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A theorem of Chevalley type for prehomogeneous vector spaces

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Introduction.

Let G be a complex reductive group, acting linearly on a vector space V. Assume that V is G-prehomogeneous, i.e., V has a dense G-orbit. Let $\boldsymbol{C}[V]_{G,\phi} =\{f\in\boldsymbol{C}[V]\mid f(gv)=\phi(g)f(v)\}$ and f be its non-zero element. Then it is known [1] that there exists a unique G-orbit O_1 which is closed in $\Omega:=f^{-1}(\boldsymbol{C}^{\times})$. Let T be a maximal torus of the isotropy subgroup $H:=G_{v_1}$ of G at $v_1\in O_1$, $N:=N_G(T)$ the normalizer of T in G, $G':=N/T$, and $V':=V_T=\{v\in V\mid tv=v$ for any $t\in T\}$. We can show that ϕ induces a character of G', which we shall denote by the same letter ϕ. Define $\boldsymbol{C}[V']_{G',\phi}$ in the same way as above.

The purpose of this note is to prove the following two theorems.

THEOREM A. (1) *V' is G'-prehomogeneous. More precisely, the G'-orbit of v_1 is open dense in V'.*

(2) *The isotropy subgroup G'_{v_1} of G' at v_1 is finite.*

(3) *The restriction $\boldsymbol{C}[V]\to\boldsymbol{C}[V']$ induces an isomorphism $\boldsymbol{C}[V]_{G,\phi}\tilde{\to}\boldsymbol{C}[V']_{G',\phi}$.*

THEOREM B. *Assume that H is finite. Take $h\in H$, and let $\langle h\rangle$ be the finite cyclic group generated by h. Put $N'':=N_G(\langle h\rangle)$, $G'':=N''/\langle h\rangle$, and $V'':=V_{\langle h\rangle}$. Then*

(1) *V'' is G''-prehomogeneous. More precisely, $G''\cdot v_1$ is open dense in V''.*

(2) *If $h\neq 1$, $|G''_{v_1}|<|G_{v_1}|$.*

(3) *Take a rational character ϕ of G and a non-zero relative invariant $f\in \boldsymbol{C}[V]_{G,\phi}$. Then ϕ induces a character of G'', which we shall denote by the same letter ϕ, and the restriction $\boldsymbol{C}[V]\to\boldsymbol{C}[V'']$ induces an isomorphism $\boldsymbol{C}[V]_{G,\phi}\tilde{\to}\boldsymbol{C}[V'']_{G'',\phi}$.*

NOTATION. If a group Γ acts on a set X, X_{Γ} denotes the set of Γ-fixed points, and Γ_x denotes the isotropy subgroup of Γ at $x\in X$. For two subsets $A, B\subset\Gamma$, $A^B:=\{b^{-1}ab\mid a\in A, b\in B\}$. We write a^B for $\{a\}^B$. The meaning of

A^b and $a^b(a, b\in\Gamma)$ is similar. The complex number field is denoted by $\boldsymbol{C}$. For a set X, $|X|$ denotes the cardinality.

1. Preliminaries.

LEMMA 1. *Let H be a complex reductive group acting linearly on a vector space U, and S a maximal torus of H. If there is no absolute H-invariant polynomial function on U other than constants, then $U_S=\{0\}$.*

PROOF. Assume $0\neq u\in U_S$. By [**2**, p. 354] or [**3**, Corollary 3], Hu is closed in U. Hence there is an absolute H-invariant f such that $f|Hu\equiv 1$ and $f(0)=0$. Thus we get a contradiction.

Let G, V, f, ϕ, Ω, O_1, etc. be as in the introduction, and $V^\vee$ the dual vector space of V. It is known that there exists a non-zero $f^\vee\in\boldsymbol{C}[V^\vee]_{G,\phi^{-1}}$. See [**5**, p. 71] or [**1**, 1.5, (2)]. Let $\Omega^\vee:=f^{\vee-1}(\boldsymbol{C}^\times)$ and $O_1^\vee$ be the unique G-orbit in $\Omega^\vee$ which is closed in $\Omega^\vee$. It is known [**1**, 1.18, (3)] that $F:=\mathrm{grad}\log f$ gives an isomorphism $O_1\tilde{\to}O_1^\vee$. Put $v_1^\vee=F(v_1)$. Then $G_{v_1^\vee}=G_{v_1}$.

LEMMA 2. $\Omega_T=(O_1)_T$.

