H}.$$

We define real linear transformations $\tilde{I}$, $\tilde{J}$, $\tilde{K}$ on $\boldsymbol{H}^n$ by

$$\tilde{I}\xi=\xi\cdot(-i), \quad \tilde{J}\xi=\xi\cdot(-j), \quad \tilde{K}\xi=\xi\cdot(-k), \qquad \text{for } \xi\in\boldsymbol{H}^n,$$

which satisfy $\tilde{I}^2=\tilde{J}^2=\tilde{K}^2=-id$, $\tilde{I}\tilde{J}=-\tilde{J}\tilde{I}=\tilde{K}$, $\tilde{J}\tilde{K}=-\tilde{K}\tilde{J}=\tilde{I}$, $\tilde{K}\tilde{I}=-\tilde{I}\tilde{K}=\tilde{J}$. In this paper we frequently use the notation in identifying $\boldsymbol{H}^n$ with $\boldsymbol{C}^{2n}$. For a vector $\xi=(z_1+w_1 j, \cdots, z_n+w_n j)\in\boldsymbol{H}^n$ we denote by $\xi=z+wj$, where $z=(z_1, \cdots, z_n)$, $w=(w_1, \cdots, w_n)\in\boldsymbol{C}^n$. Since the right action and left action are the same on a complex vector space, we some times mix these actions: For example, for $\alpha\in\boldsymbol{C}$ and $z\in\boldsymbol{C}^n$ we use both αz and $z\alpha$. Under the identification $\boldsymbol{H}^n$ with $\boldsymbol{C}^{2n}$, the actions of $\tilde{I}$, $\tilde{J}$, $\tilde{K}$ are expressed as follows;

$$\tilde{I}(z+wj)=(-iz)+(iw)j,$$

$$(1\text{-}1) \qquad \tilde{J}(z+wj)=w+(-z)j,$$

$$\tilde{K}(z+wj)=(-iw)+(-iz)j.$$

Let $(\!(\ ,\ )\!)$ denote the quaternion Hermitian inner product on the right vector space $\boldsymbol{H}^n$ defined by

$$(\!(\xi, \eta)\!)=\sum_{l=1}^{n}\bar{\xi}_l\eta_l \qquad \text{for } \xi=(\xi_1, \cdots, \xi_n),\ \eta=(\eta_1, \cdots, \eta_n)\in\boldsymbol{H}^n.$$

Clearly we have $(\!(\xi\cdot\varepsilon, \eta\cdot\delta)\!)=\bar{\varepsilon}(\!(\xi, \eta)\!)\delta$ for $\varepsilon, \delta\in\boldsymbol{H}$ and $(\!(\eta, \xi)\!)=\overline{(\!(\xi, \eta)\!)}$. The following basic lemmas will be useful in our argument.

LEMMA 1. *If unit vectors* $\xi, \eta, \zeta \in \boldsymbol{H}^n$ *satisfy* $(\!(\xi, \eta)\!) = (\!(\xi, \zeta)\!) = 0$ *and* $|(\!(\eta, \zeta)\!)| \neq 1$, *then they are* $\boldsymbol{H}$*-linearly independent.*

Let $\langle\!\langle\ ,\ \rangle\!\rangle$ denote the quaternion Hermitian form on the right vector space $\boldsymbol{H}^{n+1}$ defined by

$$\langle\!\langle \xi, \eta \rangle\!\rangle = -\bar{\xi}_0 \eta_0 + \sum_{l=1}^{n} \bar{\xi}_l \eta_l$$

for $\xi = (\xi_0, \cdots, \xi_n)$, $\eta = (\eta_0, \cdots, \eta_n) \in \boldsymbol{H}^{n+1}$. We also have $\langle\!\langle \xi \cdot \varepsilon, \eta \cdot \delta \rangle\!\rangle = \bar{\varepsilon} \langle\!\langle \xi, \eta \rangle\!\rangle \delta$ for $\varepsilon, \delta \in \boldsymbol{H}$ and $\langle\!\langle \eta, \xi \rangle\!\rangle = \overline{\langle\!\langle \xi, \eta \rangle\!\rangle}$.

LEMMA 1'. *If vectors* $\xi, \eta, \zeta \in \boldsymbol{H}^{n+1}$ *satisfy* $\langle\!\langle \xi, \xi \rangle\!\rangle = -1$, $\langle\!\langle \eta, \eta \rangle\!\rangle = \langle\!\langle \zeta, \zeta \rangle\!\rangle = 1$, $\langle\!\langle \xi, \eta \rangle\!\rangle = \langle\!\langle \xi, \zeta \rangle\!\rangle = 0$ *and* $|\langle\!\langle \eta, \zeta \rangle\!\rangle| \neq 1$, *then they are* $\boldsymbol{H}$*-linearly independent.*

§ 2. Structure torsion for circles on a quaternion Kähler monifold.

A Riemannian manifold M with a 3-dimensional subbundle $\mathcal{E}$ of the bundle $\mathrm{Hom}(TM, TM) \to M$ is called a *quaternion Kähler manifold* if the following conditions hold. For each point $x \in M$, there exist a neighborhood U of x and local frame field $\{I, J, K\}$ for $\mathcal{E}$ on U such that

i) $I^2 = J^2 = K^2 = -id$, $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$,

ii) there exist 1-forms p, q and r on U satisfying

$$\begin{aligned} \nabla_X I &= \quad r(X)J - q(X)K \\ \nabla_X J &= -r(X)I + p(X)K \\ \nabla_X K &= \quad q(X)I - p(X)J \quad . \end{aligned}$$

Let γ be a circle on a quaternion Kähler manifold M; $\nabla_X X = \kappa Y$, $\nabla_X Y = -\kappa X$, $X = \dot{\gamma}$. We define the structure torsion Θ for γ by

$$\Theta = \{\langle X, IY \rangle^2 + \langle X, JY \rangle^2 + \langle X, KY \rangle^2\}^{1/2}.$$

Since $\langle X, X \rangle \equiv 1$, we have $\langle X, Y \rangle \equiv 0$. Therefore the structure torsion is the norm of the projected vector $\mathrm{Proj}(X_t)$ of X_t onto the $\boldsymbol{H}$-linear subspace $\{Y_t \varepsilon \mid \varepsilon \in \boldsymbol{H}\}$, hence it is not greater than 1. We also find that Θ does not depend upon the parameter:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \Theta^2 = &\langle X, IY \rangle \{\langle \kappa Y, IY \rangle + \langle X, r(X)JY - q(X)KY - \kappa IX \rangle\} \\ &+ \langle X, JY \rangle \{\langle \kappa Y, JY \rangle + \langle X, -r(X)IY + p(X)KY - \kappa JX \rangle\} \\ &+ \langle X, KY \rangle \{\langle \kappa Y, KY \rangle + \langle X, q(X)IY - p(X)JY - \kappa KX \rangle\} \\ = &\ 0 . \end{aligned}$$

I. Circles on a quaternion projective space.

§ 3. Quaternion projective space.

Let $\langle\!\langle\ ,\ \rangle\!\rangle$ denote the quaternion Hermitian inner product on the right vector space $\boldsymbol{H}^{n+1}$ and $\langle\ ,\ \rangle$ denote the real scalar product; $\langle\xi, \eta\rangle=\mathrm{Re}\,\langle\!\langle\xi, \eta\rangle\!\rangle=\{\langle\!\langle\xi, \eta\rangle\!\rangle+\langle\!\langle\eta, \xi\rangle\!\rangle\}/2$. We therefore have

$$\langle\!\langle\xi, \eta\rangle\!\rangle=\langle\xi, \eta\rangle+\langle\xi, \tilde{I}\eta\rangle i+\langle\xi, \tilde{J}\eta\rangle j+\langle\xi, \tilde{K}\eta\rangle k\,.$$

The *quaternion projective space* $\boldsymbol{HP}^n$ is defined as the orbit space on $S^{4n+3}=\{\xi\in\boldsymbol{H}^{n+1}\,|\,\langle\!\langle\xi, \xi\rangle\!\rangle=1\}$ under the action of $S\boldsymbol{H}=\{\lambda\in\boldsymbol{H}\,|\,\bar{\lambda}\lambda=1\}$. We denote by $\pi: S^{4n+3}\to\boldsymbol{HP}^n$ the S^3-principal fiber bundle. When it admits the Riemannian metric $g(\ ,\ )=4\langle\ ,\ \rangle/c$, we denote as $\boldsymbol{HP}^n(c)$. Let $T_\xi S^{4n+3}=\mathcal{H}_\xi S^{4n+3}\oplus\mathcal{V}_\xi S^{4n+3}$ denote the orthogonal decomposition of the tangent space of S^{4n+3} at ξ into horizontal and vertical subspaces, where

$$\mathcal{H}_\xi S^{4n+3}=\{(\xi, \eta)\,|\,\eta\in\boldsymbol{H}^{n+1},\ \langle\!\langle\xi, \eta\rangle\!\rangle=0\}$$

$$\mathcal{V}_\xi S^{4n+3}=\{(\xi, \xi\lambda)\,|\,\lambda\in\boldsymbol{H},\ \mathrm{Re}\,\lambda=0\}\,.$$

For given ξ we can identify $T_{\pi(\xi)}\boldsymbol{HP}^n$ with $\mathcal{H}_\xi S^{4n+3}$. We shall denote $\rho_\xi(X)$ the identified vector for $X\in T_{\pi(\xi)}\boldsymbol{HP}^n$. In case we can see the base point ξ, we some times just denote it by $\tilde{X}$. For more detail see Besse [**5**] and also Tsukada [**9**].

We first study the relationship of connections. Let ∇, $\tilde{\nabla}$ and $\bar{\nabla}$ denote the Riemannian connection on $\boldsymbol{HP}^n(4)$, S^{4n+1} and $\boldsymbol{H}^{n+1}$ respectively. If we denote by N the outward unit normal of S^{4n+3} in $\boldsymbol{H}^{n+1}$, we have for any vector fields $\tilde{X}$ and $\tilde{Y}$ on S^{4n+3} that

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y}=\bar{\nabla}_{\tilde{X}}\tilde{Y}+\langle\tilde{X}, \tilde{Y}\rangle N\,.$$

For the relation between ∇ and $\tilde{\nabla}$ we have the following.

