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On the quantization of a coherent family of representations at roots of unity

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1.

Given a coherent family of virtual representations of a complex semisimple Lie algebra we associate a coherent family of virtual representations of the corresponding quantum group at roots of unity. The latter family depends on the given family in a precise fashion described below.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and let U be its universal enveloping algebra. Lusztig considered a certain $\boldsymbol{C}[v, v^{-1}]$ algebra $U_{\mathcal{A}}$, $\{\mathcal{A}=\boldsymbol{C}[v, v^{-1}]\}$ which is an $\mathcal{A}$-form of the 'quantum group' $U_{\mathcal{A}'}$, $\{\mathcal{A}'=\boldsymbol{C}(v)$, the field of fractions of $\mathcal{A}\}$; the latter are some Hopf-algebra deformations of U, defined by Drinfeld and Jimbo generalizing the case of $\mathfrak{sl}_2$.

Let $\lambda\in\boldsymbol{C}^*$ and suppose that λ is a primitive l-th root of unity where l, $(\geqq 3)$, is an odd positive integer (not divisible by 3 if G_2 is a factor of $\mathfrak{g}$).

Let $\varphi_\lambda : \mathcal{A}\to\boldsymbol{C}$ be the $\boldsymbol{C}$-algebra homomorphism obtained by sending v to λ. The algebras $U_\lambda := U_{\mathcal{A}}\otimes_{\mathcal{A}}\boldsymbol{C}$, (scalar multiplication by $u\in\mathcal{A}$ in the first factor corresponds to scalar multiplication by $\varphi_\lambda(u)$ in the second factor) are called 'quantum groups at roots of unity'; these are different from those considered by Kac, Processi and De-Concini. The algebras U_λ are also Hopf algebras.

In [**L1**, Prop. 7.5(a) and **L2**, 8.16] Lusztig defines a 'Frobenius' morphism $\psi: U_\lambda\to U$; ψ is a surjection and respects the Hopf-algebra structure.

2. Coherent family of virtual representations of U, U_λ.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Δ^+ a system of positive roots. Let $S=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the set of simple roots in Δ^+. Let $\Lambda\subseteqq\mathfrak{h}^*$ $(=\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{h}, \boldsymbol{C}))$ be the integral lattice defined by

$$\nu \in \Lambda \Longleftrightarrow 2(\nu, \alpha)/(\alpha, \alpha) \in \boldsymbol{Z}, \qquad \forall\alpha\in\Delta$$

where the pairing is induced by the Killing form in the usual way.

DEFINITION. A family of virtual (not necessarily finite dimensional) representations $\{\pi(\nu)\}_{\nu\in\Lambda}$ of U is called a *coherent family* if for every finite dimensional module F of U (in the Grothendieck group)

$$\pi(\nu)\otimes F=\sum_{\mu\in\Delta(F)} m(\mu, F)\pi(\nu+\mu)$$

where the summation is over the weights $\Delta(F)$ of F and for $\mu\in\Delta(F)$, $m(\mu, F)$ denotes the multiplicity of μ as a weight of F.

REMARK. The Grothendieck group is formed in the usual way from any subcategory of modules with finite composition series, stable under tensor products with finite dimensional modules. Depending upon the context, (see for e. g., [**BV**, Definition 2.2]) one often assumes extra information about the coherent family, e. g., that $\pi(\nu)$ has an infinitesimal character parametrized by the orbit of ν and also an irreducibility property for $\pi(\nu)$, when ν satisfies positivity conditions.

Interesting examples of coherent families arise by considering Harish-Chandra modules (generally infinite-dimensional) for a real form of $\mathfrak{g}$. Given any such irreducible Harish-Chandra module there is a coherent family it belongs to (see [**Vo**, Theorem 7.2.7]).

Finite dimensional representations of U have been quantized by Lusztig at all U_λ. Their 'weights' can be defined as elements of Λ (see [**L1**, 5.2]); they admit a weight space decomposition (see [**APW**, 9.12] and [**A**]). If F is an irreducible finite dimensional module for U, its quantization F' for U_λ is called a Weyl module; for $\mu\in\Lambda$, μ is a weight of F' if and only if it is a weight of F and then the multiplicity $m(\mu, F')$ equals $m(\mu, F)$. This allows us to define a coherent family $\{\tilde{\pi}(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U_λ exactly as in the case of U. We will assume throughout this article, without further mention, that the U_λ modules considered are all of type 1 [**L1**, 4.6].