PROOF. Let $H:=G_{v_1}$, and $\Lambda:=(TO_1^\vee)^\perp$ be the conormal bundle of $O_1^\vee$, i.e.,

$$\Lambda=(TO_1^\vee)^\perp=\{(v, v^\vee)\in V\times O_1^\vee\,|\,v\perp T_{v^\vee}O_1^\vee\}.$$

By [**1**, 1.18, (4)], Λ is G-prehomogeneous, and hence $U:=(T_{v_1^\vee}O_1^\vee)^\perp(\subset V)$ is H-prehomogeneous. Since a prehomogeneous space does not have an absolute invariant other than constants, $U_T=\{0\}$ by lemma 1. (It is known [**1**, 1.18, (1)] that G_{v_1} is reductive.) Again by [**1**, 1.18, (4)], $\Lambda_T\cong\Omega_T=V_T\cap\Omega$ is irreducible. Let $\pi^\vee: \Lambda\to O_1^\vee$ be the natural projection. Then $\dim\Lambda_T\leqq\dim(O_1^\vee)_T$, since the fibre of $\pi^\vee|\Lambda_T$ at $v_1^\vee\in O_1^\vee$ is $U_T=\{0\}$, and since $\pi^\vee(\Lambda_T)\subset(O_1^\vee)_T$. Thus

$$\dim\Omega_T=\dim\Lambda_T\leqq\dim(O_1^\vee)_T=\dim(O_1)_T\leqq\dim\Omega_T\,.$$

Hence $(O_1)_T=O_1\cap V_T$ is in the same time dense and closed in $\Omega_T=\Omega\cap V_T$. Thus we get the assertion.

2. Proof of Theorem A.

Let $t\in T$ be a topological generator with respect to the Zariski topology. Define a mapping $a: G\to G$ by $a(g)=t^g$. For $g\in G$,

$$gv_1\in V_T\Longleftrightarrow tgv_1=gv_1\Longleftrightarrow a(g)\in G_{v_1}=H\,.$$

Hence $(O_1)_T=V_T\cap Gv_1\cong(a^{-1}H)/H$. By Lemma 2, $(O_1)_T=\Omega_T=\Omega\cap V_T$ is open dense in V_T. Hence

(2.1) $$\dim V_T = \dim a^{-1}H - \dim H\,.$$

Let us show that

(2.2) $$t^G \cap H = (t^G \cap T)^H\,.$$

One inclusion is obvious; $(t^G \cap T)^H \subset (t^G)^H \cap T^H \subset t^G \cap H$. Conversely, take an element $t^g \in t^G \cap H$. Since t^g generates T^g topologically, T^g and T are both maximal tori of H. Take $h \in H$ so that $T^g = T^h$. Define $t' \in T$ by $t^g = t'^h$. Then $t' \in t^G \cap T$ and $t^g \in (t^G \cap T)^h \subset (t^G \cap T)^H$.

Next, for $g \in G$, $t^g \in T \Leftrightarrow T^g = T \Leftrightarrow g \in N$. Hence

(2.3) $$t^G \cap T = t^N\,.$$

Let $Z = Z_G(T)$ $(= Z_G(t))$ denote the centralizer. Then

(2.4) $$|t^N| = |N/Z| < +\infty\,.$$

Since for any $n \in N$, t^n is also a topological generator of T, $Z_{H^0}(t^n) = Z_{H^0}(T) = T$, where H^0 is the identity component of $H = G_{v_1}$. Hence

(2.5) $$\dim t^{nH} = \dim (t^n)^{H^0} = \dim H^0 - \dim T\,.$$

By (2.4) and (2.5),

(2.6) $$\dim t^{NH} = \dim H - \dim T\,.$$

Hence

$$\begin{aligned} \dim (t^G \cap H) &= \dim (t^G \cap T)^H && \text{by (2.2)} \\ &= \dim (t^N)^H && \text{by (2.3)} \\ &= \dim H - \dim T && \text{by (2.6)}. \end{aligned}$$

On the other hand, every fibre of a is isomorphic to $Z_G(t) = Z_G(T)$. Hence

$$\begin{aligned} \dim a^{-1}(H) &= \dim Z_G(T) + \dim (a(G) \cap H) \\ &= \dim Z_G(T) + \dim H - \dim T\,. \end{aligned}$$