LEMMA 2. *Let γ be a smooth curve on $\boldsymbol{HP}^n(4)$ and $\tilde{\gamma}$ be one of its horizontal lift onto S^{4n+3}. For a vector field Y along γ we define a vector field along $\tilde{\gamma}$ by $\tilde{Y}(t)=\rho_{\tilde{\gamma}(t)}(Y)$. Put $X=\dot{\gamma}$ and $\tilde{X}=\dot{\tilde{\gamma}}$, then we have*

$$\rho_{\tilde{\gamma}(t)}(\nabla_X Y)=\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\langle\tilde{X}, \tilde{I}\tilde{Y}\rangle\tilde{I}N-\langle\tilde{X}, \tilde{J}\tilde{Y}\rangle\tilde{J}N-\langle\tilde{X}, \tilde{K}\tilde{Y}\rangle\tilde{K}N.$$

PROOF.

$$\begin{aligned}\rho_{\tilde{\gamma}(t)}(\nabla_X Y)&=\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\langle\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{I}N\rangle\tilde{I}N-\langle\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{J}N\rangle\tilde{J}N-\langle\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{K}N\rangle\tilde{K}N\\&=\tilde{\nabla}_{\tilde{X}}\tilde{Y}+\langle\tilde{Y}, \tilde{\nabla}_{\tilde{X}}(\tilde{I}N)\rangle\tilde{I}N+\langle\tilde{Y}, \tilde{\nabla}_{\tilde{X}}(\tilde{J}N)\rangle\tilde{J}N+\langle\tilde{Y}, \tilde{\nabla}_{\tilde{X}}(\tilde{K}N)\rangle\tilde{K}N\\&=\tilde{\nabla}_{\tilde{X}}\tilde{Y}+\langle\tilde{Y}, \bar{\nabla}_{\tilde{X}}(\tilde{I}N)\rangle\tilde{I}N+\langle\tilde{Y}, \bar{\nabla}_{\tilde{X}}(\tilde{J}N)\rangle\tilde{J}N+\langle\tilde{Y}, \bar{\nabla}_{\tilde{X}}(\tilde{K}N)\rangle\tilde{K}N\end{aligned}$$

$$= \tilde{\nabla}_{\tilde{X}}\tilde{Y}+\langle\tilde{Y}, \tilde{I}\tilde{X}\rangle\tilde{I}N+\langle\tilde{Y}, \tilde{J}\tilde{X}\rangle\tilde{J}N+\langle\tilde{Y}, \tilde{K}\tilde{X}\rangle\tilde{K}N$$

$$= \tilde{\nabla}_{\tilde{X}}\tilde{Y}-\langle\tilde{X}, \tilde{I}\tilde{Y}\rangle\tilde{I}N-\langle\tilde{X}, \tilde{J}\tilde{Y}\rangle\tilde{J}N-\langle\tilde{X}, \tilde{K}\tilde{Y}\rangle\tilde{K}N .$$

Let γ be a circle on $\boldsymbol{HP}^n(4)$ with geodesic curvature κ; $\nabla_X X=\kappa Y$, $\nabla_X Y=-\kappa X$. For a horizontal lift $\tilde{\gamma}$ of γ, we set

$$\theta = \theta(\tilde{\gamma}) = \langle\tilde{X}, \tilde{I}\tilde{Y}\rangle, \quad \varphi = \varphi(\tilde{\gamma}) = \langle\tilde{X}, \tilde{J}\tilde{Y}\rangle, \quad \phi = \phi(\tilde{\gamma}) = \langle\tilde{X}, \tilde{K}\tilde{Y}\rangle .$$

Since we have

$$\begin{aligned}\frac{d}{dt}\theta &= \langle\bar{\nabla}_{\tilde{X}}\tilde{X}, \tilde{I}\tilde{Y}\rangle+\langle\tilde{X}, \tilde{I}(\bar{\nabla}_{\tilde{X}}\tilde{Y})\rangle = \langle\tilde{\nabla}_{\tilde{X}}\tilde{X}-N, \tilde{I}\tilde{Y}\rangle+\langle\tilde{X}, \tilde{I}(\tilde{\nabla}_{\tilde{X}}\tilde{Y})\rangle \\ &= \langle\widetilde{(\nabla_X X)}, \tilde{I}\tilde{Y}\rangle+\langle\tilde{X}, \tilde{I}\{\widetilde{(\nabla_X Y)}+\theta\tilde{I}N+\varphi\tilde{J}N+\phi\tilde{K}N\}\rangle \\ &= \langle\kappa\tilde{Y}, \tilde{I}\tilde{Y}\rangle+\langle\tilde{X}, -\kappa\tilde{I}\tilde{X}-\theta N+\varphi\tilde{K}N-\phi\tilde{J}N\rangle = 0 ,\end{aligned}$$

the index θ does not depend on the parameter. Similarly φ and ϕ do not depend on the parameter. Though they depend on the choice of horizontal lifts, $(\theta^2+\varphi^2+\phi^2)^{1/2}$ does not depend on it, hence it is the structure torsion for the circle γ. By applying Lemma 2 we have

PROPOSITION 1. *Let γ denote a circle on $\boldsymbol{HP}^n(4)$ with geodesic curvature κ and structure torsion $\Theta : \nabla_X X=\kappa Y$ and $\nabla_X Y=-\kappa X$. Then its horizontal lift $\tilde{\gamma}$ on S^{4n+3} is a helix of order 2, 3 or 5 corresponding to $\Theta=0$, $\Theta=1$ and $0<\Theta<1$. It satisfies the following differential equations*:

$$\begin{aligned}\tilde{\nabla}_{\tilde{X}}\tilde{X} &= \kappa\tilde{Y} ,\\ \tilde{\nabla}_{\tilde{X}}\tilde{Y} &= -\kappa\tilde{X} + \Theta S ,\\ \tilde{\nabla}_{\tilde{X}}S &= -\Theta\tilde{Y} + \sqrt{1-\Theta^2}U ,\\ \tilde{\nabla}_{\tilde{X}}U &= -\sqrt{1-\Theta^2}S + \kappa V ,\\ \tilde{\nabla}_{\tilde{X}}V &= -\kappa U ,\end{aligned}$$

where $\tilde{X}=\rho_{\tilde{\gamma}}(X)=\dot{\tilde{\gamma}}$, $\tilde{Y}=\rho_{\tilde{\gamma}}(Y)$, $S=1/\Theta\cdot(\theta\tilde{I}N+\varphi\tilde{J}N+\phi\tilde{K}N)$, $U=1/\Theta\sqrt{1-\Theta^2}\cdot(\theta\tilde{I}\tilde{X}+\varphi\tilde{J}\tilde{X}+\phi\tilde{K}\tilde{X}+\Theta^2\tilde{Y})$, *and* $V=1/\Theta\sqrt{1-\Theta^2}\cdot(\theta\tilde{I}\tilde{Y}+\varphi\tilde{J}\tilde{Y}+\phi\tilde{K}\tilde{Y}-\Theta^2\tilde{X})$.

By solving these equations we shall show the following.

THEOREM 1. *Let γ be a circle with geodesic curvature κ and structure torsion Θ on a quaternion projective space $\boldsymbol{HP}^n(c)$.*

(1) *If $\Theta=0$, then it is a simple closed curve with prime period $4\pi/\sqrt{4\kappa^2+c}$.*

(2) *If $\Theta=1$, then it is a simple closed curve with prime period $2\pi/\sqrt{\kappa^2+c}$.*

(3) *When $0<\Theta<1$, we denote by a, b and d $(a<b<d)$ the nonzero solutions for $c\lambda^3-(4\kappa^2+c)\lambda-2\sqrt{c}\kappa\Theta=0$.*

(i) *If one of the three ratios* a/b, b/d *and* d/a *is rational, it is a simple closed curve. Its prime period is the least common multiple of* $4\pi/\sqrt{c}(b-a)$ *and* $4\pi/\sqrt{c}(d-a)$.

(ii) *If each of the three ratios* a/b, b/d *and* d/a *is irrational, it is a simple open curve.*

Before we start the proof, we mention for the relationship between homothetic change on metrics and the geodesic curvature of circles. Let γ be a circle on a Riemannian manifold (M, g) with geodesic curvature κ. When we change the metric homothetically $g \to m^2 \cdot g$ for some positive constant m, the curve $\sigma(t)=\gamma(t/m)$ is a circle on $(M, m^2 \cdot g)$ with curvature κ/m. Under the operation $g \to m^2 g$, the prime period of a closed curve changes to m-times of the original prime period. We therefore treat only for the case $c=4$.

For a geodesic on $\boldsymbol{H}P^n(4)$, which is the case with $\kappa=0$ in Proposition 1, we get that a horizontal lift $\tilde{\gamma}$ of γ satisfies $\ddot{\tilde{\gamma}}(t)+\tilde{\gamma}(t)=0$. Hence we have the following expression.

PROPOSITION 2. *The geodesic* γ *on a quaternion projective space* $\boldsymbol{H}P^n(4)$ *with* $\dot{\gamma}(0)=d\pi((\xi, \eta))$ *is given by*

$$\gamma(t) = \pi(\xi \cos t + \eta \sin t).$$

§4. Circles on $\boldsymbol{H}P^n(4)$ with $\Theta=1$.

We first treat the case that the structure torsion Θ is 1. In this case we can write dawn $\tilde{Y}$ as $\tilde{Y}=-(\theta \tilde{I}\tilde{X}+\varphi \tilde{J}\tilde{X}+\phi \tilde{K}\tilde{X})$. Therefore, by Proposition 1, our equation for a horizontal lift $\tilde{\gamma}$ is

$$\tilde{\nabla}_{\tilde{X}}\tilde{X} = -\kappa(\theta \tilde{I}\tilde{X}+\varphi \tilde{J}\tilde{X}+\phi \tilde{K}\tilde{X}), \tag{4-1}$$

or equivalently,

$$\overline{\nabla}_{\tilde{X}}\tilde{X} = -\kappa(\theta \tilde{I}\tilde{X}+\varphi \tilde{J}\tilde{X}+\phi \tilde{K}\tilde{X})-N. \tag{4-1'}$$

We shall express $\tilde{\gamma}$ as $\tilde{\gamma}(t)=Z(t)+W(t)j$, $Z(t), W(t) \in \boldsymbol{C}^{n+1}$. By (1-1) we can rewrite (4-1′) into the following linear differential equations;

$$\begin{cases} \ddot{Z} = \kappa(\theta i\dot{Z}-\varphi \dot{W}+\phi i\dot{W})-Z \\ \ddot{W} = \kappa(-\theta i\dot{W}+\varphi \dot{Z}+\phi i\dot{Z})-W. \end{cases} \tag{4-2}$$

The eigenvalues of the associated matrix of the linear equation

$$\frac{d}{dt}\begin{pmatrix} Z \\ \dot{Z} \\ W \\ \dot{W} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \kappa\theta i & 0 & -\kappa(\varphi-\psi i) \\ 0 & 0 & 0 & 1 \\ 0 & \kappa(\varphi+\psi i) & -1 & -\kappa\theta i \end{pmatrix}\begin{pmatrix} Z \\ \dot{Z} \\ W \\ \dot{W} \end{pmatrix}$$

are $\pm ai$, $\pm bi$, where

$$a=\frac{1}{2}(\kappa+\sqrt{\kappa^2+4}), \qquad b=\frac{1}{2}(\kappa-\sqrt{\kappa^2+4}).$$