Let $\rho=1/2\sum_{\alpha\in\Delta^+}\alpha$ (half the sum of the positive roots). Recall that λ is a primitive l-th root of 1. If $\nu\in\Lambda$ is dominant integral (i. e., $2(\nu, \alpha)/(\alpha, \alpha)\in \boldsymbol{Z}^+$, $\forall\alpha\in\Delta^+$), let F_ν denote the irreducible finite dimensional representation of U with highest weight ν. We have then a representation $U_\lambda\to\mathrm{End}(F'_\nu)$ of the quantum group U_λ on the corresponding Weyl module F'_ν. Recall that if $\nu=(l-1)\rho$, the Weyl module $F'_{(l-1)\rho}$ is called the 'Steinberg module'; it is irreducible (see [**AW**, 2.2] and [**A**]). We let St denote the Steinberg module.

If $\pi: U\to\mathrm{End}(V)$ is a representation of U, we define a representation $\tilde{\pi}: U_\lambda\to\mathrm{End}(V)$ by $\tilde{\pi}=\pi\circ\phi$ where $\phi: U_\lambda\to U$ is the Frobenius morphism defined by Lusztig [**L1**, 7.5 and **L2**, 8.16].

Given any $\nu\in\Lambda$, we can uniquely write $\nu=\nu'+l\nu''$ where i) $\nu', \nu''\in\Lambda$ and ii) $2(\nu', \alpha)/(\alpha, \alpha)\in\{0, 1, \cdots, l-1\}$ for every simple root α in Δ^+.

Given a coherent family $\{\pi(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U we proceed to construct a coherent family $\{\tilde{\pi}(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U_λ such that for any $\nu''\in\Lambda$

$$\tilde{\pi}(l\nu'') = \tilde{\pi}(\nu'')\otimes St$$

where St is the Steinberg representation of U_λ.

For this we introduce some notation mainly following [**V**] and [**H**].

Let $\delta_1, \delta_2, \cdots, \delta_n$ be the fundamental weights; i.e., $2(\delta_i, \alpha_j)/(\alpha_j, \alpha_j)=\delta_{ij}$ (Kronecker delta) where $S=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is the set of simple roots. Let W be the Weyl group, $W\subset \mathrm{Aut}(\mathfrak{h}^*)$, generated by the Coxeter generators $s_i(i=1, \cdots, n)$ defined by $s_i(\nu)=\nu-[2(\nu, \alpha_i)/(\alpha_i, \alpha_i)]\alpha_i$. For $\sigma\in W$, let $I_\sigma=\{i\,|\,1\leqq i\leqq n, l(\sigma s_i)<l(\sigma)\}$. Here $l(\ )$ denotes the length function on W with respect to the Coxeter generators $s_1, s_2, \cdots, s_n$. Put $\delta_\sigma=\sum_{i\in I_\sigma}\delta_i$ and define $\varepsilon_\sigma=\sigma(\delta_\sigma)$. Let $\mathfrak{R}$ be the ring of formal integral combinations $\sum_{\eta\in\Lambda} m_\eta e^\eta$. Since the action of W on $\mathfrak{h}^*$ leaves Λ stable, W obviously acts as automorphisms of the ring $\mathfrak{R}$. We let $\mathfrak{R}^W$ denote the subring of invariants. We now summarize some key observations of Hulsurkar in [**H**] which were reinforced by Verma [**V**].

PROPOSITION ([**H**], [**V**]). (i) *For $\sigma\in W$, $-\varepsilon_{\sigma\sigma_0}+\varepsilon_\sigma=\sigma\rho$, where σ_0 is the unique element of W of maximum length. If $m\geqq 2$ and $\tau_1, \cdots, \tau_m$ are distinct elements of W, then at least one of the elements $-\varepsilon_{\tau_1\sigma_0}+\varepsilon_{\tau_2}$, $-\varepsilon_{\tau_2\sigma_0}+\varepsilon_{\tau_3}, \cdots$, $-\varepsilon_{\tau_{m-1}\sigma_0}+\varepsilon_{\tau_m}$, $-\varepsilon_{\tau_m\sigma_0}+\varepsilon_{\tau_1}$ is singular. (ν is non-singular$\Leftrightarrow(\nu, \alpha)\neq 0$ for any $\alpha\in\Delta\Leftrightarrow$"$w\in W$, $w\nu=\nu\Rightarrow w=1$").*

