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Self-similar diffusions on a class of infinitely ramified fractals

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§ 0. Introduction.

In this paper we construct self-similar diffusions on a class of infinitely ramified (self-similar) fractals.

Construction of self-similar diffusions on finitely ramified fractals has been done by Goldstein [7], Kusuoka [12], Barlow-Perkins [6] and Kumagai [11] for the Sierpinski gasket, and Lindstrøm [14] for the nested fractals.

As for infinitely ramified fractals, Barlow-Bass [1, 2, 3, 4] and Barlow-Bass-Sherwood [5] studied the two dimensional Sierpinski carpet. Although some strong estimates of transition probability densities were obtained, the self-similarity and uniqueness of their Brownian motions were not known. Recently Kusuoka-Zhou [13] have constructed self-similar diffusions on the recurrent fractals; a class of fractals containing the two dimensional Sierpinski carpet.

One crucial step to study the infinitely ramified fractals in the above works was to use the quantities such as resistance and Poincaré constants. We however take a different approach: We first consider the equation (2.9) of hitting probabilities in the fasion of Lindstrøm [14];

$$\Phi(\mathbf{q}) = \mathbf{q} \qquad (\mathbf{q} \in \mathbf{Q}_{G,H}(\boldsymbol{F})). \tag{2.9}$$

Here $\mathbf{Q}_{G,H}(\boldsymbol{F})$ is the set consisting of hitting probabilities. Then we construct self-similar diffusions from its solutions.

In the case of nested fractals, $\mathbf{Q}_{G,H}(\boldsymbol{F})$ can be regarded as a compact convex set in $\boldsymbol{R}^n$ and the map Φ is continuous by the geometrical symmetry of fractals. Accordingly Lindstrøm could solve (2.9) by applying Brouwer's fixed point theorem. If the fractal is infinitely ramified, $\mathbf{Q}_{G,H}(\boldsymbol{F})$ becomes infinitely dimensional and it seems difficult at present to use fixed point theorems.

To solve the equation (2.9), we reduce the problem to the existence of an approximate solution $\mathbf{q}\in\mathbf{Q}_{G,H}(\boldsymbol{F};\ \mathcal{F})$ and obtain the solution $\mathbf{r}$ by taking $\mathbf{r}=\lim_{n\to\infty}\Phi^n(\mathbf{q})$ (Theorem 2.1). We also prove that such an approximate solution exists if the fractal has a nice surjection to another fractal where a self-similar diffusion exists (Theorem 3.1).

The main result is Theorem 4.8. By this theorem we can construct self-similar diffusions from known self-similar diffusions (see Examples (5.1) and (5.4)). Indeed Theorem 4.8 provides a procedure to lift self-similar diffusions on a fractal F_2 to ones on a more complicated fractal F_1. Here even if F_2 is finitely ramified, F_1 becomes infinitely ramified in general (see Figs. 5.1 and 5.2 in Section 5 for examples of F_2 and F_1, respectively). Example (5.4) does not satisfy the assumption (R) in [**13**] for at least $d \geqq 3$. Hence our results are not contained by [**13**].

The fractals in the present paper are quotient spaces of symbol dynamics and not necessary imbedded in $\boldsymbol{R}^n$ (cf. [**10**]). We restrict our attention to compact space.

The organization of this paper is as follows. In Section 1 we prepare definitions and notations to be used throughout this paper. In Section 2 we solve the equations of hitting probabilities (2.9) and transition times (2.27) under the assumptions (2.10), (2.11) and (2.26). In Section 3 we present sufficient conditions for the above assumptions to be true. In Section 4 we construct self-similar diffusions from the solutions obtained in Section 2. In Section 5 we present examples of self-similar diffusions on infinitely ramified fractals.

§1. Definitions of cell fractals and $(G, \boldsymbol{H})$-self similar diffusions.

Let I be a finite set endowed with a discrete topology and $\mathbf{I}=I^N$ the countable product with a product topology. We denote by θ^i the shift operator on $\mathbf{I}$ such that $\theta^i((i_1, i_2, i_3, \cdots))=(i, i_1, i_2, \cdots)$.

Let F be a topological space. Let $f^i: F \to F$ be an injection for each $i \in I$ and $\pi: \mathbf{I} \to F$ be a surjection. Then $(F, I, \{f^i\}, \pi)$ is said to be *self-similar set* if it satisfies that

$$\pi \circ \theta^i = f^i \circ \pi \qquad \text{for each} \quad i \in I, \tag{1.1}$$

and that F is endowed with the quotient topology induced by π.

Let $\boldsymbol{I}^0=\{\emptyset\}$, $\boldsymbol{I}^n=\{(i_1, \cdots, i_n); i_j \in I\}$ and $\boldsymbol{I}^\infty=\bigcup_{n=0}^\infty \boldsymbol{I}^n$. For $\boldsymbol{i}=(i(1), \cdots, i(n)) \in \boldsymbol{I}^n$, we set $f^{\boldsymbol{i}}=f^{i(n)} \circ \cdots \circ f^{i(1)}$, $\theta^{\boldsymbol{i}}=\theta^{i(n)} \circ \cdots \circ \theta^{i(1)}$, where f^ϕ and θ^ϕ denote the identity on F and $\boldsymbol{I}$ respectively.

A self-similar set $\boldsymbol{F}=(F, I, \{f^i\}, \pi)$ is called *a quasi fractal* with $(B, \boldsymbol{B})$ if $(B, \boldsymbol{B})$ satisfies (1.2), $\cdots$, (1.6):

(1.2) $\boldsymbol{B}$ is a finite set of subsets of F such that $f^{\boldsymbol{i}}(b)$ are closed for all $b \in \boldsymbol{B}$ and $\boldsymbol{i} \in \boldsymbol{I}^\infty$.

$$B = \bigcup_{b \in \boldsymbol{B}} b \quad \text{and} \quad F - B \neq \emptyset. \tag{1.3}$$

(1.4) $$b\cap f^{i}(F)\subset f^{i}(b) \quad \text{for each } b\in \boldsymbol{B} \text{ and } i\in I.$$

(1.5) $$f^{\boldsymbol{i}}(F)\cap f^{\boldsymbol{j}}(F)=f^{\boldsymbol{i}}(B)\cap f^{\boldsymbol{j}}(B) \quad \text{for each } \boldsymbol{i},\boldsymbol{j}\in I^{n} \text{ with } \boldsymbol{i}\neq \boldsymbol{j}.$$

These conditions are slightly restrictive than, but essentially same as those in [**15**]. Condition (1.4) is necessary for (4.7). The conditions (1.2) and (1.3) imply that B is a closed set. We call B *the boundary* of $\boldsymbol{F}$ and an element of $\boldsymbol{B}$ *a boundary cell*. B may be empty in general. By (1.4), we see

(1.6) $$B\subset \bigcup_{i\in I} f^{i}(B),$$

from which the open set condition ([**9**]) follows.

We next introduce the 1-cell condition. Let $\boldsymbol{F}=(F, I, \{f^{i}\}, \pi)$ be a quasi fractal with $(B, \boldsymbol{B})$. We say that $\boldsymbol{F}$ satisfies *the 1-cell condition* if either B is empty or $(B, \boldsymbol{B})$ satisfies (1.7) and (1.8):

(1.7) For each $b\in \boldsymbol{B}$, there exist $I_{b}\subsetneqq I$ and a surjection π_{b}: $I_{b}^{N}\to b$ and a continuous map ι_{b}: $I_{b}^{N}\to \mathbf{I}$ such that $\boldsymbol{b}=(b, I_{b}, \{f^{i}|_{b}\}, \pi_{b})$ is a quasi fractal with $(B_{b}, \boldsymbol{B}_{b})$, and that $\pi\circ\iota_{b}=\pi_{b}$.

Here B_{b} is a subset of b and $\boldsymbol{B}_{b}$ is a set of subsets of B_{b}; $(B_{b}, \boldsymbol{B}_{b})$ satisfies (1.2), $\cdots$, (1.6) for $\boldsymbol{b}=(b, I_{b}, \{f^{i}|_{b}\}, \pi_{b})$.

(1.8) For each $b, b'\in \boldsymbol{B}$ and $\boldsymbol{i}, \boldsymbol{i}'\in \boldsymbol{I}^{\infty}$ such that $f^{\boldsymbol{i}}(b)\neq f^{\boldsymbol{i}'}(b')$ and that $\boldsymbol{i}, \boldsymbol{i}'\in \boldsymbol{I}^{n}$ for some $n\geqq 0$,

$$f^{\boldsymbol{i}}(b)\cap f^{\boldsymbol{i}'}(b')=f^{\boldsymbol{i}}(B_{b})\cap f^{\boldsymbol{i}'}(B_{b'}).$$

Let $\boldsymbol{B}(1)=\boldsymbol{B}$. We call an element of $\boldsymbol{B}(1)$ a *1-bouudary cell*. We define k-cell condition and $\boldsymbol{B}(k)$ inductively as follows:

A quasi fractal $\boldsymbol{F}=(F^{i}, I, \{f^{i}\}, \pi)$ satisfies *k-cell condition* if $\boldsymbol{F}$ satisfies $(k-1)$-cell condition, and for each $b\in\boldsymbol{B}(k-1)$, $\boldsymbol{b}=(b, I_{b}, \{f^{i}|_{b}\}, \pi_{b})$ is a quasi fractal with $(B_{b}, \boldsymbol{B}_{b})$. Here

(1.9) $$\boldsymbol{B}(k)=\bigcup_{a\in\boldsymbol{B}(k-1)}\boldsymbol{B}_{a}.$$

DEFINITION. A quasi fractal $\boldsymbol{F}=(F, I, \{f^{i}\}, \pi)$ is *a cell fractal* if it satisfies k-cell condition for all $k=1, 2, \cdots$.

REMARK. There exists a k such that $\boldsymbol{B}(k)=\{\emptyset\}$. Hence we set

(1.10) $$k_{0}=\min\{k\,;\,\boldsymbol{B}(k)=\{\emptyset\}\}.$$

We quote the following lemma from [**15**].

LEMMA 1.1. *A cell fractal is a compact metric space.*

REMARK. Since F is a Hausdorff space, f^{i} is continuous. Moreover, each

f^{i} has a unique fixed point and π is determined by $(F, I, \{f^{i}\})$ uniquely. Also, $\{f^{i}\}$ is determined uniquely from (F, I, π). Hence we often write $\boldsymbol{F}=(F, I, \{f^{i}\})$ and $\boldsymbol{F}=(F, I, \pi)$.

To help reader's understanding we give a simple example:

EXAMPLE. Let $F=[0, 1]\times[0, 1]$ and $I=\{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Then we can regard (F, I) as a cell fractal as follows: (See Fig. 2.1).

$$\pi(\mathbf{i}) = \left(\sum_{n=1}^{\infty} i_n 2^{-n},\ \sum_{n=1}^{\infty} j_n 2^{-n}\right), \qquad \mathbf{i} = ((i_n, j_n)) \in I^{N}.$$

$$\boldsymbol{B} \equiv \boldsymbol{B}(1) = \{b_i\}_{1\leq i\leq 4},$$

where $b_1 = \{0\}\times[0, 1]$, $b_2 = \{1\}\times[0, 1]$, $b_3 = [0, 1]\times\{0\}$, $b_4 = [0, 1]\times\{1\}$.

$$\boldsymbol{B}(2) = \{\{b_i^2\}\}_{1\leq i\leq 4},$$

where $b_1^2 = (0, 0)$, $b_2^2 = (1, 0)$, $b_3^2 = (0, 1)$, $b_4^2 = (1, 1)$.

$$\boldsymbol{B}(k) = \{\emptyset\} \qquad \text{for} \quad k \geqq 3.$$

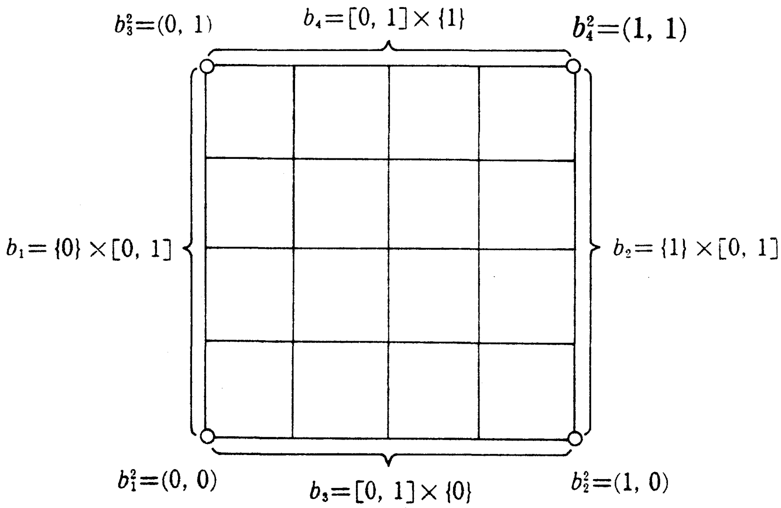


Fig. 2.1.

