

## Uniqueness of stable minimal surfaces with partially free boundaries

Dedicated to Robert Finn on the occasion of his seventieth birthday

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### 1. Introduction.

The aim of this paper is to prove a uniqueness theorem for stable minimal surfaces  $X: B \rightarrow \mathbf{R}^3$  of the type of the disk which are stationary in a boundary configuration  $\langle \Gamma, S \rangle$  consisting of a surface  $S$  and of a Jordan arc  $\Gamma$  with endpoints on  $S$ . The existence of such surfaces for a prescribed configuration  $\langle \Gamma, S \rangle$  was established by Courant under fairly general assumptions on  $\Gamma$  and  $S$ , while H. Lewy proved the first basic results on boundary regularity of minimizers. A detailed investigation of this problem with regard to existence, boundary regularity and properties of the free trace can be found in the recent monograph [3]; cf. also [2] and [9].

It is well-known that in general a configuration  $\langle \Gamma, S \rangle$  bounds more than one stationary minimal surface of disk-type and even more than one minimizer. In fact, uniqueness seems to be a rather rare phenomenon, and not much is known about as to when it will occur. To our knowledge the question of uniqueness of minimal surfaces solving a free boundary value problem was only studied in the papers [4]-[6]. Here we want to prove a restricted uniqueness result applying only to stable minimal surfaces, whereas [5] and [6] require no restrictions of this kind. On the other hand, the method of this paper, derived from ideas of [11], applies to more general configurations  $\langle \Gamma, S \rangle$  than [5] and [6], and also applications to  $H$ -surfaces seem possible. In [7] the results and techniques of this paper will be used to study existence and uniqueness for a singular problem, of which [4] is in some sense a limit case.

Let us now fix some notation to be used in the sequel. We denote by

$$X(u, v) = (X^1(u, v), X^2(u, v), X^3(u, v))$$

a minimal surface defined on the parameter domain  $B = \{(u, v) \in \mathbf{R}^2 : u^2 + v^2 < 1, v > 0\}$ . This is to say,  $X: B \rightarrow \mathbf{R}^3$  is a harmonic mapping,

$$(1.1) \quad \Delta X = 0,$$

satisfying the conformality relations

$$(1.2) \quad E = G, \quad F = 0,$$

where  $E, F, G$  denote the coefficients of the first fundamental form  $X$ ,

$$(1.3) \quad E := |X_u|^2, \quad F := X_u \cdot X_v, \quad G := |X_v|^2.$$

We assume that  $S$  is a complete  $C_*^3$ -submanifold of  $\mathbf{R}^3$  which means that  $S \in C^3$  and that  $S$  satisfies some "uniformity condition at infinity" (in the sense of [3], vol. 2, 7.6, Definitions 1 and 2); the manifold  $S$  is called a support surface. Secondly we suppose that  $\Gamma$  is a regular Jordan arc of class  $C^{2,\alpha}$ ,  $0 < \alpha < 1$ , which has no points in common with  $S$  except for its endpoints  $P_1$  and  $P_2$  where  $\Gamma$  meets  $S$  with a positive angle  $\beta$ .

We say that  $X$  is of class  $\mathcal{C}(\Gamma, S)$  (or:  $X$  is bounded by  $\langle \Gamma, S \rangle$ ) if its Dirichlet integral

$$\mathcal{D}(X) := \frac{1}{2} \int_B |DX|^2 du dv$$

is finite,  $X(u, 0) \in S$  for almost all  $u \in I := (-1, 1)$ , and  $X$  has a continuous Sobolev trace  $X|_C$  on the circular arc  $C := \{(u, v) \in \mathbf{R}^2 : u^2 + v^2 = 1, v \geq 0\}$  mapping  $C$  monotonically onto  $\Gamma$ . Furthermore, a minimal surface  $X \in \mathcal{C}(\Gamma, S)$  is said to be *stationary in the configuration*  $\langle \Gamma, S \rangle$  if  $(d/d\varepsilon)\mathcal{D}(X_\varepsilon)|_{\varepsilon=0} = 0$  holds true for every *admissible variation*  $\{X_\varepsilon\}_{|\varepsilon| < \varepsilon_0}$  of  $X$ , i. e., for every differentiable one-parameter family of surfaces  $X_\varepsilon \in H^{1,2}(B, \mathbf{R}^3) \cap \mathcal{C}(\Gamma, S)$  satisfying  $X_0 = X$  (see [3], vol. 1, pp. 328-335, for a precise definition).

Well-known regularity results ensure that  $X$  is continuous on  $\bar{B}$ , of class  $C^{2,\alpha}$  on  $\bar{B} - \{-1, 1\}$ , and that  $\int_I |dX| < \infty$ . If  $\beta = \pi/2$  we even have  $X \in C^1(\bar{B}, \mathbf{R}^3)$ .

To prove uniqueness for stable stationary surfaces in  $\langle \Gamma, S \rangle$ , we consider a very special situation. Firstly, we assume that  $S$  is a cylinder surface  $\Sigma_0 \times \mathbf{R}$  in the three-dimensional space  $\mathbf{R}^3$  of points  $x = (x^1, x^2, x^3)$  which has a planar curve  $\Sigma_0$  as directrix and whose generating lines are parallels to the  $x^3$ -axis. The directrix  $\Sigma_0$  is supposed to be an embedded curve contained in the  $x^1, x^2$ -plane  $\Pi$  which is given by a representation  $\sigma: \mathbf{R} \rightarrow \Pi$  of class  $C^{2,\alpha}$  satisfying

$$|\sigma'(s)| = 1 \quad \text{and} \quad |\sigma''(s)| \leq c \quad \text{for all } s \in \mathbf{R}$$

