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Surfaces of section for expansive flows on three-manifolds

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0. Introduction.

A continuous non singular flow $\phi_t : X \to X$ on the metric space X is said to be *expansive* if $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) < \delta$ $\forall t \in \boldsymbol{R}$ for some continuous function $h : \boldsymbol{R} \to \boldsymbol{R}$, $h(0)=0$, then $y = \phi_s(x)$ for some $s \in (-\varepsilon, \varepsilon)$ (see [**B-W**], where there are some other equivalent definitions). A classical example of expansive flow is given by an Anosov flow ([**Ano**]); another example is the suspension of a pseudo-Anosov diffeomorphism ([**F-L-P**]).

We shall be concerned with expansive flows on a 3-dimensional closed manifold M, and we will assume the transitivity of the flows under consideration: $\Omega(\phi_t) = M$, where $\Omega(\phi_t)$ is the non-wandering set. We will suppose that M is connected, even if not explicitly stated.

In [**Fri**] D. Fried proved that any transitive Anosov flow ϕ_t on a closed 3-manifold M has a *surface of section*: there exists an embedding $j : \Sigma \hookrightarrow M$, $\Sigma =$ compact surface with boundary, such that $j(\partial\Sigma)$ is a union of closed orbits of ϕ_t, $j(\operatorname{int}\Sigma)$ is transverse to ϕ_t, and every orbit of ϕ_t intersects $j(\Sigma)$ in a uniformly bounded time (the case $\partial\Sigma = \emptyset$ is allowed). The flow ϕ_t then induces a first return map $f : \Sigma \to \Sigma$, which is topologically conjugate to a pseudo-Anosov diffeomorphism with 1-prong singularities on $\partial\Sigma$ ([**F-L-P**]).

We shall prove a similar statement in the case of expansive flows:

THEOREM 1. *Any transitive expansive flow on a closed* 3-*manifold M has a surface of section $\Sigma \hookrightarrow M$.*

The first return map $f : \Sigma \to \Sigma$ associated to such a surface of section is then topologically conjugate to a pseudo-Anosov diffeomorphism, with 1-prong singularities on $\partial\Sigma$ and k-prong singularities ($k \geqq 3$) in $\operatorname{int}\Sigma$. Using the language of [**Fri**] we could restate theorem 1 in terms of "surgery" along closed orbits of suspensions of pseudo-Anosov diffeomorphisms.

The proof of Theorem 1 is very close to that of Fried, and the main tool is the existence of stable and unstable foliations with circle-singularities, established by [**I-M**] and [**Pat**]. We recall in § 1 such a result (which relies on the theorem of Lewowicz and Hiraide, [**Lew**], [**Hir**]), and in § 2 we prove Theorem 1 and we verify that the first return map is conjugate to a pseudo-Anosov one. Our construction of "local" surfaces of section has also some resemblance with the "daisy chains lemma" of J. Christy ([**Chr**]).

Using the fact that a pseudo-Anosov diffeomorphism has a Markov partition ([**F-S**]) and using Theorem 1, we will prove in § 3 that any expansive transitive flow on a closed three manifold has Markov partitions, and we will deduce some consequences at the level of symbolic dynamics. Remark that D. Fried assumes in [**Fri**] the existence of Markov partitions, however we shall see that in order to prove the existence of surfaces of section it is sufficient only to show the existence of a "local product structure" given by the stable and unstable foliations.

1. Stable and unstable foliations.

We will denote by $\mathcal{H}_k$ and $\bar{\mathcal{H}}_k$ the singular foliations on $\boldsymbol{D}^2 \subset \boldsymbol{C}$ whose leaves are respectively $\{\mathfrak{Re}\, z^{k/2} = \text{constant}\}$ and $\{\mathfrak{Im}\, z^{k/2} = \text{constant}\}$, with $k \in \boldsymbol{Z}$, $k \geqq 3$. We will consider the singular foliations on $\boldsymbol{D}^2 \times (0, 1)$ defined by $\mathcal{G}_k = \mathcal{H}_k \times (0, 1)$ and $\bar{\mathcal{G}}_k = \bar{\mathcal{H}}_k \times (0, 1)$.

Let M be a closed 3-manifold and let $C_1, \cdots, C_N \subset M$ be disjoint closed curves.

DEFINITION ([**I-M**]). A *foliation with circle-prong singularities* $C_1, \cdots, C_N$ is a singular C^0 foliation $\mathcal{F}$ on M whose singular set is $S = \bigcup_j C_j$ and such that $\forall x \in S$ there exist a neighborhood $U \subset M$ of x and a homeomorphism $h: U \to \boldsymbol{D}^2 \times (0, 1)$ such that $\mathcal{F}|_U = h^*(\mathcal{G}_k)$ for some k.