The following notation will be used throughout this paper: Let $\boldsymbol{F}=(F, I, \{f^{i}\}, \pi)$ be a cell fractal with $(B, \boldsymbol{B})$ and k-boundary cells $\boldsymbol{B}(k)$. For $\boldsymbol{F}$ we set

$$\boldsymbol{C}^{n} = \{c\,;\ c = f^{\boldsymbol{i}}(F)\ (\boldsymbol{i} \in \boldsymbol{I}^{n})\}, \qquad \boldsymbol{C}^{\infty} = \bigcup_{n=0}^{\infty} \boldsymbol{C}^{n}.$$

$$\boldsymbol{B}^{n}(k) = \{f^{\boldsymbol{i}}(b)\,;\ \boldsymbol{i} \in \boldsymbol{I}^{n},\ b \in \boldsymbol{B}(k)\}\ (k \geqq 1), \qquad \boldsymbol{B}^{n}(0) = \boldsymbol{C}^{n}\ (k = 0). \tag{1.11}$$

$$B^{n} = \bigcup_{\boldsymbol{i}\in\boldsymbol{I}^{n}} f^{\boldsymbol{i}}(B), \qquad B^{\infty} = \bigcup_{n=0}^{\infty} B^{n}.$$

Further we set $\boldsymbol{A}(k)=\phi$ $(k\geqq k_0)$, where k_0 is defined by (1.10), and

$$A(k)=\{b-\bigcup_{i>k}\bigcup_{a\in\boldsymbol{A}(i)} a\,;\ b\in\boldsymbol{B}(k)\}\qquad\text{for}\quad k_0>k\geqq 1$$

$$\boldsymbol{A}^n(k)=\{f^{\boldsymbol{i}}(a)\,;\ a\in\boldsymbol{A}(k),\ \boldsymbol{i}\in\boldsymbol{I}^n\}\tag{1.12}$$

$$\boldsymbol{A}=\bigcup_{k=1}^{\infty}\boldsymbol{A}(k),\qquad \boldsymbol{A}^n=\{f^{\boldsymbol{i}}(a)\,;\ a\in\boldsymbol{A},\ \boldsymbol{i}\in\boldsymbol{I}^n\},\qquad \boldsymbol{A}^{\infty}=\bigcup_{n=0}^{\infty}\boldsymbol{A}^n.$$

REMARK. $\boldsymbol{A}$ is a partition of B: $B=\sum_{a\in\boldsymbol{A}}a$. Moreover for each $b\in\boldsymbol{B}(k)$ there exist $a_1,\cdots,a_n\in\bigcup_{j\geq k}\boldsymbol{A}(j)$ such that $b=\sum_{1\leq i\leq n}a_i$.

We set $a^n[x]\in\boldsymbol{A}^n$ such that $x\in a^n[x]$ for $x\in B^n$, and $a^n[x]=\emptyset$ for $x\in F-B^n$.

For $a\in\boldsymbol{A}^n$ and $x\in F$, we set

$$\begin{aligned} C^n[a]&=\bigcup_{a\subset c\in\boldsymbol{C}^n} c, &\quad B^n[a]&=\Big(\bigcup_{b\cap a=\emptyset,\,b\in\boldsymbol{B}^n} b\Big)\cap C^n[a],\\ C^n[x]&=\bigcup_{x\in c\in\boldsymbol{C}^n} c, &\quad B^n[x]&=\Big(\bigcup_{b\cap a^n[x]=\emptyset,\,b\in\boldsymbol{B}^n} b\Big)\cap C^n[x]. \end{aligned}\tag{1.13}$$

For $a\in\boldsymbol{A}^1$ with $a\subset B^1-B$, let $\mathbf{f}_a\colon C^2[a]\to C^1[a]$ be the map defined by

$$\mathbf{f}_a(x)=f^i\circ(f^j)^{-1}\circ(f^i)^{-1}(x)\qquad\text{if}\quad x\in c\in\boldsymbol{C}^2.\tag{1.14}$$

Here $i,j\in I$ are such that $f^i\circ f^j(F)=c$. It is easy to see that $\mathbf{f}_a$ are well-defined. $\mathbf{f}_a$ are not injective in general.

We next turn to the definition of self-similar diffusions. Let $\sigma(x,n)=\inf\{t>0\,;\ X_t\in B^n[x]\}$, where $\{X_t\}\in C\{[0,\infty)\to F\}$.

DEFINITION. A system of diffusion measures $\{P_x\}_{x\in F}$ is a *self-similar diffusion* with *a time scaling factor* λ if

$$P_x(f^i(X_{\lambda t\wedge\sigma(x,0)})\in\cdot)=P_{f^i(x)}(X_{t\wedge\sigma(f^i(x),1)}\in\cdot)\tag{1.15}$$

for $i\in I$, $x\in F-B$.

DEFINITION. A diffusion $\{P_x\}$ is *strongly self-similar* if, for $a\in\boldsymbol{A}^1$ such that $a\subset B^1-B$ and $x\in C^1[a]-B^1[a]$,

$$P_x(X_{\lambda t\wedge\sigma(x,0)}\in\cdot)=P_y(\mathbf{f}_a(X_{t\wedge\sigma(y,1)})\in\cdot)\tag{1.16}$$

for each $y\in\mathbf{f}_a^{-1}(x)$.

REMARK. We note that (1.15) does not imply (1.16). Brownian motions on nested fractals [**14**] and p-stream diffusions on the Sierpinski gasket [**12**] are strongly self-similar diffusions.

Let $\boldsymbol{F}_m=(F_m,I_m,\{f_m^i\},\pi_m)$ $(m=1,2)$ be cell fractals and S_m $(m=1,2)$ their subsets respectively. A homeomorphism $h\colon S_1\to S_2$ is a cell homeomorphism if

$$\{h(c)\,;\ c\in\boldsymbol{B}_1^n(k),\ c\subset S_1\}=\{c\,;\ c\in\boldsymbol{B}_2^n(k),\ c\subset S_2\}\qquad\text{for all}\quad k,\ n.$$

Here $\boldsymbol{B}_m^n(k)$ is $\boldsymbol{B}^n(k)$ for $\boldsymbol{F}_m$.

We next introduce groups acting on cell fractals: We set

$$G(\boldsymbol{F}) = \{g\,;\ g\colon F \to F \text{ is a cell homeomorphism}\}.$$

We set for $a \in \boldsymbol{A}^n$ such that $a \subset B^1 - B$,

$$H_a^n(\boldsymbol{F}) = \{h\colon C^n[a] \to C^n[a]\,;\ h \text{ is a cell homeomorphism, } h|_a = id.\},$$

and for $x \in B^1 - B$, $H_x^n(\boldsymbol{F}) = H_{a^n[x]}(\boldsymbol{F})$. We regard $G(\boldsymbol{F})$ and $H_x^n(\boldsymbol{F})$ as groups under the operation of composition of maps. Let for $x \in B^1 - B$

$$\boldsymbol{H}_x(\boldsymbol{F}) = \prod_{n=1}^{\infty} H_x^n(\boldsymbol{F}) \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{F}) = \prod_{x \in B^1 - B} \boldsymbol{H}_x(\boldsymbol{F}). \tag{1.17}$$

Here $\prod$ denotes the direct product of groups.

DEFINITION. We call $\boldsymbol{F}$ a $(G, \boldsymbol{H})$*-cell fractal* and also $(G, \boldsymbol{H})$ is *the structure group* of $\boldsymbol{F}$, if G and $\boldsymbol{H} = \prod_{x \in B^1 - B} \boldsymbol{H}_x$ are subgroups of $G(\boldsymbol{F})$ and $\boldsymbol{H}(\boldsymbol{F})$ respectively, and satisfy the following:

(1.18) H_x^n is a subgroup of $H_x^n(\boldsymbol{F})$ such that $H_x^n = H_y^n$ if $y \in a^n[x]$, where $\boldsymbol{H}_x = \prod_{n=1}^{\infty} H_x^n$.

$$\{g|_{C^n[x]}\,;\ g \in G,\ g|_{a^n[x]} = id.\} \subset H_x^n. \tag{1.19}$$

$$\{h|_{C^{n+1}[x]}\,;\ h \in H_x^n\} \subset H_x^{n+1} \quad \text{for} \quad x \in B^1 - B. \tag{1.20}$$

(1.21) $H_x^n \cong H_{g(x)}^n$ with the isomorphism $h \mapsto g \circ h \circ g^{-1}$ for ${}^\forall g \in G$.

Here $\cong$ denotes the group isomorphism.

DEFINITION. Let $\boldsymbol{F}$ be a $(G, \boldsymbol{H})$-cell fractal. A self-similar diffusion $\{P_x\}_{x \in F}$ is *a* $(G, \boldsymbol{H})$*-self similar diffusion* if $\{P_x\}_{x \in F}$ is $(G, \boldsymbol{H})$-invariant;

$$P_x(g(X_{t \wedge \sigma(x,0)}) \in \cdot) = P_{g(x)}(X_{t \wedge \sigma(g(x),0)} \in \cdot) \quad \text{for all} \quad g \in G. \tag{1.22}$$

$$P_y(h(X_{t \wedge \sigma(x,n)}) \in \cdot) = P_{h(y)}(X_{t \wedge \sigma(x,n)} \in \cdot) \tag{1.23}$$

for all $x \in B^1 - B$, $h \in H_x^n$, $y \in C^n[x]$.

§ 2. Equations of hitting probabilities and transition times.

Let $\boldsymbol{F} = (F, I, \{f^i\}, \pi)$ be a $(G, \boldsymbol{H})$-cell fractal with a boundary B and k-boundary cells $\boldsymbol{B}(k)$. We denote by $\mathscr{B}(*)$ the Borel σ-field of a topological space $*$.

We set for $n \geq 1$

$$Q^n(\boldsymbol{F}) = \{q\colon (B^n - B) \times \mathscr{B}(B^\infty) \to [0, 1]\,;\ q \text{ satisfies (2.1), (2.2)}\}$$

(2.1) $q(x, \cdot)$ is a measure such that $q(x, B^n[x]^c)=0$ for each x.

(2.2) $q(\cdot, A)$ is $\mathcal{B}(B^n-B)$-measurable for each $A \in \mathcal{B}(B^\infty)$.

Let

$$Q^n_{G,H}(\boldsymbol{F}) = \{q \in Q^n(\boldsymbol{F});\ q(x, B^n[x]) = 1\ (^\forall x),\ q \text{ satisfies (2.3), (2.4)}\}.$$

(2.3) $q(\cdot, *) = q(g(\cdot), g(*))$ for all $g \in G$.

(2.4) $q(x, A) = q(x, h(A))$ for $^\forall h \in H^n_x$ and $^\forall A \in \mathcal{B}(B^n)$ such that $A \subset B^n[x]$.

$Q^n[\boldsymbol{F}]$, $n=1, 2, \cdots$ are sets of hitting probabilities and $Q^n_{G,H}(\boldsymbol{F})$ are their $(G, \boldsymbol{H})$-invariant elements. If a self-similar diffusion $\{P_x\}$ exists, then $q^n(x, dy) = P_x(X_{\sigma(x,n)} \in dy)$ is an element of $Q^n(\boldsymbol{F})$. By the self-similarity of $\{P_x\}$, we see

$$q^{n+1}(f^i(\cdot), f^i(*)) = q^n(\cdot, *) \quad \text{for } i \in I,\ n \geqq 1. \tag{2.5}$$

Moreover if $\{P_x\}$ is a strongly self-similar diffusion, then by (1.16)

$$q^{n+1}(y, \mathbf{f}_a^{-1}(*)) = q^n(x, *) \quad (y \in \mathbf{f}_a^{-1}(x)) \tag{2.6}$$

for $a \in \boldsymbol{A}^1$ with $a \subset B^1 - B$, $n \geqq 1$.

Taking (2.5) and (2.6) into account, we set

$$\begin{aligned} \boldsymbol{Q}(\boldsymbol{F}) &= \{\mathbf{q} = (q^n) \in \prod_{n=1}^\infty Q^n(\boldsymbol{F});\ \mathbf{q} \text{ satisfies (2.5)}\}, \\ \mathbf{Q}^R(\boldsymbol{F}) &= \{\mathbf{q} = (q^n) \in \mathbf{Q}(\boldsymbol{F});\ \mathbf{q} \text{ satisfies (2.6)}\}, \\ \mathbf{Q}_{G,H}(\boldsymbol{F}) &= \{\mathbf{q} = (q^n) \in \mathbf{Q}(\boldsymbol{F});\ q^n \in Q^n_{G,H}(\boldsymbol{F})\}. \end{aligned} \tag{2.7}$$

REMARK. From (2.5) $\mathbf{q}=(q^n)$ is determined by $q^n(x, \cdot)$, where $x \in B^1 - B$ and $n=1, 2, \cdots$. Moreover if $\mathbf{q} \in \mathbf{Q}^R(\boldsymbol{F})$ and $\mathbf{f}_a$ are injective, then $\mathbf{q}=(q^n)$ is determined by q^1.