Together with (2.1), we get

(2.7) $$\dim V_T = \dim Z_G(T) - \dim T = \dim N_G(T) - \dim T\,.$$

Consider the N-action on V_T. Then

(2.8) $$\begin{aligned} N_{v_1} &= N \cap G_{v_1} \cong Z \cap G_{v_1} \quad \text{(locally)} \\ &= Z_H(T) \cong T \quad \text{(locally)}. \end{aligned}$$

Hence $\dim N_{v_1} = \dim T$, i.e., $\dim G'_{v_1} = 0$. Thus we get (2) and also

$$\dim G'v_1 = \dim G' = \dim N - \dim T = \dim V_T$$

by (2.7). Thus we get (1). By [**5**, §4, Prop. 3], $\boldsymbol{C}[V]_{G,\phi}=\boldsymbol{C}f$, $\boldsymbol{C}[V']_{G',\phi}=\boldsymbol{C}f|V'$, and we get (3).

3. Proof of Theorem B.

(2) is obvious; $|G''_{v_1}|=|N''_{v_1}/\langle h\rangle|\leqq|G_{v_1}/\langle h\rangle|\lneqq|G_{v_1}|$.

(1) Since $|H|<\infty$ and V is G-prehomogeneous, $\dim V\leqq\dim G=\dim G\cdot v_1\leqq\dim V$ and hence $G\cdot v_1$ is open dense in V. Define $a: G\to G$ by $a(g)=h^g$. For $g\in G$,

$$gv_1\in V_{\langle h\rangle}\Longleftrightarrow hgv_1=gv_1\Longleftrightarrow h^g\in H\Longleftrightarrow g\in a^{-1}H\,.$$

Hence $Gv_1\cap V_{\langle h\rangle}\cong(a^{-1}H)/H$. Moreover

$$\dim V_{\langle h\rangle}=\dim a^{-1}H\,, \tag{3.1}$$

since $Gv_1\cap V_{\langle h\rangle}$ is open dense in $V_{\langle h\rangle}$ and $|H|<+\infty$. Since every fibre of a is isomorphic to $Z_G(h)=Z_G(\langle h\rangle)$ and $a(a^{-1}H)=a(G)\cap H$ is a finite set,

$$\dim a^{-1}H=\dim Z_G(\langle h\rangle)\,. \tag{3.2}$$

Since $N_G(\langle h\rangle)/Z_G(\langle h\rangle)\subset\mathrm{Aut}(\langle h\rangle)$,

$$|N_G(\langle h\rangle)/Z_G(\langle h\rangle)|<+\infty\,. \tag{3.3}$$

By (2) and (3.1)-(3.3),

$$\dim V''=\dim V_{\langle h\rangle}=\dim N_G(\langle h\rangle)=\dim G''=\dim G''-\dim G''_{v_1}\,,$$

which implies (1). (3) is proved in the same way as Theorem A, (3).

4.

Let notation be as in the introduction, and assume that $f\in\boldsymbol{C}[V]_{G,\phi}$ and $f'\in\boldsymbol{C}[V']_{G',\phi}$ correspond to each other by the isomorphism of Theorem A, (3). Let $b(s)$ and $b'(s)$ be the b-functions of f and f', respectively. (See [**5**, p. 72] or [**1**, 1.6] for the b-functions.) Suppose that $b(s)=b_0\prod_{j=1}^d(s+\alpha_j)$ and $b'(s)=b'_0\prod_{j=1}^d(s+\alpha'_j)$ (b_0, b'_0, α_j, $\alpha'_j\in\boldsymbol{C}$), and put $b^{\exp}(t)=\prod_{j=1}^d(t-\exp(2\pi\sqrt{-1}\alpha_j))$ and $b'^{\exp}(t)=\prod_{j=1}^d(t-\exp(2\pi\sqrt{-1}\alpha'_j))$.

CONJECTURE. $b^{\exp}(t)=b'^{\exp}(t)$.

We can prove this equality when (G, V) is among those listed in [**5**, §7].