Two foliations with circle-prong singularities $\mathcal{F}_1$, $\mathcal{F}_2$ are said to be *transverse* if they have the same singular set S, they are transverse (in the usual sense) in $M \setminus S$, and if $x \in S$ then there exist a neighborhood $U \subset M$ of x and a homeomorphism $h: U \to \boldsymbol{D}^2 \times (0, 1)$ such that $\mathcal{F}_1 = h^*(\mathcal{G}_k)$, $\mathcal{F}_2 = h^*(\bar{\mathcal{G}}_k)$, for some k.

A *separatrix at* $C_j \subset S$ is a leaf with an end on C_j; the union of C_j and all the separatrices at C_j is called *extended leaf*.

Let now $\phi_t: M \to M$ be an expansive flow of class C^r, $r \geqq 0$. Define for any $\varepsilon > 0$ and any $x \in M$:

$$W_\varepsilon^s(x) = \{y \in M \mid \exists \omega \in \text{Homeo}([0, +\infty)) \text{ s.t. } d(\phi_t(x), \phi_{\omega(t)}(y)) < \varepsilon \ \ \forall t \geqq 0\}$$

$$W_\varepsilon^u(x) = \{y \in M \mid \exists \omega \in \text{Homeo}((-\infty, 0]) \text{ s.t. } d(\phi_t(x), \phi_{\omega(t)}(y)) < \varepsilon \ \ \forall t \leqq 0\}.$$

If ε is sufficiently small then $y \in W_\varepsilon^s(x) \Rightarrow \exists$ a homeomorphism $\omega: [0, +\infty) \to [0, +\infty)$ such that $d(\phi_t(x), \phi_{\omega(t)}(y)) \to 0$ as $t \to +\infty$, and $y \in W_\varepsilon^u(x) \Rightarrow \exists$ a homeomorphism $\omega:(-\infty, 0]\to(-\infty, 0]$ such that $d(\phi_t(x), \phi_{\omega(t)}(y))\to 0$ as $t\to-\infty$ ([**K-S**]).

PROPOSITION ([**I-M**], [**Pat**]). *There exist two transverse foliations with circle prong singularities* $\mathcal{F}^s$, $\mathcal{F}^u$ *such that*:

1) *if* $x\in M\setminus S$ *and* $L^s(x)\in\mathcal{F}^s$, $L^u(x)\in\mathcal{F}^u$ *are the leaves through* x, *then for* ε *sufficiently small* $W_\varepsilon^s(x)$ *is a neighborhood of* x *in* $L^s(x)$ *and* $W_\varepsilon^u(x)$ *is a neighborhood of* x *in* $L^u(x)$;

2) *if* $x\in S$ *and* $\bar{L}^s(x)$, $\bar{L}^u(x)$ *are the extended leaves of* $\mathcal{F}^s$, $\mathcal{F}^u$ *through* x, *then for* ε *sufficiently small* $W_\varepsilon^s(x)$ *is a neighborhood of* x *in* $\bar{L}^s(x)$ *and* $W_\varepsilon^u(x)$ *is a neighborhood of* x *in* $\bar{L}^u(x)$.

Observe that $\mathcal{F}^s\cap\mathcal{F}^u$ defines a one dimensional foliation, coincident with that given by the flow ϕ_t. The closed orbits of ϕ_t corresponding to the singular circles of these foliations will be called *singular closed orbits*. The *number of prongs* of a singular closed orbit is the number of separatrices of the singular foliation induced by $\mathcal{F}^s$ (or $\mathcal{F}^u$) on a small disk transverse to the closed orbit; this number may be greater than the number of separatrices of $\mathcal{F}^s$ (or $\mathcal{F}^u$) ending on the closed orbit.

$\mathcal{F}^s$ and $\mathcal{F}^u$ will be called *stable and unstable foliations*. The name is appropriate, because (for example) if $y\in L^s(x)\in\mathcal{F}^s$ (or $y\in\bar{L}^s(x)$) then there exists a homeomorphism $\omega:[0, +\infty)\to[0, +\infty)$ such that $d(\phi_t(x), \phi_{\omega(t)}(y))\to 0$ as $t\to+\infty$.

These foliations enjoy many properties of Anosov foliations: the leaves are open cylinders or open Moebius strips or planes, every leaf injects its fundamental group ([**I-M**]), the holonomy representation of every leaf is injective, etc..

If the flow is transitive, then the same argument as in the context of Anosov flows shows that the periodic orbits are dense in M.