To define the function $\Phi: \mathbf{Q}(\boldsymbol{F}) \to \mathbf{Q}(\boldsymbol{F})$ below, we prepare several notations. We set

$$\mathbf{W}^n = \{\mathbf{w} = (w^a, w^b);\ \mathbf{w} \in \boldsymbol{A}^n \times \boldsymbol{A}^n \text{ such that } w^b \subset B^n[w^a]\}.$$

We call $\boldsymbol{w}=(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m)$ an n-walk and m its length, denoted by $m=|\boldsymbol{w}|$, if $\mathbf{w}_i \in \mathbf{W}^n$ $(1 \leqq i \leqq m)$ and $w^b_{i+1} = w^a_i$ $(1 \leqq i \leqq m-1)$, where $\mathbf{w}_i = (w^a_i, w^b_i)$. We denote by $\boldsymbol{W}^n$ the set of n-walks. We set

$$\begin{aligned} &\boldsymbol{W}^n[x] \\ &= \{\boldsymbol{w} \in \boldsymbol{W}^n;\ x \in w^a_1,\ w^a_k \cap B^{n-1}[x] = \phi\ (1 \leqq {}^\forall k \leqq |\boldsymbol{w}|),\ w^b_{|\boldsymbol{w}|} \subset B^{n-1}[x]\}, \end{aligned}$$

$$A(\boldsymbol{w}) = w^b_1 \times w^b_2 \times \cdots \times w^b_{|\boldsymbol{w}|} \quad \text{for } \boldsymbol{w} = (\mathbf{w}_i),\ \mathbf{w}_i = (w^a_i, w^b_i).$$

Let $\Phi: Q^n(\boldsymbol{F}) \to Q^{n-1}(\boldsymbol{F})$ be the map defined by

$$\Phi(q)(x, A) = \sum_{W^n[x]} \int_{A(w)} \prod_{m=1}^{|w|} q(x_{m-1}, dx_m) \cdot 1_A(x_{|w|}), \quad (x_0 = x). \tag{2.8}$$

We regard Φ as the map $\Phi: \prod_{n=1}^{\infty} Q^n(\boldsymbol{F}) \to \prod_{n=1}^{\infty} Q^n(\boldsymbol{F})$ by

$$\Phi(\mathbf{q}) = (\Phi(q^{n+1}))_{n=1,2,3,\dots} \qquad (\mathbf{q} = (q^n)).$$

The key step to construct $(G, \boldsymbol{H})$-self similar diffusions is to solve the following equations:

$$\Phi(\mathbf{q}) = \mathbf{q} \qquad (\mathbf{q} \in \mathbf{Q}_{G,H}(\boldsymbol{F})), \tag{2.9}$$

and

$$\Phi(\mathbf{q}) = \mathbf{q} \qquad (\mathbf{q} \in \mathbf{Q}_{G,H}(\boldsymbol{F}) \cap \mathbf{Q}^R(\boldsymbol{F})). \tag{2.9'}$$

REMARK. a) If a (resp. strongly) $(G, \boldsymbol{H})$-self similar diffusion $\{P_x\}$ exists, then $(P_x(X_{\sigma(x,n)} \in \cdot))_{n=1,2,\dots}$ satisfies (2.9) ((2.9′)) by the strong Markov property.

b) The equation discussed in 48 p. [**14**] essentially corresponds to (2.9′). See also [**8**] and [**10**].

Let $\mathcal{F}$ be a sub σ-field of $\mathcal{B}(B)$ and $\mathcal{F}^n$ a sub σ-field of $\mathcal{B}(B^n)$ such that $\mathcal{F}^n = \sigma[\{f^{\boldsymbol{i}}(A);\ A \in \mathcal{F}, \boldsymbol{i} \in \boldsymbol{I}^n\}]$. We set

$$\mathbf{Q}(\boldsymbol{F}: \mathcal{F}) = \{\mathbf{q} \in \mathbf{Q}(\boldsymbol{F});\ q^n(\cdot, A) \text{ is } \mathcal{F}^n\text{-measurable for } A \in \mathcal{F}^n,\ n \in \boldsymbol{N}\},$$

$$\mathbf{Q}_{G,H}(\boldsymbol{F};\ \mathcal{F}) = \mathbf{Q}(\boldsymbol{F}: \mathcal{F}) \cap \mathbf{Q}_{G,H}(\boldsymbol{F}).$$

We now consider a reduction of the equation (2.9).

THEOREM 2.1. *Suppose that there exist* $\mathcal{F}$ *and* $\mathbf{q}$ *satisfying* (2.10) *and* (2.11):

$$\mathbf{q} \in \mathbf{Q}_{G,H}(\boldsymbol{F};\ \mathcal{F}) \text{ and } \boldsymbol{A} \subset \mathcal{F}, \quad (\boldsymbol{A} \text{ is defined in } (1.12)), \tag{2.10}$$

$$\Phi(q^{n+1})(\cdot, A) = q^n(\cdot, A) \qquad \text{for all} \quad A \in \mathcal{F}^n,\ n \geqq 1. \tag{2.11}$$

Then there exists a solution $\mathbf{r} = (r^m) \in \mathbf{Q}_{G,H}(\boldsymbol{F};\ \mathcal{F})$ *of* (2.9) *such that*

$$\lim_{n\to\infty} \Phi^n(q^{m+n})(x, A) = r^m(x, A) \qquad \text{for} \quad x \in B^n - B,\ A \in \boldsymbol{A}^{\infty}. \tag{2.12}$$

Moreover if $\mathbf{q} \in \mathbf{Q}^R(\boldsymbol{F})$, *then* $\mathbf{r} \in \mathbf{Q}^R(\boldsymbol{F})$.

The following lemmas are analogous to the Lemma 3.2 and Lemma 3.3 in [**15**]. Hence we omit the proofs.

LEMMA 2.2. *Suppose* $\mathbf{q} \in \mathbf{Q}_{G,H}(\boldsymbol{F})$. *Then* $\Phi^n(\mathbf{q}) \in \mathbf{Q}_{G,H}(\boldsymbol{F})$ *for* $^{\forall} n \geqq 1$. *Moreover, if* $\mathbf{q} \in \mathbf{Q}^R(\boldsymbol{F})$. *Then* $\Phi^n(\mathbf{q}) \in \mathbf{Q}^R(\boldsymbol{F})$ *for all* $n \geqq 1$.

LEMMA 2.3. *Let* $\mathbf{q} = (q^n) \in \mathbf{Q}(\boldsymbol{F}: \mathcal{F})$ *satisfy* (2.10) *and* (2.11). *Then*

$$(2.13)\qquad \Phi^{m}(q^{n+m})(\cdot, A)=\Phi^{l}(q^{n+l})(\cdot, A) \quad \text{for} \quad A \in \mathcal{I}^{n+l},\ m \geqq l.$$

PROOF OF THEOREM 2.1. Since $B^{m}[x]$ is compact and $\Phi^{n}(q^{m+n})(x, B^{m}[x])=1$, $\{\Phi^{n}(q^{m+n})(x, *)\}_{n\geq 0}$ is tight for all x and m. Hence for $x \in B^{m}-B$ there exists a probability measure $r^{m}(x, \cdot)$ and a sub sequence $n(x)$ depending on x and m such that $r^{m}(x, B^{m}[x])=1$ and that

$$(2.14)\qquad \lim_{n(x)\to\infty} \Phi^{n(x)}(q^{m+n(x)})(x, \cdot)=r^{m}(x, \cdot) \quad \text{weakly}.$$

Let m be fixed. We will prove r^{m} satisfies (2.12). Let $\mathcal{Y}_{k}^{n}=\{a \in \boldsymbol{A}^{n}(k);\ a \subset B^{m}\}$ and $\mathcal{Y}_{k}^{\infty}=\bigcup_{n=m}^{\infty} \mathcal{Y}_{k}^{n}$. Then $\mathcal{Y}_{k}^{n} \subset \mathcal{I}^{n}$. From Lemma 2.3, we see

$$(2.15)\qquad \lim_{n\to\infty} \Phi^{n}(q^{m+n})(x, A) \text{ exists} \quad \text{for} \quad x \in B^{m}-B,\ A \in \mathcal{Y}_{k}^{\infty}.$$

Let $B^{m}(k)=\bigcup_{\boldsymbol{B}^{m}(k)} b$, where $\boldsymbol{B}^{m}(k)$ is defined by (1.11). Then $B^{m}(k)$ is compact, and

$$(2.16)\qquad \lim_{n(x)\to\infty} \Phi_{k}^{n(x)}(q^{m+n(x)})(x, \cdot)=r_{k}^{m}(x, \cdot)$$

weakly as measures on $B^{m}(k)$. Here $\Phi_{k}^{n(x)}(q^{m})(x, \cdot)$ and r_{k}^{m} denote the restrictions of $\Phi^{n(x)}(q^{m})(x, \cdot)$ and r^{m} on $\mathcal{B}(B^{m}(k))$ respectively.

Now we see easily that $B^{m}(k)=\bigcup_{l=k}^{k_0} \mathcal{Y}_{l}^{\infty}$, where k_0 is defined by (1.10). Since $\boldsymbol{B}(k_0)=\{\emptyset\}$, each elements of $\mathcal{Y}_{k_0}^{\infty}$ are open and closed in $B^{m}(k_0)$, endowed with the relative topology. Hence by (2.15) and (2.16)

$$(2.17)\qquad \lim_{n\to\infty} \Phi^{n}(q^{m+n})(x, b)=r^{m}(x, b) \quad \text{for} \quad {}^{\forall} b \in \mathcal{Y}_{k_0}^{\infty}.$$

We next prove (2.17) for $\mathcal{Y}_{k_0-1}^{\infty}$. Let $a_0 \in \mathcal{Y}_{k_0-1}^{\infty}$. Then a_0 is open in $B^{m}(k_0-1)$. Hence by (2.16)

$$(2.18)\qquad \varliminf_{n(x)\to\infty} \Phi^{n(x)}(q^{m+n(x)})(x, a_0) \geqq r^{m}(x, a_0).$$

For a_0, there exist $b \in \boldsymbol{B}^{m}(k_0-1)$ and $a_1, \cdots, a_l \in \mathcal{Y}_{k_0}^{\infty}$ such that $b=\bigcup_{i=0}^{l} a_i$, and that $a_i \cap a_j=\emptyset$ if $i \neq j$. Note that b is closed in $B^{m}(k_0-1)$. Hence

$$\begin{aligned}
r^{m}(x, a_0) &= r^{m}(x, b)-\sum_{i=1}^{l} r^{m}(x, a_i) \\
&\leqq \varlimsup_{n(x)\to\infty} \Phi^{n(x)}(q^{m+n(x)})(x, b)-\lim_{n(x)\to\infty} \sum_{i=1}^{l} \Phi^{n(x)}(q^{m+n(x)})(x, a_i) \\
&= \varlimsup_{n(x)\to\infty} \Phi^{n(x)}(q^{m+n(x)})(x, a_0),
\end{aligned}$$

which together with (2.18) implies (2.17) for $\mathcal{Y}_{k_0-1}^{\infty}$. We can prove (2.17) for $1 \leqq k \leqq k_0-2$ by induction with respect to k similarly, which together with $\Phi^{n}(q^{m+n})(x, B^{m}[x]^{c})=r^{m}(x, B^{m}[x]^{c})=0$ yields (2.12).

By (2.12), $r^{m}(\cdot, A)$ is $\mathcal{B}(B^{m}-B)$-measurable for ${}^{\forall} A \in \boldsymbol{A}^{\infty}$. Hence by using the monotone class theorem we conclude $r^m(\cdot, A)$ is $\mathscr{B}(B^m-B)$-measurable for ${}^\forall A\in\mathscr{B}(B^\infty)$. Moreover by Lemma 2.2, we see $\mathbf{r}=(r^m)\in\mathbf{Q}_{G,\boldsymbol{H}}(\boldsymbol{F})$, which completes the proof of Theorem 2.1. $\square$

We next turn to the second equation (2.27) on *transition times*. In the rest of this section, we assume that $\mathbf{q}\in\mathbf{Q}_{G,\boldsymbol{H}}(\boldsymbol{F}:\mathscr{F})$ is the solution of (2.9) constructed by Theorem 2.1. We write $\mathbf{q}$ instead of $\mathbf{r}$.

Let

$$T^n(\boldsymbol{F})=\{t\colon (B^n-B)\times B^n\times\mathscr{B}([0,\infty))\to[0,1]\,;\ t \text{ satisfies (2.19), (2.20)}\},$$

(2.19) $\qquad t(x, y, \cdot)$ is a probability measure for all x and y.

(2.20) $\qquad t(\cdot, *, S)$ is $\mathscr{B}((B^n-B)\times B^n)$-measurable for all S.

Here t and t' are identified if $t(x, y, S)=t'(x, y, S)$ a.e. y with respect to $q^n(x, dy)$ for all x and $S\in\mathscr{B}([0,\infty))$.

We set

$$T^n_{G,\boldsymbol{H}}(\boldsymbol{F})=\{t\in T^n(\boldsymbol{F})\,;\ t \text{ satisfies (2.21), (2.22)}\}.$$

(2.21) $\quad t(x, y, S)=t(g(x), g(y), S)\quad$ for all $g\in G$ and $S\in\mathscr{B}([0,\infty))$.

(2.22) $\quad t(x, y, S)=t(x, h(y), S)\quad$ for all $h\in H_x$, $y\in B^n[x]$ and $S\in\mathscr{B}([0,\infty))$.

$T^n(\boldsymbol{F})$ are sets of transition times and $T^n_{G,\boldsymbol{H}}(\boldsymbol{F})$ are their $(G, \boldsymbol{H})$-invariant elements. Indeed for a diffusion $\{P_x\}$ $t^n(x, y, dt)=P_x(\sigma(x, n)\in dt\,|\,X_{\sigma(x,n)}=y)$ is an element of $T^n(\boldsymbol{F})$. If $\{P_x\}$ is a self-similar diffusion with a time scaling facter λ, then

(2.23) $\quad t^{n+1}(f^i(\cdot), f^i(*), S)=t^n(\cdot, *, \lambda S)\quad$ for all $i\in I$, n, $S\in\mathscr{B}([0,\infty))$.

Moreover if $\{P_x\}$ is a strongly self-similar diffusion, then

(2.24) $\qquad t^{n+1}(y, \mathbf{f}_a^{-1}(*), S)=t^n(x, *, \lambda S)\qquad (y\in\mathbf{f}_a^{-1}(x))$

for $a\in\boldsymbol{A}^1$ with $a\subset B^1-B$.