Computing associated eigenvectors we obtain the following explicit expression for $\tilde{\gamma}$ under the initial condition that

$$\tilde{\gamma}(0)=\xi=x+yj, \quad \dot{\tilde{\gamma}}(0)=\eta=u+vj, \qquad x, y, u, v\in \boldsymbol{C}^{n+1}.$$

When $\theta\neq 1$ we have by putting $\omega=(\psi+i\varphi)/(1-\theta)$ that

$$Z(t)=\omega(Ae^{iat}+Be^{ibt})-Ce^{-iat}-De^{-ibt},$$

$$W(t)=Ae^{iat}+Be^{ibt}+\bar{\omega}(Ce^{-iat}+De^{-ibt}),$$

where

$$A=(1-\theta)(-b\bar{\omega}x-by-i\bar{\omega}u-iv)/2\sqrt{\kappa^2+4},$$

$$B=(1-\theta)(a\bar{\omega}x+ay+i\bar{\omega}u+iv)/2\sqrt{\kappa^2+4},$$

$$C=(1-\theta)(bx-b\omega y-iu+i\omega v)/2\sqrt{\kappa^2+4},$$

$$D=(1-\theta)(-ax+a\omega y+iu-i\omega v)/2\sqrt{\kappa^2+4}.$$

Hence we get

(4-3) $$\tilde{\gamma}(t)=\{(A+Cj)e^{iat}+(B+Dj)e^{ibt}\}(\omega+j),$$

$$=\Big\{\{\xi b(-\bar{\omega}+j)+\eta(-i)(\bar{\omega}+j)\}e^{iat} + \{\xi a(\bar{\omega}-j)+\eta i(\bar{\omega}+j)\}e^{ibt}\Big\}\frac{1-\theta}{2\sqrt{\kappa^2+4}}(\omega+j).$$

When $\theta=1$ we have

(4-4) $$\tilde{\gamma}(t)=\{\xi(-be^{iat}+ae^{ibt})+\eta i(-e^{iat}+e^{ibt})\}(\kappa^2+4)^{-1/2}.$$

PROPOSITION 3. *Let γ be a circle with geodesic curvature κ on $\boldsymbol{HP}^n(4)$. If its structure torsion Θ is 1, then it is a simple closed curve with prime period $2\pi/\sqrt{\kappa^2+4}$.*

PROOF. Let $\tilde{\gamma}$ be a horizontal lift of γ. The condition that $\gamma(t_0)=\gamma(0)$ is equivalent to the condition that

(4-5) $$\tilde{\gamma}(t_0)=\tilde{\gamma}(0)\varepsilon \quad \text{for some } \varepsilon\in S\boldsymbol{H}.$$

We may suppose $\theta\neq 1$. Since $\langle\langle\tilde{\gamma}(0), \dot{\tilde{\gamma}}(0)\rangle\rangle=0$ we have ξ, η are $\boldsymbol{H}$-linearly independent. Therefore by using the expression (4-3) we find (4-5) is equivalent to

$$\{-b(\bar{\omega}-j)e^{iat_0}+a(\bar{\omega}-j)e^{ibt_0}\}(\omega+j)=(a-b)\frac{2}{1-\theta}\varepsilon$$

$$e^{iat_0}-e^{ibt_0}=0,$$

which hold if and only if

$$(a-b)t_0/\pi=\sqrt{\kappa^2+4}\,t_0/\pi\in\boldsymbol{Z} \quad \text{and} \quad \varepsilon=(\omega+j)^{-1}e^{ibt_0}(\omega+j).$$

Since

$$\begin{aligned}\dot{\tilde{\gamma}}(t_0)&=\{(A+Cj)iae^{iat_0}+(B+Dj)ibe^{ibt_0}\}(\omega+j)\\&=\{(A+Cj)iae^{i(a-b)t_0}+(B+Dj)ib\}e^{ibt_0}(\omega+j)=\dot{\tilde{\gamma}}(0)\varepsilon,\end{aligned}$$

we can conclude γ is a simple closed curve with prime period $2\pi/\sqrt{\kappa^2+4}$.

§5. Circles on $HP^n(4)$ with $\Theta=0$.

Next we shall concern ourselves with circles having null structure torsion. In this case the equations for a horizontal lift $\tilde{\gamma}$ are

(5-1) $$\tilde{\nabla}_{\tilde{X}}\tilde{X}=\kappa\tilde{Y}, \qquad \tilde{\nabla}_{\tilde{X}}\tilde{Y}=-\kappa\tilde{X}$$

which are equivalent to

(5-1′) $$\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{X}}\tilde{X}=-(\kappa^2+1)\tilde{X}.$$

As in §4 denoting $\tilde{\gamma}(t)=Z(t)+W(t)j$ we can rewrite this into the following;

(5-2) $$\begin{cases} Z^{(3)}=-(\kappa^2+1)\dot{Z}\\ W^{(3)}=-(\kappa^2+1)\dot{W},\end{cases}$$

which are the same type equations as the equation for circles with null complex torsion. Solving these equations under the initial conditions

$$\tilde{\gamma}(0)=\xi=x+yj,\quad \dot{\tilde{\gamma}}(0)=\eta=u+vj,\quad \ddot{\tilde{\gamma}}(0)+\tilde{\gamma}(0)=\zeta\kappa=(z+wj)\kappa,$$
$$x,\ y,\ u,\ v,\ z,\ w\in\boldsymbol{C}^{n+1},$$

we get the following explicit expression of $\tilde{\gamma}$.

$$Z(t)=A\exp(i\sqrt{\kappa^2+1}\,t)+B+C\exp(-i\sqrt{\kappa^2+1}\,t),$$

$$W(t)=D\exp(i\sqrt{\kappa^2+1}\,t)+B+F\exp(-i\sqrt{\kappa^2+1}\,t),$$

where

$$A = \{2(\kappa^2+1)\}^{-1}(x-\kappa z)-i(2\sqrt{\kappa^2+1})^{-1}u$$

$$D = \{2(\kappa^2+1)\}^{-1}(y-\kappa w)-i(2\sqrt{\kappa^2+1})^{-1}v$$

$$B = \kappa(\kappa^2+1)^{-1}(\kappa x+z), \quad E = \kappa(\kappa^2+1)^{-1}(\kappa y+w)$$

$$C = \{2(\kappa^2+1)\}^{-1}(x-\kappa z)+i(2\sqrt{\kappa^2+1})^{-1}u$$

$$F = \{2(\kappa^2+1)\}^{-1}(y-\kappa w)+i(2\sqrt{\kappa^2+1})^{-1}v\,.$$

Hence we have

$$\begin{aligned}(5\text{-}3)\qquad \tilde{\gamma}(t) &= (A+Fj)\exp(i\sqrt{\kappa^2+1}\,t)+(B+Ej)+(C+Dj)\exp(-i\sqrt{\kappa^2+1}\,t),\\ &= \{(\xi-\zeta\kappa-\eta i\sqrt{\kappa^2+1})\exp(i\sqrt{\kappa^2+1}\,t)+\xi\cdot 2\kappa^2+\zeta\cdot 2\kappa\\ &\quad +(\xi-\zeta\kappa+\eta i\sqrt{\kappa^2+1})\exp(-i\sqrt{\kappa^2+1}\,t)\}\,\{2(\kappa^2+1)\}^{-1}.\end{aligned}$$

PROPOSITION 4. *Let γ be a circle with geodesic curvature κ on $\boldsymbol{H}P^n(4)$. If its structure torsion Θ is 0, then it lies on a totally geodesic $\boldsymbol{R}P^2(1)$, hence is a simple closed curve with prime period $2\pi/\sqrt{\kappa^2+1}$.*

§ 6. Circles on $\boldsymbol{H}P^n(4)$ with $0<\Theta<1$.

The rest of this chapter is devoted to study the case $0<\Theta<1$. In this case the equations for a horizontal lift $\tilde{\gamma}$ of a circle are

$$(6\text{-}1)\qquad \begin{cases} \tilde{\nabla}_{\tilde{X}}\tilde{X} = \kappa\tilde{Y} \\ \tilde{\nabla}_{\tilde{X}}\tilde{Y} = -\kappa\tilde{X}+(\theta\tilde{I}N+\varphi\tilde{J}N+\phi\tilde{K}N) \end{cases}$$

which are equivalent to

$$(6\text{-}1')\qquad \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{X}}\tilde{X} = -(\kappa^2+1)\tilde{X}+\kappa(\theta\tilde{I}N+\varphi\tilde{J}N+\phi\tilde{K}N)\,.$$

As usual, denoting $\tilde{\gamma}(t)=Z(t)+W(t)j$ we can rewrite this into the following;

$$(6\text{-}2)\qquad \begin{cases} Z^{(3)} = -(\kappa^2+1)\dot{Z}-\kappa(i\theta Z-\varphi W+i\phi W) \\ W^{(3)} = -(\kappa^2+1)\dot{W}+\kappa(i\theta W-\varphi Z-i\phi Z)\,. \end{cases}$$

Consider a cubic equation $\tau^3-(\kappa^2+1)\tau-\kappa\Theta=0$. This equation has three nonzero real solutions a, b, d $(a<b<d)$, which satisfy $a+b+d=0$, $ab+bd+ba=-(\kappa^2+1)$ and $abd=\kappa\Theta$. The eigenvalues of the matrix associated to the differential equations (6-2) are the solutions of one of the cubic equations $\lambda^3+(\kappa^2+1)\lambda\pm\kappa\Theta i=0$, hence $\pm ai$, $\pm bi$ and $\pm di$. Under the initial conditions

$$\tilde{\gamma}(0) = \xi = x+yj, \quad \dot{\tilde{\gamma}}(0) = \eta = u+vj, \quad \dddot{\tilde{\gamma}}(0)+\dot{\tilde{\gamma}}(0) = \zeta\kappa = (z+wj)\kappa\,,$$

$$x, y, u, v, z, w\in \boldsymbol{C}^{n+1},$$

we get the following solutions of (6-2). When $\theta\neq\Theta$ we have

$$Z(t) = \omega(Ae^{ait}+Be^{bit}+Ce^{dit})-De^{-ait}-Ee^{-bit}-Fe^{-dit},$$

$$W(t) = Ae^{ait}+Be^{bit}+Ce^{dit}+\bar{\omega}(De^{-ait}+Ee^{-bit}+Fe^{-dit}),$$

where $\omega=\omega_\Theta=(\psi+i\varphi)/(\Theta-\theta)$ and

$$A = \frac{\Theta-\theta}{2(d-a)(a-b)\Theta}\{-(1+bd)(\bar{\omega}x+y)+ai(\bar{\omega}u+v)+\kappa(\bar{\omega}z+w)\},$$

$$B = \frac{\Theta-\theta}{2(a-b)(b-d)\Theta}\{-(1+da)(\bar{\omega}x+y)+bi(\bar{\omega}u+v)+\kappa(\bar{\omega}z+w)\},$$

$$C = \frac{\Theta-\theta}{2(b-d)(d-a)\Theta}\{-(1+ab)(\bar{\omega}x+y)+di(\bar{\omega}u+v)+\kappa(\bar{\omega}z+w)\},$$

$$D = \frac{\Theta-\theta}{2(d-a)(a-b)\Theta}\{(1+bd)(x-\omega y)+ai(u-\omega v)-\kappa(z-\omega w)\},$$

$$E = \frac{\Theta-\theta}{2(a-b)(b-d)\Theta}\{(1+da)(x-\omega y)+bi(u-\omega v)-\kappa(z-\omega w)\},$$

$$F = \frac{\Theta-\theta}{2(b-d)(d-a)\Theta}\{(1+ab)(x-\omega y)+di(u-\omega v)-\kappa(z-\omega w)\}.$$