(ii) *For any $\nu\in\Lambda$, there exist unique W-invariant elements $\chi_{\nu,\tau}$, $(\tau\in W)$, $\in\mathfrak{R}^W$ such that*

$$e^\nu = \sum_{\tau\in W}\chi_{\nu,\tau}\cdot e^{\varepsilon_\tau}.$$

(iii) *For any $\nu\in\Lambda$, there exist unique W-invariant elements $\eta_{\nu,\tau}$, $(\tau\in W)$, $\in\mathfrak{R}^W$ such that*

$$e^\nu\cdot\chi(St) = \sum_{\tau\in W}\eta_{\nu,\tau}\cdot e^{l\varepsilon_\tau},$$

$$\eta_{0,w} = \begin{cases}\chi(St) & \text{if } w=1\\ 0 & \text{otherwise.}\end{cases}$$

(Here $\chi(St)$, which lies in $\mathfrak{R}^W$ denotes the character of the Steinberg representation.) The statement (i) *is the Main Lemma of* [**H**]. *For statement* (ii) *see also* [**J**, Satz 1].

INDICATION OF PROOF (following [**H**] and [**V**]) OF (ii). Define an operator $c: \mathfrak{R}\to\mathfrak{R}$ by

$$c(e^\eta) = \frac{\sum_{\tau\in W}(-1)^{l(\tau)}e^{\tau\eta}}{\sum_{\tau\in W}(-1)^{l(\tau)}e^{\tau\rho}}.$$

The operator c is $\mathfrak{R}^W$-linear.

The main idea of the proof by Hulsurkar and Verma is to solve the system of linear equations by inverting a $|W|\times|W|$ matrix $(a_{\sigma,\tau})$ where $a_{\sigma,\tau}=c(e^{-\varepsilon_\sigma\sigma_0+\varepsilon_\tau})$ [which is essentially guaranteed to be 'upper triangular' unipotent by (i)].

To find $(\chi_{\nu,\tau})_{\tau\in W}$ which solves

$$e^\nu=\sum_\tau \chi_{\nu,\tau}\cdot e^{\varepsilon_\tau}$$

multiply both sides by $e^{-\varepsilon_\sigma\sigma_0}$ to get

$$e^{\nu-\varepsilon_\sigma\sigma_0}=\sum_\tau \chi_{\nu,\tau}\cdot e^{-\varepsilon_\sigma\sigma_0+\varepsilon_\tau}.$$

Applying c to both sides

$$c(e^{\nu-\varepsilon_\sigma\sigma_0})=\sum_\tau \chi_{\nu,\tau}\cdot c(e^{-\varepsilon_\sigma\sigma_0+\varepsilon_\tau}) \qquad (\sigma\in W).$$

The left side of this system of equations is a column vector (whose $|W|$ entries belong to the ring $\mathfrak{R}^W$). Multiply this column vector on the left by the $|W|\times|W|$ matrix $(\beta_{\sigma,\tau})$ (whose entries are in the same ring) which is the inverse of the matrix $(a_{\sigma,\tau})$ where $a_{\sigma,\tau}=c(e^{-\varepsilon_\sigma\sigma_0+\varepsilon_\tau})$ to solve for the unknown column vector $(\chi_{\nu,\tau})_{\tau\in W}$.

The proof of (iii) is similar. In fact one can see the following:—

Let $\Phi:\mathfrak{R}\to\mathfrak{R}$ be the ring homomorphism defined by $\Phi(e^\theta)=e^{l\theta}$. Observe that

$$(\Phi\circ c(e^\eta))\chi(St)=c\circ\Phi(e^\eta) \qquad \forall\eta\in\Lambda.$$

Define a $|W|\times|W|$ matrix $(\beta'_{\sigma,\tau})$ by $\beta'_{\sigma,\tau}=\Phi(\beta_{\sigma,\tau})$, where $\beta_{\sigma,\tau}$ are as above. Then the column vector $(\eta_{\nu,\tau})_{\tau\in W}$ required in (iii) is obtained by multiplying the column vector $(c(e^{\nu-l\varepsilon_\sigma\sigma_0}))_{\sigma\in W}$ on the left by the matrix $(\beta'_{\sigma,\tau})_{\sigma,\tau\in W}$.

