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Sutherland-Takesaki invariants of dual actions of finite abelian group actions on type III factors

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§0. Introduction.

By works of Sutherland-Takesaki [7] and Kawahigashi-Sutherland-Takesaki [3], actions of a discrete amenable group on an approximately finite dimensional factor of type III have been classified up to cocycle conjugacy in terms of four invariants: a certain normal subgroup, the module, the characteristic invariant and the modular invariant.

For a given action of an abelian group on a von Neumann algebra, since the dual action on the crossed product algebra is canonically defined, the invariants of the dual action should be completely determined by those of the original action. In this note, we shall compute the invariants of the dual action of an action of a finite abelian group in two cases, one is that the module is trivial and the other is that the normal subgroup is trivial. Similar calculations have been done in Kosaki-Sano [5] for $\boldsymbol{Z}_2$-case. In that case, one of the above assumptions is automatically satisfied.

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§1. Preliminaries.

We recall the definition of the cocycle conjugacy invariant of an action of a discrete group of a factor of type III. For details, see Sutherland-Takesaki [7].

Let $\alpha: G \to \mathrm{Aut}\, M$ be an action of a discrete group G on a type III factor M. The cocycle conjugacy invariant of α is then given by $(N(\alpha), \mathrm{mod}, \chi, \nu)$, where each of them is defined as follows: Let φ be a dominant weight on M and denote the continuous decomposition of M by $M = M_\varphi \rtimes_\theta \boldsymbol{R}$. Note that the flow of weights $(P_M, \{F_t^M\}_{t\in \boldsymbol{R}})$ of M is identified with $(Z(M_\varphi), \{\theta_t|_{Z(M_\varphi)}\}_{t\in\boldsymbol{R}})$. For each automorphism α of M, the Connes-Takesaki module $\mathrm{mod}\,\alpha$ in $\mathrm{Aut}\, Z(M_\varphi)$,

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which commutes with θ, is given by

$$\text{(1.1)} \qquad \operatorname{mod}\alpha = \operatorname{Ad}u\circ\alpha|_{Z(M_\varphi)}.$$

Here u is a unitary in M satisfying $\varphi\circ\alpha^{-1}=\varphi\circ\operatorname{Ad}u$. The normal subgroup $N(\alpha)$ of G is defined by

$$\text{(1.2)} \qquad N(\alpha)=\alpha^{-1}(\{\operatorname{Ad}u\circ\bar{\sigma}_c^\varphi \mid u\in U(M),\ c\in Z_\theta^1(\boldsymbol{R},\ U(Z(M_\varphi)))\}).$$

By expressing $\alpha_h=\operatorname{Ad}u_h\circ\bar{\sigma}_{c(h)}^\varphi$, $h\in N(\alpha)$, the characteristic invariant $\chi=[\lambda, \mu]$ in $\Lambda(G, N(\alpha), U(Z(M_\varphi)))$ is defined by

$$\text{(1.3)} \qquad u_h\bar{\sigma}_{c(h)}^\varphi(u_k)=u_{hk}\mu(h, k), \quad h, k\in N(\alpha),$$

$$\text{(1.4)} \qquad \alpha_g(u_{g^{-1}hg})(D\varphi\circ\alpha_g^{-1}: D\varphi)(\operatorname{mod}\alpha_g)(c(g^{-1}hg))=u_h\lambda(g, h), \quad g\in G,\ h\in N(\alpha),$$

where we choose $u_e=1$ and $c(e)=1$ for the unit e of G, and λ and μ satisfy the following relations:

$$\text{(1.5)} \qquad \mu(f, h)\mu(fh, k)=\mu(h, k)\mu(f, hk), \quad f, h, k\in N(\alpha),$$

$$\text{(1.6)} \qquad \lambda(g, hk)\lambda(g, h)^*\lambda(g, k)^*=\mu(h, k)(\operatorname{mod}\alpha_g)(\mu(g^{-1}hg, g^{-1}kg))^*, \quad g\in G,\ h, k\in N(\alpha),$$