and some constant  $c > 0$ . It is also required that  $\sigma'(s)$  tends to a limit as  $s \rightarrow \infty$  or  $-\infty$  respectively. Let  $p_1$  and  $p_2$  be the orthogonal projections of the endpoints  $P_1$  and  $P_2$  to the plane  $\Pi$ , and suppose that  $p_1 = \sigma(s_1)$ ,  $p_2 = \sigma(s_2)$  and  $s_1 < s_2$ ; the closed subarc  $\{\sigma(s) : s_1 \leq s \leq s_2\}$  is denoted by  $\Sigma$ . Secondly, we assume that the orthogonal projection  $\Gamma$  of  $\Gamma$  to the  $x_1, x_2$ -plane is a regular Jordan arc of

class  $C^{2,\alpha}$ , and that  $\Gamma$  is given as a graph  $\{(x^1, x^2, \gamma(x^1, x^2)) : (x^1, x^2) \in \underline{\Gamma}\}$  of a height function  $\gamma$  above  $\underline{\Gamma}$  of class  $C^{2,\alpha}$ . Then the closed contour  $\underline{\Gamma} \cup \Sigma$  bounds a bounded domain  $\mathcal{Q}$  in  $E$ , and we, thirdly, assume that  $\underline{\Gamma}$  is convex with respect to  $\mathcal{Q}$ . Finally, let  $\underline{t}(s) := \sigma'(s)$  be the unit tangent and  $\underline{\nu}(s)$  the unit normal field to  $\Sigma_0$  which on  $[s_1, s_2]$  points to the exterior of  $\mathcal{Q}$ ; set  $t(s) = (\underline{t}(s), 0) \in \mathbf{R}^3$  and  $\nu(s) = (\underline{\nu}(s), 0) \in \mathbf{R}^3$ . Then  $\nu(s)$  is the surface normal of  $S$  along the line  $\{\sigma(s)\} \times \mathbf{R}$ . It can be assumed that  $t(s) = e_3 \wedge \nu(s)$ .

We also introduce the following condition on  $\underline{\Gamma}$  and  $\Sigma_0$ :

CONDITION (B). For each  $s \in (-\infty, s_1) \cup (s_2, \infty)$  the normal line  $\mathcal{L}(s) := \{p \in E; [p - \sigma(s)] \cdot \sigma'(s) = 0\}$  meets the set  $\bar{\mathcal{Q}} \cup \Sigma_0$  only at the point  $\sigma(s)$ .

**2. Properties of the Gauss map.**

Let now  $X: B \rightarrow \mathbf{R}^3$  be a stationary minimal surface in  $\langle \Gamma, S \rangle$ . Since  $X(u, v) \neq \text{const}$ , it follows from well-known results that the branch points of  $X$  (i. e., the zeros of  $W := |X_u \wedge X_v| = E$ ) are isolated in  $\bar{B}$ . Therefore the surface normal

$$(2.1) \quad N = (N^1, N^2, N^3) = \frac{X_u \wedge X_v}{W}$$

is well-defined on  $\bar{B}$  except for finitely many points. From well-known asymptotic expansions one then infers that  $N$  can be extended to a continuous map  $N: \bar{B} \rightarrow \mathbf{R}^3$ , the Gauss map of  $X$ . It turns out that  $N(u, v)$  is real analytic in  $B$  and satisfies both

$$(2.2) \quad |N_u|^2 = |N_v|^2, \quad N_u \cdot N_v = 0 \quad \text{in } B$$

as well as

$$(2.3) \quad -\Delta N + 2EKN = 0 \quad \text{in } B$$

where  $K$  denotes the Gauss curvature of  $X$ . Because of

$$(2.4) \quad |K|W = -KE = \frac{1}{2} |\nabla N|^2$$

it follows that (2.3) is equivalent to

$$(2.5) \quad -\Delta N = N|\nabla N|^2 \quad \text{in } B,$$

which means that  $N$  is a harmonic mapping of  $B$  into the unit sphere  $S^2$ . Moreover, (2.4) implies that

$$\frac{1}{2} |\nabla N|^2 du dv = |K|W du dv = |K| dA$$

where  $dA = W du dv$  is the area element on  $X$ , and an appropriate variant of

the Gauss-Bonnet theorem yields that

$$(2.6) \quad \mathcal{D}(N) := \frac{1}{2} \int_B |\nabla N|^2 du dv = \int_X |K| dA < \infty,$$

i. e.,  $N \in H^{1,2}(B, \mathbf{R}^3) \cap C^0(\bar{B}, \mathbf{R}^3)$ , and  $|N(u, v)| \equiv 1$  (see [3], Chapters 7 and 8 for pertinent results). Let  $\mathcal{L}, \mathcal{M}, \mathcal{N}$  be the coefficients of the second fundamental form of  $X$ . Then we have

$$(2.7) \quad 2H = \frac{\mathcal{L} + \mathcal{N}}{E} = 0, \quad K = \frac{\mathcal{L}\mathcal{N} - \mathcal{M}^2}{E^2}$$

and

$$(2.8) \quad N_u = -\frac{\mathcal{L}}{E} X_u - \frac{\mathcal{M}}{E} X_v, \quad N_v = -\frac{\mathcal{M}}{E} X_u - \frac{\mathcal{N}}{E} X_v.$$

It follows that

$$(2.9) \quad X_u \wedge N_v + N_u \wedge X_v = 0,$$

$$(2.10) \quad N_u = N \wedge N_v, \quad N_v = -N \wedge N_u,$$

$$(2.11) \quad N_u \wedge N_v = EKN,$$

$$(2.12) \quad X_u \wedge N = -X_v, \quad N \wedge X_v = -X_u.$$

Now we want to show that  $N^3 = N \cdot e_3$  satisfies a boundary condition on the free boundary  $I$ , which will be useful later on. Here  $e_3$  denotes the unit vector  $(0, 0, 1)$ . To this end we define the vector fields  $\tau(x)$  and  $n(x)$  on  $S$  by

$$(2.13) \quad \tau(x) := t(s), \quad n(x) := \nu(s) \quad \text{if } x \in \{\sigma(s)\} \times \mathbf{R},$$

and similarly we define the curvature function  $\kappa(x)$  by

$$(2.14) \quad \kappa(x) := k(s) \quad \text{for } x \in \{\sigma(s)\} \times \mathbf{R}$$

where  $k(s) = \sigma''(s) \cdot \nu(s)$  is the curvature function of  $\sigma(s)$ . We can assume that  $\tau, n$ , and  $\kappa$  are smoothly extended to all of  $\mathbf{R}^3$ . Then we obtain