Let us also remark that in general an expansive flow has not strong stable and unstable foliations, i.e., the sets

$$W_\varepsilon^{ss}(x)=\{y\in\boldsymbol{B}(x, \varepsilon)\mid d(\phi_t(x), \phi_t(y))\to 0 \text{ as } t\to+\infty\}\subset W^s(x)$$

$$W_\varepsilon^{uu}(x)=\{y\in\boldsymbol{B}(x, \varepsilon)\mid d(\phi_t(x), \phi_t(y))\to 0 \text{ as } t\to-\infty\}\subset W^u(x)$$

are not parts of leaves of one-dimensional foliations (with singularities). For example, an expansive flow may have a closed orbit γ in a neighborhood of which the flow is given by the differential equation

$$\dot{\theta}=1+r^2+s^2 \quad \dot{r}=-r^3 \quad \dot{s}=s^3 \quad \theta\in\boldsymbol{S}^1, (r, s)\in\boldsymbol{R}^2$$

it is easy to see, e.g. by explicit integration ($\theta(t)=\theta(0)+t+(1/2)\ln[(1+2r(0)^2t)\cdot(1-2s(0)^2t)]$), that the sets $W_\varepsilon^{ss}((\theta, 0, 0))$ and $W_\varepsilon^{uu}((\theta, 0, 0))$ are both reduced only to $\{(\theta, 0, 0)\}$. Remark also that if we reparametrize this flow by multiplying the corresponding vector field by $(1+r^2+s^2)^{-1}$ we obtain

$$\dot{\theta}=1 \qquad \dot{r}=-\frac{r^3}{1+r^2+s^2} \qquad \dot{s}=\frac{s^3}{1+r^2+s^2}$$

and this flow has $W^{ss}((\theta, 0, 0))=\{(\theta, r, 0)|r\in\boldsymbol{R}\}$. $W^{uu}((\theta, 0, 0))=\{(\theta, 0, s)|s\in\boldsymbol{R}\}$. Briefly, the existence or not of strong stable and unstable foliations with singularities strictly depends on the parametrization of the flow. As another example, it is easy to reparametrize in a continuous way an Anosov flow to obtain a flow without strong stable and unstable foliations; such a phenomenon cannot appear if the reparametrization is a little more than continuous, for example Holderian.

2. Proof of Theorem 1.

Let $\phi_t: M\to M$ be a transitive expansive flow and let $\mathcal{F}^s$, $\mathcal{F}^u$ be its stable and unstable foliations, with singular set $S=\bigcup_{j=1}^N C_j$. In order to simplify statements about transversality we shall assume that ϕ_t is at least of class C^1, even if this is not really necessary for the proof.

LEMMA. *$\forall x\in M$ there exists an immersion $j: D\to M$ of a compact surface with boundary D such that:*
a) *$j|_{\partial D}$ is injective and $j(\partial D)$ is a union of closed orbits of ϕ_t*
b) *$j(\operatorname{int} D)$ is transverse to the flow*
c) *$x\in j(\operatorname{int} D)\setminus[j(\operatorname{int} D)\cap j(\partial D)]$.*

PROOF. Suppose firstly that $x\in C_j\subset S$, and let $i: \boldsymbol{D}^2\hookrightarrow M$ be an embedding of the disk transverse to ϕ_t, with $i(0)=x$. Choosing i with image sufficiently small, we may assume that the pair of singular foliations $i^*(\mathcal{F}^s)$, $i^*(\mathcal{F}^u)$ is C^0-conjugate to the pair $\mathcal{H}_k$, $\tilde{\mathcal{H}}_k$ for some $k\in\boldsymbol{Z}$, $k\geqq 3$ (see §1 for the definitions of $\mathcal{H}_k$ and $\tilde{\mathcal{H}}_k$).

The separatrices of $\mathcal{H}_k$ and $\tilde{\mathcal{H}}_k$ divide $\boldsymbol{D}^2$ in $2k$ open regions $A_1, B_1, \cdots, A_k, B_k$ (Fig. 1), with $(\bar{A}_j\cap\bar{B}_j)$=a separatrix of $\mathcal{H}_k$, $(\bar{B}_j\cap\bar{A}_{j+1})$=a separatrix of $\tilde{\mathcal{H}}_k$, $\forall j=1, \cdots, k$ $(k+1=1)$. Take points $p_j\in A_j$, $j=1, \cdots, k$, such that $i(p_j)$ is a point on a closed orbit γ_j of ϕ_t $\forall j=1, \cdots, k$, and assume that $\gamma_j\neq\gamma_i$ for $j\neq i$. Assume also that the leaves $\tilde{\mathcal{H}}_k(p_j)$ and $\mathcal{H}_k(p_{j+1})$ intersect in a point $r_j\in B_j$, $\forall j=1, \cdots, k$. These choices are possible because of the transitivity of ϕ_t.