Hence

$$\text{(6-3)}\qquad \begin{aligned}\tilde{\gamma}(t) &= \{(A+Dj)e^{ait}+(B+Ej)e^{bit}+(C+Fj)e^{dit}\}(\omega+j),\\ &= \Big\{\{-\xi(1+bd)(\bar{\omega}-j)+\eta ai(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(b-d)e^{ait}\\ &\quad+\{-\xi(1+da)(\bar{\omega}-j)+\eta bi(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(d-a)e^{bit}\\ &\quad+\{-\xi(1+ab)(\bar{\omega}-j)+\eta di(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(a-b)e^{dit}\Big\}\\ &\quad\times\frac{\Theta-\theta}{2(a-b)(b-d)(d-a)\Theta}(\omega+j).\end{aligned}$$

When $\theta=\Theta$ we obtain

$$\text{(6-4)}\qquad \begin{aligned}\tilde{\gamma}(t) &= \Big\{\{-\xi(1+bd)+\eta ai+\zeta\kappa\}(b-d)e^{ait}\\ &\quad+\{-\xi(1+da)+\eta bi+\zeta\kappa\}(d-a)e^{bit}\\ &\quad+\{-\xi(1+ab)+\eta di+\zeta\kappa\}(a-b)e^{dit}\Big\}\frac{1}{(a-b)(b-d)(d-a)}.\end{aligned}$$

By these expressions we have

PROPOSITION 5. *Every circle on* $\boldsymbol{HP}^n(4)$ *with geodesic curvature* κ *and structure torsion* Θ $(0<\Theta<1)$ *is a simple curve. Further more, let* a, b *and* d $(a<b<d)$ *be the nonzero real solutions for the cubic equation* $\tau^3-(\kappa^2+1)\tau-\kappa\Theta$

$=0$. *Then it is closed if and only if one of the three ratios a/b, b/d and d/a is rational. In that case, its prime period is the least common multiple of $2\pi/(b-a)$ and $2\pi/(d-a)$.*

PROOF. Let $\tilde{\gamma}$ be a horizontal lift of γ. We may suppose $\theta \neq \Theta$ so we use the expression (6-3). Since $\langle\langle\tilde{\gamma}, \tilde{X}\rangle\rangle = \langle\langle\tilde{\gamma}, \tilde{Y}\rangle\rangle = 0$ and $|\langle\langle\tilde{X}, \tilde{Y}\rangle\rangle| = \Theta < 1$, we find $\tilde{\gamma}(0) = \xi$, $\tilde{X}_0 = \eta$ and $\tilde{Y}_0 = \zeta$ are $\boldsymbol{H}$-linearly independent. Therefore we can rewrite the condition $\gamma(t_0) = \gamma(0)$ into

$$\{(b-d)(1+bd)(\bar{\omega}-j)e^{ait_0}+(d-a)(1+da)(\bar{\omega}-j)e^{bit_0}$$
$$+(a-b)(1+ab)(\bar{\omega}-j)e^{dit_0}\}(\omega+j)$$
$$= \{(b-d)bd+(d-a)da+(a-b)ab\}\frac{2\Theta}{\Theta-\theta}\varepsilon$$

$$a(b-d)e^{ait_0}+b(d-a)e^{bit_0}+d(a-b)e^{dit_0} = 0 \tag{6-5}$$
$$(b-d)e^{ait_0}+(d-a)e^{bit_0}+(a-b)e^{dit_0} = 0\,.$$

for some $\varepsilon \in S\boldsymbol{H}$. The second and third equalities in (6-5) imply $(a-d)/(a-b)$ and $(a-d)/(b-d)$ are rational numbers. Since $a+b+d=0$, this implies a/b, $b/d \in \boldsymbol{Q}$. Hence if $a/b \notin \boldsymbol{Q}$, then γ is a simple open curve. (Of course, by $a+b+d=0$ the condition $a/b \in \boldsymbol{Q}$ is equivalent to $b/d \in \boldsymbol{Q}$ or $d/a \in \boldsymbol{Q}$.)

When $a/b \in \boldsymbol{Q}$, or equivalently $(a-d)/(a-b) \in \boldsymbol{Q}$, if we choose t_0 as a common multiple of $2\pi/(b-a)$ and $2\pi/(d-a)$, then the equalities (6-5) hold with $\varepsilon = (\omega+j)^{-1}e^{idt_0}(\omega+j)$. One can easily get $\dot{\tilde{\gamma}}(t_0) = \dot{\tilde{\gamma}}(0)\varepsilon$ and $\ddot{\tilde{\gamma}}(t_0) = \ddot{\tilde{\gamma}}(0)\varepsilon$. Hence we find γ is simply closed if $a/b \in \boldsymbol{Q}$.

Summarizing up Proposition 3, 4, 5, we get Theorem 1.

II. Circles on a quaternion hyperbolic space.

§ 7. Quaternion hyperbolic space.

We shall start with recalling on a quaternion hyperbolic space. Let $\langle\langle\,,\rangle\rangle$ denote the quaternion Hermitian form on the right vector space $\boldsymbol{H}^{n+1}$. We set $\langle\,,\rangle = \mathrm{Re}\langle\langle\,,\rangle\rangle$, which is invariant under the action of $S\boldsymbol{H}$; $\langle\xi\lambda, \eta\lambda\rangle = \langle\xi, \eta\rangle$ for $\lambda \in S\boldsymbol{H}$. The *quaternion hyperbolic space* $\boldsymbol{H}H^n$ is defined as the orbit space on $H_1^{4n+3} = \{\xi \in \boldsymbol{H}^{n+1} \mid \langle\langle\xi, \xi\rangle\rangle = -1\}$ under the action of $S\boldsymbol{H}$. We denote by $\pi: H_1^{4n+3} \to \boldsymbol{H}H^n$ the S^3-principal fiber bundle. When it admits the Riemannian metric $g(\,,)$ $=4\langle\,,\rangle/c$, we denote as $\boldsymbol{H}H^n(-c)$. Let $D_n(\boldsymbol{H})$ denote the unit disk in $\boldsymbol{H}^n$;

$$D_n(\boldsymbol{H}) = \{\mu \in \boldsymbol{H}^n \mid \langle\langle\mu, \mu\rangle\rangle < 1\}\,.$$

We can identify $\boldsymbol{H}H^n$ with $D_n(\boldsymbol{H})$ by the map $\Phi: \boldsymbol{H}H^n \to D_n(\boldsymbol{H})$ which is given by

$$\Phi(\pi(\xi)) = (\xi_1\xi_0^{-1}, \cdots, \xi_n\xi_0^{-1}).$$

We denote by $T_\xi H_1^{4n+3} = \mathcal{H}_\xi H_1^{4n+3} \oplus \mathcal{V}_\xi H_1^{4n+3}$ the orthogonal decomposition of the tangent space of H_1^{4n+3} at ξ into horizontal and vertical subspaces, where

$$\mathcal{H}_\xi H_1^{4n+3} = \{(\xi, \eta) \mid \eta \in \boldsymbol{H}^{n+1}, \langle\langle \xi, \eta \rangle\rangle = 0\}$$

$$\mathcal{V}_\xi H_1^{4n+3} = \{(\xi, \xi\lambda) \mid \lambda \in \boldsymbol{H}, \operatorname{Re}\lambda = 0\}.$$

For given ξ we can identify $T_{\pi(\xi)}\boldsymbol{HH}^n$ with $\mathcal{H}_\xi H_1^{4n+3}$. We shall denote $\rho_\xi(X)$ the identified vector for $X \in T_{\pi(\xi)}\boldsymbol{HH}^n$.

We first study the relationship of connections. Let ∇, $\tilde{\nabla}$ and $\bar{\nabla}$ denote the Riemannian connection on $\boldsymbol{HH}^n(-4)$, H_1^{4n+3} and $\boldsymbol{H}^{n+1}$ respectively. If we denote by N the position vector of H_1^{4n+3} in $\boldsymbol{H}^{n+1}$, which satisfies $\langle N, N \rangle = -1$, we have for any vector fields $\tilde{X}$ and $\tilde{Y}$ on H_1^{4n+3} that

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \bar{\nabla}_{\tilde{X}}\tilde{Y} - \langle \tilde{X}, \tilde{Y} \rangle N.$$

For the relation between ∇ and $\tilde{\nabla}$ we get the following by a same argument as in Lemma 1, except $\langle N, N \rangle = -1$.

LEMMA 3. *Let γ be a smooth curve on $\boldsymbol{HH}^n(-4)$ and $\tilde{\gamma}$ be one of its horizontal lift onto H_1^{4n+3}. For a vector field Y along γ we define a vector field along $\tilde{\gamma}$ by $\tilde{Y}(t) = \rho_{\tilde{\gamma}(t)}(Y)$. Put $X = \dot{\gamma}$ and $\tilde{X} = \dot{\tilde{\gamma}}$, then we have*

$$\rho_{\tilde{\gamma}(t)}(\nabla_X Y) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} + \langle \tilde{X}, \tilde{I}\tilde{Y} \rangle \tilde{I}N + \langle \tilde{X}, \tilde{J}\tilde{Y} \rangle \tilde{J}N + \langle \tilde{X}, \tilde{K}\tilde{Y} \rangle \tilde{K}N.$$

Let γ be a circle on $\boldsymbol{HH}^n(-4)$ with geodesic curvature $\kappa : \nabla_X X = \kappa Y$, $\nabla_X Y = -\kappa X$. For a horizontal lift $\tilde{\gamma}$ of γ, we set

$$\theta = \theta(\tilde{\gamma}) = \langle \tilde{X}, \tilde{I}\tilde{Y} \rangle, \quad \varphi = \varphi(\tilde{\gamma}) = \langle \tilde{X}, \tilde{J}\tilde{Y} \rangle, \quad \psi = \psi(\tilde{\gamma}) = \langle \tilde{X}, \tilde{K}\tilde{Y} \rangle,$$

which do not depend on the parameter. The structure torsion Θ coincides with $(\theta^2 + \varphi^2 + \psi^2)^{1/2}$. Applying Lemma 3 we can conclude the following.