REMARK 1. Let $\mu\in\Lambda$. Applying Proposition (iii) to $\nu+l\mu$ in place of ν we find that $\exists\eta_{\nu+l\mu,\tau}\in\mathfrak{R}^W$, $(\tau\in W)$, such that

$$e^{\nu+l\mu}\cdot\chi(St)=\sum_\tau \eta_{\nu+l\mu,\tau}\cdot e^{l\varepsilon_\tau}.$$

Therefore

$$\begin{aligned}e^\nu\cdot\chi(St)&=\sum_\tau \eta_{\nu,\tau}\cdot e^{l\varepsilon_\tau}\\&=\sum_\tau \eta_{\nu+l\mu,\tau}\cdot e^{l\varepsilon_\tau-l\mu}.\end{aligned}$$

We denote by $\mathcal{F}$ the Grothendieck group of formal integral combinations of finite dimensional representations of U. If $\omega\in\mathcal{F}$, the character $\chi(\omega)\in\mathfrak{R}^W$ has an obvious meaning and $\chi:\mathcal{F}\to\mathfrak{R}^W$ is an isomorphism. We also have to introduce the corresponding Grothendieck group $\mathcal{F}'$ for U_λ-modules. Again if $\omega\in\mathcal{F}'$, the character $\chi(\omega)\in\mathfrak{R}^W$ has an obvious meaning and $\chi:\mathcal{F}'\to\mathfrak{R}^W$ is an isomorphism. Sometimes, if convenient, we use the same symbol to denote an element of $\mathcal{F}'$ and its character in $\mathcal{R}^W$.

THEOREM. *Suppose a coherent family $\{\pi(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U is given. Given $\nu\in\Lambda$, write $\nu=\nu'+l\nu''$ where $\nu''\in\Lambda$ and $2(\nu',\alpha)/(\alpha,\alpha)\in\{0,1,\cdots,l-1\}$ for each simple root α. Let $e^{\nu'}\cdot St=\sum\eta_{\nu',\tau}\cdot e^{l\varepsilon_\tau}$ in the notation of Proposition* (iii). *Choose $\rho(\nu',\tau)\in\mathcal{F}'$ whose character is $\eta_{\nu',\tau}$. Set*

$$\bar{\pi}(\nu)=\sum_{\tau}\rho(\nu',\tau)\otimes\tilde{\pi}(\nu''+\varepsilon_\tau)$$

(in the Grothendieck group of representations of U_λ). Then $\{\bar{\pi}(\nu)\}_{\nu\in\Lambda}$ is a coherent family of representations of U_λ with $\bar{\pi}(l\nu'')=\tilde{\pi}(\nu'')\otimes St$.

PROOF. We have $\eta_{\nu',\tau}\in\mathcal{R}^W$ and

$$e^{\nu'}\cdot St=\sum\eta_{\nu',\tau}\cdot e^{l\varepsilon_\tau}.$$

REMARK 2. By Remark 1, for any $\mu\in\Lambda$, we can also write (uniquely)

$$e^{\nu'}\cdot St=\sum_{\tau}\eta_{\nu'+l\mu,\tau}\cdot e^{-l\mu+l\varepsilon_\tau}$$

where $\eta_{\nu'+l\mu,\tau}\in\mathcal{R}^W$. In the course of the proof, it will be established that in the statement of the Theorem, the rightside of $\bar{\pi}(\nu)$, i.e., $\sum_\tau\rho(\nu',\tau)\otimes\tilde{\pi}(\nu''+\varepsilon_\tau)$ equals

$$\sum\rho(\nu'+l\mu,\tau)\otimes\tilde{\pi}(\nu''-\mu+\varepsilon_\tau)$$

where $\rho(\nu'+l\mu,\tau)$, $(\in\mathcal{F}')$, is chosen so as to have character $\eta_{\nu'+l\mu,\tau}$. The main ingredient in the proof of the theorem is the following lemma.

LEMMA. *Suppose*

$$\sum\chi_i e^{l\beta_i}=\sum\phi_j e^{l\gamma_j}$$

where χ_i, $(i=1,\cdots,m)$, and ϕ_j, $(j=1,\cdots,n)$, $\in\mathcal{R}^W$ and $\beta_i,\gamma_j\in\Lambda$. Assume, as in the Theorem, that $\{\pi(\nu)\}_{\nu\in\Lambda}$ is a coherent family of representations of U. Let $\rho_i,\tau_j\in\mathcal{F}'$ such that $\chi(\rho_i)=\chi_i$ and $\chi(\tau_j)=\phi_j$. Then,

$$\sum\rho_i\otimes\tilde{\pi}(\beta_i)=\sum\tau_j\otimes\tilde{\pi}(\gamma_j).$$

(Both sides lie in the Grothendieck group obtained from U_λ-modules.)