For any $j=1, \cdots, k$ let $\mathcal{P}_j: \mathcal{D}_j\to\tilde{\mathcal{D}}_j$ be the first return map corresponding to the closed orbit γ_j and to the transversal $i: \boldsymbol{D}^2\to M$; $\mathcal{D}_j$ and $\tilde{\mathcal{D}}_j$ are subset of $\boldsymbol{D}^2$, $\mathcal{D}_j$=maximal connected domain of definition of $\mathcal{P}_j$ on which the first return time is continuous, $\tilde{\mathcal{D}}_j=\mathcal{P}_j(\mathcal{D}_j)$. We may assume that $\mathcal{P}_j$ preserves the orientations of the stable and unstable leaves through p_j, because the closed orbits with this property are dense in M; we postpone the verification of this fact to the end of the proof.

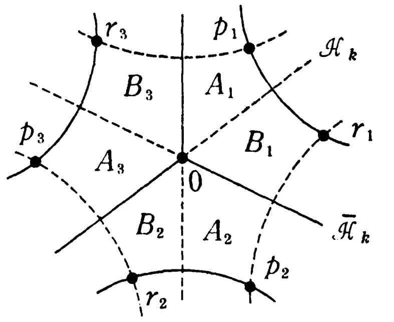

Fig. 1.

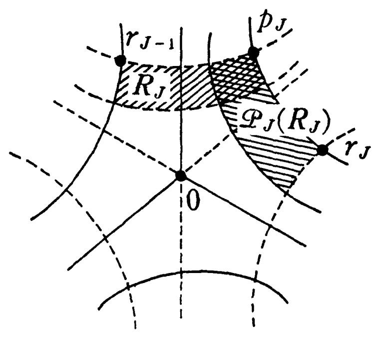

Fig. 2.

Clearly, the segment of $\mathscr{H}_k(p_j)$ between p_j and r_{j-1} is contained in $\mathscr{D}_j$, and the segment of $\bar{\mathscr{H}}_k(p_j)$ between p_j and r_j is contained $\tilde{\mathscr{D}}_j$. Moreover, $\mathscr{P}_j^{-1}(r_j)$ belongs to A_j (and not to B_j), since otherwise the point $\bar{\mathscr{H}}_k(p_j)\cap\mathscr{H}_k(0)$ would belong to $\mathscr{D}_j$ and would be mapped by $\mathscr{P}_j$ to a point in B_j, which is impossible because points of $\mathscr{H}_k(0)$ correspond to orbits of ϕ_t positively asimptotic to the closed orbit C_j and so the positive semitrajectory $\phi_{[0,+\infty)}(i(\bar{\mathscr{H}}_k(p_j)\cap\mathscr{H}_k(0)))$ intersects $i(\boldsymbol{D}^2)$ only in points belonging to $i(\mathscr{H}_k(0))$. A similar argument (with time reversed) show that $\mathscr{P}_j(r_{j-1})\in A_j$.

We deduce the existence of a "rectangle" $R_j\subset\mathscr{D}_j$, bounded by leaves of $\mathscr{H}_k$ and $\bar{\mathscr{H}}_k$, and with r_{j-1}, p_j, $\mathscr{P}_j^{-1}(r_j)$ as vertices (see Fig. 2). Remark that $\forall j=1,\cdots,k$ $\mathscr{P}_j(R_j)\cap R_{j+1}$ is a non-empty rectangle.

Define:

$$Q_1=\mathscr{P}_1^{-1}(\mathscr{P}_1(R_1)\cap R_2)$$

and for $l=2,\cdots,k-1$:

$$Q_l=(\mathscr{P}_l\circ\mathscr{P}_{l-1}\circ\cdots\circ\mathscr{P}_1)^{-1}((\mathscr{P}_l\circ\mathscr{P}_{l-1}\circ\cdots\circ\mathscr{P}_1)(Q_{l-1})\cap R_{l+1})$$

then Q_{k-1} is a non-empty subrectangle of R_1 and $\mathscr{P}_k\circ\cdots\circ\mathscr{P}_1$ is defined on it. Moreover, $(\mathscr{P}_k\circ\cdots\circ\mathscr{P}_1)(Q_{k-1})$ is a subrectangle of $\mathscr{P}_k(R_k)$ which intersects Q_{k-1} along a subrectangle Q of $\mathscr{P}_k(R_k)\cap R_1$ as in Fig. 3.