**PROPOSITION 1.** *Let  $I'$  be the open set on  $\mathbf{R}$  obtained by removing the finite set of boundary branch points from  $I$ . Then  $N(u, v) = (N^1(u, v), N^2(u, v), N^3(u, v))$  is of class  $C^{1,\mu}(B \cup I', \mathbf{R}^3)$  for any  $\mu \in (0, 1)$  and satisfies*

$$(2.15) \quad N_v^3 = \kappa(X) [N \cdot \tau(X)]^2 \{X_v \cdot n(X)\} N^3 \quad \text{on } I'.$$

**PROOF.** On account of (2.10) it follows that

$$N_v^3 = (N \cdot e_3)_v = N_v \cdot e_3 = -(N \wedge N_u) \cdot e_3 = -(N_u \wedge e_3) \cdot N$$

whence

$$(2.16) \quad N_v^3 = -(N \wedge e_3)_u \cdot N.$$

Since  $X_v(u, 0)$  is perpendicular to  $S$  at  $X(u, 0)$ , the vectors  $N(u, 0)$  and  $e_3$  are tangent to  $S$ , and thus  $N(u, 0) \wedge e_3$  is perpendicular to  $S$ . Hence there is a function  $\rho(u)$  on  $I$  such that

$$(2.17) \quad N(u, 0) \wedge e_3 = \rho(u)n(X(u, 0)).$$

It follows that

$$(2.18) \quad \rho = (N \wedge e_3) \cdot n(X) = N \cdot (e_3 \wedge n(X)) = N \cdot \tau(X) \quad \text{on } I,$$

and we infer from (2.16)-(2.17) that

$$N_v^3 = -[\rho n(X)]_u \cdot N \quad \text{on } I'.$$

Because of

$$n(X) \cdot N = 0 \quad \text{on } I$$

it follows that

$$N_v^3 = -\rho[n(X)]_u \cdot N \quad \text{on } I'$$

whence

$$(2.19) \quad N_v^3 = -\rho N \cdot \nabla n(X) \cdot X_u \quad \text{on } I.$$

Now we interpret  $\nabla n(X)$  as Weingarten map, which is a symmetric linear operator on the tangent space  $T_X S$  to  $S$  at  $X$ . Because of the special structure of  $S$  it follows that we have

$$(2.20) \quad N \cdot \nabla n(X) \cdot X_u = \rho \tau(X) \cdot \nabla n(X) \cdot \tau(X) [X_u \cdot \tau(X)]$$

and

$$(2.21) \quad \tau(X) \cdot \nabla n(X) \cdot \tau(X) = \kappa(X)$$

on  $I$ . Finally, by virtue of

$$(2.22) \quad \tau(x) = e_3 \wedge n(x) \quad \text{for } x \in S,$$

the conformality relations  $|X_u| = |X_v|$ ,  $X_u \cdot X_v = 0$ , and the free boundary condition for  $X$ , we see that

$$(2.23) \quad X_u \cdot \tau(X) = (N \cdot e_3)[X_v \cdot (-n(X))] \quad \text{on } I.$$

Relation (2.15) is now a consequence of equations (2.18)-(2.23). □

### 3. The second variation of the area functional for stationary minimal surfaces of class $\mathcal{C}(I, S)$ .

The area  $A(X)$  of a smooth surface  $X: B \rightarrow \mathbf{R}^3$  is defined by

$$(3.1) \quad A(X) := \int_B |X_u \wedge X_v| du dv.$$

If  $X$  is a minimal surface spanning a closed Jordan arc  $\Gamma$  in  $\mathbf{R}^3$ , one usually considers so-called *normal variations*  $X_\varepsilon$  of  $X$  defined by

$$(3.2) \quad X_\varepsilon(w) := X(w) + \varepsilon \lambda(w) N(w), \quad |\varepsilon| < \varepsilon_0,$$

where  $w$  denotes a point  $w = (u, v) \in B$ ,  $N: B \rightarrow \mathbf{R}^3$  is the Gauss map of  $X$ , and  $\lambda$  is a smooth scalar function vanishing at the boundary of  $B$ . Clearly  $X_\varepsilon$  spans  $\Gamma$  if  $X$  is bounded by  $\Gamma$ , and therefore we obtain  $(d^2/d\varepsilon^2)A(X_\varepsilon)|_{\varepsilon=0} \geq 0$  if  $X$  is area minimizing among all surfaces spanning  $\Gamma$ . One usually calls the expression

$$(3.3) \quad \delta^2 A(X, \lambda) := \frac{d^2}{d\varepsilon^2} A(X_\varepsilon)|_{\varepsilon=0}$$

for variations  $X_\varepsilon$  of type (3.2) the *second variation of the area functional (with respect to normal displacements  $\lambda N$ ,  $\lambda|_{\partial B} = 0$ )*. A well-known computation\*) shows that for variations (3.2) the second variation can be written as

$$(3.4) \quad \delta^2 A(X, \lambda) := \int_B (|\nabla \lambda|^2 + 2EK\lambda^2) du dv.$$

The Euler equation of (3.4) is

$$(3.5) \quad -\Delta \lambda + 2EK\lambda = 0,$$

and by (2.3) we know that (3.5) is satisfied by  $\lambda = N^1, N^2, N^3$ , i. e., by the components of the Gauss map  $N$  of  $X$ .

It has become customary to call a minimal surface  $X: B \rightarrow \mathbf{R}^3$  *stable*, if the expression (3.4) is nonnegative for all  $\lambda \in C_c^1(B)$ . In this sense each minimal surface  $X$  minimizing area among all surfaces (of disk-type) spanning  $\Gamma$  is stable.