PROOF OF LEMMA. Write

$$e^{\beta_i}=\sum_{t\in W}\theta_{i,t}e^{\varepsilon_t}\tag{*}$$

and

$$e^{\gamma_j}=\sum_{s\in W}\kappa_{j,s}e^{\varepsilon_s}$$

as in Proposition ii). By abuse of notation, we also let $\theta_{i,t}$ and $\kappa_{j,s}$ denote elements of $\mathfrak{F}$, having character $\theta_{i,t}$ and $\kappa_{j,s}$ respectively. Lifting by the Frobenius map, $\tilde{\theta}_{i,t}$ and $\tilde{\kappa}_{j,s}$ denote elements of $\mathfrak{F}'$. It is clear that if $\theta_{i,t}$ has character $\sum_{\nu\in\Lambda} p_\nu e^\nu$, then $\tilde{\theta}_{i,t}$ has character $\sum_{\nu\in\Lambda} p_\nu e^{l\nu}$. A similar statement holds for $\tilde{\kappa}_{j,s}$.

We have, from (*),

$$e^{l\beta_i} = \sum_t \tilde{\theta}_{i,t} e^{l\varepsilon_t}$$

and

$$e^{l\gamma_j} = \sum_s \tilde{\kappa}_{j,s} e^{l\varepsilon_s}.$$

Since $\{\pi(\nu)\}_{\nu\in\Lambda}$ is a coherent family, we have

$$\tilde{\pi}(\beta_i) = \sum_t \tilde{\theta}_{i,t} \otimes \tilde{\pi}(\varepsilon_t)$$

and

$$\tilde{\pi}(\gamma_j) = \sum_s \tilde{\kappa}_{j,s} \otimes \tilde{\pi}(e_s).$$

By the assumption in the lemma,

$$\sum \chi_i e^{\iota\beta_i} = \sum \psi_j e^{l\gamma_j}.$$

Therefore, (in $\mathfrak{R}$)

$$\sum_i \chi_i \sum_t \tilde{\theta}_{i,t} e^{l\varepsilon_t}$$

$$= \sum_j \psi_j \sum_s \tilde{\kappa}_{j,s} e^{l\varepsilon_s}$$

from which follows

$$(\flat) \qquad \sum_i \chi_i \tilde{\theta}_{i,t} = \sum_j \psi_j \tilde{\kappa}_{t,j}$$

($\forall t \in W$). For the validity of the assertion in the lemma, observe that

$$\sum_i \chi_i \otimes \tilde{\pi}(\beta_i) = \sum_i \chi_i \otimes (\sum_t \tilde{\theta}_{i,t} \otimes \tilde{\pi}(\varepsilon_t))$$

$$= \sum_t \sum_i \chi_i \otimes \tilde{\theta}_{i,t} \otimes \tilde{\pi}(\varepsilon_t)$$

while

$$\sum_j \psi_j \otimes \tilde{\pi}(\gamma_j) = \sum_j \psi_j \otimes (\sum_s \tilde{\kappa}_{s,i} \otimes \tilde{\pi}(\varepsilon_s))$$

$$= \sum_s \sum_j \psi_j \otimes \tilde{\kappa}_{j,s} \otimes \tilde{\pi}(\varepsilon_s).$$

The lemma follows from ($\flat$).

To continue with the proof of the theorem, we remark that it is an immediate consequence of the lemma that if

$$e^\nu St = \sum_j \psi_j e^{l\gamma_j},$$

where $\psi_j \in \mathcal{R}^W$, then $\tilde{\pi}(\nu)$ defined in the theorem also equals $\sum_j \tau_j \otimes \tilde{\pi}(\gamma_j)$, where $\tau_j \in \mathcal{F}'$ has character ψ_j.

Let F' be a finite dimensional U_λ-module. For $\mu \in \Lambda$, let $m(\mu, F')$ be the multiplicity of μ as a weight of F'. We have to show that for any $\nu \in \Lambda$

$$F' \otimes \tilde{\pi}(\nu) = \sum_{\mu} m(\mu, F') \tilde{\pi}(\mu + \nu) \tag{†}$$

(in the Grothendieck group). Writing as in the statement of the theorem

$$e^{\nu'} \cdot St = \sum \eta_{\nu', \tau} \cdot e^{l \varepsilon_\tau}$$

multiply both sides by $\chi(F') \cdot e^{l\nu''}$. We obtain,

$$\sum_{\mu} m(\mu, F') e^{\nu + \mu} \cdot St = \sum_{\tau \in W} \chi(F') \cdot \eta_{\nu', \tau} e^{l(\nu'' + \varepsilon_\tau)} .$$