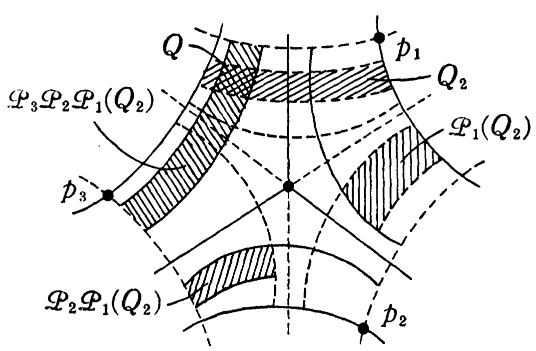

Fig. 3.

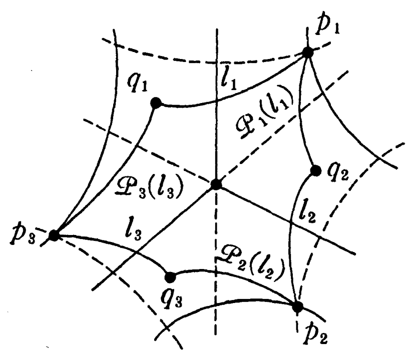

Fig. 4.

We deduce the existence in Q of a fixed point q_1 of $\mathcal{P}_k\circ\cdots\circ\mathcal{P}_1$. Let us remark that $q_j=(\mathcal{P}_{j-1}\circ\cdots\circ\mathcal{P}_1)(q_1)$ belongs to $R_j\cap\mathcal{P}_{j-1}(R_{j-1})$ $\forall j=1,\cdots,k$ and so, intuitively, the closed orbit γ of ϕ_t that corresponds to q_1 "follows" cyclically $\gamma_1,\cdots,\gamma_k$.

Now, as in [**Fri**], we consider segments $l_j\subset \boldsymbol{D}^2$ joining q_j with p_j, transverse to $\mathcal{H}_k$ and $\tilde{\mathcal{H}}_k$, such that $l_1\cup\mathcal{P}_1(l_1)\cup\cdots\cup l_k\cup\mathcal{P}_k(l_k)$ bounds a $2k$-gon $D_0\subset\boldsymbol{D}^2$ (Fig. 4). The union of $i(D_0)$ and the segments of trajectories of ϕ_t from $y\in i(l_j)$ to $i(\mathcal{P}_j(y))\in i(\mathcal{P}_j(l_j))$ may be deformed to an immersed surface $D\to M$ with the required properties ([**Fri**]). Observe that $\partial D=\{\gamma_1,\cdots,\gamma_k,\gamma\}$ and D is a disk with k holes.

This completes the proof in the case $x\in S$; the case $x\in M\setminus S$ is completely similar (in fact, simpler), and formally corresponds to the case $k=2$.

It remains only to prove the above statement about the density of closed orbits with first return map which preserves the orientations of the stable and unstable leaves. We repeat the above construction with only the following changement: if $\mathcal{P}_j:\mathcal{D}_j\to\tilde{\mathcal{D}}_j$ does not preserve the orientations of $\mathcal{H}_k(p_j)$ and $\tilde{\mathcal{H}}_k(p_j)$, then we substitute it with $\mathcal{P}_j^2$. Then we obtain again a closed orbit γ for ϕ_t, which "follows" $\gamma_1,\cdots,\gamma_k$ but some γ_j are now "followed" twice; this orbit γ has first return map with the desired property, and the arbitrarity in the choice of the initial embedding $i:\boldsymbol{D}^2\to M$ shows the density of the orbits of this type. $\triangle$

The proof of Theorem 1 is achieved as in [**Fri**]: we take, by compactness, a finite union of immersed surfaces $D_1,\cdots,D_m$ as in the lemma, such that ∂D_j are all disjoint and every flowline of ϕ_t intersects $\bigcup_{j=1}^m D_j$ in a uniformly bounded time; $\bigcup_{j=1}^m D_j$ is then an "immersed surface of section", and a surgery along its self-intersections produces an embedded surface of section for ϕ_t. $\triangle$

Remark that by the above proof any non-singular closed orbit with first return map preserving the orientations of the stable and unstable leaves (i.e., any closed orbit with stable and unstable leaves of cilindrical type) may be a component of the boundary of a surface of section. On the other hand, it is easy to see that the boundary of a surface of section is formed only by closed orbit with trivial normal bundle, and this gives some restriction.