PROPOSITION 6. *Let γ denote a circle on $\boldsymbol{HH}^n(-4)$ with geodesic curvature κ and structure torsion $\Theta : \nabla_X X = \kappa Y$ and $\nabla_X Y = -\kappa X$. Then its horizontal lift $\tilde{\gamma}$ on H_1^{4n+3} is a helix of order* 2, 3 *or* 5 *corresponding to* $\Theta = 0$, $\Theta = 1$ *and* $0 < \Theta < 1$. *It satisfies the following differential equations*:

$$\begin{aligned}
\tilde{\nabla}_{\tilde{X}}\tilde{X} &= \kappa\tilde{Y}, \\
\tilde{\nabla}_{\tilde{X}}\tilde{Y} &= -\kappa\tilde{X} - \Theta S, \\
\tilde{\nabla}_{\tilde{X}}S &= -\Theta\tilde{Y} + \sqrt{1-\Theta^2}\,U, \\
\tilde{\nabla}_{\tilde{X}}U &= \sqrt{1-\Theta^2}\,S + \kappa V, \\
\tilde{\nabla}_{\tilde{X}}V &= -\kappa U,
\end{aligned}$$

where $\tilde{X}=\rho_{\tilde{\gamma}}(X)=\dot{\tilde{\gamma}}$, $\tilde{Y}=\rho_{\tilde{\gamma}}(Y)$, $S=1/\Theta\cdot(\theta\tilde{I}N+\varphi\tilde{J}N+\phi\tilde{K}N)$, $U=1/\Theta\sqrt{1-\Theta^2}\cdot(\theta\tilde{I}\tilde{X}+\varphi\tilde{J}\tilde{X}+\phi\tilde{K}\tilde{X}+\Theta^2\tilde{Y})$, *and* $V=1/\Theta\sqrt{1-\Theta^2}\cdot(\theta\tilde{I}\tilde{Y}+\varphi\tilde{J}\tilde{Y}+\phi\tilde{K}\tilde{Y}-\Theta^2\tilde{X})$.

A quaternion hyperbolic space is a typical example of a Hadamard manifold, a simply connected Riemannian manifold with nonpositive curvature. For a Hadamard manifold M we have an important notion of the ideal boundary ∂M. Under the identification $\boldsymbol{H}H^n$ with $D_n(\boldsymbol{H})$, the ideal boundary can be identified with $\partial D_n(\boldsymbol{H})=\{\zeta\in\boldsymbol{H}^n\,|\,\langle\!\langle\zeta,\zeta\rangle\!\rangle=1\}$, and the compactification of $\boldsymbol{H}H^n$ with its ideal boundary coincides with the topological compactification of $D_n(\boldsymbol{H})$. For a curve γ on a Hadamard manifold M, we call γ two-sides unbounded if both $\{\gamma(t)\,|\,t>0\}$ and $\{\gamma(t)\,|\,t<0\}$ are unbounded sets. We put

$$\gamma(\infty)=\lim_{t\to\infty}\gamma(t),\quad \gamma(-\infty)=\lim_{t\to-\infty}\gamma(t),$$

if the limits exist in $\bar{M}=M\cup\partial M$. We call $\gamma(\infty)$ and $\gamma(-\infty)$ the points at infinity for γ. We shall call γ *horocyclic* if

i) it has single point at infinity; $\gamma(\infty)=\gamma(-\infty)$,

ii) it crosses orthogonally at $\gamma(t)$ to the geodesic joining $\gamma(t)$ and $\gamma(\infty)$.

Our theorem which we shall show in this chapter is the following.

THEOREM 2. *Let γ be a circle with geodesic curvature κ and structure torsion Θ on a quaternion hyperbolic space $\boldsymbol{H}H^n(-c)$. We denote by κ_Θ the unique positive solution for $27\lambda^2\Theta^2-4(\lambda^2-1)^3=0$. Then the following hold:*

(1) *When $\kappa\leqq\sqrt{c}\kappa_\Theta/2$, it is a simple two-sides unbounded open curve. If $\kappa=\sqrt{c}\kappa_\Theta/2$, it is horocyclic, and if $\kappa<\sqrt{c}\kappa_\Theta/2$, it has two distinct points at infinity.*

(2) *When $\Theta=0$ and $\kappa>\sqrt{c}\kappa_0/2=\sqrt{c}/2$, it is a simple closed curve with prime period $4\pi/\sqrt{4k^2-c}$.*

(3) *When $\Theta=1$ and $\kappa>\sqrt{c}\kappa_1/2=\sqrt{c}$, it is a simple closed curve with prime period $2\pi/\sqrt{k^2-c}$.*

(4) *When $0<\Theta<1$ and $\kappa>\sqrt{c}\kappa_\Theta/2$, we denote by a, b and d $(a<b<d)$ the non-zero real solutions for $c\lambda^3-(4\kappa^2-c)\lambda-2\sqrt{c}\kappa\Theta=0$.*

(i) *If one of the three ratios a/b, b/d and d/a is rational, it is a simple closed curve. Moreover, its prime period is the least common multiple of $4\pi/\sqrt{c}(b-a)$ and $4\pi/\sqrt{c}(d-a)$.*

(ii) *If each of the three ratios a/b, b/d and d/a is irrational, it is a simple bounded open curve.*

We first point out the expression of geodesics on $\boldsymbol{H}H^n(-4)$. By Proposition 6, a horizontal lift $\tilde{\gamma}$ of a geodesic γ satisfies $\ddot{\tilde{\gamma}}-\tilde{\gamma}=0$. We hence get

PROPOSITION 7. *The geodesic γ on a quaternion hyperbolic space $\boldsymbol{H}H^n(-4)$ with $\dot{\gamma}(0)=d\pi((\xi,\eta))$ is expressed as*

$$\gamma(t) = \pi(\xi \cosh t + \eta \sinh t).$$

§ 8. Circles on $HH^n(-4)$ with $\Theta=1$.

We first treat the case that the structure torsion Θ is 1, hence $\tilde{Y} = -(\theta \tilde{I}\tilde{X} + \varphi \tilde{J}\tilde{X} + \psi \tilde{K}\tilde{X})$. Our equation for a horizontal lift is

$$\tilde{\nabla}_{\tilde{X}} \tilde{X} = -\kappa(\theta \tilde{I}\tilde{X} + \varphi \tilde{J}\tilde{X} + \psi \tilde{K}\tilde{X}), \tag{8-1}$$

or equivalently,

$$\overline{\nabla}_{\tilde{X}} \tilde{X} = -\kappa(\theta \hat{I}\tilde{X} + \varphi \hat{J}\tilde{X} + \psi \hat{K}\tilde{X}) + N. \tag{8-1'}$$

Putting $\tilde{\gamma}(t) = Z(t) + W(t)j$, $Z(t), W(t) \in \boldsymbol{C}^{n+1}$, we obtain the following linear differential equations.

$$\begin{cases} \ddot{Z} = \kappa(\theta i \dot{Z} - \varphi \dot{W} + \psi i \dot{W}) + Z \\ \ddot{W} = \kappa(-\theta i \dot{W} + \varphi \dot{Z} + \psi i \dot{Z}) + W. \end{cases} \tag{8-2}$$

The eigenvalues of the matrix associated to these equations are $\pm\alpha$, $\pm\beta$, where

$$\alpha = \frac{1}{2}(\kappa i + \sqrt{4-\kappa^2}), \quad \beta = \frac{1}{2}(\kappa i - \sqrt{4-\kappa^2}).$$

So the type of the solution for (8-2) changes according to $\kappa > 2$, $\kappa = 2$ and $\kappa < 2$. Solving this equation under the initial conditions that

$$\tilde{\gamma}(0) = \xi = x + yj, \quad \dot{\tilde{\gamma}}(0) = \eta = u + vj, \qquad x, y, u, v \in \boldsymbol{C}^{n+1},$$

we get the following. When $\theta \neq 1$ we get by putting $\omega = (\psi + i\varphi)/(1-\theta)$ the following.

$$Z(t) = \begin{cases} \omega(Ae^{\alpha t} + Be^{\beta t}) - (Ce^{-\alpha t} + De^{-\beta t}), & \text{if } \kappa \neq 2, \\ \dfrac{1-\theta}{2}\Big\{\omega\{(1-it)(\bar{\omega}x + y) + t(\bar{\omega}u + v)\}e^{it} \\ \qquad + \{(1+it)(x - \omega y) + t(u - \omega v)\}e^{-it}\Big\} & \text{if } \kappa = 2, \end{cases}$$

$$W(t) = \begin{cases} Ae^{\alpha t} + Be^{\beta t} + \bar{\omega}(Ce^{-\alpha t} + De^{-\beta t}), & \text{if } \kappa \neq 2, \\ \dfrac{1-\theta}{2}\Big\{\{(1-it)(\bar{\omega}x + y) + t(\bar{\omega}u + v)\}e^{it} \\ \qquad - \bar{\omega}\{(1+it)(x - \omega y) + t(u - \omega v)\}e^{-it}\Big\} & \text{if } \kappa = 2, \end{cases}$$

where

$$A = (1-\theta)(-\beta\bar{\omega}x - \beta y + \bar{\omega}u + v)/2\sqrt{4-\kappa^2},$$

$$B = (1-\theta)(\alpha\bar{\omega}x + \alpha y - \bar{\omega}u - v)/2\sqrt{4-\kappa^2},$$

$$C=(1-\theta)(\beta x-\beta\omega y+u-\omega v)/2\sqrt{4-\kappa^2},$$

$$D=(1-\theta)(-\alpha x+\alpha\omega y-u+\omega v)/2\sqrt{4-\kappa^2}.$$

Hence we have by putting $a=(\kappa+\sqrt{\kappa^2-4})/2$, $b=(\kappa-\sqrt{\kappa^2-4})/2$, when $\kappa>2$, that

(8-3a)
$$\begin{aligned}\tilde{\gamma}(t) &= \{(A+Cj)e^{at}+(B+Dj)e^{\beta t}\}(\omega+j)\\ &= \Big\{\{-\xi b(\bar{\omega}-j)-\eta i(\bar{\omega}+j)\}e^{ait}\\ &\quad+\{\xi a(\bar{\omega}-j)+\eta i(\bar{\omega}+j)\}e^{bit}\Big\}\frac{(1-\theta)(\omega+j)}{2\sqrt{\kappa^2-4}} \qquad \text{if } \kappa>2,\end{aligned}$$

(8-3b)
$$\tilde{\gamma}(t)=\frac{1-\theta}{2}\Big\{\xi(\bar{\omega}-j)+t\{\eta(\bar{\omega}-j)-\xi i(\bar{\omega}+j)\}\Big\}e^{it}(\omega+j), \quad \text{if } \kappa=2,$$

(8-3c)
$$\begin{aligned}\tilde{\gamma}(t) &= \{(A+Dj)e^{\alpha t}+(B+Cj)e^{\beta t}\}(\omega+j)\\ &= \Big\{\{-\xi(\beta\bar{\omega}+\alpha j)+\eta(\bar{\omega}-j)\}e^{\alpha t}\\ &\quad+\{\xi(\alpha\bar{\omega}+\beta j)-\eta(\bar{\omega}-j)\}e^{\beta t}\Big\}\frac{(1-\theta)(\omega+j)}{2\sqrt{4-\kappa^2}}, \qquad \text{if } \kappa<2.\end{aligned}$$

When $\theta=1$ we obtain

(8-4)
$$\tilde{\gamma}(t)=\begin{cases}\{(-\xi b-\eta i)e^{ait}+(\xi a+\eta i)e^{bit}\}(\kappa^2-4)^{-1/2}, & \text{if } \kappa>2,\\ \xi e^{it}+\{\xi(-i)+\eta\}te^{it}, & \text{if } \kappa=2,\\ \{(-\xi\beta+\eta)e^{\alpha t}+(\xi\alpha-\eta)e^{\beta t}\}(4-\kappa^2)^{-1/2}, & \text{if } \kappa<2.\end{cases}$$

Using these expressions we show the following.