Stable minimal surfaces and, more generally, stable  $H$ -surfaces have in the recent past become an attractive field of investigations where many beautiful results were found; we only mention the work of Barbosa and do Carmo [1]. However, for minimal surfaces with partially free boundary this notion of stability is not anymore appropriate, as it only refers to variations (3.2) of  $X$  which keep the boundary values  $X|_{\partial B}$  fixed, i. e.,  $\lambda|_{\partial B} = 0$ . Yet for stationary surfaces  $X: B \rightarrow \mathbf{R}^3$  in  $\langle \Gamma, S \rangle$  it is natural to consider variations  $X_\varepsilon$  of  $X$  which change the boundary values of  $X$  along the free boundary  $I$ , under the provision that  $X_\varepsilon(I) \subset S$ , i. e., that  $X_\varepsilon \in \mathcal{C}(\Gamma, S)$ , and one might call  $X$  *freely stable* if  $(d^2/d\varepsilon^2)A(X_\varepsilon) \geq 0$  holds true for any admissible family of variations  $X_\varepsilon$  of  $X$ .

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\*) Following a suggestion by the referee we should like to point out that the classical formula for the second variation has been computed for arbitrary variations of  $X$ , and that it is, of course, irrelevant that the parameter domain is a disk of [9], §§ 101-103, pp. 91-94.

Unfortunately one cannot anymore operate with purely normal variations (3.2) if one wants to allow for  $\lambda \in C^1(\bar{B})$  with  $\lambda|_C=0$ , e.g., for  $\lambda \in C^1_0(B \cup I)$ , because in this case it will in general not be true that  $X_\varepsilon(I) \subset S$ . In order to transform  $X_\varepsilon$  into an admissible variation we have to add correction terms of higher order ensuring that  $X_\varepsilon \in \mathcal{C}(\Gamma, S)$ . Therefore we consider now variations  $X_\varepsilon$  of the type

$$(3.6) \quad X_\varepsilon(w) = X(w) + \varepsilon \lambda(w) N(w) + \frac{\varepsilon^2}{2} Z(w) + o(\varepsilon^2)$$

which we assume to be admissible, i.e.,  $X_\varepsilon \in \mathcal{C}(\Gamma, S)$ , and we compute  $(d^2/d\varepsilon^2)A(X_\varepsilon)|_{\varepsilon=0}$ . Thereafter we indicate how the additional term  $Z(w)$  is to be chosen so that  $X_\varepsilon$  becomes admissible.

Let us write  $\tilde{X}(w, \varepsilon)$  for  $X_\varepsilon(w)$ , i.e., we consider a differentiable one-parameter family of surfaces

$$(3.6') \quad \tilde{X}(w, \varepsilon) = X(w) + \varepsilon \lambda(w) N(w) + \frac{\varepsilon^2}{2} Z(w) + o(\varepsilon^2),$$

$w \in \bar{B}$ ,  $|\varepsilon| < \varepsilon_0$ ,  $w = (u, v)$ . As usual we write  $X_u, X_v, \dots$  for  $\partial X / \partial u, \partial X / \partial v, \dots$ . Then we have

$$\tilde{X}_u = X_u + \varepsilon \lambda_u N + \varepsilon \lambda N_u + \frac{1}{2} \varepsilon^2 Z_u + o(\varepsilon^2),$$

$$\tilde{X}_v = X_v + \varepsilon \lambda_v N + \varepsilon \lambda N_v + \frac{1}{2} \varepsilon^2 Z_v + o(\varepsilon^2),$$

whence

$$\begin{aligned} \tilde{X}_u \wedge \tilde{X}_v &= EN + \varepsilon [\lambda(X_u \wedge N_v + N_u \wedge X_v) + (\lambda_v X_u \wedge N + \lambda_u N \wedge X_v)] \\ &\quad + \frac{1}{2} \varepsilon^2 \{ (X_u \wedge Z_v + Z_u \wedge X_v) + 2(\lambda \lambda_u N \wedge N_v + \lambda \lambda_v N_u \wedge N) \\ &\quad + 2\lambda^2 N_u \wedge N_v \} + o(\varepsilon^2). \end{aligned}$$

On account of (2.9)-(2.11) it follows that

$$\begin{aligned} \tilde{X}_u \wedge \tilde{X}_v &= EN - \varepsilon (\lambda_u X_u + \lambda_v X_v) + \varepsilon^2 \lambda^2 EKN \\ &\quad + \varepsilon^2 (\lambda \lambda_u N_u + \lambda \lambda_v N_v) + \frac{1}{2} \varepsilon^2 (X_u \wedge Z_v - X_v \wedge Z_u) + o(\varepsilon^2) \end{aligned}$$

whence

$$\begin{aligned} |\tilde{X}_u \wedge \tilde{X}_v|^2 &= E^2 + 2\varepsilon^2 \lambda^2 E^2 K + \varepsilon^2 E |\nabla \lambda|^2 \\ &\quad + \varepsilon^2 EN \cdot (X_u \wedge Z_v - X_v \wedge Z_u) + o(\varepsilon^2) \end{aligned}$$

by virtue of the conformality relations (1.2) and of  $N \perp X_u, X_v, N_u, N_v$ . By (2.12) and  $\Delta X = 0$  we obtain

$$\begin{aligned} N \cdot (X_u \wedge Z_v - X_v \wedge Z_u) &= (N \wedge X_u) \cdot Z_v - (N \wedge X_v) \cdot Z_u \\ &= X_v \cdot Z_v + X_u \cdot Z_u = (X_u \cdot Z)_u + (X_v \cdot Z)_v. \end{aligned}$$

Thus we obtain for  $\tilde{W}$ , defined by

$$\tilde{W} := |\tilde{X}_u \wedge \tilde{X}_v|,$$

that

$$\tilde{W}^2 = E^2 + \varepsilon^2 \{E|\nabla\lambda|^2 + 2E^2K\lambda^2 + E[(X_u \cdot Z)_u + (X_v \cdot Z)_v]\} + o(\varepsilon^2)$$

whence

$$\left(\frac{\partial}{\partial \varepsilon} \tilde{W}^2\right)\Big|_{\varepsilon=0} = 0$$

and

$$\left(\frac{\partial^2}{\partial \varepsilon^2} \tilde{W}\right)\Big|_{\varepsilon=0} = \frac{(\partial^2/\partial \varepsilon^2) \tilde{W}^2}{2\sqrt{\tilde{W}^2}}\Big|_{\varepsilon=0} = \frac{1}{2E} \left(\frac{\partial^2}{\partial \varepsilon^2} \tilde{W}^2\right)\Big|_{\varepsilon=0}.$$