Let λ be a fixed constant such that $1<\lambda$. Taking (2.23) and (2.24) into account, we consider the following;

$$\mathbf{T}(\boldsymbol{F})=\Big\{\mathbf{t}=(t^n)\in\prod_{n=1}^{\infty}T^n(\boldsymbol{F})\,;\ \mathbf{t} \text{ satisfies (2.23)}\Big\},$$

$$\mathbf{T}^R(\boldsymbol{F})=\{\mathbf{t}=(t^n)\in\mathbf{T}(\boldsymbol{F})\,;\ \mathbf{t} \text{ satisfies (2.24)}\},$$

$$\mathbf{T}_{G,\boldsymbol{H}}(\boldsymbol{F})=\{\mathbf{t}=(t^n)\in\mathbf{T}(\boldsymbol{F})\,;\ t^n\in T^n_{G,\boldsymbol{H}}(\boldsymbol{F})\}.$$

For $t\in T^n(\boldsymbol{F})$, let $\tilde{\Psi}(t)\colon (B^{n-1}-B)\times\mathscr{B}(B^{n-1}\times[0,\infty))\to\boldsymbol{R}^+$ such that

$$(2.25)\qquad \tilde{\Psi}(t)(x, A\times S) = \sum_{W^n[x]} \int_{A(w)} t_w(\boldsymbol{x}, S)\cdot \prod_{m=1}^{|w|} q^n(x_{m-1}, dx_m)\cdot 1_A(x_{|w|}),$$

where $\boldsymbol{x}=(x_0, \cdots, x_{|w|})$, $x_0=x$, and

$$t_w(\boldsymbol{x}, ds_m) = t(x_0, x_1, ds_1)*\cdots*t(x_{m-1}, x_m, ds_m).$$

Here $*$ denotes the convolution on $ds_1, \cdots, ds_{|w|}$.

Note that $\tilde{\Psi}(t^n)(x, A\times[0, \infty)) = q^{n-1}(x, A)$. Hence $\tilde{\Psi}(t^n)(x, dy, S)$ has a Radon-Nykodim density, denoted by $\Psi(t^n)(x, y, S)$, with respect to $q^{n-1}(x, dy)$ for all x and $S\in\mathcal{B}([0, \infty))$. We regard Ψ as the map Ψ: $\mathbf{T}(\boldsymbol{F})\to\mathbf{T}(\boldsymbol{F})$ by $\Psi(\mathbf{t})=(\Psi(t^{n+1}))_{n=1,2,3,\dots}$.

For $t\in T^n(\boldsymbol{F})$, we set $\tilde{t}(x, A, S)=\int_A t(x, y, S)\, q^n(x, dy)$, and consider the condition

$$(2.26)\qquad \tilde{t}(\cdot, A, S) \text{ is } \mathcal{F}^n\text{-measurable for } A\in\mathcal{F}^n,\ S\in\mathcal{B}([0, \infty)).$$

We set $\mathbf{T}(\boldsymbol{F}: \mathcal{F})=\{\mathbf{t}=\{t^n\}\in\mathbf{T}(\boldsymbol{F});\ t^n \text{ satisfies (2.25)}\}$ and

$$\mathbf{T}_{G,H}(\boldsymbol{F}: \mathcal{F}) = \mathbf{T}_{G,H}(\boldsymbol{F})\cap\mathbf{T}(\boldsymbol{F}: \mathcal{F}).$$

The proof of the following theorem is similar to that of Theorem 2.1. Hence we omit it.

THEOREM 2.4. *Suppose that there exists* $\mathbf{t}=(t^n)\in\mathbf{T}_{G,H}(\boldsymbol{F};\ \mathcal{F})$ *such that*

$$(2.27)\qquad \tilde{\Psi}(t^{n+1})(x, A\times *) = \tilde{t}^n(x, A, *)\quad \textit{for all}\quad A\in\mathcal{F}^n \textit{ and } n\geqq 1.$$

Then there exists $\mathbf{u}=(u^n)\in\mathbf{T}_{G,H}(\boldsymbol{F})$ *such that*

$$(2.28)\qquad \Psi(\mathbf{u}) = \mathbf{u}.$$

Moreover, if $\mathbf{t}\in\mathbf{T}^R(\boldsymbol{F})$, *then* $\mathbf{u}\in\mathbf{T}^R(\boldsymbol{F})$.

REMARK. When $\boldsymbol{F}$ is finitely ramified, for each solution $\mathbf{q}$ of (2.9) there exists a unique $\lambda>1$ such that for this λ, (2.28) has a unique solution $\mathbf{u}$ of (2.28) (up to a constant multiplication) satisfying $\int_0^\infty s^2\mathbf{u}(ds)<\infty$ in componentwise. This result is a generalization of Lindstrøm's result, Theorem VI. 5 in [**14**] for nested fractals. In [**10**] Kigami construct *Laplace operators* from *harmonic structures* for P.C.F. self-similar sets. Here harmonic structure is a solution of an equation similar to (2.7′).

§ 3. Sufficient conditions for solvability.

Before proceeding to the construction of self-similar diffusions, we present in this section a sufficient condition for (2.10) (2.11) and (2.27) to hold.

Let $\boldsymbol{F}_j=(F_j, I_j, \{f_j^i\}, \pi_j)\,(j=1, 2)$ be $(G_j, \boldsymbol{H}_j)$-cell fractals. Throughout this section the subscripts j $(j=1, 2)$ of $\boldsymbol{C}_j^n$, $\boldsymbol{A}_j^n$, $\cdots$ will indicate that they are related to $\boldsymbol{F}_j$. For example $\boldsymbol{C}_1^n$ is $\boldsymbol{C}^n$ for $\boldsymbol{F}_1$, and $\boldsymbol{A}_2^n$ is $\boldsymbol{A}^n$ for $\boldsymbol{F}_2$.

DEFINITION. A map ζ: $F_1 \to F_2$ is called *a fractal covering map* if ζ satisfies (3.1), (3.2) and (3.3).

(3.1) ζ is a surjection such that, for ${}^\forall i \in I_1$, ${}^\exists j \in I_2$ satisfying $\zeta \circ f_1^i = f_2^j \circ \zeta$.

(3.2) $\zeta(a) \in \boldsymbol{A}_2 \quad$ for ${}^\forall a \in \boldsymbol{A}_1$.

(3.3) $\zeta(B_1^1[x]) = B_2^1[\zeta(x)] \quad$ for ${}^\forall x \in B_1^1$.

REMARK. We immediately see the following.

(3.1′) $\quad \zeta(c) \in \boldsymbol{C}_2^n \quad$ for ${}^\forall c \in \boldsymbol{C}_1^n$ and ${}^\forall n \geqq 0$.

(3.2′) $\quad \zeta(a) \in \boldsymbol{A}_2^n \quad$ for ${}^\forall a \in \boldsymbol{A}_1^n$ and ${}^\forall n \geqq 1$.

(3.3′) $\quad \zeta(B_1^n[x]) = B_2^n[\zeta(x)] \quad$ for ${}^\forall x \in B_1^n$ and ${}^\forall n \geqq 1$.

Moreover ζ is continuous by (3.1′) and an open map from (3.3′). However ζ is not a covering map in the usual sense.

We consider a condition such that ζ is compatible with $(G_i, \boldsymbol{H}_i)$.

(3.4) $\quad$ For ${}^\forall g \in G_1$, there exists $g' \in G_2$ such that $g' \circ \zeta = \zeta \circ g$.

For ${}^\forall h \in \boldsymbol{H}_1$, there exists $h' \in \boldsymbol{H}_2$ such that $h' \circ \zeta = \zeta \circ h$.

Since ζ is a surjection, g' and h' in (3.4) are unique. Hence we can define maps ζ^G: $G_1 \to G_2$ and $\zeta^{H(n,x)}$: $H_x^n(\boldsymbol{F}_1) \to H_{\zeta(x)}^n(\boldsymbol{F}_2)$ by

$$\zeta^G(g) = g' \quad \text{and} \quad \zeta^{H(n,x)}(h) = h'.$$

It is easy to see that ζ^G and $\zeta^{H(n,x)}$ are group homomorphisms.

For $a' \in \boldsymbol{A}_2^n$ and $b \in \boldsymbol{A}_1^n$, we set $\langle a' \rangle_b = \{a \in \boldsymbol{A}_1^n;\ \zeta(a) = a',\ a \subset B_1^n[b]\}$. Let

$$\langle\!\langle a' \rangle\!\rangle_b = \{a \in \langle a' \rangle_b;\ a \in \boldsymbol{A}_1^n(i_0)\}, \qquad i_0 = \min\{i;\ \boldsymbol{A}_1^n(i) \cap \langle a' \rangle_b \neq \emptyset\},$$

and $\langle\!\langle a' \rangle\!\rangle_b = \emptyset$ if $\langle a' \rangle_b = \emptyset$ (see (1.12) for the definition of $A^n(i)$), and

$$\langle\!\langle a' \rangle\!\rangle = \bigcup_{b \in \boldsymbol{A}_1^1} \langle\!\langle a' \rangle\!\rangle_b.$$

By (3.2) and (3.3), we can regard ζ as the map from $\mathbf{W}_1^n$ to $\mathbf{W}_2^n$ such that $\zeta(\mathbf{w}) = (\zeta(w^a), \zeta(w^b))$, where $\mathbf{w} = (w_a, w_b)$. For $x \in B^\infty$, we set

$$[x]^n = \{\mathbf{w} = (w^a, w^b) \in \mathbf{W}_1^n;\ x \in w^a,\ w^b \in \langle\!\langle \zeta(w^b) \rangle\!\rangle_{w^a}\}.$$

$$[x, \mathbf{w}] = \{\hat{\mathbf{w}} \in [x]^n;\ \zeta(\hat{\mathbf{w}}) = \zeta(\mathbf{w})\}, \qquad \text{where} \quad \mathbf{w} \in \mathbf{W}_1^n.$$

Let $(\mathbf{q}_2, \mathbf{t}_2)$ be a solution of (2.9) and (2.28) with λ. For $(\mathbf{q}_2, \mathbf{t}_2)$ we define $(\mathcal{F}_1, \mathbf{q}_1, \mathbf{t}_1)$, where $\mathbf{q}_1=(q_1^n)$ and $\mathbf{t}_1=(t_1^n)$ by

$$\mathcal{F}_1 = \sigma[A_1\cup\zeta^{-1}(\mathcal{B}(B_2))],$$

$$q_1^n(x, A) = \sum_{\mathbf{w}\in[x]^n} q_2^n(\zeta(x), \zeta(w^b\cap A))/{}^{\#}[x, \mathbf{w}] \qquad (\mathbf{w} = (w^a, w^b)),$$

$$t_1^n(x, y, \cdot) = t_2^n(\zeta(x), \zeta(y), \cdot).$$

Here $q_1^n: (B_1^n-B_1)\times\mathcal{F}_1^n\to \boldsymbol{R}$ and $\mathcal{F}_1^n=\sigma[\{f^{\boldsymbol{i}}(A);\ A\in\mathcal{F}_1,\ \boldsymbol{i}\in\boldsymbol{I}_1^n\}]$.

For a technical reason we need the following condition (3.5):

(3.5) A finite measure on $\mathcal{F}_1$ can be extend to a measure on $\mathcal{B}(B_1^\infty)$.

By (3.5) we can extend the domain of $q_1^n(x, \cdot)$ from $\mathcal{F}_1^n$ to $\mathcal{B}(B_1^\infty)$ to obtain $q_1^n\in Q^n(\boldsymbol{F}_1)$. Since $\mathbf{q}_2$ satisfies (2.5), so does $\mathbf{q}_1$ by (3.1). Hence $\mathbf{q}_1=(q_1^n)\in\mathbf{Q}(\boldsymbol{F})$. It is clear that $\mathbf{t}_1\in\mathbf{T}(\boldsymbol{F})$.

We can easily verify the condition (3.5) to the examples in Section 5. We conjecture that (3.5) always holds.

We now state the main result of this section.

THEOREM 3.1. *Let $\boldsymbol{F}_j$ $(j=1, 2)$ be $(G_j, \boldsymbol{H}_j)$-cell fractals and $\zeta: F_1\to F_2$ be a map satisfying* (3.1), $\cdots$, (3.5). *Suppose that ζ satisfies* (3.6) *and* (3.7):

(3.6) $\mathrm{Ker}\,\zeta^G$ *is transitive on $\langle\!\langle a'\rangle\!\rangle$ for each $a'\in A_2^1$; for all a and $\hat{a}\in\langle\!\langle a'\rangle\!\rangle$ there exists $g\in\mathrm{Ker}\,\zeta^G$ such that $g(a)=\hat{a}$.*

(3.7) $\mathrm{Ker}\,\zeta^{H(n,x)}$ *is transitive on $[x, \mathbf{w}]$ for each $\mathbf{w}\in\mathbf{W}_1^n$; for all $\hat{\mathbf{w}}$, $\mathbf{w}''\in[x, \mathbf{w}]$ there exists $h\in\mathrm{Ker}\,\zeta^{H(n,x)}$ such that $h(\hat{\mathbf{w}})=\mathbf{w}''$.*

Here Ker $*$ *denotes the kernel of a group homomorphism* $*$.

a) *Suppose that $(\mathbf{q}_2, \mathbf{t}_2)$ is a solution of* (2.9) *and* (2.28) *with λ for $\boldsymbol{F}_2$. Then $(\mathcal{F}_1, \mathbf{q}_1, \mathbf{t}_1)$ satisfies* (2.10), (2.11) *and* (2.27) *with λ.*

b) *Moreover if $\mathbf{q}_2\in\mathbf{Q}^R(\boldsymbol{F}_2)$ and $\mathbf{t}_2\in\mathbf{T}^R(\boldsymbol{F}_2)$, then $\mathbf{q}_1\in\mathbf{Q}^R(\boldsymbol{F}_1)$ and $\mathbf{t}_1\in\mathbf{T}^R(\boldsymbol{F}_1)$.*

To prove Theorem 3.1, we prepare three lemmas. In the following we assume ζ satisfies (3.1), $\cdots$, (3.7).