$$\text{(1.7)} \qquad \lambda(gg', h)=\lambda(g, h)(\operatorname{mod}\alpha_g)(\lambda(g', g^{-1}hg)), \quad g, g'\in G,\ h\in N(\alpha),$$

$$\text{(1.8)} \qquad \lambda(h, k)=\mu(h, h^{-1}kh)\mu(k, h)^*, \quad h, k\in N(\alpha),$$

$$\text{(1.9)} \qquad \lambda(e, h)=\lambda(g, e)=\mu(e, h)=\mu(h, e)=1, \quad g\in G,\ h\in N(\alpha).$$

The modular invariant ν, which is a homomorphism from $N(\alpha)$ into the first cohomology group of the flow of weights, is defined by

$$\text{(1.10)} \qquad \nu(h)=[c(h)], \quad h\in N(\alpha).$$

Furthermore, these are not independent of each other and satisfy the following (Sutherland-Takesaki [7; Theorem 5.14]):

$$\text{(1.11)} \qquad c(h)c(k)=(\partial\mu(h, k))c(hk), \quad h, k\in N(\alpha),$$

$$\text{(1.12)} \qquad (\operatorname{mod}\alpha_g)(c(g^{-1}hg))=(\partial\lambda(g, h))c(h), \quad g\in G,\ h\in N(\alpha).$$

Here, for a unitary u in $Z(M_\varphi)$, ∂u means the coboundary given by

$$\text{(1.13)} \qquad (\partial u)(t)=u^*\theta_t(u), \quad t\in\boldsymbol{R}.$$

The invariant is independent of the choice of φ and depends only on the cocycle conjugacy class of α.

§ 2. Sutherland-Takesaki invariants of dual actions.

Let $\alpha: G\rightarrow \mathrm{Aut}\, M$ be an (outer) action of a discrete group G on a type III factor M and let φ be a dominant weight on M such that

$$\varphi\circ\alpha_g = \varphi, \quad g\in G,$$

and for the continuous decomposition $M=M_\varphi \rtimes_\theta \boldsymbol{R}$

$$(\alpha_0)_g\circ\theta_t = \theta_t\circ(\alpha_0)_g, \quad g\in G,\ t\in\boldsymbol{R},$$

$$\begin{cases} \alpha_g(\pi_\theta(x)) = \pi((\alpha_0)_g(x)), & x\in M_\varphi, \\ \alpha_g(\lambda(t)) = \lambda(t), & t\in\boldsymbol{R}. \end{cases}$$

Here α_0 is the action of G on M_φ induced by α. (See Sutherland-Takesaki [**7**; Lemma 5.11].)

LEMMA 2.1. *Let $\tilde{\varphi}$ be the dual weight of φ on $M\rtimes_\alpha G$. Then $\tilde{\varphi}$ is a dominant weight and the continuous decomposition of $M\rtimes_\alpha G$ is given by*

$$M\rtimes_\alpha G = (M_\varphi\rtimes_{\alpha_0} G)\rtimes_F \boldsymbol{R},$$

where F is the action of $\boldsymbol{R}$ on $M_\varphi\rtimes_{\alpha_0}G$ defined by

$$\begin{cases} F_t(\pi_{\alpha_0}(x)) = \pi_{\alpha_0}(\theta_t(x)), & x\in M_\varphi, \\ F_t(\lambda(g)) = \lambda(g), & g\in G. \end{cases}$$

PROOF. By Connes-Takesaki [**1**], $\tilde{\varphi}$ is a dominant weight. The rest follows from Haagerup-Størmer [**2**] or Sekine [**6**]. q.e.d.