Therefore we obtain

$$(3.7) \quad \left(\frac{\partial^2}{\partial \varepsilon^2} \tilde{W}\right)\Big|_{\varepsilon=0} = |\nabla\lambda|^2 + 2EK\lambda^2 + (X_u \cdot Z)_u + (X_v \cdot Z)_v,$$

and hence

$$(3.8) \quad \frac{d^2}{d\varepsilon^2} A(\tilde{X}(\cdot, \varepsilon))\Big|_{\varepsilon=0} = \int_B [|\nabla\lambda|^2 + 2EK\lambda^2 + (X_u \cdot Z)_u + (X_v \cdot Z)_v] du dv.$$

Note that this expression differs from (3.4) only by the divergence term

$$\int_B [(X_u \cdot Z)_u + (X_v \cdot Z)_v] du dv$$

which by partial integration can be transformed into the boundary integral

$$\int_{\partial B} (X_u \cdot Z) dv - (X_v \cdot Z) du.$$

Suppose now that  $\lambda \in C^1(\bar{B})$  and  $Z \in C^1(\bar{B}, \mathbf{R}^3)$ , and that  $\lambda|_C=0$  and  $Z|_C=0$ . Then it follows that

$$(3.9) \quad \frac{d^2}{d\varepsilon^2} A(\tilde{X}(\cdot, \varepsilon))\Big|_{\varepsilon=0} = \int_B [|\nabla\lambda|^2 + 2EK\lambda^2] du dv - \int_I X_v \cdot Z du,$$

and we also obtain

$$(3.10) \quad \frac{d}{d\varepsilon} A(\tilde{X}(\cdot, \varepsilon))\Big|_{\varepsilon=0} = 0$$

if  $X$  is a stationary minimal surface in the boundary configuration  $\langle \Gamma, S \rangle$ . Hence we have for stationary surfaces in  $\langle \Gamma, S \rangle$  that

$$(3.11) \quad A(\tilde{X}(\cdot, \varepsilon)) = A(X) + \frac{\varepsilon^2}{2} \int_B (|\nabla\lambda|^2 + 2EK\lambda^2) du dv - \frac{\varepsilon^2}{2} \int_I X_v \cdot Z du + o(\varepsilon^2),$$

and for local minimizers of  $A$  in  $C(I, S)$  we even have

$$(3.12) \quad \int_B (|\nabla\lambda|^2 + 2EK\lambda^2) du dv - \int_I X_v \cdot Z du \geq 0.$$

Now we want to investigate how the additional term  $Z(w)$  in (3.6) or (3.6') has to be chosen if we want  $X_\varepsilon = \tilde{X}(\cdot, \varepsilon)$  to be an admissible variation of  $X$  for  $|\varepsilon| \ll 1$ . Clearly we have to add a second-order correction term in direction of  $n(X)$  if we want to correct a possible lift-off of the trace  $\tilde{X}(I, \varepsilon)$  from the support surface  $S$  since  $n|_S$  is the surface normal of  $S$ . Thus we write  $Z$  in the form

$$(3.13) \quad Z(w) = \mu(w)n(X(w))$$

with a scalar factor  $\mu(w)$  which is to be determined. Let us denote differentiation with respect to  $\varepsilon$  by  $\dot{\cdot}$ ,  $\partial/\partial\varepsilon = \dot{\cdot}$ . For fixed  $u \in I$  the mapping  $\varepsilon \rightarrow X_\varepsilon(u, 0)$  describes a curve on  $S$ , and thus we have necessarily

$$(3.14) \quad \dot{X}_\varepsilon(u, 0) \cdot n(X_\varepsilon(u, 0)) = 0 \quad \text{for } |\varepsilon| \ll 1.$$

Differentiating this equation with respect to  $\varepsilon$  we arrive at

$$(3.15) \quad \ddot{X}_\varepsilon(u, 0) \cdot n(X_\varepsilon(u, 0)) + \dot{X}_\varepsilon(u, 0) \cdot \nabla n(X_\varepsilon(u, 0)) \cdot \dot{X}_\varepsilon(u, 0) = 0,$$

and for  $\varepsilon=0$  it follows that

$$(3.16) \quad Z \cdot n(X) + \lambda N \cdot \nabla n(X) \cdot \lambda N = 0 \quad \text{on } I.$$

Because of (3.13) this relation is equivalent to

$$(3.17) \quad \mu = -\lambda^2 [N \cdot \nabla n(X) \cdot N] \quad \text{on } I.$$

Here  $\nabla n(x)$  denotes the Weingarten operator on the tangent space  $T_x S$  for  $x \in S$ , and the special structure of  $S$  yields that

$$(3.18) \quad N \cdot \nabla n(X) \cdot N = \kappa(X) [N \cdot \tau(X)]^2 \quad \text{on } I.$$

Thus we infer that

$$(3.19) \quad \mu = -\lambda^2 \kappa(X) [N \cdot \tau(X)]^2 \quad \text{on } I,$$

and therefore the correction term  $Z$  is of the form

$$(3.20) \quad Z = -\lambda^2 \kappa(X) [N \cdot \tau(X)]^2 n(X)$$

on  $I$ .