Now we regard ζ as the map from $\boldsymbol{W}_1^n$ to $\boldsymbol{W}_2^n$, by setting $\zeta(\boldsymbol{w})=(\zeta(\mathbf{w}_i))$, where $\boldsymbol{w}=(\mathbf{w}_i)\in\boldsymbol{W}_1^n$. For $x\in B_1^n$ and $\boldsymbol{w}'\in\boldsymbol{W}_2^n$, let

$$\langle\!\langle \boldsymbol{w}'\rangle\!\rangle_x = \{\boldsymbol{w} = (\mathbf{w}_i) \in \boldsymbol{W}_1^n;\ x \in w_1^a,\ \zeta(\boldsymbol{w}) = \boldsymbol{w}',\ w_i^b \in \langle\!\langle\zeta(w_1^b)\rangle\!\rangle_{w_i^a}\ ({}^{\forall}i)\}.$$

LEMMA 3.2. *Let $\boldsymbol{w}\in\boldsymbol{W}_1^n$ and $x\in B_1^n$. Then*

$${}^{\#}\langle\!\langle\zeta(\boldsymbol{w})\rangle\!\rangle_x = {}^{\#}\langle\!\langle\zeta(\boldsymbol{w})\rangle\!\rangle_{g(x)} \qquad \text{for} \quad {}^{\forall}g\in\mathrm{Ker}\,\zeta^G.$$

PROOF. Let g be regarded as the map $g;\ \boldsymbol{W}_1^n\to\boldsymbol{W}_1^n$ naturally. Then g is a bijection. Since $g\in\mathrm{Ker}\,\zeta^G$, $g(\langle\!\langle\zeta(\boldsymbol{w})\rangle\!\rangle_x)=\langle\!\langle\zeta(\boldsymbol{w})\rangle\!\rangle_{g(x)}$. $\square$

LEMMA 3.3. *Let* $\boldsymbol{w}=(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m)\in \boldsymbol{W}_1^n$. *Then*

$$\#\langle\!\langle \zeta(\boldsymbol{w})\rangle\!\rangle_{x_1} = \prod_{i=1}^{m} \#[x_i, \mathbf{w}_i], \qquad \textit{where} \quad \mathbf{w}_i = (w_i^a, w_i^b) \textit{ and } x_i \in w_i^a. \tag{3.8}$$

PROOF. First suppose that $\boldsymbol{w}$ is included by a cell $c\in \boldsymbol{C}^{n-1}$. Let $f=f_1^{\boldsymbol{i}}(\boldsymbol{i}\in \boldsymbol{I}_1^{n-1})$ such that $f_1^{\boldsymbol{i}}(F_1)=c$. Then

$$\#\langle\!\langle \zeta(\boldsymbol{w})\rangle\!\rangle_{x_1} = \#\langle\!\langle \zeta(f^{-1}(\boldsymbol{w}))\rangle\!\rangle_{f^{-1}(x_1)}, \qquad \#[x_i, \mathbf{w}_i] = \#[f^{-1}(x_i), f^{-1}(\mathbf{w}_i)].$$

Hence it is sufficient to prove the case $n=1$. Suppose (3.8) holds for $m-1$. Then

$$\#\langle\!\langle \zeta(\boldsymbol{w})\rangle\!\rangle_{x_1} = \sum_{\mathbf{v}\in[x_1, \mathbf{w}_1]} \#\langle\!\langle \zeta(\hat{\boldsymbol{w}})\rangle\!\rangle_{x(\mathbf{v})} \qquad (\mathbf{v} = (v^a, v^b)),$$

where $\hat{\boldsymbol{w}}=(\mathbf{w}_2, \cdots, \mathbf{w}_m)$ and $x(\mathbf{v})$ is a point such that $x(\mathbf{v})\in v^b$. By (3.6) and Lemma 3.2, $\#\langle\!\langle \zeta(\hat{\boldsymbol{w}})\rangle\!\rangle_{x(\mathbf{v})}$ is independent of $\mathbf{v}\in[x_1, \mathbf{w}_1]$, which yields (3.8) for m. Hence (3.8) holds for all m by induction.

Next suppose $\boldsymbol{w}$ is included by $\bigcup_{i=1}^N c_i$, where $c_i\in \boldsymbol{C}_1^{n-1}$. Then we divide $\boldsymbol{w}$ to $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_N$ such that each $\boldsymbol{w}_i$ is included by c_i to reduce the above case. □

LEMMA 3.4. a) $\mathbf{q}_1\in \mathbf{Q}_{G_1, H_1}(\boldsymbol{F}_1 : \mathcal{F}_1)$ *and* $t_1\in \mathbf{T}_{G_1, H_1}(\boldsymbol{F}_1 : \mathcal{F}_1)$.

b) *Suppose* $\mathbf{q}_2\in \mathbf{Q}^R(\boldsymbol{F}_2)$ *and* $\mathbf{t}_2\in \mathbf{T}^R(\boldsymbol{F}_2)$. *Then* $\mathbf{q}_1\in \mathbf{Q}^R(\boldsymbol{F}_1)$ *and* $t_1\in \mathbf{T}^R(\boldsymbol{F}_1)$.

PROOF. We first check (2.3). Let $g\in G_1$ and $A\in \mathcal{B}(B_1^n)$. Then

$$\begin{aligned} q_1^n(g(x), g(A)) &= \sum_{\mathbf{w}\in[g(x)]^n} q_2^n(\zeta(g(x)), \zeta(w^b\cap g(A)))/\#[g(x), \mathbf{w}] \\ &= \sum_{\mathbf{w}\in[x]^n} q_2^n(\zeta(g(x)), \zeta(g(w^b)\cap g(A)))/\#[g(x), g(\mathbf{w})] \\ &= \sum_{\mathbf{w}\in[x]^n} q_2^n(\zeta^G(g)(\zeta(x)), \zeta^G(g)(\zeta(w^b\cap A)))/\#[x, \mathbf{w}] = q_1^n(x, A). \end{aligned}$$

We can check (2.4) similarly.

For $\mathbf{w}\in[x]^n$ and $A\in \mathcal{F}_1^n$, $\zeta(w^b\cap A)\in \mathcal{F}_2^n$. Then by $\mathbf{q}_2\in \mathbf{Q}_{G_2, H_2}(\boldsymbol{F}_2 : \mathcal{F}_2)$, $q_2^n(\cdot, \zeta(w^b\cap A))$ is $\mathcal{F}_2^n$-measurable, which implies $q_2^n(\zeta(\cdot), \zeta(w^b\cap A))$ is $\mathcal{F}_1^n$-measurable. Clearly, $\#[\cdot, \mathbf{w}]$ is $\sigma[A_1^n]$-measurable. Combining these we see $q_1^n(\cdot, A)$ is $\mathcal{F}_1^n$-measurable. Hence we obtain $\mathbf{q}_1\in \mathbf{Q}_{G_1, H_1}(\boldsymbol{F}_1 : \mathcal{F}_1)$. The proofs of $\mathbf{t}_1\in \mathbf{T}_{G_1, H_1}(\boldsymbol{F}_1 : \mathcal{F}_1)$ and b) are similar to the above. Hence we omit them. □

PROOF OF THEOREM 3.1. $\boldsymbol{A}_1\subset \mathcal{F}_1$ is clear and $\mathbf{q}\in \mathbf{Q}_{G_1, H_1}(\boldsymbol{F}_1 : \mathcal{F}_1)$ follows from Lemma 3.4. We next check (2.11). Let $A\in \mathcal{F}_1^{n-1}$. Note that

$$\Phi_1(q_1^n)(x, A) = \sum_{\mathbf{w}\in[x]^{n-1}} \Phi_1(q_1^n)(x, A\cap w^b) \qquad (\mathbf{w} = (w^a, w^b)).$$

Hence we assume $A\subset w^b$ for some $\mathbf{w}\in[x]^{n-1}$. Let

$$\bar{A} = \zeta^{-1}(\zeta(A))\cap \{ \bigcup_{\hat{\mathbf{w}}\in[x, \mathbf{w}]} \hat{w}^b\} \qquad (\hat{\mathbf{w}} = (\hat{w}^a, \hat{w}^b)).$$

We see by (3.7) that

$$\Phi_1(q_1^n)(x, \bar{A}) = \Phi_1(q_1^n)(x, A)\cdot{}^{\#}[x, \mathbf{w}] \tag{3.9}$$

Let

$$\mathbf{w}' = \zeta(\mathbf{w}), \qquad x_i' = \zeta(x_i), \qquad x' = \zeta(x) \quad \text{and} \quad A' = \zeta(A).$$

Then

$$\begin{aligned} &\Phi_1(q_1^n)(x, \bar{A}) \\ &= \sum_{w' \in W_2^n[x']} \sum_{w \in \langle\langle w' \rangle\rangle_x} \int_{A(w)} \prod_{i=1}^{|w|} q_1^n(x_{i-1}, dx_i)\cdot 1_{\bar{A}}(x_{|w|}) \\ &= \sum_{w' \in W_2^n[x']} \sum_{w \in \langle\langle w' \rangle\rangle_x} \int_{A(w')} \prod_{i=1}^{|w'|} \{q_2^n(x'_{i-1}, dx'_i)/{}^{\#}[x_{i-1}, \mathbf{w}_i]\} \cdot 1_{A'}(x'_{|w'|}) \\ &= \sum_{w' \in W_2^n[x']} \int_{A(w')} \prod_{i=1}^{|w'|} q_2^n(x'_{i-1}, dx'_i)\cdot 1_{A'}(x'_{|w'|}) \\ &= \Phi_2(q_2^n)(x', A') = q_2^{n-1}(x', A'). \end{aligned} \tag{3.10}$$

Here we used Lemma 3.3 to pass from the third line to the forth, and the fact that $\mathbf{q}_2=(q_2^n)$ is a solution of (2.9) for the last equality.

Combining (3.9) and (3.10) proves that $\mathbf{q}_1=(q_1^n)$ satisfies (2.11). We can prove similarly that $\mathbf{t}_1$ satisfies (2.26). Hence we obtain a). b) is already proved in Lemma 3.4 b). □

§ 4. Construction of self-similar diffusions.

Let $\boldsymbol{F}=(F, I, \{f^i\}, \pi)$ be a $(G, \boldsymbol{H})$-cell fractals. In this section we assume that equations (2.9) and (2.28) have solutions $\mathbf{q}=(q^n)\in\mathbf{Q}_{G,H}(\boldsymbol{F}\,;\,\mathcal{F})$ and $\mathbf{t}=(t^n)\in \mathbf{T}_{G,H}(\boldsymbol{F}\,;\,\mathcal{F})$ with time scaling factor $\lambda>1$, (($\mathbf{q}$, $\mathbf{t}$) is written ($\mathbf{r}$, $\mathbf{u}$) in Theorems 2.1 and (2.4)). We construct $(G, \boldsymbol{H})$-self-similar diffusions from this ($\mathbf{q}$, $\mathbf{t}$). Our argument follows the line in Lindstrøm. Indeed it is a standard analysis version of Section VII in [**14**] but with appropriate modifications due to the infinite ramifiedness of fractals.

To construct Brownian motions, we first define a Markov chain induced by ($\mathbf{q}$, $\mathbf{t}$). Let $\{Q_{x,s}^n\}$ be a Markov chain with the state space $B^n\times[0, \infty)$ whose transition probability is given by

$$\begin{aligned} P\{\Upsilon_{k+1} \in A\times T \mid \Upsilon_k = (x', s')\} &= \int_A q^n(x', dy)\int_{T-s'} t^n(x', y', du) \\ &\qquad\qquad \text{if} \quad x' \in B^n - B \\ &= \delta_{(x',s')}(A\times T) \qquad \text{if} \quad x' \in B \end{aligned}$$

and $Q^n_{x,s}(\Upsilon_0=(x, s))=1$. Here $\Upsilon_k=(Y_k, T_k)\in B^n\times[0, \infty)$ and δ_* is the point mass at $*$.

Let χ: $\{B^\infty\times[0, \infty)\}^N\to D\{[0, \infty)\to B^\infty\times[0, \infty)\}$ such that $\chi(\{\Upsilon_k\})=\{X_t\}$, where $X_t=Y_k$ if $T_k\leqq t<T_{k+1}$. We set

$$P^n_x=\chi\circ Q^n_{x,0}.$$

Here $\chi\circ Q^n_{x,0}(\cdot)=Q^n_{x,0}(\chi\in\cdot)$; $\chi\circ Q^n_{x,0}$ is the image measure of $Q^n_{x,0}$ induced by χ. We shall use the same convention in the rest of this section.

We shall construct the self-similar diffusion as the limit of $\{P^n_x\}$.

LEMMA 4.1.

$$(4.1)\quad P^n_x(X_t\in a)\geqq P^{n+1}_x(X_t\in a) \qquad \textit{for}\quad {}^\forall a\in \boldsymbol{A}^m,\quad {}^\forall x\in B^m,\quad {}^\forall n\geqq m.$$

$$(4.2)\quad P^n_x(X_t\in u)\leqq P^{n+1}_x(X_t\in u) \qquad \textit{for}\quad {}^\forall u\in \mathcal{U}^m,\quad {}^\forall x\in B^m,\quad {}^\forall n\geqq m,$$

where $\mathcal{U}^n=\{u\,;\,u=c-B^n,\ c\in \boldsymbol{C}^n\}$.