LEMMA 2.2. *Assume that G is abelian and let $\hat{\alpha}$ be the dual action of α on $M\rtimes_\alpha G=(M_\varphi\rtimes_{\alpha_0}G)\rtimes_F\boldsymbol{R}$. Then we have*

$$\tilde{\varphi}\circ\hat{\alpha}_p = \tilde{\varphi}, \quad p\in\hat{G},$$

$$(\hat{\alpha}_0)_p\circ F_t = F_t\circ(\hat{\alpha}_0)_p, \quad p\in\hat{G},\ t\in\boldsymbol{R},$$

$$\begin{cases} \hat{\alpha}_p(\pi_F(x)) = \pi_F((\hat{\alpha}_0)_p(x)), & x\in M_\varphi\rtimes_{\alpha_0}G, \\ \hat{\alpha}_p(\lambda(t)) = \lambda(t), & t\in\boldsymbol{R}. \end{cases}$$

Here $\hat{\alpha}_0$ means the dual action of α_0 on $M_\varphi\rtimes_{\alpha_0}G$.

PROOF. We notice that the action on $M_\varphi\rtimes_{\alpha_0}G$ induced by $\hat{\alpha}$ is exactly the dual action of α_0 (Haagerup-Størmer [**2**] or Sekine [**6**]). The conclusions follow from direct computations. q.e.d.

In what follows let $\alpha: G\rightarrow\mathrm{Aut}\, M$ be an (outer) action of a finite abelian group G on a factor M of type III. We choose and fix a dominant weight φ

on M with the above properties and denote by $(N(\alpha)$, mod, $[\lambda, \mu]$, $\nu)$ the invariant of α given by using φ.

2.1. The case of ker (mod)$=G$.

Throughout this subsection, we assume

$$\operatorname{mod} \alpha_g = 1, \quad g \in G.$$

The relation (1.12) says

$$c(h) = (\partial\lambda(g, h))c(h), \quad g \in G, \ h \in N(\alpha).$$

This means

$$\theta_t(\lambda(g, h)) = \lambda(g, h), \quad t \in \boldsymbol{R}.$$

From the ergodicity of θ, we have

$$\lambda(g, h) \in \boldsymbol{T} = \{c \in \boldsymbol{C} \mid |c| = 1\}.$$

The condition (1.7) means

$$\lambda(gg', h) = \lambda(g, h)\lambda(g', h), \quad g, g' \in G, \ h \in N(\alpha).$$

hence, for a fixed $h \in N(\alpha)$, the map

$$g \in G \longrightarrow \lambda(g, h) \in \boldsymbol{T}$$

defines a character of G. We get a map π from $N(\alpha)$ into $\hat{G}$ such that

$$\lambda(g, h) = \langle g, \pi(h) \rangle, \quad g \in G, \ h \in N(\alpha),$$

where $\langle g, \pi(h) \rangle$ denotes the value of $\pi(h) \in \hat{G}$ at $g \in G$. Furthermore, by (1.6) and (1.9), π is a homomorphism.

THEOREM 2.3. *The normal subgroup $N(\hat{\alpha})$ arising from the dual action $\hat{\alpha}$ is given by*

$$N(\hat{\alpha}) = \pi(N(\alpha)).$$

PROOF. Let $p = \pi(h)$, $h \in N(\alpha)$. By definition, there exist a unitary u_h in M_φ (because of the invariance $\varphi \circ \alpha_h = \varphi$) and a unitary one-cocycle $c(h)$ such that $\alpha_h = \operatorname{Ad} u_h \circ \bar{\sigma}^{\varphi}_{c(h)}$, in particular, $(\alpha_0)_h = \operatorname{Ad} u_h$. Let $\{\lambda(g)\}_{g \in G}$ denote the usual generator of $M_\varphi \rtimes_{\alpha_0} G$ coming from G. Then $u_h \lambda(h)^*$ is a unitary in $M_\varphi \rtimes_{\alpha_0} G$ and it is trivial that