Conversely, if we define  $Z: \bar{B} \rightarrow \mathbf{R}^3$  by the right-hand side of (3.20), a simple

“flow argument” shows that we can construct an admissible variation  $X_\varepsilon = \tilde{X}(\cdot, \varepsilon)$  of  $X$  which is of type (3.6) or (3.6'). Summarizing we have found:

PROPOSITION 2. *Let  $X \in \mathcal{C}(\Gamma, S)$  be a stationary minimal surface in  $\langle \Gamma, S \rangle$ . Then, for any  $\lambda \in C^1(\bar{B})$  satisfying  $\lambda=0$  on  $C$ , there exists a differentiable one-parameter family  $\{X_\varepsilon\}_{|\varepsilon| < \varepsilon_0}$  of admissible variations*

$$(3.21) \quad X_\varepsilon(w) = X(w) + \varepsilon \lambda(w) N(w) + \frac{\varepsilon^2}{2} \lambda^2(w) T(w) + o(\varepsilon^2)$$

where  $T(w)$  satisfies

$$(3.22) \quad T = -\kappa(X)[N \cdot \tau(X)]^2 n(X) \quad \text{on } I.$$

The second variation

$$\bar{\delta}^2 A(X, \lambda) := \frac{d^2}{d\varepsilon^2} A(X_\varepsilon)|_{\varepsilon=0}$$

of the area functional with respect to these variations is given by

$$(3.23) \quad \bar{\delta}^2 A(X, \lambda) = \int_B (|\nabla \lambda|^2 + 2EK\lambda^2) du dv \\ + \int_I \kappa(X)[N \cdot \tau(X)]^2 \{X_v \cdot n(X)\} \lambda^2 du.$$

Since the expression  $\bar{\delta}^2 A(X, \lambda)$  is nonnegative for any  $\lambda \in C^1(\bar{B})$  with  $\lambda=0$  on  $C$  provided that  $X$  is area minimizing in  $\mathcal{C}(\Gamma, S)$ , the following terminology seems reasonable.

DEFINITION 1. A stationary minimal surface  $X: B \rightarrow \mathbf{R}^3$  in  $\mathcal{C}(\Gamma, S)$  is said to be freely stable if

$$(3.24) \quad \bar{\delta}^2 A(X, \lambda) \geq 0 \quad \text{for each } \lambda \in C^1(\bar{B}) \text{ with } \lambda=0 \text{ on } C$$

where  $\bar{\delta}^2 A(X, \lambda)$  is defined by (3.23).

Note that each freely stable minimal surface  $X \in \mathcal{C}(\Gamma, S)$  is necessarily stable since the boundary integral on the right-hand side of (3.23) vanishes, but the converse is in general not true. Thus free stability is a stronger property than stability, but each area-minimizing surface  $X$  in  $\mathcal{C}(\Gamma, S)$  is necessarily freely stable.

#### 4. The Gauss map of freely stable minimal surfaces.

Let  $N = (N^1, N^2, N^3): \bar{B} \rightarrow S^2 \subset \mathbf{R}^3$  be the Gauss map of a stationary minimal surface  $X = (X^1, X^2, X^3)$  in  $\mathcal{C}(\Gamma, S)$ , and let  $f: \bar{B} \rightarrow E = \mathbf{R}^2$  be the orthogonal projection of  $X$  into the plane  $E$ , that is,

$$(4.1) \quad f(w) = (X^1(w), X^2(w)) \quad \text{for } w=(u, v) \in \bar{B}.$$

The Jacobian  $J_f(w)$  of the planar map  $f$ ,

$$(4.2) \quad J_f = \det(\nabla X^1, \nabla X^2) = X_u^1 X_v^2 - X_v^1 X_u^2,$$

is connected with the third component  $N^3 = N \cdot e_3$  of  $N$  by the relation

$$(4.3) \quad J_f = EN^3.$$

LEMMA 1. *If  $\Gamma$  and  $\Sigma_0$  satisfy Condition (B), then we have  $f(\bar{I}) = \Sigma$ , and therefore  $f(\partial B) = \partial \mathcal{G}$ .*

The proof of this result essentially follows from the maximum principle for harmonic functions and from E. Hopf's boundary point lemma and can be taken from the proof of Proposition 2.1 in [5] or of Proposition 1 in [6].

The next lemma is more or less a special case of Satz 2 in [11, p. 49].

LEMMA 2. *If  $\Gamma$  and  $\Sigma_0$  satisfy Condition (B), then there are no branch points of  $X$  on the open circular arc  $\mathring{C} = C - \{\pm 1\}$ , and we have*

$$(4.4) \quad N^3 > 0 \quad \text{on } \mathring{C}.$$

PROOF. By the maximum principle we infer from Lemma 1 that  $f(\bar{B})$  is contained in the convex hull  $\mathcal{K}$  of  $\partial \mathcal{G}$ . Since  $\underline{\Gamma} = f(C)$  is convex with respect to  $\mathcal{G}$ , it follows that  $\underline{\Gamma} \subset \partial \mathcal{K}$ .

Consider now an arbitrary point  $w_0$  of  $\mathring{C}$  and let  $p_0 = f(w_0)$  be its image point on  $\underline{\Gamma} = \underline{\Gamma} - \{p_1, p_2\}$ . Then there is a linear function  $l: E \rightarrow \mathbf{R}$ ,

$$l(p) := a \cdot p + b, \quad p \in E,$$

with  $a \in \mathbf{R}^2$ ,  $|a|=1$ , and  $b \in \mathbf{R}$ , such that the line

$$\mathcal{L} := \{p \in E : l(p) = 0\}$$

is a support line of the convex set  $\mathcal{K}$  at  $p_0$ , in the sense that  $\mathcal{K}$  is contained in the halfspace  $\mathcal{H} := \{p \in E : l(p) \leq 0\}$  and  $l(p_0) = 0$ . Then the function  $\phi := l \circ f$  is continuous in  $\bar{B}$ , harmonic in  $B$ , and satisfies  $\phi(w_0) = 0$ ,  $\phi(w) \leq 0$  for all  $w \in \bar{B}$ , and  $\phi(w) \neq \text{const.}$ . From E. Hopf's lemma we infer that the normal derivative  $(\partial/\partial r)\phi(w_0)$  of  $\phi$  at  $w_0 \in \mathring{C}$  satisfies

$$(4.5) \quad \frac{\partial}{\partial r} \phi(w_0) > 0.$$

Here we use polar coordinates  $r, \phi$  about the origin, connected with  $u$  and  $v$  by  $u + iv = re^{i\theta}$ . Since  $\partial \phi / \partial r = a \cdot \partial f / \partial r$ , it follows that

$$(4.6) \quad |X_\theta(w_0)| = |X_r(w_0)| \geq |f_r(w_0)| > 0$$

whence  $E(w_0) > 0$ . Hence there are no branch points of  $X$  on  $\mathring{C}$ .