PROOF. Let $\tau_{n,k}=\inf\{t>\tau_{n,k-1}\,;\ X_t\in B^n-a[X_{\tau_{n,k-1}}]\}$ for n, $k\geqq 1$, where $\tau_{n,0}=0$. Then

$$\begin{aligned}
&P^{n+1}_x(X_t\in a)\\
&=\sum_{k,l=0}^{\infty}P^{n+1}_x(\{X_{\tau_{n,k}}\in a,\ \tau_{n,k}\leqq t<\tau_{n,k+1},\ X_{\tau_{n+1,l}}\in a,\ \tau_{n+1,l}\leqq t<\tau_{n+1,l+1}\})\\
&\leqq\sum_{k,l=0}^{\infty}P^{n+1}_x(\{X_{\tau_{n,k}}\in a,\ \tau_{n,k}\leqq t<\tau_{n,k+1},\ \tau_{n+1,l}\leqq t<\tau_{n+1,l+1}\})\\
&=\sum_{k=0}^{\infty}P^{n+1}_x(\{X_{\tau_{n,k}}\in a,\ \tau_{n,k}\leqq t<\tau_{n,k+1}\})\\
&=\sum_{k=0}^{\infty}P^{n}_x(X_{\tau_{n,k}}\in a,\ \tau_{n,k}\leqq t<\tau_{n,k+1})=P^n_x(X_t\in a).
\end{aligned}$$

Here we used the fact that $\mathbf{q}$ and $\mathbf{t}$ are solutions of (2.9) and (2.27) to pass from the forth line to the fifth.

Let $c\in \boldsymbol{C}^n$ such that $u=c-B^n$. Then

$$\begin{aligned}
&P^{n+1}_x(X_t\in u)\\
&=\sum_{k,l=0}^{\infty}P^{n+1}_x(\{X_{\tau_{n,k}}\in c,\ \tau_{n,k}\leqq t<\tau_{n,k+1},\ X_{\tau_{n+1,l}}\in u,\ \tau_{n+1,l}\leqq t<\tau_{n+1,l+1}\})\\
&\geqq\sum_{k,l=0}^{\infty}P^{n+1}_x(\{X_{\tau_{n,k}}\in u,\ \tau_{n,k}\leqq t<\tau_{n,k+1},\ X_{\tau_{n+1,l}}\in u,\ \tau_{n+1,l}\leqq t<\tau_{n+1,l+1}\})\\
&=\sum_{k=0}^{\infty}P^{n+1}_x(\{X_{\tau_{n,k}}\in u,\ \tau_{n,k}\leqq t<\tau_{n,k+1}\}).
\end{aligned}$$

The rest of the proof is similar to that of (4.1). Hence we omit it. □

Let $\mu_{x,t}^{n}=X_{t}\circ P_{x}^{n}$.

LEMMA 4.2. *There exists a family of probability measures* $\{\mu_{x,t}\}$, $x\in B^{\infty}$, $t\in[0, \infty)$, *such that*

$$\lim_{n\to\infty}\mu_{x,t}^{n}=\mu_{x,t} \quad \textit{weakly for all } x \textit{ and } t. \tag{4.3}$$

PROOF. Since F is compact, $\{\mu_{x,t}^{n}\}$ is tight for all x and t. Hence there exist a subsequence $\{\mu_{x,t}^{n'}\}$ and a probability measure $\mu_{x,t}$ such that $\underline{\lim}_{n'\to\infty}\mu_{x,t}^{n'}(\mathcal{O})\geqq\mu_{x,t}(\mathcal{O})$ for all open sets $\mathcal{O}$. By Lemma 4.1, $\lim_{n\to\infty}\mu_{x,t}^{n}(v)$ exists for ${}^{\forall}v\in\bigcup_{n=1}^{\infty}\mathcal{V}^{n}\cup\boldsymbol{A}^{\infty}$. Since open sets $\mathcal{O}$ in F are countable disjoint unions of elements of $\bigcup_{n=1}^{\infty}\mathcal{V}^{n}\cup\boldsymbol{A}^{\infty}$, we obtain $\underline{\lim}_{n'\to\infty}\mu_{x,t}^{n'}(\mathcal{O})=\lim_{n\to\infty}\mu_{x,t}^{n}(\mathcal{O})$. Combining these we see $\lim_{n\to\infty}\mu_{x,t}^{n}(\mathcal{O})\geqq\mu_{x,t}(\mathcal{O})$, which means (4.3). $\square$

Now we set for $x\in F$

$$D^{n}[x]=\bigcup_{c\in\boldsymbol{C}^{n},\, c\cap\boldsymbol{C}^{n}[x]\neq\emptyset} c. \tag{4.4}$$

and

$$\partial D^{n}[x]=\{y\in D^{n}[x]\,;\ {}^{\exists}c\in\boldsymbol{C}^{n} \text{ such that } y\in c \text{ and } c\cap(D^{n}[x])^{c}\neq\emptyset\}.$$

Then we easily see

$$D^{n}[x]\supset D^{n+1}[x], \qquad \bigcap_{n}D^{n}[x]=\{x\}. \tag{4.5}$$

(4.6) For $x\in F-B$, there exists an n such that $D^{n}[x]\cap B=\emptyset$.

$$D^{n+1}[x]\cap\partial D^{n}[x]=\emptyset \qquad \text{for } x \text{ and } n. \tag{4.7}$$

EXAMPLE. Let $F=[0, 1]\times[0, 1]$. We regard F as a cell fractal as the example in Section 1. Let $x=(1/8, 1/8)$. Then $D^{3}[x]=[0, 3/8]\times[0, 3/8]$ and $\partial D^{2}[x]$ is the bold lines in Fig. 4.1; $\partial D^{2}[x]=\{1/2\}\times[0, 1/2]\cup[0, 1/2]\times\{1/2\}$.

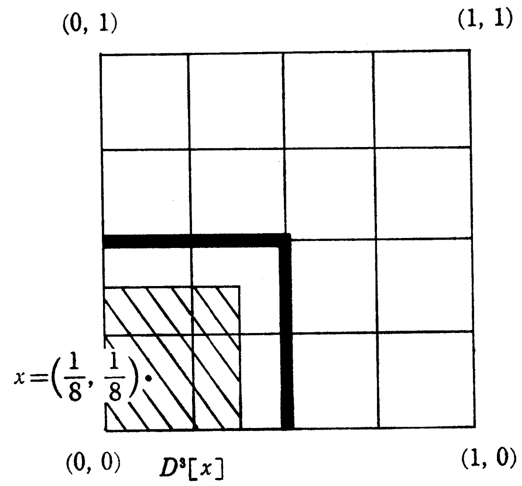


Fig. 4.1.

We note $(0, 0)\notin\partial D^2[x]$.

Let $\tau(x, n)=\inf\{t>0\,;\, X_t\in\partial D^n[x]\}$. In the rest of this section we assume (4.8), $\cdots$, (4.11).

(4.8) For $x\in B^1-B$ and $m\in\boldsymbol{N}$ such that $D^{m-1}[x]\cap B=\emptyset$, there exists $\{\varepsilon_n\}_{n\geqq m}$ satisfying $0<\varepsilon_n<1$, $\prod_{n\geqq m}(1-\varepsilon_n)=0$ and

$$\varepsilon_n\cdot P_y^n(X_{\tau(x,n-1)}\in a)\leqq P_z^n(X_{\tau(x,n-1)}\in a)\leqq \varepsilon_n^{-1}\cdot P_y^n(X_{\tau(x,n-1)}\in a)$$

for $y, z\in\partial D^n[x]$, $a\in\mathcal{F}^{n-1}$ and $n\geqq m$.

(4.9) There exist $C, \delta>0$ such that

$$\sup_{y\in\partial D^n[x]} P_y^n(\tau(x, n-1)\leqq \lambda^{-n}\cdot 2^{-m})\leqq C\cdot 2^{-(1+\delta)m}$$

for $x\in B^1-B$, $n, m\in\boldsymbol{N}$ such that $D^{n-1}[x]\cap B=\emptyset$.

(4.10) $P_x^1(\tau_B<\infty)=1$ for $x\in B^1$, where $\tau_B=\inf\{t>0\,;\, X_t\in B\}$.

(4.11) $\lim_{n\to\infty}\sup_{x\in B^1-B} P_x^n(\sigma\geqq\varepsilon)=0$ for $\varepsilon>0$ $(\sigma=\inf\{t>0\,;\, X_t\neq X_0\})$.

REMARK. These assumptions are satisfied, for example, if $(\mathbf{q}, \mathbf{t})$ is obtained by Theorems 2.1 and 2.4 from an approximate solution $(\mathbf{q}_1, \mathbf{t}_1)$, and $(\mathbf{q}_1, \mathbf{t}_1)$ is obtained in Theorem 3.1 from $(\mathbf{q}_2, \mathbf{t}_2)$ that satisfies (4.8), $\cdots$, (4.11). We see this fact from the following:

$$\zeta\circ P_{1,x}^n=P_{2,\zeta(x)}^n.$$

Here $P_{i,x}^n (i=1, 2)$ are P_x^n for $(\mathbf{q}_i, \mathbf{t}_i)$. See the remark after Theorem 4.8.

Let E_x^n denote the expectation with respect to P_x^n. Recall that $\mathcal{F}^n$ is the σ-field such that $\mathcal{F}^n=\sigma[\{f^{\boldsymbol{i}}(A)\,;\, A\in\mathcal{F}, \boldsymbol{i}\in I^n\}]$.

LEMMA 4.3. *Let $x\in B^\infty$ and $l\in\boldsymbol{N}$ such that $D^l[x]\cap B=\emptyset$. Let h be a $\mathcal{F}^k$-measurable function on B^l. Set $\bar{h}=\sup h(x)$ and $\underline{h}=\inf h(x)$ Then for $n\geqq m>k\geqq l$*

$$|E_y^n[h(X_{\tau(x,l)})]-E_z^n[h(X_{\tau(x,l)})]|\leqq\Big(\prod_{i=k+1}^m(1-\varepsilon_i)\Big)\cdot\{\bar{h}-\underline{h}\},$$

for y and $z\in D^m[x]\cup B^n$.

PROOF. Let $g(y)=E_y^n[h(X_{\tau(x,l)})]$. Then by (4.10), $\underline{h}\leqq g(y)\leqq\bar{h}$ for each $y\in D^m[x]$. Hence we can suppose $\underline{h}=0$. By $\mathbf{q}\in\mathbf{Q}(\boldsymbol{F}\colon\mathcal{F})$ and $\mathbf{t}\in\mathbf{T}(\boldsymbol{F}\colon\mathcal{F})$, g is $\mathcal{F}^j$-measurable on B^j for $j\geqq k$. Hence the problem is reduced to the case $n=m=k+1=l+1$. So we suppose this. Let ν be the signed measure on $\mathcal{F}^l$ such that

$$\nu=P_y^n(X_{\tau(x,l)}\in\cdot)-P_z^n(X_{\tau(x,l)}\in\cdot).$$

Then there exist (positive) measures ν_1, ν_2 and S_1, $S_2 \in \mathcal{F}^l$ such that $\nu=\nu_1-\nu_2$ and $\nu_i(S_i^c)=0$ and $S_1 \cap S_2=\emptyset$. By (4.8)

$$\int_{S_1} h\, d\nu_1 \leqq (1-\varepsilon_n)\cdot E_y^n[h\cdot 1_{S_1}(X_{\tau(x,l)})] \leqq (1-\varepsilon_n)\cdot \bar{h},$$

and

$$\int_{S_2} h\, d\nu_2 \leqq (1-\varepsilon_n)\cdot E_z^n[h\cdot 1_{S_2}(X_{\tau(x,l)})] \leqq (1-\varepsilon_n)\cdot \bar{h}.$$

Hence $|g(x)-g(y)| \leqq (1-\varepsilon_n)\cdot \bar{h}$, which completes the proof. □

LEMMA 4.4. a) *For* $\varepsilon>0$, $z \in B^\infty - B$ *and* $f \in C(F)$, *there exists an* m *such that*

$$\varlimsup_{n\to\infty}\left|\int_F f\, d\mu_{x,t}^n - \int_F f\, d\mu_{y,t}^n\right| < 4\varepsilon \qquad \text{for} \quad x, y \in D^m[z] \cap B^\infty.$$

b) *For* s, $\varepsilon>0$, $x \in B^\infty - B$ *and* $f \in C(F)$, *there exists a* δ *such that*

$$\varlimsup_{n\to\infty}\left|\int_F f\, d\mu_{x,s}^n - \int_F f\, d\mu_{x,t}^n\right| < 4\varepsilon \qquad \text{for} \quad s \leqq t \leqq s+\delta.$$

c) $\int_F f\, d\mu_{x,t}$ *is continuous on* $B^\infty - B$ *for* $t>0$ *and* $f \in C(F)$.

PROOF. There exists $n(1)$ such that $|f(x)-f(y)|<\varepsilon$ for $^\forall x$, $y \in c \in \boldsymbol{C}^{n(1)}$. Let $\tau_t = \inf\{u>t;\ X_u \in B^{n(1)}\}$. Then by (4.10) and self-similarity, we see $\tau_t<\infty$ a.s.. Hence $|f(X_t)-f(X_{\tau_t})|<\varepsilon$ a.s., which yields

$$|E_x^n[f(X_t)]-E_x^n[f(X_{\tau_t})]| < \varepsilon \qquad \text{for} \quad n \geqq n(1). \tag{4.12}$$

Let $n(2) \geqq n(1)$ and $g:\ B^{n(1)} \to \boldsymbol{R}$ such that g is $\mathcal{F}^{n(2)}$-measurable and that $\sup\{|f(x)-g(x)|\,;\ x \in B^{n(1)}\}<\varepsilon$. Then

$$|E_x^n[f(X_{\tau_t})]-E_x^n[g(X_{\tau_t})]| < \varepsilon \qquad \text{for} \quad n \geqq n(1). \tag{4.13}$$

By Lemma 4.3, there exists an m such that

$$\varlimsup_{n\to\infty}|E_x^n[g(X_{\tau_t})]-E_y^n[g(X_{\tau_t})]| < \varepsilon \qquad \text{for} \quad x, y \in D^m[z] \cap B^\infty. \tag{4.14}$$

Combining (4.12), (4.13) and (4.14) yields a).