$$u_h \lambda(h)^* x = x u_h \lambda(h)^*, \quad x \in M_\varphi.$$

Since

$$\begin{aligned} u_h\lambda(h)^*\lambda(g)(u_h\lambda(h)^*)^* &= u_h\lambda(g)u_h^* \\ &= u_h\alpha_g(u_h)^*\lambda(g) \\ &= \overline{\lambda(g, h)}u_h u_h^*\lambda(g) \\ &= \overline{\langle g, \pi(h)\rangle}\lambda(g), \end{aligned}$$

we get

$$(\hat{\alpha}_0)_p = \mathrm{Ad}\, u_h\lambda(h)^*.$$

Hence we have $\pi(N(\alpha))\subset N(\hat{\alpha})$.

Conversely, let $p\in N(\hat{\alpha})$. Then there exists a unitary $V=\sum_{g\in G} v_g\lambda(g)$ in $M_\varphi \rtimes_{\alpha_0} G$ such that $(\hat{\alpha}_0)_p=\mathrm{Ad}\, V$. By the definition of the dual action, we know

$$xV = Vx, \quad x\in M_\varphi.$$

A standard computation says that V must be of the form

$$V = \sum_{h\in N(\alpha)} c_h u_h \lambda(h)^*$$

for some $c_h\in Z(M_\varphi)$. Moreover, we know

$$(\hat{\alpha}_0)_p(\lambda(g)) = \overline{\langle g, p\rangle}\lambda(g), \quad g\in G,$$

that is,

$$V\lambda(g) = \overline{\langle g, p\rangle}\lambda(g)V, \quad g\in G.$$

Since

$$V\lambda(g) = \sum_{h\in N(\alpha)} c_h u_h \lambda(h^{-1}g)$$

and

$$\overline{\langle g, p\rangle}\lambda(g)V = \sum_{h\in N(\alpha)} \overline{\langle g, p\rangle}\lambda(g, h)c_h u_h\lambda(h^{-1}g),$$

we have

$$c_h = \overline{\langle g, p\rangle}\lambda(g, h)c_h, \quad g\in G, \ h\in N(\alpha).$$

Because V is a unitary, there exists an element $h\in N(\alpha)$ such that $c_h\neq 0$. Hence we have

$$\lambda(g, h) = \langle g, p\rangle, \quad g\in G,$$

and $N(\hat{\alpha})\subset\pi(N(\alpha))$. q. e. d.

We set

$$N_0(\alpha) = \ker \pi \quad (\subset N(\alpha)),$$

namely,

$$N_0(\alpha) = \{h\in N(\alpha) \quad \lambda(g, h)=1,\ g\in G\}.$$

LEMMA 2.4. *The center* $Z(M_{\varphi}\rtimes_{\alpha_0}G)$ *of* $M_{\varphi}\rtimes_{\alpha_0}G$ *is isomorphic to* $Z(M_{\varphi})\rtimes_{id,\mu}N_0(\alpha)$.

PROOF. By Kawahigashi-Takesaki [4] or Sekine [6], we know that the anti-isomorphism Φ from $(M_{\varphi})'\cap(M_{\varphi}\rtimes_{\alpha_0}G)$ onto $Z(M_{\varphi})\rtimes_{id,\mu}N(\alpha)$ defined by

$$\Phi:\ \sum_{h\in N(\alpha)} c_h u_h \lambda(h)^* \longrightarrow \sum_{h\in N(\alpha)} c_h z_h,\quad c_h\in Z(M_{\varphi})$$

induces the isomorphism between $Z(M_{\varphi}\rtimes_{\alpha_0}G)$ and $(Z(M_{\varphi})\rtimes_{id,\mu}N(\alpha))^{\gamma}$. Here γ is the action of G on $Z(M_{\varphi})\rtimes_{id,\mu}N(\alpha)$ given by

$$\gamma_g\Big(\sum_{h\in N(\alpha)} c_h z_h\Big)=\sum_{h\in N(\alpha)} \lambda(g,h)c_h z_h,\quad g\in G.$$

So we have the assertion. q. e. d.