Now we set

$$\begin{aligned} \xi(\theta) &:= X^1(\cos \theta, \sin \theta), \\ \eta(\theta) &:= X^2(\cos \theta, \sin \theta), \\ \zeta(\theta) &:= X^3(\cos \theta, \sin \theta) = \gamma(\xi(\theta), \eta(\theta)), \quad 0 \leq \theta \leq \pi. \end{aligned}$$

Then we have

$$\zeta_\theta = \gamma_x(\xi, \eta)\xi_\theta + \gamma_y(\xi, \eta)\eta_\theta$$

and for some suitable constant  $c$  it follows that

$$\zeta_\theta^2 \leq c[\xi_\theta^2 + \eta_\theta^2]$$

whence

$$(4.7) \quad |X_\theta(\cos \theta, \sin \theta)|^2 \leq (1+c)[\xi_\theta^2(\theta) + \eta_\theta^2(\theta)].$$

Hence we obtain that

$$(4.8) \quad |f_\theta(w_0)| > 0.$$

Finally we have

$$(4.9) \quad \phi_\theta(w_0) = a \cdot f_\theta(w_0) = 0$$

since  $\phi|_c$  assumes its maximum at  $w=w_0$ , while (4.5) implies that

$$(4.10) \quad \phi_r(w_0) = a \cdot f_r(w_0) > 0.$$

From (4.6), (4.8)-(4.10) we derive that

$$\det(f_r, f_\theta) > 0 \quad \text{on } \mathring{C}$$

since  $f(\cos \theta, \sin \theta) = (\xi(\theta), \eta(\theta))$  yields a parametrization of  $\underline{\Gamma}$  which is positively oriented with respect to  $\underline{g}$ , and this implies  $J_f > 0$  on  $\mathring{C}$ .  $\square$

Adapting the reasoning of [11], Hilfssatz 6 and Hilfssatz 7 to freely stable minimal surfaces we now obtain

**PROPOSITION 3.** *Let  $X$  be a stationary minimal surface in  $\mathcal{C}(\Gamma, S)$  which is freely stable, and suppose that  $\Gamma, \Sigma_0$  satisfy Condition (B). Then the third component  $N^3 = N \cdot e_3$  of the Gauss map  $N$  of  $X$  satisfies*

$$(4.11) \quad N^3(w) > 0 \quad \text{for all } w \in \bar{B} - \{\pm 1\}.$$

*Moreover, there are no branch points of  $X$  in  $\bar{B} - \{\pm 1\}$ , and we have  $J_f(w) > 0$  on  $\bar{B} - \{\pm 1\}$ . Finally,  $f$  yields a homeomorphism of  $\bar{B}$  onto  $\bar{g}$  and a diffeomorphism of  $B$  onto  $\underline{g}$ .*

PROOF. Since  $X$  is assumed to be freely stable, property (3.24) holds true. Then a well-known approximation device leads to the relation

$$(4.12) \quad \bar{\delta}^2 A(X, \lambda) \geq 0 \quad \text{for all } \lambda \in H^{1,2}(B) \cap L^\infty(B) \text{ satisfying } \lambda|_C = 0.$$

Set  $\omega := N^3$  and define  $\omega^-$  by

$$\omega^-(u, v) := \begin{cases} \omega(u, v) & \text{if } \omega(u, v) < 0 \\ 0 & \text{if } \omega(u, v) \geq 0. \end{cases}$$

Then we have  $\omega^- \in H^{1,2}(B) \cap C^0(\bar{B})$ , and Lemma 2 implies that

$$(4.13) \quad \text{supp } \omega^- \subset B \cup \bar{I}$$

whence in particular

$$(4.14) \quad \omega^- = 0 \quad \text{on } C.$$

Let us introduce the open subsets  $B^+$  and  $B^-$  of  $B$  where  $\omega(u, v)$  is positive or negative respectively. By Lemma 2, the set  $B^+$  is nonempty. We now want to show that, therefore,  $B^-$  has to be void. To this end we first note that a well-known approximation device leads (via integration by parts) to the identity

$$\begin{aligned} & \int_B (|\nabla \omega^-|^2 + 2EK|\omega^-|^2) du dv \\ &= \int_{B^-} \omega^- [-\Delta \omega + 2EK\omega] du dv - \int_{I \cap \partial B^-} \omega \omega_\nu du. \end{aligned}$$

By virtue of (2.3) we obtain

$$\int_B (|\nabla \omega^-|^2 + 2EK|\omega^-|^2) du dv = - \int_I \omega^- \omega_\nu du,$$

and thus we derive from (3.23) that

$$\bar{\delta}^2 A(X, \omega^-) = - \int_I \omega^- [\omega_\nu - \kappa(X)(N \cdot \tau(X))^2 (X_\nu \cdot n(X)) \omega] du.$$

Then we infer from (2.15) that

$$(4.15) \quad \bar{\delta}^2 A(X, \omega^-) = 0.$$

Let now  $\varphi$  be an arbitrary function of class  $C_c^\infty(B)$  and set  $\lambda = \omega^- + \varepsilon\varphi$ ,  $\varepsilon \in \mathbf{R}$ . Then  $\lambda \in H^{1,2} \cap L^\infty(B)$  and  $\lambda = 0$  on  $C$ , and we conclude on account of (4.12) that  $\Phi(\varepsilon) := \bar{\delta}^2 A(X, \omega^- + \varepsilon\varphi) \geq 0$ , whereas (4.15) implies that  $\Phi(0) = 0$ . Thus we obtain  $\Phi'(0) = 0$ , which means that

$$\int_B (\nabla \omega^- \cdot \nabla \varphi + 2EK\omega^- \varphi) du dv = 0$$