PROPOSITION 8. *Let γ be a circle with geodesic curvature κ on $\boldsymbol{HH}^n(-4)$. If its structure torsion Θ is* 1, *then the following hold.*

(1) *When $\kappa>2$, it is a simple closed curve with prime period $2\pi/\sqrt{\kappa^2-4}$.*

(2) *When $\kappa\leqq 2$, it is a simple two-sides unbounded open curve.*

(3) *When $\kappa<2$, it has two distinct points at infinity, and when $\kappa=2$, it is horocyclic.*

PROOF. The second assertion follows from the expression. We can show the first assertion along the same lines as Proposition 3. We here just show the third assertion. Let $\tilde{\gamma}$ be a horizontal lift of γ with $\theta\neq 1$. When $\kappa<2$, by the expression (8-3c) of $\tilde{\gamma}$ we have

$$\Phi\circ\gamma(t)=\Big(\Big\{\{-\xi_l(\beta\bar{\omega}+\alpha j)+\eta_l(\bar{\omega}-j)\}\exp(\sqrt{4-\kappa^2}\,t/2)$$
$$+\{\xi_l(\alpha\bar{\omega}+\beta j)-\eta_l(\bar{\omega}-j)\}\exp(-\sqrt{4-\kappa^2}\,t/2)\Big\}$$
$$\times\Big\{\{-\xi_0(\beta\bar{\omega}+\alpha j)+\eta_0(\bar{\omega}-j)\}\exp(\sqrt{4-\kappa^2}\,t/2)$$
$$+\{\xi_0(\alpha\bar{\omega}+\beta j)-\eta_0(\bar{\omega}-j)\}\exp(-\sqrt{4-\kappa^2}\,t/2)\Big\}^{-1}\Big)_{1\leq l\leq n},$$

hence get that γ has points of infinity;

$$\lim_{t\to\infty}\Phi\circ\gamma(t)=\Big(\{-\xi_l(\beta\bar{\omega}+\alpha j)+\eta_l(\bar{\omega}-j)\}\{-\xi_0(\beta\bar{\omega}+\alpha j)+\eta_0(\bar{\omega}-j)\}^{-1}\Big)_{1\leq l\leq n},$$

$$\lim_{t\to-\infty}\Phi\circ\gamma(t)=\Big(\{\xi_l(\alpha\bar{\omega}+\beta j)-\eta_l(\bar{\omega}-j)\}\{\xi_0(\alpha\bar{\omega}+\beta j)-\eta_0(\bar{\omega}-j)\}^{-1}\Big)_{1\leq l\leq n}.$$

These lead us to $\gamma(\infty)\neq\gamma(-\infty)$ in the following manner. Suppose $\lim_{t\to\infty}\Phi\circ\gamma(t)=\lim_{t\to-\infty}\Phi\circ\gamma(t)$;

$$\{\xi_l(\beta\bar{\omega}+\alpha j)-\eta_l(\bar{\omega}-j)\}\{\xi_0(\beta\bar{\omega}+\alpha j)-\eta_0(\bar{\omega}-j)\}^{-1}$$
$$=\{\xi_l(\alpha\bar{\omega}+\beta j)-\eta_l(\bar{\omega}-j)\}\{\xi_0(\alpha\bar{\omega}+\beta j)-\eta_0(\bar{\omega}-j)\}^{-1},\quad 1\leqq l\leqq n.$$

Multiply both sides of the above equalities by $\bar{\xi}_l$ from left and sum up with respect to l $(1\leqq l\leqq n)$. We then have

$$(\beta\bar{\omega}+\alpha j)\{\xi_0(\beta\bar{\omega}+\alpha j)-\eta_0(\bar{\omega}-j)\}^{-1}=(\alpha\bar{\omega}+\beta j)\{\xi_0(\beta\bar{\omega}+\alpha j)-\eta_0(\bar{\omega}-j)\}^{-1}.$$

Similarly by multiplying $\bar{\eta}_l$ we get

$$\{\xi_0(\beta\bar{\omega}+\alpha j)-\eta_0(\bar{\omega}-j)\}^{-1}=\{\xi_0(\beta\bar{\omega}+\alpha j)-\eta_0(\bar{\omega}-j)\}^{-1}.$$

These lead us to $\alpha=\beta$, which is a contradiction. We get $\gamma(\infty)\neq\gamma(-\infty)$ when $\kappa<\kappa_\Theta$.

When $\kappa=2$ we find

$$\lim_{t\to\infty}\Phi\circ\gamma(t)=\lim_{t\to-\infty}\Phi\circ\gamma(t)$$
$$=\Big(\{\eta_l(\bar{\omega}-j)-\xi_l i(\bar{\omega}+j)\}\{\eta_l(\bar{\omega}-j)-\xi_l i(\bar{\omega}+j)\}^{-1}\Big)_{1\leq l\leq n}$$
$$=\Big(\{\xi_l+\eta_l(\theta i+\varphi j+\psi k)\}\{\xi_0+\eta_0(\theta i+\varphi j+\psi k)\}^{-1}\Big)_{1\leq l\leq n}.$$

Consider the geodesic σ on $\boldsymbol{HH}^n(-4)$ with $\dot{\sigma}(0)=d\pi\Big((\xi,\ -(\theta\tilde{I}\eta+\varphi\tilde{J}\eta+\psi\tilde{K}\eta))\Big)$. We then have

$$\lim_{t\to\infty}\Phi\circ\sigma(t)=\Big(\{\xi_l+\eta_l(\theta i+\varphi j+\phi k)\}\{\xi_0+\eta_0(\theta i+\varphi j+\phi k)\}^{-1}\Big)_{1\leq l\leq n}$$
$$=\lim_{t\to\infty}\varphi\circ\gamma(t)\,.$$

Since γ and σ cross orthogonally at $\gamma(0)$, we get the conclusion.

§ 9. Circles on $HH^n(-4)$ with $\Theta=0$.

Next we shall concern ourselves with circles having null structure torsion. In this case the equations for a horizontal lift $\tilde{\gamma}$ are

(9-1) $$\tilde{\nabla}_{\dot{\tilde{X}}}\tilde{X}=\kappa\tilde{Y},\qquad \tilde{\nabla}_{\dot{\tilde{X}}}\tilde{Y}=-\kappa X$$

which are equivalent to

(9-1′) $$\tilde{\nabla}_{\dot{\tilde{X}}}\tilde{\nabla}_{\dot{\tilde{X}}}\tilde{X}=-(\kappa^2-1)\tilde{X}\,.$$

Denoting $\tilde{\gamma}(t)=Z(t)+W(t)j$ we can rewrite this into the followinγ;

(9-2) $$\begin{cases} Z^{(3)}=-(\kappa^2-1)\dot{Z}\\ W^{(3)}=-(\kappa^2-1)\dot{W}\,,\end{cases}$$

which are the same type equations as the equation for circles on a complex hyperbolic space $CH^n(-4)$ with null complex torsion. Solving these equations under the initial conditions

$$\tilde{\gamma}(0)=\xi=x+yj,\quad \dot{\tilde{\gamma}}(0)=\eta=u+vj,\quad \ddot{\tilde{\gamma}}(0)-\tilde{\gamma}(0)=\zeta\kappa=(z+wj)\kappa,$$
$$x,\,y,\,u,\,v,\,z,\,w\in C^{n+1}\,,$$

we get the following;

$$Z(t)=\begin{cases} A\exp(\sqrt{1-\kappa^2}\,t)+B+C\exp(-\sqrt{1-\kappa^2}\,t), & \text{if } \kappa\neq 1,\\ \left(1+\dfrac{t^2}{2}\right)x+tu+\dfrac{t^2\kappa}{2}z, & \text{if } \kappa=1,\end{cases}$$

$$W(t)=\begin{cases} D\exp(\sqrt{1-\kappa^2}\,t)+E+F\exp(-\sqrt{1-\kappa^2}\,t), & \text{if } \kappa\neq 1,\\ \left(1+\dfrac{t^2}{2}\right)y+tv+\dfrac{t^2\kappa}{2}w & \text{if } \kappa=1,\end{cases}$$

where

$$A=\{2(1-\kappa^2)\}^{-1}(x+\kappa z)+\{2\sqrt{1-\kappa^2}\}^{-1}u$$

$$D=\{2(1-\kappa^2)\}^{-1}(y+\kappa w)+\{2\sqrt{1-\kappa^2}\}^{-1}v$$

$$B=-\kappa(1-\kappa^2)^{-1}(\kappa x+z),\quad E=-\kappa(1-\kappa^2)^{-1}(\kappa y+w)$$

$$C=\{2(1-\kappa^2)\}^{-1}(x+\kappa z)-\{2\sqrt{1-\kappa^2}\}^{-1}u$$

$$F=\{2(1-\kappa^2)\}^{-1}(y+\kappa w)-\{2\sqrt{1-\kappa^2}\}^{-1}v\,.$$

We hence have the following expression of $\tilde{\gamma}$.

$$\text{(9-3a)}\qquad \tilde{\gamma}(t)=(A+Fj)\exp(i\sqrt{\kappa^2-1}\,t)+(B+Ej)+(C+Dj)\exp(-i\sqrt{\kappa^2-1}\,t)$$
$$=\{-(\xi+\zeta\kappa+\eta i\sqrt{\kappa^2-1})\exp(i\sqrt{\kappa^2-1}\,t)+2(\xi\kappa^2+\zeta\kappa)$$
$$-(\xi+\zeta\kappa-\eta i\sqrt{\kappa^2-1})\exp(-i\sqrt{\kappa^2-1}\,t)\}\frac{1}{2(\kappa^2-1)},\qquad \text{if } \kappa>1,$$

$$\text{(9-3b)}\qquad \tilde{\gamma}(t)=\xi\Big(1+\frac{t^2}{2}\Big)+\eta t+\zeta\cdot\frac{t^2\kappa}{2},\qquad \text{if } \kappa=1,$$

$$\text{(9-3c)}\qquad \tilde{\gamma}(t)=(A+Dj)\exp(\sqrt{1-\kappa^2}t)+(B+Ej)+(C+Fj)\exp(-\sqrt{1-\kappa^2}t)$$
$$=\{(\xi+\zeta\kappa+\eta\sqrt{1-\kappa^2})\exp(\sqrt{1-\kappa^2}t)-2(\xi\kappa^2+\zeta\kappa)$$
$$+(\xi+\zeta\kappa-\eta\sqrt{1-\kappa^2})\exp(-\sqrt{1-\kappa^2}t)\}\frac{1}{2(1-\kappa^2)},\qquad \text{if } \kappa<1.$$

We therefore find out the following.