The flow ϕ_t induces on the interior of any surface of section $\Sigma\hookrightarrow M$ a first return map, which extends to a homeomorphism $f:\Sigma\to\Sigma$. We may assume, up to topological equivalence, that the stable and unstable manifolds of the closed orbits in $\partial\Sigma$ are smooth near $\partial\Sigma$ and their different branches intersect transversally, and that the angle of incidence between Σ and these branches varies with non-zero velocity along these closed orbits. Then the foliations $\mathcal{F}^s$, $\mathcal{F}^u$ restricted to Σ give foliations $\mathcal{G}^s$, $\mathcal{G}^u$ with 1-prongs on $\partial\Sigma$ and k-prongs ($k\geqq 3$)

on int Σ, transverse and invariant by f. Any component of $\partial\Sigma$ contains at least one 1-prong of $\mathcal{G}^s$ and of $\mathcal{G}^u$; it may happen the case where there is only one 1-prong, if the stable and unstable manifolds of the corresponding closed orbit are Moebius strips (we are considering here a general surface of section, not necessarily the one constructed in the proof of Theorem 1). Observe that $f|_{\mathrm{int}\,\Sigma}$ is expansive with respect to a distance which degenerates on $\partial\Sigma$.

PROPOSITION. *The first return map $f:\Sigma\to\Sigma$ is topologically conjugate to a pseudo-Anosov diffeomorphism.*

PROOF. We first prove by a classical argument ([**Fra**]) that $\mathcal{G}^s$ and $\mathcal{G}^u$ are minimal, in the sense that every leaf in int Σ is dense in Σ. Let $L_0\in\mathcal{G}^u$ be the unstable leaf through a periodic point of period k and let $L_j=f^j(L_0)$, $L=\bigcup_{j=0}^{k-1}L_j$. Take $x\in\bar{L}$ a regular point and let $U\subset\mathrm{int}\,\Sigma$ be a product neighborhood for $\mathcal{G}^s$, $\mathcal{G}^u$; if $y\in U$ is l-periodic then its stable leaf $\mathcal{G}^s(y)$ intersects $\mathcal{G}^u(x)\subset\bar{L}$ in a point $z\in U$ and the f-invariance of $\bar{L}$, together with the property $f^{ln}(z)\to y$ as $n\to+\infty$, implies that $y\in\bar{L}$; the density of periodic orbits and the closedness of $\bar{L}$ imply that $U\subset\bar{L}$, hence $\bar{L}\cap\mathrm{int}\,\Sigma$ is open, i.e., $\bar{L}=\Sigma$. This means that every L_j is also dense in Σ, i.e., the leaves of $\mathcal{G}^u$ through periodic points are dense in Σ. Similarly, the leaves of $\mathcal{G}^s$ through periodic points are dense in Σ.

Let now $V\subset\mathrm{int}\,\Sigma$ be any open set and take $y\in V$ k-periodic; let $l\subset\mathcal{G}^s(y)\cap V$ be a segment containing y. Then for N sufficiently large $f^{-kN}(l)$ is a segment in $\mathcal{G}^s(y)$ with the property that every leaf of $\mathcal{G}^u|_{\mathrm{int}\,\Sigma}$ intersects $f^{-kN}(l)$ (because $\mathcal{G}^s(y)$ is dense in Σ and $\mathcal{G}^s(y)=\bigcup_{n=1}^{+\infty}f^{-kn}(l)$). For any $x\in\mathrm{int}\,\Sigma$, $\mathcal{G}^u(f^{-kN}(x))$ intersects $f^{-kN}(l)$ and hence $\mathcal{G}^u(x)$ intersects l, i.e., $\mathcal{G}^u(x)\cap V\neq\emptyset$. This shows that every leaf of $\mathcal{G}^u|_{\mathrm{int}\,\Sigma}$ is dense in Σ, and $\mathcal{G}^u$ is minimal. Similarly, $\mathcal{G}^s$ is minimal.

Let now Σ' denote the closed surface obtained from Σ by collapsing to a point every component of $\partial\Sigma$, let $f':\Sigma'\to\Sigma'$ be the homeomorphism naturally induced by f, and let $\mathcal{G}^{s\prime}$, $\mathcal{G}^{u\prime}$ be the f'-invariant foliations induced by $\mathcal{G}^s$, $\mathcal{G}^u$. Clearly every leaf of $\mathcal{G}^{s\prime}$, $\mathcal{G}^{u\prime}$ is dense in Σ'. Let $\pi:\Sigma\to\Sigma'$ be the natural projection.