By the remark we made following the proof of the lemma, the right side of the above equality can be used to get $\sum_\mu m(\mu, F') \tilde{\pi}(\nu + \mu)$, namely,

$$\sum_{\mu} m(\mu, F') \tilde{\pi}(\nu + \mu) = \sum_{\tau \in W} (F' \otimes \rho(\nu', \tau)) \otimes \tilde{\pi}(\nu'' + \varepsilon_\tau) .$$

But the right side equals $F' \otimes \{\sum_{\tau \in W} \rho(\nu', \tau) \otimes \tilde{\pi}(\nu'' + \varepsilon_\tau)\}$ which is nothing but $F' \otimes \tilde{\pi}(\nu)$. This completes the proof of the theorem.

Suppose the coherent family $\pi(\nu)_{\nu \in \Lambda}$ has the property (see [**BV**, Definition 2.2]) that

i) $\pi(\nu)$ has infinitesimal character parametrized by the W-orbit of ν

ii) $\pi(\nu)$ is zero or irreducible when ν is dominant with respect to a fixed positive system, and $\pi(\nu) \neq 0$ if ν is dominant regular.

Then it can be expected that $\tilde{\pi}(\nu)$ for dominant ν (with respect to the positive system in ii) above) is represented in the Grothendieck group by a U_λ-module (as opposed to an arbitrary element of the Grothendieck group, which in general is a virtual module, i.e., a difference of two modules). The author has verified this (see [**P1**]) for A_2 and B_2, using Lusztig's formula for the multiplicity of irreducibles in Weyl modules of quantum groups at roots of unity. More generally, we can also relax the conditions i) and ii) above to allow families $\pi(\nu)_{\nu \in \Lambda}$ which do not necessarily have integral infinitesimal characters. For G_2, there is enough evidence though the verification is still incomplete.

If in the theorem we take Verma modules for the coherent family $\pi(\nu)_{\nu \in \Lambda}$ then the expression for $\tilde{\pi}(\nu)$ given in the theorem can be used to deduce the multiplicities of the irreducible subquotients occuring in a composition series for the quantized Verma modules at roots of unity. The formula so obtained, of course, involves

i) the multiplicities of the irreducibles occuring in $\pi(\nu)$ for various ν and

ii) the multiplicities of irreducibles occuring in the Weyl modules for U_λ. In addition the formula involves the knowledge of the coefficients $\eta_{\nu,\tau}$; the explicit determination of $\eta_{\nu,\tau}$ was indicated in the proof of Proposition (iii).

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References

[A] H.H. Andersen, Finite dimensional representations of quantum groups, to appear in Proc. Sympos. Pure Math..

[APW] H.H. Andersen, P. Polo and W. Kexin, Representations of quantum algebras, Invent. Math., **104** (1991), 1-59.

[AW] H.H. Andersen and W. Kexin, Representations of quantum algebras, The mixed case, J. Reine Angew. Math., **427** (1992), 35-50.

[BV] D. Barbasch and D. Vogan, Weyl Group Representations and Nilpotent Orbits, In Representation Theory of Reductive Groups, (ed. P.C. Trombi), Birkhäuser, 1983, pp. 21-33.

[L1] G. Lusztig, Modular representations and quantum groups, in Classical groups and related topics, Contemp. Math., **82** (1989), 59-77.

[L2] G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata, **35** (1990), 89-114.

[H] S.G. Hulsurkar, Proof of Verma's conjecture on Weyl's dimension polynomial, Invent. Math., **27** (1974), 45-52.

[J] J.C. Jantzen, Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren, J. Algebra, **49** (1977), 441-469, Satz 1.

[P1] R. Parthasarathy, Quantum analogues of a coherent family of modules at roots of unity: A_2, B_2, In Current Trends in Mathematics and Physics- A Tribute to Harish-Chandra, (ed. S.D. Adhikari), Narosa, New Delhi, 1995. (http://e-math.ams.org/web/preprints/17/199410/199410-17-001/199410-17-001.ps)

[P2] R. Parthasarathy, Quantum analogues of a coherent family of modules at roots of unity: A_3 (preprint).

[V] D.-N. Verma, The role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, In Lie Groups and Their Representations, Proc. Budapest 1971, (ed. I.M. Gel'fand), pp. 653-705.

[Vo] D.A. Vogan, Representations of Real Reductive Groups, Birkhäuser, Boston, 1981.

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