PROPOSITION 9. *Let γ be a circle on $\boldsymbol{HH}^n(-4)$ with geodesic curvature κ and structure torsion $\Theta=0$.*

(1) *If $\kappa>1$, it is a simple closed curve with prime period $2\pi/\sqrt{\kappa^2-1}$.*

(2) *If $\kappa\leqq 1$, it is two-sides unbounded simple curve.*

(3) *If $\kappa=1$, it is horocyclic, and if $\kappa<1$, it has two distinct points at infinity.*

PROOF. By the expressions (9-3) the assertion (1), (2) are trivial. We shall concern the last assertion. When $\kappa<1$ we have

$$\lim_{t\to\infty}\Phi\circ\gamma(t)=\Big((\xi_l+\zeta_l\kappa+\eta_l\sqrt{1-\kappa^2})(\xi_0+\zeta_0\kappa+\eta_0\sqrt{1-\kappa^2})^{-1}\Big)_{1\leqq l\leqq n},$$

$$\lim_{t\to-\infty}\Phi\circ\gamma(t)=\Big((\xi_l+\zeta_l\kappa-\eta_l\sqrt{1-\kappa^2})(\xi_0+\zeta_0\kappa-\eta_0\sqrt{1-\kappa^2})^{-1}\Big)_{1\leqq l\leqq n}.$$

Here since ξ, η, ζ are $\boldsymbol{H}$-linearly independent, we see that $\xi_0+\zeta_0\kappa\pm\eta_0\sqrt{1-\kappa^2}\neq 0$. Now suppose they coincide;

$$(\xi_l+\zeta_l\kappa+\eta_l\sqrt{1-\kappa^2})(\xi_0+\zeta_0\kappa+\eta_0\sqrt{1-\kappa^2})^{-1}$$
$$=(\xi_l+\zeta_l\kappa-\eta_l\sqrt{1-\kappa^2})(\xi_0+\zeta_0\kappa-\eta_0\sqrt{1-\kappa^2})^{-1},\qquad 1\leqq l\leqq n.$$

We multiply both sides of the above equalities by $\bar{\xi}_l$ from left and sum up with respect to l, then get $\eta_0=0$, by use of $\langle\!\langle\xi,\xi\rangle\!\rangle=-1$ and $\langle\!\langle\xi,\eta\rangle\!\rangle=\langle\!\langle\xi,\zeta\rangle\!\rangle=0$. Similarly by multiplying $\bar{\eta}_l$ we get $\xi_0+\zeta_0\kappa=0$. This is a contradiction. We therefore have $\gamma(\infty)\neq\gamma(-\infty)$, if $\kappa<1$.

When $\kappa=1$, we find

$$\lim_{t\to\infty}\Phi\circ\gamma(t)=\lim_{t\to-\infty}\Phi\circ\gamma(t)=\Big((\xi_l+\zeta_l)(\xi_0+\zeta_0)^{-1}\Big)_{1\le l\le n}.$$

Consider the geodesic σ on $HH^n(-4)$ with $\dot\sigma(0)=d\pi((\xi,\zeta))$. It crosses orthogonally to γ at $\gamma(0)$, and $\lim_{t\to\infty}\Phi\circ\sigma(t)=\lim_{t\to\infty}\Phi\circ\gamma(t)$. Hence we get the conclusion.

§ 10. Circles on $HH^n(-4)$ with $0<\Theta<1$.

We finally study circles on $HH^n(-4)$ with structure torsion $0<\Theta<1$. In this case the equations for a horizontal lift $\tilde\gamma$ of a circle are

(10-1)
$$\begin{cases}\tilde\nabla_{\tilde X}\tilde X=\kappa\tilde Y\\ \tilde\nabla_{\tilde X}\tilde Y=-\kappa\tilde X+(\theta\tilde IN+\varphi\tilde JN+\phi\tilde KN)\end{cases}$$

which are equivalent to

(10-1′)
$$\tilde\nabla_{\tilde X}\tilde\nabla_{\tilde X}\tilde X=-(\kappa^2-1)\tilde X+\kappa(\theta\tilde IN+\varphi\tilde JN+\phi\tilde KN).$$

As usual, denoting $\tilde\gamma(t)=Z(t)+W(t)j$ we can rewrite this into the following;

(10-2)
$$\begin{cases}Z^{(3)}=-(\kappa^2-1)\dot Z-\kappa(i\theta Z-\varphi W+i\phi W)\\ W^{(3)}=-(\kappa^2-1)\dot W+\kappa(i\theta W-\varphi Z-i\phi Z).\end{cases}$$

For given Θ $(0<\Theta<1)$ we denote by κ_Θ the unique positive solution of the equation $27\lambda^2\Theta^2=4(\lambda^2-1)^3$. We consider a cubic equation $\tau^3-(\kappa^2-1)\tau-\kappa\Theta=0$. This equation has

(i) three nonzero real solutions a, b, d $(a<b<0<d)$, when $\kappa>\kappa_\Theta$,

(ii) a negative double root a and a positive solution d, when $\kappa=\kappa_\Theta$,

(iii) two complex solutions $f+ig$, $f-ig$ $(g>0)$ and a positive solution d, when $\kappa<\kappa_\Theta$.

The eigenvalues of the matrix associated to the differential equations (10-2) are the solutions of one of the cubic equations $\lambda^3+(\kappa^2-1)\lambda\pm\kappa\Theta i=0$. Hence they are

(i) 6 distinct pure imaginary numbers $\pm\alpha$, $\pm\beta$ $(\alpha=ai,\ \beta=bi)$ and $\pm di$, when $\kappa>\kappa_\Theta$,

(ii) 4 pure imaginary numbers $\pm ai$ and $\pm di$ $(a=-d/2=-\sqrt{(\kappa_\Theta^2-1)/3}$, when $\kappa=\kappa_\Theta$,

(iii) $\pm\alpha$, $\pm\beta$ $(\alpha=g+if,\ \beta=-g+if,\ g>0)$ and distinct pure imaginary numbers $\pm di$, when $\kappa<\kappa_\Theta$.

Under the initial conditions

$$\tilde\gamma(0)=\xi=x+yj,\quad \dot{\tilde\gamma}(0)=\eta=u+vj,\quad \ddot{\tilde\gamma}(0)-\tilde\gamma(0)=\boldsymbol{\zeta}\kappa=(z+wj)\kappa,$$

$$x,y,u,v,z,w\in\boldsymbol{C}^{n+1},$$

we get the following solutions of (10-2). When $\theta \neq \Theta$ we have

$$Z(t)=\begin{cases}\omega(Ae^{\alpha t}+Be^{\beta t}+Ce^{dit})-(De^{-\alpha t}+Ee^{-\beta t}+Fe^{-dit}), & \text{if } \kappa\neq\kappa_\Theta,\\ \omega(\tilde{A}e^{ait}+\tilde{B}te^{ait}+\tilde{C}e^{dit})-(\tilde{D}e^{-ait}+\tilde{E}te^{-ait}+\tilde{F}e^{-dit}), & \text{if } \kappa=\kappa_\Theta,\end{cases}$$

$$W(t)=\begin{cases}Ae^{\alpha t}+Be^{\beta t}+Ce^{dit}+\bar{\omega}(De^{-\alpha t}+Ee^{-\beta t}+Fe^{-dit}), & \text{if } \kappa\neq\kappa_\Theta,\\ \tilde{A}e^{ait}+\tilde{B}te^{ait}+\tilde{C}e^{dit}+\bar{\omega}(\tilde{D}e^{-ait}+\tilde{E}te^{-ait}+\tilde{F}a^{-dit}) & \text{if } \kappa=\kappa_\Theta,\end{cases}$$

where $\omega=\omega_\Theta=(\psi+i\varphi)/(\Theta-\theta)$ and

$$A=\frac{\Theta-\theta}{2(di-\alpha)(\alpha-\beta)\Theta}\{-(1+\beta di)(\bar{\omega}x+y)-\alpha(\bar{\omega}u+v)-\kappa(\bar{\omega}z+w)\},$$

$$B=\frac{\Theta-\theta}{2(\alpha-\beta)(\beta-di)\Theta}\{-(1+di\alpha)(\bar{\omega}x+y)-\beta(\bar{\omega}u+v)-\kappa(\bar{\omega}z+w)\},$$

$$C=\frac{\Theta-\theta}{2(\beta-di)(di-\alpha)\Theta}\{-(1+\alpha\beta)(\bar{\omega}x+y)-di(\bar{\omega}u+v)-\kappa(\bar{\omega}z+w)\},$$

$$D=\frac{\Theta-\theta}{2(di-\alpha)(\alpha-\beta)\Theta}\{-(1+\beta di)(\omega y-x)+\alpha(\omega v-u)-\kappa(\omega w-z)\},$$

$$E=\frac{\Theta-\theta}{2(\alpha-\beta)(\beta-di)\Theta}\{-(1+di\alpha)(\omega y-x)+\beta(\omega v-u)-\kappa(\omega w-z)\},$$

$$F=\frac{\Theta-\theta}{2(\beta-di)(di-\alpha)\Theta}\{-(1+\alpha\beta)(\omega y-x)+di(\omega v-u)-\kappa(\omega w-z)\},$$

$$\tilde{A}=\{(8a^2+1)(\bar{\omega}x+y)-2ai(\bar{\omega}u+v)+\kappa(\bar{\omega}z+w)\}(\Theta-\theta)/18\Theta a^2,$$

$$\tilde{B}=\{-i(2a^2+1)(\bar{\omega}x+y)+a(\bar{\omega}u+v)-\kappa i(\bar{\omega}z+w)\}(\Theta-\theta)/6\Theta a,$$

$$\tilde{C}=\{(a^2-1)(\bar{\omega}x+y)+2ai(\bar{\omega}u+v)-\kappa(\bar{\omega}z+w)\}(\Theta-\theta)/18\Theta a^2,$$

$$\tilde{D}=\{(8a^2+1)(\omega y-x)+2ai(\omega v-u)+\kappa(\omega w-z)\}(\Theta-\theta)/18\Theta a^2,$$

$$\tilde{E}=\{i(2a^2+1)(\omega y-x)+a(\omega v-u)+\kappa i(\omega w-z)\}(\Theta-\theta)/6\Theta a,$$

$$\tilde{F}=\{(a^2-1)(\omega y-x)-2ai(\omega v-u)-\kappa(\omega w-z)\}(\Theta-\theta)/18\Theta a^2.$$

Hence we can conclude the following.