5. Example.

Let $G_1=GL_n(\boldsymbol{C})$ and T_1 be the totality of diagonal matrices in G_1. Define the action of $G:=G_1\times G_1$ on $V:=M_n(\boldsymbol{C})$ by $(g, g')v=gvg'^{-1}$. Let $f=\det$ and $v_1\in V$ be the identity matrix. Then $H=\{(g, g)\mid g\in G_1\}$ and we may take $\{(t, t)\mid t\in T_1\}$ as T. Then

$$V'=\{\operatorname{diag}(x_1, \cdots, x_n)\mid x_i\in\boldsymbol{C}\}$$

and N is generated by

$$(\operatorname{diag}(t_1, \cdots, t_n), \operatorname{diag}(t'_1, \cdots, t'_n)) \quad (t_i, t'_i\in\boldsymbol{C}^\times) \quad \text{and}$$

$$((\delta_{i,\sigma(j)}), (\delta_{i,\sigma(j)})) \quad (\sigma\in\mathfrak{S}_n),$$

where δ_{ij} denotes the Kronecker delta. The former acts on V' by $x_i\mapsto t_i t_i'^{-1}x_i$, and the latter by $x_i\mapsto x_{\sigma^{-1}(i)}$. The restriction f' of f to V' is the monomial $\prod_{i=1}^n x_i$. The b-function $b(s)$ (resp. $b'(s)$) of f (resp. f') is given by

$$b(s)=(s+1)(s+2)\cdots(s+n) \quad (\text{resp. } b'(s)=(s+1)^n).$$

6. Example.

Let $G:=SL_3(\boldsymbol{C})\times SL_3(\boldsymbol{C})\times GL_2(\boldsymbol{C})$. Define the action of $(g_1, g_2, g_3)\in G$ on $V:=M_3(\boldsymbol{C})\oplus M_3(\boldsymbol{C})$ by

$$(g_1, g_2)\cdot(X, Y)=(g_1X{}^tg_2, g_1Y{}^tg_2), \qquad g_1, g_2\in SL_3(\boldsymbol{C}) \quad \text{and}$$

$$g_3\cdot(X, Y)=(aX+cY, bX+dY), \qquad g_3=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(\boldsymbol{C}).$$

Let f be the discriminant of the binary cubic form $\det(Xu+Yv)$, where $(X, Y)\in V$ and u, v are variables. Put

$$v_1=\left(\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}\right).$$

Then $H=G_{v_1}$ is generated by

$$T=\left\{\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}\times\begin{pmatrix} t_1^{-1} & & \\ & t_2^{-1} & \\ & & t_3^{-1} \end{pmatrix}\times\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\,\middle|\, t_1t_2t_3=1\right\},$$

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\times\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\times\begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\times\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\times\begin{pmatrix} 1 & \\ 1 & -1 \end{pmatrix}.$$

(The kernel of $G\to GL(V)$ is $\{\mathrm{diag}(t, t, t)\times\mathrm{diag}(t^{-1}, t^{-1}, t^{-1})\,|\,t^3=1\}$.) Then

$$V' = V^T = \{(\mathrm{diag}(x_1, x_2, x_3), \mathrm{diag}(y_1, y_2, y_3))\},$$

and N is generated by

$$\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}\times\begin{pmatrix} t_1' & & \\ & t_2' & \\ & & t_3' \end{pmatrix}\times\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \left(t_i, t_i'\in \boldsymbol{C}^{\times}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(\boldsymbol{C})\right) \text{ and}$$

$$(\delta_{i,\sigma(j)})\times(\delta_{i,\sigma(j)})\times\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\sigma\in\mathfrak{S}_3).$$

The former acts on V' by

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \longmapsto \begin{pmatrix} t_1t_1' & & \\ & t_2t_2' & \\ & & t_3t_3' \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the latter by $(x_i, y_i)\mapsto(x_{\sigma^{-1}(i)}, y_{\sigma^{-1}(i)})$. The restriction f' of f to V' is given by

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}^2 \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}^2.$$

The b-function $b(s)$ (resp. $b'(s)$) of f (resp. f') is given by

$$b(s) = (s+1)^4\left(s+\frac{3}{2}\right)^4\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$$

$$\left(\text{resp. } b'(s)=(s+1)^4\left(s+\frac{1}{2}\right)^4\left(s+\frac{1}{3}\right)\left(s+\frac{2}{3}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\right).$$

7. Remark.

In our construction, naturally appears a prehomogeneous vector space (G, V) such that the isotropy group at a generic point of V is finite. A typical example of such (G, V) can be obtained from a cuspidal pair in the sense of G. Lusztig [**4**] via the Dynkin-Kostant theory. See [**4**, 2.8]. For example the prehomogeneous vector spaces of type (4), (8) and (11) in Table I of [**5**, §7] come from the unique cuspidal pair of the simple algebraic group of type G_2, F_4 and E_8, respectively. The generic isotropy group are isomorphic to $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$, respectively, if we assume $G\subset GL(V)$.

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