PROPOSITION 2.5. *The characteristic invariant* $\hat{\chi}=[\hat{\lambda},\hat{\mu}]$ *arising from* $N(\hat{\alpha})$ *is given by*

$$\hat{\mu}(p,q)=\mu(k,h),\quad p=\pi(h),\ q=\pi(k)\in N(\hat{\alpha}),$$

$$\hat{\lambda}(r,p)=\langle h,r\rangle,\quad r\in\hat{G},\ p=\pi(h)\in N(\hat{\alpha}).$$

PROOF. Let $p=\pi(h)$, $q=\pi(k)\in N(\hat{\alpha})$, $h,k\in N(\alpha)$. From the proof of Proposition 2.3, we may assume

$$(\hat{\alpha}_0)_p=\mathrm{Ad}\,\hat{u}_p,\quad \hat{u}_p=u_h\lambda(h)^*,$$

$$(\hat{\alpha}_0)_q=\mathrm{Ad}\,\hat{u}_q,\quad \hat{u}_q=u_k\lambda(k)^*,$$

$$(\hat{\alpha}_0)_{pq}=\mathrm{Ad}\,\hat{u}_{pq},\quad \hat{u}_{pq}=u_{hk}\lambda(hk)^*.$$

Then the conclusions follow from simple calculations. q. e. d.

PROPOSITION 2.6. *The module* $\mathrm{mod}\,\hat{\alpha}_r$ $(r\in\hat{G})$ *in* $\mathrm{Aut}(Z(M_{\varphi})\rtimes_{id,\mu}N_0(\alpha))$ *is given by*

$$(\mathrm{mod}\,\hat{\alpha}_r)\Big(\sum_{h\in N_0(\alpha)} c_h z_h\Big)=\sum_{h\in N_0(\alpha)}\langle h,r\rangle c_h z_h,$$

where $\{z_h\}_{h\in N_0(\alpha)}$ *is the generator coming from* $N_0(\alpha)$.

PROOF. The assertion follows from Lemma 2.4 and its proof. q. e. d.

REMARK 2.7. By Proposition 2.6,

$$\mathrm{Ker}(\mathrm{mod}\,\hat{\alpha})=\{r\in\hat{G}\mid\langle h,r\rangle=1,\ h\in N_0(\alpha)\}.$$

PROPOSITION 2.8. *Let* $p=\pi(h)\in N(\hat{\alpha})$ $(h\in N(\alpha))$. *Then the unitary one-cocycle* $\hat{c}(p)$ *arising from* $\hat{\alpha}_p$ *is given by*

$$\hat{c}(p)=c(h).$$

PROOF. Expressing $(\hat{\alpha}_0)_p = \mathrm{Ad}\,\hat{u}_p$, $\hat{u}_p = u_h\lambda(h)^*$, we compute

$$\begin{aligned}\hat{c}(p, t) &= \hat{u}_p{}^* F_t(\hat{u}_p) \quad ([\mathbf{7}\,;\ \text{Lemma 5.12. (5.28)}]) \\ &= (u_h\lambda(h)^*)^* F_t(u_h\lambda(h)^*) \\ &= \lambda(h)u_h{}^*\theta_t(u_h)\lambda(h)^* \\ &= \lambda(h)c(h, t)\lambda(h)^* \quad ([\mathbf{7}\,;\ \text{Lemma 5.12. (5.28)}]) \\ &= c(h, t),\ t\in\boldsymbol{R}.\end{aligned}$$

q. e. d.