Let $n(3) \geqq n(2)$ such that $x \in B^{n(3)}$. Then for $n \geqq n(3)$

$$|E_x^n[g(X_{\tau_s})]-E_x^n[g(X_{\tau_t})]| = |E_x^{n(3)}[g(X_{\tau_s})]-E_x^{n(3)}[g(X_{\tau_t})]|. \tag{4.15}$$

Here we used $\mathbf{q} \in \mathbf{Q}(\boldsymbol{F}:\ \mathcal{F})$ and $\mathbf{t} \in \mathbf{T}(\boldsymbol{F}:\ \mathcal{F})$. Since $g(X_{\tau_t})$ is right continuous for a.s. ω, there exists δ such that

$$|E_x^{n(3)}[g(X_{\tau_s})]-E_x^{n(3)}[g(X_{\tau_t})]| < \varepsilon \qquad \text{for} \quad s \leqq t \leqq s+\delta. \tag{4.16}$$

By (4.13), (4.15) and (4.16) we obtain b).

c) follows from a) and b) immediately. □

By Lemma 4.2, we obtain probability measures $\mu_{x,t}$ for $x\in B^{\infty}$. We next define $\mu_{x,t}$ for $x\in F-B^{\infty}$.

LEMMA 4.5. a) *For each* $x\in F-B^{\infty}$ *and* $t>0$, *there exists a unique probability measure* $\mu_{x,t}$ *on* F *satisfying* $\mu_{x,t}=\lim_{k\to\infty}\mu_{x_k,t_k}$ *weakly. Here* $\{(x_k, t_k)\}$ *is a sequence in* $B^{\infty}\times(0, \infty)$ *converging to* (x, t) *with* $t_k\geqq t$.

b) *For* $A\in\mathcal{B}(F)$ *and* $t\geqq 0$, $\mu_{x,t}(A)$ *is* $\mathcal{B}(F)$*-measurable.*

c) *For* x, y, s, t, $\int_F \mu_{x,s}(dy)\cdot\mu_{y,t}(dz)=\mu_{x,s+t}(dz)$.

PROOF. a) follows from (4.3) and Lemma 4.4 immediately.

b) follows from Lemma 4.4 by the monotone class theorem. Next we prove c): For a measure μ on F and $f\in C(F)$ we set $\mu(f)=\int_F fd\mu$. Let $\iota(s)=\min\{i\geqq 0\,;\, T_i>s\}$ and $\hat{\mu}^n_{x,s}=Y_{\iota(s)}\circ Q^n_{x,0}$. Then by the strong Markov property of $\{Q^n_{x,0}\}$,

$$\int_{F\times[s,t]}\Upsilon_{\iota(s)}\circ Q^n_{x,0}(dy, du)\cdot\hat{\mu}^n_{y,t-u}(f)=\hat{\mu}^n_{x,t}(f)\qquad (f\in C(F)). \tag{4.17}$$

Noting $Y_{\iota(s)-1}\in C^n[Y_{\iota(s)}]$ a.s. $Q^n_{x,0}$ and $\mu^n_{x,s}=Y_{\iota(s)-1}\circ Q^n_{x,0}$, we see

$$\lim_{n\to\infty}\hat{\mu}^n_{x,s}(f)=\lim_{n\to\infty}\mu^n_{x,s}(f)=\mu_{x,s}(f)\qquad \text{for}\quad f\in C(F). \tag{4.18}$$

Moreover since $\lim_{n\to\infty}\sup\{|f(y)-f(z)|\,;\, y, z\in C^n[x], x\in F\}=0$,

$$\lim_{n\to\infty}\sup_{x\in F}|\hat{\mu}^n_{x,s}(f)-\mu^n_{x,s}(f)|=0. \tag{4.19}$$

From (4.11), $\lim_{n\to\infty}Q^n_{x,0}(T_{\iota(s)}-s\geqq\varepsilon)=0$ for $^\forall\varepsilon>0$. Hence

$$\lim_{n\to\infty}\Upsilon_{\iota(s)}\circ Q^n_{x,0}=\mu^n_{x,s}\times\delta_s\qquad \text{weakly}. \tag{4.20}$$

Here δ_s is the point mass at s. By (4.17), (4.18), (4.19), (4.20) and c) of Lemma 4.4 we obtain c) of Lemma 4.5. □

By Lemma 4.5 there exists a family of probability measures $\{P^{\infty}_x\}$ on $F^{[0,\infty)}$ such that

$$P^{\infty}_x(X_{t_1}\in A_1, \cdots, X_{t_m}\in A_m)=\int_{A_1\times\cdots\times A_m}\prod_{k=1}^m \mu_{x_{i-1}, t_i-t_{i-1}}(dx_i)\qquad (x_0=x)$$

for $A_i\in\mathcal{B}(F)$, $0<t_1<\cdots<t_m$.

LEMMA 4.6. P^{∞}_x *has a continuous modification* P_x; *there exists a probability measure* P_x *on* $C\{[0, \infty)\mapsto F\}$ *whose finitely dimensional distributions are equal to* P^{∞}_x.

PROOF. We prove the continuous modification only on $[0, 1)$, since the general case follows from the scaling immediately.

Let $X_{n,k}=X_{k/(2\lambda)^n}$, and ρ^n; $F^n \times F^n \to \boldsymbol{R}$ be the function such that $\rho^n(x, y) = \min\{m \geqq 1$; there exist $z_0, \cdots, z_m$ such that

$$D^n[z_i] \cap D^n[z_{i+1}] \neq \emptyset \qquad \text{for all } \ 0 \leqq i \leqq m-1,\ z_0 = x,\ z_m = y\}.$$

Then

$$P_x^\infty(\{\rho^{n-1}(X_{n,k-1}, X_{n,k}) \geqq 3\}) \leqq P_x^n(\{\rho^{n-1}(X_{n,k-1}, X_{n,k}) \geqq 2\}).$$

Hence by (4.9) there exist positive constants α and β with $\alpha<\beta$ such that

$$P_x^\infty(\{\rho^{n-1}(X_{n,k-1}, X_{n,k}) \geqq 3\}) \leqq C \cdot 2^{n\alpha} \cdot 2^{-n(\beta+1)}$$

for ${}^\forall n$, k with $0<k \leqq (2\lambda)^n$. By the Borel-Cantelli lemma we see

$$P_x^\infty(\underline{\lim}_{n\to\infty}\{\rho^{n-1}(X_{n,k-1}, X_{n,k}) \leqq 2,\ 0 \leqq {}^\forall k \leqq (2\lambda)^n\}) = 1.$$

This implies Lemma 4.6. □

Collecting the above results we obtain

THEOREM 4.7. *Suppose that the solutions* $\mathbf{q} \in \mathbf{Q}_{G,H}(\boldsymbol{F}; \mathcal{F})$ *and* $\mathbf{t} \in \mathbf{T}_{G,H}(\boldsymbol{F}; \mathcal{F})$ *of* (2.9) *and* (2.27) *satisfy* (4.8), (4.9), (4.10) *and* (4.11). *Then there exists a* $(G, \boldsymbol{H})$-*self similar diffusion* $\{P_x\}$. *Moreover* $\{P_x\}$ *is strongly self-similar if* $\mathbf{q} \in \mathbf{Q}^R(\boldsymbol{F})$ *and* $\mathbf{t} \in \mathbf{T}^R(\boldsymbol{F})$.

PROOF. By Lemma 4.4 c), Lemma 4.5 and Lemma 4.6, $\{P_x\}$ is a diffusion. The self-similarity and $(G, \boldsymbol{H})$-invariance are clear from those of $\{P_x^n\}$. □

We call $(\mathbf{q}, \mathbf{t})$ is induced by $\{P_x\}$ if $\mathbf{q} = (X_{\sigma(x,n)} \circ P_x)$ and $\mathbf{t} = (\sigma(x, n) \circ P_x(\cdot \mid X_{\sigma(x,n)}=y))$.

THEOREM 4.8. *Let* $\boldsymbol{F}_j$ $(j = 1, 2)$ *be* $(G_j, \boldsymbol{H}_j)$-*cell fractals and* ζ *be a map satisfying* (3.1), $\cdots$, (3.7). *Suppose that there exists a* $(G_2, \boldsymbol{H}_2)$-*self-similar diffusion* $\{P_{2,x}\}$ *on* F_2 *with a time scaling factor* λ. *Suppose that* $(\mathbf{q}_2, \mathbf{t}_2)$ *is induced by* $\{P_{2,x}\}$ *and satisfies* (4.8), (4.9), (4.10) *and* (4.11). *Then there exists a* $(G_1, \boldsymbol{H}_1)$-*self-similar diffusion* $\{P_{1,x}\}$ *on* F_1 *with the same time scaling facter* λ. *Moreover if* $\{P_{2,x}\}$ *is strongly self-similar then so is* $\{P_{1,x}\}$.

PROOF. Theorem 4.8 is an immediate consequence of Theorems 3.1 and 4.7. □

REMARK. In the following cases $(\mathbf{q}_2, \mathbf{t}_2)$ satisfies (4.8), (4.9), (4.10) and (4.11).

a) F_2 is a nested fractal and $\{P_{2,x}\}$ is Brownian motion.

b) F_2 is the n-dimensional cube $[0, 1]^d$ and $\{P_{2,x}\}$ is the standard Brownian motion.

§ 5. Examples of self-similar diffusion on infinitely ramified cell fractals.

Let I be a finite set and $\mathbf{I}=I^{N}$. We set θ^{i} as in Section 1. An equivalence relation $\sim$ on $\mathbf{I}$ is called the self-similar equivalence relation (s.s.e.r.) if $\sim$ commutes with θ^{i}; $\mathbf{i}\sim\mathbf{i}'$ if and only if $\theta^{\boldsymbol{i}}(\mathbf{i})\sim\theta^{\boldsymbol{i}}(\mathbf{i}')$ for all $\boldsymbol{i}\in\bigcup_{n=-\infty}^{\infty}\boldsymbol{I}^{n}$. Here $\boldsymbol{I}^{-n}=\{-\boldsymbol{i}\,;\,\boldsymbol{i}\in\boldsymbol{I}^{n}\}(n\geqq 1)$ and for $-\boldsymbol{i}\in\boldsymbol{I}^{-n}$, $\theta^{-\boldsymbol{i}}$ is the inverse map of $\theta^{\boldsymbol{i}}$. We call $\boldsymbol{F}=(F, I, \{f^{i}\}, \pi)$ the self-similar set associated with $\sim$ if $F=\mathbf{I}/\sim$, $\pi:\mathbf{I}\to F$ is the canonical surjection and $f^{i}: F\to F$ is the map defined by $f^{i}\circ\pi=\pi\circ\theta^{i}(i\in I)$. We call $\boldsymbol{F}$ the cell fractal associated with $\sim$ if in addition $\boldsymbol{F}$ is a cell fractal.

For a subgroup G of $G(\boldsymbol{F})$, we denote by $\boldsymbol{H}[G]$ the maximal subgroup of $\boldsymbol{H}(\boldsymbol{F})$ such that $(G, \boldsymbol{H}[G])$ is a structure group of $\boldsymbol{F}$.

For $\mathbf{i}=(i_m)$ we write $\mathbf{i}=(i_1, i_2, \cdots, \dot{i}_n)$ if $i_m=i_n$ for all $m\geqq n$. For a finite set I_*, we set $\mathbf{I}_*$, $\boldsymbol{I}_*$, $\theta_*^{\boldsymbol{i}}$, $\cdots$ similarly as in Section 1. The subscripts $*$ of $\mathbf{I}_*$, π_*, θ_*, $\cdots$ mean that they are related to I_*.

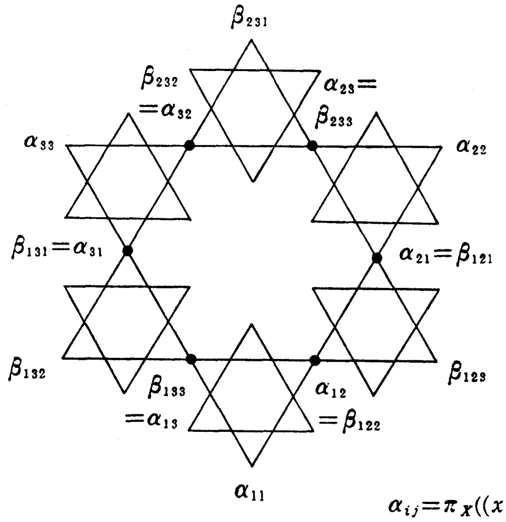
EXAMPLE (5.1). For $d(d\geqq 2)$ we set

$$I_X=\{x_1, \cdots, x_{d+1}\}\cup\{x_{jk}\,;\,1\leqq j<k\leqq d+1\}.$$

Let $x_{kj}=x_{jk}$ for $j<k$. We consider the following relation $\sim_X$ on $\mathbf{I}_X$;

$$\mathbf{x}\underset{X}{\sim}\mathbf{x}' \quad \text{if} \quad \mathbf{x}=(x_i, \dot{x}_j), \quad \mathbf{x}'=(x_{ij}, \dot{x}_i) \text{ and } i\neq j.$$

We write the s.s.e.r. generated by $\sim_X$ with the same symbol $\sim_X$, and denote the cell fractal associated with $\sim_X$ by $\boldsymbol{F}_X=(F_X, I_X, \{f_X^x\}, \pi_X)$. F_X with $d=2$ is homeomorphic to Fig. 5.1. $\boldsymbol{F}_X$ can be regarded as a nested fractal and $G(\boldsymbol{F}_X)\cong\mathfrak{S}_{d+1}$, where $\mathfrak{S}_d$ is the symmetric group of order d.