(10-3a)
$$\begin{aligned}\gamma(t)&=\{(A+Dj)e^{ait}+(B+Ej)e^{bit}+(C+Fj)e^{dit}\}(\omega+j)\\&=\Big\{\{\xi(1-bd)(\bar{\omega}-j)+\eta ai(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(b-d)e^{ait}\\&\quad+\{\xi(1-da)(\bar{\omega}-j)+\eta bi(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(d-a)e^{bit}\\&\quad+\{\xi(1-ab)(\bar{\omega}-j)+\eta di(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(a-b)e^{dit}\Big\}\\&\quad\times\frac{\Theta-\theta}{2(a-b)(b-d)(d-a)\Theta}(\omega+j),\qquad \text{if } \kappa>\kappa_\Theta,\end{aligned}$$

(10-3b)
$$\begin{aligned}\tilde{\gamma}(t) &= \{(\tilde{A}+\tilde{D}j)e^{ait}+(\tilde{B}+\tilde{E}j)te^{ait}+(\tilde{C}+\tilde{F}j)e^{dit}\}(\omega+j)\\ &= \Big\{\{\xi(8a^2+1)(\bar{\omega}-j)-\eta\cdot 2ai(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}\,e^{ait}\\ &\quad -3a\,\{\xi(2a^2+1)i(\bar{\omega}+j)-\eta a(\bar{\omega}-j)+\zeta\kappa i(\bar{\omega}+j)\}\,te^{ait}\\ &\quad +\{\xi(a^2-1)(\bar{\omega}-j)+\eta\cdot 2ai(\bar{\omega}+j)-\zeta\kappa(\bar{\omega}-j)\}\,e^{dit}\Big\}\frac{\omega+j}{18\Theta a^2}\end{aligned}$$
if $\kappa=\kappa_\Theta$,

(10-3c)
$$\begin{aligned}\tilde{\gamma}(t) &= \{(A+Ej)e^{\alpha t}+(B+Dj)e^{\beta t}+(C+Fj)e^{dit}\}(\omega+j)\\ &= \Big\{-\{\xi(\bar{\omega}-j)(1+\beta di)+\eta(\alpha\bar{\omega}+\beta j)+\zeta\kappa(\bar{\omega}-j)\}(\beta-di)e^{\alpha t}\\ &\quad -\{\xi(\bar{\omega}-j)(1+\alpha di)+\eta(\beta\bar{\omega}+\alpha j)+\zeta\kappa(\bar{\omega}-j)\}(di-\alpha)e^{\beta t}\\ &\quad -\{\xi(1+\alpha\beta)(\bar{\omega}-j)+\eta di(\bar{\omega}+j)+\zeta\kappa(\bar{\omega}-j)\}(\alpha-\beta)e^{dit}\Big\}\\ &\quad \times\frac{\Theta-\theta}{2(\alpha-\beta)(\beta-di)(di-\alpha)\Theta}(\omega+j),\qquad \text{if } \kappa<\kappa_\Theta.\end{aligned}$$

When $\theta=\Theta$ we obtain

(10-4a)
$$\begin{aligned}\tilde{\gamma}(t) &= \Big\{\{\xi(1-bd)+\eta ai+\zeta\kappa\}(b-d)e^{ait}\\ &\quad +\{\xi(1-da)+\eta bi+\zeta\kappa\}(d-a)e^{bit}\\ &\quad +\{\xi(1-ab)+\eta di+\zeta\kappa\}(a-b)e^{dit}\Big\}\frac{1}{(a-b)(b-d)(d-a)},\end{aligned}$$
if $\kappa>\kappa_\Theta$,

(10-4b)
$$\begin{aligned}\tilde{\gamma}(t) &= \Big\{\{\xi(8a^2+1)-\eta\cdot 2ai+\zeta\kappa\}\,e^{ait}-\{\xi i(2a^2+1)-\eta a+\zeta\kappa i\}\,3ate^{ait}\\ &\quad +\{\xi(a^2-1)+\eta\cdot 2ai-\zeta\kappa\}\,e^{dit}\Big\}\frac{1}{9a^2},\qquad \text{if } \kappa=\kappa_\Theta,\end{aligned}$$

(10-4c)
$$\begin{aligned}\tilde{\gamma}(t) &= \Big\{-\{\xi(1+\beta di)+\eta\alpha+\zeta\kappa\}(\beta-di)e^{\alpha t}\\ &\quad -\{\xi(1+\alpha di)+\eta\beta+\zeta\kappa\}(di-\alpha)e^{\beta t}\\ &\quad -\{\xi(1+\alpha\beta)+\eta di+\zeta\kappa\}(\alpha-\beta)e^{dit}\Big\}\frac{1}{(\alpha-\beta)(\beta-di)(di-\alpha)}\end{aligned}$$
if $\kappa<\kappa_\Theta$.

By these expressions we have

PROPOSITION 10. *Let γ be a circle on $HH^n(-4)$ with geodesic curvature κ and structure torsion Θ* ($0<\Theta<1$).

(1) *If* $\kappa \leqq \kappa_{\Theta}$, *then it is two-sides unbounded simple curve.*

(2) *When* $\kappa=\kappa_{\Theta}$, *it is horocyclic, and when* $\kappa<\kappa_{\Theta}$, *it has two distinct points at infinity.*

(3) *When* $\kappa>\kappa_{\Theta}$, *it is bounded simple curve. Further more, let* a, b *and* d $(a<b<d)$ *denote nonzero real solutions for the cubic equation* $\tau^3-(\kappa^2-1)\tau-\kappa\Theta=0$. *Then it is closed if and only if one of the three ratios* a/b, b/d *and* d/a *is rational. In that case, its prime period is the least common multiple of* $2\pi/(b-a)$ *and* $2\pi/(d-a)$.

PROOF. Since α, β have their real part, we get the first assertion by the expressions (10-4). The last assertion follows by the same argument as in Proposition 5. We show the second assertion. When $\kappa=\kappa_{\Theta}$ we have

$$\begin{aligned}\lim_{t\to\infty}\Phi\circ\gamma(t) &= \lim_{t\to-\infty}\Phi\circ\gamma(t)\\ &= \Big(\{\xi_l(2a^2+1)i(\bar{\omega}+j)-\eta_l a(\bar{\omega}-j)+\zeta_l\kappa i(\bar{\omega}-j)\}\\ &\qquad\times\{\xi_0(2a^2+1)i(\bar{\omega}+j)-\eta_0 a(\bar{\omega}-j)+\zeta_0\kappa i(\bar{\omega}-j)\}^{-1}\Big)_{1\leq l\leq n}\\ &= \Big(\{\xi_l(2a^2+1)+\eta_l(\theta i+\varphi j+\psi k)a/\Theta+\zeta_l\kappa\}\\ &\qquad\times\{\xi_l(2a^2+1)+\eta_l(\theta i+\varphi j+\psi k)a/\Theta+\zeta_0\kappa\}^{-1}\Big)_{1\leq l\leq n}.\end{aligned}$$

Consider the geodesic σ on $\boldsymbol{HH}^n(-4)$ with $\dot{\sigma}(0)=d\pi((\xi,\mu))$, where

$$\mu=\{\eta(\theta i+\varphi j+\psi k)a/\Theta+\zeta\kappa_{\Theta}\}(2a^2+1)^{-1},\qquad a=\sqrt{(\kappa_{\Theta}^2-1)/3}.$$

We then have $\langle\!\langle\xi,\mu\rangle\!\rangle=0$, $\langle\!\langle\eta,\mu\rangle\!\rangle=0$ and

$$\begin{aligned}(2a^2+1)^2\langle\!\langle\mu,\mu\rangle\!\rangle &= a^2\langle\!\langle\eta,\eta\rangle\!\rangle-\frac{2a}{\Theta}\kappa_{\Theta}\{\theta\langle\tilde{I}\eta,\zeta\rangle+\varphi\langle\tilde{J}\eta,\zeta\rangle+\psi\langle\tilde{K}\eta,\zeta\rangle\}+\kappa_{\Theta}^2\langle\!\langle\zeta,\zeta\rangle\!\rangle\\ &= a^2+2a\Theta\kappa_{\Theta}+\kappa_{\Theta}^2=a^2-2a\{a^3-(\kappa_{\Theta}^2-1)a\}+\kappa_{\Theta}^2\\ &= (2a^2+1)^2.\end{aligned}$$

Also we have

$$\lim_{t\to\infty}\Phi\circ\sigma(t)=\Big((\xi_l+\mu_l)(\xi_0+\mu_0)^{-1}\Big)_{1\leq l\leq n}=\lim_{t\to\infty}\Phi\circ\gamma(t),$$

hence we find γ is horocyclic.

When $\kappa<\kappa_{\Theta}$, we have

$$\lim_{t\to\infty} \Phi\circ\gamma(t) = \Big(\{\xi_l(\bar{\omega}-j)(1+\beta di)+\eta_l(\alpha\bar{\omega}+\beta j)+\zeta_l\kappa(\bar{\omega}-j)\}$$

$$\times\{\xi_0(\bar{\omega}-j)(1+\beta di)+\eta_0(\alpha\bar{\omega}+\beta j)+\zeta_0\kappa(\bar{\omega}-j)\}^{-1}\Big)_{1\leq l\leq n},$$

$$\lim_{t\to-\infty} \Phi\circ\gamma(t) = \Big(\{\xi_l(\bar{\omega}-j)(1+\alpha di)+\eta_l(\beta\bar{\omega}+\alpha j)+\zeta_l\kappa(\bar{\omega}-j)\}$$

$$\times\{\xi_0(\bar{\omega}-j)(1+\alpha di)+\eta_0(\beta\bar{\omega}+\alpha j)+\zeta_0\kappa(\bar{\omega}-j)\}^{-1}\Big)_{1\leq l\leq n}.$$

By the same argument as in Proposition 9 we find they are distinct.

REMARK. The bound κ_Θ can be interpret in terms of curvature.

$$\kappa_\Theta^2 = -\mathrm{Riem}\Big(d\pi(\xi,\eta),\ d\pi(\xi,\mu)\Big),$$

where μ is the direction of the point of infinity for horocyclic circle; $\mu=\{\eta(\theta i+\varphi j+\phi k)a/\Theta+\zeta\kappa_\Theta\}(2a^2+1)^{-1}$, $a=-\{(\kappa_\Theta^2-1)/3\}^{1/2}$.

Summarizing up Proposition 8, 9, 10, we obtain Theorem 2.

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