2.2. The case of $N(\alpha)=\{0\}$.
Throughout this subsection, we assume

$$N(\alpha) = \{0\}.$$

Therefore, the characteristic invariant $\chi=[\lambda, \mu]$ and the modular invariant ν arising from $N(\alpha)$ are trivial. We notice

$$(M_\varphi)' \cap (M_\varphi \rtimes_{\alpha_0} G) = Z(M_\varphi)$$

and

$$Z(M_\varphi \rtimes_{\alpha_0} G) = Z(M_\varphi)^{\mathrm{mod}\,\alpha}: \text{ fixed point subalgebra under the module action of } \alpha,$$

in particular,

$$\mathrm{mod}\,\hat{\alpha}_r = 1, \quad r\in\hat{G}.$$

The argument in 2.1 shows that

$$\hat{\lambda}(r, p) \in \boldsymbol{T}, \quad r\in\hat{G}, \ p\in N(\hat{\alpha})$$

and there exists a homomorphism $\hat{\pi}$ from $N(\hat{\alpha})$ into G such that

$$\hat{\lambda}(r, p) = \langle\hat{\pi}(p), r\rangle, \quad r\in\hat{G},\ p\in N(\hat{\alpha}),$$

$$N(\hat{\hat{\alpha}}) = \hat{\pi}(N(\hat{\alpha})).$$

On the other hand, the Takesaki duality says

$$N(\hat{\hat{\alpha}}) = N(\alpha).$$

Hence we conclude

$$\hat{\lambda}(r, p) = 1, \quad r\in\hat{G},\ p\in N(\hat{\alpha}).$$

For each $p\in\hat{G}$, we define an eigen-operator space $(P_M)^{\mathrm{mod}\,\alpha}(p)$ of the flow of weights for α to be

$$\{x\in P_M \mid (\mathrm{mod}\,\alpha_g)(x) = \langle g, p\rangle x,\ g\in G\}.$$

PROPOSITION 2.9. *Let $p\in\hat{G}$. Then the following conditions are equivalent.*
(i) $p\in N(\hat{\alpha})$.
(ii) *The eigen-operator space $(P_M)^{\mathrm{mod}\,\alpha}(p)$ contains a unitary operator.*

PROOF. (i)→(ii): If p belongs to $N(\hat{\alpha})$, then there exists a unitary $V=\sum_{g\in G} v_g\lambda(g)\in M_\varphi\rtimes_{\alpha 0}G$ such that $(\hat{\alpha}_0)_p=\mathrm{Ad}\,V$. By definition, since

$$xV=Vx,\quad x\in M_\varphi,$$

it follows from the assumption that $v_g=0$ $(g\neq 0)$, that is, V is a unitary in $Z(M_\varphi)$. Furthermore,

$$(\hat{\alpha}_0)_p(\lambda(g))=\overline{\langle g\cdot p\rangle}\lambda(g),\quad g\in G,$$

means that V must be an element in $(P_M)^{\mathrm{mod}\,\alpha}(p)$.

(ii)→(i): If a unitary u is in $(P_M)^{\mathrm{mod}\,\alpha}(p)$, then it is trivial that $(\hat{\alpha}_0)_p=\mathrm{Ad}\,u$ from the above argument. q.e.d.

PROPOSITION 2.10. *Let $p\in N(\hat{\alpha})$. Then the unitary one-cocycle $\hat{c}(p)$ of the flow of weights of $M\rtimes_\alpha G$ arising from $\hat{\alpha}_p$ is calculated by*

$$\hat{c}(p,t)=u^*\theta_t(u),\quad t\in\boldsymbol{R},$$

where u is a unitary in $(P_M)^{\mathrm{mod}\,\alpha}(p)$.

PROOF. The assertion follows from the proof of Proposition 2.9 and Sutherland-Takesaki [7; Lemma 5.12. (5.28)]. q.e.d.

REMARK 2.11.

$$\mathrm{Ker}\,(\mathrm{mod}\,\alpha)=\{g\in G \mid \langle g,\,p\rangle=1,\ p\in N(\hat{\alpha})\}.$$

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