Remark that f' is not necessarily expansive, because $\mathcal{G}^{s\prime}$ and $\mathcal{G}^{u\prime}$ may have 1-prong singularities in points of Σ' arising from components of $\partial\Sigma$; however, proposition B of [**Hir**] applies also to this situation and gives two transverse Borel measures $\mu^{s\prime}$, $\mu^{u\prime}$ on $\mathcal{G}^{s\prime}$, $\mathcal{G}^{u\prime}$, which are non-atomic, positive on open non-empty sets, and such that for some $\lambda>1$:

$$f'(\mathcal{G}^{u\prime},\mu^{u\prime})=(\mathcal{G}^{u\prime},\lambda\mu^{u\prime})\qquad f'(\mathcal{G}^{s\prime},\mu^{s\prime})=(\mathcal{G}^{s\prime},\lambda^{-1}\mu^{s\prime})$$

$\mu^s\overset{\mathrm{def}}{=}\pi^*(\mu^{s\prime})$ and $\mu^u\overset{\mathrm{def}}{=}\pi^*(\mu^{u\prime})$ are then transverse Borel measures on $\mathcal{G}^s$, $\mathcal{G}^u$, non-atomic, positive on open non-empty sets, and such that:

$$f(\mathcal{G}^u, \mu^u) = (\mathcal{G}^u, \lambda\mu^u) \qquad f(\mathcal{G}^s, \mu^s) = (\mathcal{G}^s, \lambda^{-1}\mu^s)$$

this implies easily that f is topologically conjugate to a pseudo-Anosov diffeomorphism: the local C^0-coordinates given by μ^s, μ^u define a smooth structure on Σ with respect to which f is a pseudo-Anosov diffeomorphism. $\triangle$

Let us remark the following trivial corollary of Theorem 1: if the transitive expansive flow $\phi_t: M \to M$ is without singular closed orbits, then it is topologically equivalent to an Anosov flow.

Another simple consequence of Theorem 1 is that, starting from [**G-K**], [**Ger**] and [**L-L**], we may construct smooth or analytic models (up to topological equivalence) of transitive expansive flows, which moreover are conditionally stable ([**Ger**]), i.e., they are structurally stable with respect to perturbations with vanishing k-jet along the singular closed orbits (k depending on the number of prongs of the closed orbit). These models admit strong stable and unstable foliations with singularities, the local models of which are given by the foliations on $\boldsymbol{D}^2 \times (0, 1)$ whose leaves are $\{(z, t) \mid \Re e z^{k/2} = \text{constant}, t = \text{constant}\}$, $k \in \boldsymbol{Z}$, $k \geqq 3$.

3. Markov partitions and symbolic dynamics.

Let $\phi_t: M \to M$ be an expansive flow on a closed 3-manifold, with stable and unstable foliations $\mathcal{F}^s$, $\mathcal{F}^u$. The following definitions are given in analogy with [**F-S**] and [**Bow**].

DEFINITION. A *rectangle* is a closed subset $R \subset M$, contained in the image of an embedding $\boldsymbol{D}^2 \hookrightarrow M$ transverse to ϕ_t, such that there exists a homeomorphism $h: [0, 1] \times [0, 1] \to R$ mapping $\{0\} \times (0, 1)$, $\{1\} \times (0, 1)$, $\{s\} \times [0, 1]$ $\forall s \in (0, 1)$ to leaves of $\mathcal{F}^s$, and $(0, 1) \times \{0\}$, $(0, 1) \times \{1\}$, $[0, 1] \times \{t\}$ $\forall t \in (0, 1)$ to leaves of $\mathcal{F}^u$.

If $R = h([0, 1] \times [0, 1])$ is a rectangle, define $\mathring{R} = h((0, 1) \times (0, 1))$, and if $x = h(s, t) \in R$ define $W^s(x, R) = h(\{s\} \times [0, 1])$, $W^u(x, R) = h([0, 1] \times \{t\})$.

DEFINITION. A *Markov partition* is a finite union of disjoint rectangles $\mathcal{R} = \{R_1 \cdots R_m\}$ such that for some $\alpha > 0$:

1) $\phi_{[0, \alpha]}(\bigcup_j R_j) = \phi_{[-\alpha, 0]}(\bigcup_j R_j) = M$

2) if $x \in \mathring{R}_j$ and $y = \phi_t(x) \in \mathring{R}_i$ for some $t > 0$, then there exists a continuous function $\beta: W^s(x, R_j) \to \boldsymbol{R}$, $\beta(x) = t$, such that $\phi_{\beta(z)}(z) \in W^s(y, R_i)$ $\forall z \in W^s(x, R_j)$; and there exists a continuous function $\gamma: W^u(y, R_i) \to \boldsymbol{R}$, $\gamma(y) = -t$, such that $\phi_{\gamma(z)}(z) \in W^u(x, R_j)$ $\forall z \in W^u(y, R_i)$.