F_X with $d=2$

$\alpha_{ij}=\pi_X((x_i, \dot{x}_j))$
$\beta_{ijk}=\pi_X((x_{ij}, \dot{x}_k))$

Fig. 5.1.

Let $I_Y=\{y_1, \cdots, y_d\}$ and let $\sim_Y$ denote the s.s.e.r. on $\mathbf{I}_Y$ generated by

$$\mathbf{y} \underset{Y}{\sim} \mathbf{y}' \quad \text{if} \quad \mathbf{y}=(y_i, \dot{y}_j), \quad \mathbf{y}'=(y_j, \dot{y}_i) \text{ and } i \neq j .$$

Let $\boldsymbol{F}_Y=(F_Y, I_Y, \{f_Y^{\mathbf{y}}\}, \pi_Y)$ be the cell fractal associated with $\sim_Y$. It is easy to see that F_Y is homeomorphic to the $(d-1)$-dimensional Sierpinski gasket for $d \geqq 3$ and the segment $[0, 1]$ for $d=2$. Hence $G(\boldsymbol{F}_Y) \cong \mathfrak{S}_d$.

Let $I_Z=I_X \times I_Y \equiv \{x \times y\,;\ x \in I_X,\ y \in I_Y\}$. For $\mathbf{x} \in \mathbf{I}_X$ and $\mathbf{y} \in \mathbf{I}_Y$ we set $\mathbf{x} \times \mathbf{y} =(x_i \times y_i) \in \mathbf{I}_Z$, where $\mathbf{x}=(x_i)$ and $\mathbf{y}=(y_i)$. Then for each $\mathbf{z} \in \mathbf{I}_Z$ there exists unique $\mathbf{x} \in \mathbf{I}_X$ and $\mathbf{y} \in \mathbf{I}_Y$ such that $\mathbf{z}=\mathbf{x} \times \mathbf{y}$. We write $\mathbf{x} \times \mathbf{y} \sim_Z \mathbf{x}' \times \mathbf{y}'$ if one of the following holds;

(5.2) $$\mathbf{x} \underset{X}{\sim} \mathbf{x}' \quad \text{and} \quad \mathbf{y}=\mathbf{y}',$$

(5.3) $\mathbf{x}=\mathbf{x}'=(x_{ij}, \dot{x}_k)$ $(k \neq i, j)$ and $g(\pi_Y(\mathbf{y}))=\pi_Y(\mathbf{y}')$ for some $g \in G(\boldsymbol{F}_Y)$.

Let $\sim_Z$ denote the s.s.e.r. associated with $\sim_Z$ and $\boldsymbol{F}_Z=(F_Z, I_Z, \{f_Z^{\mathbf{z}}\}, \pi_Z)$ the cell fractal associated with $\sim_Z$. Let $\boldsymbol{F}_Z$ be equipped with the boundary cells $\boldsymbol{B}_Z=\{\pi_Z(x_i \times \mathbf{I}_Y)\,;\ 1 \leqq i \leqq d+1\}$. By (5.3) and the definition of s.s.e.r., $\pi_Z(x_i \times \mathbf{I}_Y)$ are homeomorphic to F_Y. Hence $\boldsymbol{F}_Z$ is infinitely ramified. F_Z with $d=2$ is homeomorphic to Fig. 5.2. Fig. 5.3 shows how 1-cells of $\boldsymbol{F}_Z$ are connected with each other.

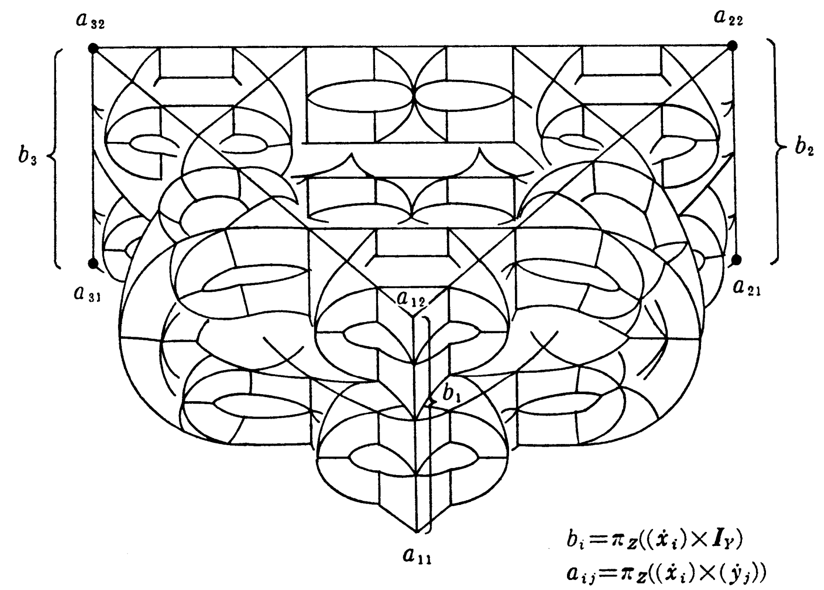


Fig. 5.2.

$\boldsymbol{F}_Z$ with the structure group $(G(\boldsymbol{F}_Z), \boldsymbol{H}[G(\boldsymbol{F}_Z)])$ and $\boldsymbol{F}_X$ with $(G(\boldsymbol{F}_X), \boldsymbol{H}[G(F_X)])$ satisfy the assumptions in Theorem 4.8. Indeed we can take ζ: $F_Z \rightarrow F_X$ by $\zeta(\mathbf{x} \times \mathbf{y})=\mathbf{x}$, $\boldsymbol{F}_1=\boldsymbol{F}_Z$ and $\boldsymbol{F}_2=\boldsymbol{F}_X$. Since $\boldsymbol{F}_X$ is a nested fractal, $\boldsymbol{F}_X$ satisfies the assumptions (4.8), $\cdots$, (4.11). Hence we can construct self-similar diffusion on F_Z.

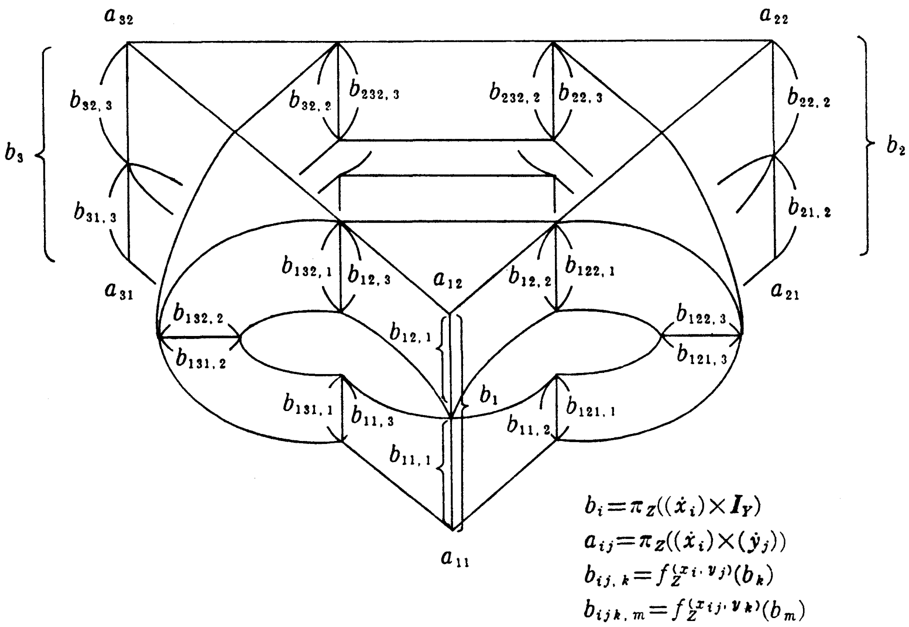


Fig. 5.3.

EXAMPLE (5.4). We next take $F_X=[0,1]^d$ $(d\geqq 2)$, that is F_X is the d-dimensional cube. We regard F_X as a cell fractal in the following way. Let

$$I_X=\{(j_1,\cdots,j_d);\ j_k=0,1\}\equiv\{0,1\}^d.$$

Let $\pi_X\colon \mathbf{I}_X\to F_X$ such that

$$\pi_X(\mathbf{j})=\Big(\sum_{m=1}^{\infty} j_{1,m}\cdot 2^{-m},\ \cdots,\ \sum_{m=1}^{\infty} j_{d,m}\cdot 2^{-m}\Big)\in \boldsymbol{R}^d,$$

where $\mathbf{j}=(j_m)$ and $j_m=(j_{k,m})\in\{0,1\}^d$. For $j\in I_X$ we define f_X^j by $\pi_X\circ\theta_X^j=f_X^j\circ\pi_X$. Then $\boldsymbol{F}_X=(F_X, I_X, \{f_X^j\}, \pi_X)$ is a cell fractal. Let B_X be the topological boundary of F_X in $\boldsymbol{R}^d$, that is $B_X=\bigcup_{i=1}^d\{x=(x_k)_{1\leq k\leq d}\,;\ x\in F_X,\ x_i=0 \text{ or } 1\}$. I_X is naturally imbedded in $\boldsymbol{R}^d$. For $i, j\in I_X$ with $|i-j|=1$ let H_{ij} be the $(d-1)$-dimensional hyperplane including i and perpendicular to the vector $\overrightarrow{i,j}$. Let $\tilde{b}_{ij}=H_{ij}\cap B_X$ and $b_{ij}\subset\tilde{b}_{ij}$ be the $(d-1)$-dimensional cube with the edge length 1/2 and a corner at i. See Fig. 5.4. We set $\boldsymbol{B}_X=\{b_{ij};\ |i-j|=1,\ i, j\in I_X\}$.

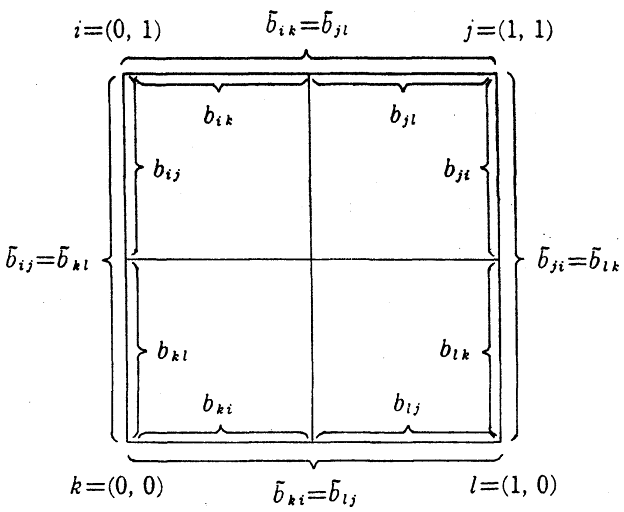
Let $I_Y=\{-1,1\}$ and $F_Y=I_Y{}^N$. Then $\boldsymbol{F}_Y=(F_Y, I_Y, \{\theta_Y^i\}_{i\in Y}, \mathrm{id})$ is a (trivial) cell fractal.

We consider an equivalence relation $\sim$ on $F_X\times F_Y$ such that

$$f_X^i(x)\times\mathbf{y}\sim f_X^j(R_{ij}(x))\times(-\mathbf{y}) \qquad \text{if } x\in b_{ji} \text{ and } \mathbf{y}\in F_Y,$$

$$f_X^i(x)\times\mathbf{y}\sim f_X^j(R_{ij}(x))\times\mathbf{y} \qquad \text{if } x\in \tilde{b}_{ji}-b_{ji} \text{ and } \mathbf{y}\in F_Y,$$

where $i, j\in I_X$ such that $|i-j|=1$ and $R_{ij}\colon F_X\to F_X$ is the reflection such that



F_X $(d=2)$

Fig. 5.4.

$R_{ij}(i)=j$. Let $I_Z=I_X\times I_Y$ and $\sim_Z$ be the s.s.e.r. on $\mathbf{I}_Z$ generated by $\sim$. Let $F_Z=\mathbf{I}_Z/\sim_Z$ and π_Z be its canonical surjection. Then $\boldsymbol{F}_Z=(F_Z, I_Z, \{f_Z^z\}, \pi_Z)$ is a cell fractal with $\boldsymbol{B}_Z=\{b_{ij}\times I_Y;\ b_{ij}\in\boldsymbol{B}_X\}$; $\boldsymbol{F}_Z$ satisfies the assumptions in Theorem 4.8. Indeed we can take $\boldsymbol{F}_2=\boldsymbol{F}_X$ and $\{P_{2,x}\}$ as the Brownian motion on F_X. Cleary if $d>2$, $P_{1,z}(\tau_a<\infty)=0$ for all $a\in F_Z-B_Z$, where $\tau_a=\inf\{t>0;\ X_t\in a\}$. We conjecture $\{P_{1,z}\}$ is symmetric and its spectral dimension is greater than 2 if $d\geqq 2$.

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