THEOREM 2. *Any transitive expansive flow ϕ_t on a closed 3-manifold M has* *a Markov partition.*

PROOF. Let $\Sigma \overset{j}{\hookrightarrow} M$ be a surface of section and let $f: \Sigma \to \Sigma$ be the first return map. Because f is conjugate to a pseudo-Anosov diffeomorphism, it admits thank to [**F-S**] a Markov partition $\mathcal{R}=\{R_1 \cdots R_m\}$ (see [**F-L-P**], exposé 11, the modifications needed for the case of surfaces with boundary). This Markov partition is formed by rectangles in $\operatorname{int}\Sigma$ and pentagons intersecting $\partial\Sigma$ along one of their sides. Let R_i be one of such pentagons, then the embedding $j: \Sigma \to M$ induces an embedding $j_i: R_i \to M$, which maps one side of R_i to a segment of a closed orbit γ and the two adiacent sides to two segments contained in the stable and the unstable leaf through γ; moreover, $j_i(R_i)$ is transverse to ϕ_t except along γ. Clearly, we may move $j_i(R_i)$ along the flowlines of ϕ_t in order to produce a rectangle $\bar{R}_i$, see Fig. 5.

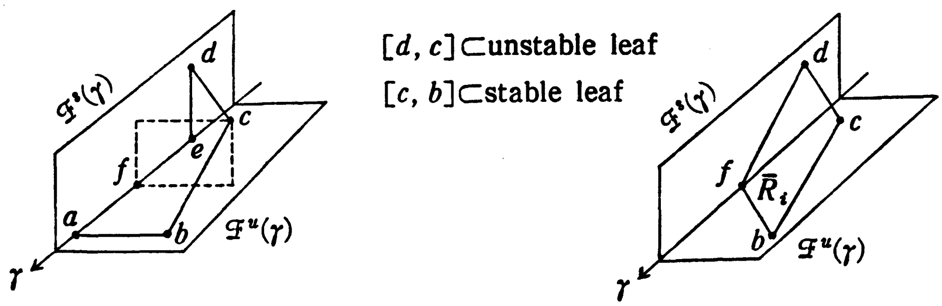

Fig. 5.

If, on the contrary, R_j is a rectangle, then its image in M is also a rectangle $\bar{R}_j$. Deforming along the flowlines all the rectangles so obtained from $\mathcal{R}$ we obtain a disjoint collection of rectangles, which is the desired Markov partition. △

Let now $\mathcal{R}=\{R_1 \cdots R_m\}$ be a Markov partition for ϕ_t and let us consider the matrix $A=(a_{ij})_{1\le i,j\le m}$ defined by

$$a_{ij}=\begin{cases} 1 & \text{if } \exists x\in \mathring{R}_i \text{ s.t. } \phi_t(x)\in \mathring{R}_j \text{ for some } t>0 \text{ and } \phi_s(x)\notin \bigcup_j R_j \ \forall s\in(0,t) \\ 0 & \text{otherwise.}\end{cases}$$

Let $\sigma_A: \Sigma_A \to \Sigma_A$ be the subshift of finite type associated to A and if $\omega: \Sigma_A \to \boldsymbol{R}$ is a continuous positive function let $\psi_t: \Sigma(\sigma_A, \omega) \to \Sigma(\sigma_A, \omega)$ be the suspension of σ_A with first return time ω, where $\Sigma(\sigma_A, \omega)=(\Sigma_A\times\boldsymbol{R})/((x, t)\cong(\sigma_A(x), t+\omega(x)))$ with the usual metric, see [**Bow**] or [**B-W**]. Then the usual arguments ([**Bow**], [**B-W**], [**F-S**]) give:

COROLLARY. *There exists for an appropriate* ω *a continuous, surjective, finite-to-one map* $h: \Sigma(\sigma_A, \omega) \to M$ *such that* $h\circ\psi_t=\phi_t\circ h$ $\forall t\in\boldsymbol{R}$. △

REMARK 1. ω and h are not necessarily Lipschitz, as in the hyperbolic case. This is related to the eventual non-existence of strong stable and unstable foliations.

REMARK 2. We do not know if it is possible to prove this corollary without Markov partitions and directly from the existence of a surface of section and the semiconjugacy results in [**F-S**]. The problem is that the surgery needed to pass from the suspension of a pseudo-Anosov diffeomorphism to a given expansive flow is a "discontinuous" operation. Observe also that the semiconjugacy result in [**F-S**] does not hold in the case of surfaces with boundary.

The above corollary is only an example. Most of the results proved in the context of hyperbolic dynamics (minimal sets, recurrence properties, distribution of closed orbits, zeta function, invariant measures and their ergodic properties$\cdots$) still hold in the context of transitive expansive flows on three-manifolds.

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