for all  $\varphi \in C_c^\infty(B)$ . Since  $EK$  is real analytic on the open set  $B$ , it follows that also  $\omega^-$  is real analytic on  $B$ . On the other hand  $\omega^-$  vanishes on the nonempty open set  $B^+$ , and thus  $\omega^-$  has to vanish on  $B$ . Therefore  $B^-$  is empty which means that  $\omega(u, v) \geq 0$  on  $B$ . Then we infer from (2.3) that

$$(4.16) \quad N^3(u, v) > 0 \quad \text{on } B.$$

Then we infer from the asymptotic expansions of  $X(w)$  at points  $w_0 \in B$  that each  $f(w_0)$  must be an inner point of  $f(B)$ . (In fact, for a sufficiently small disk  $B_\varepsilon(w_0)$  in  $B$ , the image set  $X(B_\varepsilon(w_0))$  is in first order an  $m$ -fold copy of a disk,  $m \geq 1$ , and by (4.16) the set  $f(B_\varepsilon(w_0))$  is in first order an  $m$ -fold copy of a solid ellipse centered at  $f(w_0)$ .) Thus the mapping  $f|_B$  is open, and Lemma 1 yields  $f(\partial B) = \partial \mathcal{G}$ . We then conclude that  $f(B) = \mathcal{G}$ , and so  $X(B)$  lies on the same side of  $S$  as  $\Gamma$ . On account of asymptotic expansions of  $X$  at boundary branch points we infer that no branch points of  $X$  lie in  $I$ , that is,  $I' = I$  and

$$(4.17) \quad W = E > 0 \quad \text{on } I.$$

By the reasoning used in the proof of Proposition 4.4 of [5] we then infer that

$$(4.18) \quad J_f(u, 0) > 0 \quad \text{for all } u \in I$$

and in particular  $|f_u(u, 0)| > 0$  on  $I$ . It follows that  $f|_{\bar{I}}$  provides a topological mapping of  $\bar{I}$  onto  $\Sigma$ . Moreover, it is well known that  $X|_C$  yields a topological mapping of  $C$  onto  $\Gamma$ , and therefore  $f|_C$  furnishes a topological mapping of  $C$  onto  $\underline{\Gamma}$ . Then we conclude that  $f|_{\partial B}$  yields a one-to-one map of  $\partial B$  onto  $\partial \mathcal{G}$ , and we also had  $f(B) = \mathcal{G}$ . By employing H. Kneser's artifice [8] we infer that  $f$  yields a homeomorphism of  $\bar{B}$  onto  $\bar{\mathcal{G}}$ , and that  $f|_B$  is a diffeomorphism of  $B$  onto  $\mathcal{G}$ ; in particular,  $J_f(w) > 0$  for  $w \in B$  (see [5], Proposition 4.2). Thus  $X$  has no branch points in  $\bar{B} - \{\pm 1\}$ . This completes the proof of Proposition 3.  $\square$

## 5. Uniqueness and existence results.

Now we are prepared to prove the following uniqueness result.

**THEOREM 1.** *Let  $\langle \Gamma, S \rangle$  be a projecting boundary configuration as described in Section 1, with the additional property that the orthogonal projections  $\underline{\Gamma}$  and  $\Sigma_0$  of  $\Gamma$  and  $S$  into the plane  $E$  satisfy Condition (B). Then, up to reparametrization, there is exactly one freely stable and stationary minimal surface  $X: B \rightarrow \mathbf{R}^3$  in the configuration  $\langle \Gamma, S \rangle$ . This surface is the unique minimizer of Dirichlet's integral in the class  $\mathcal{C}(\Gamma, S)$  as well as of the area functional, and it can be represented as a graph of a scalar function  $z = \zeta(x, y)$  of class  $C^0(\bar{\mathcal{G}}) \cap C^2(\bar{\mathcal{G}} - \{p_1, p_2\})$  which satisfies the minimal surface equation*

$$(5.1) \quad (1 + \zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1 + \zeta_x^2)\zeta_{yy} = 0 \quad \text{in } \mathcal{G}$$

and fulfils the boundary conditions

$$(5.2) \quad \zeta = \gamma \quad \text{on } \underline{\Gamma} \quad \text{and} \quad \frac{\partial \zeta}{\partial \nu} = 0 \quad \text{on } \overset{\circ}{\Sigma}.$$

Here we have written  $x, y, z$  for  $x^1, x^2, x^3$ , and  $\overset{\circ}{\Sigma}$  denotes the open arc  $\Sigma - \{p_1, p_2\}$ .

PROOF. Courant's existence theorem furnishes the existence of a minimizer  $X$  of Dirichlet's integral in the class  $\mathcal{C}(\Gamma, S)$  which then is also area minimizing in  $\mathcal{C}(\Gamma, S)$  (see, e.g. [3, Chapter 4]). This minimizer then is seen to be a stationary minimal surface in the configuration  $\langle \Gamma, S \rangle$ , which is also freely stable. Therefore we can apply Proposition 3 to  $X$  and obtain that  $\zeta := X^3 \circ f^{-1}$  is continuous on  $\bar{\mathcal{G}}$ , smooth (of class  $C^2$ ) in  $\bar{\mathcal{G}} - \{p_1, p_2\}$  and satisfies (5.1) and (5.2). Moreover, if  $\tilde{X}$  is another stationary minimal surface spanning  $\langle \Gamma, S \rangle$  which is freely stable, then the corresponding function  $\tilde{\zeta} := \tilde{X}^3 \circ \tilde{f}^{-1}$  has the same properties as  $\zeta$ . A standard uniqueness argument invoking E. Hopf's lemma yields that  $\zeta(x, y) \equiv \tilde{\zeta}(x, y)$ . Therefore  $X$  and  $\tilde{X}$  represent the same minimal surface, i.e., there is a conformal mapping  $\alpha$  of  $B$  onto itself, extending to a homeomorphism of  $\bar{B}$  onto itself, such that  $\tilde{X} = X \circ \alpha$ . (For details we refer to [5, p. 86]). □

REMARK 1. The usual existence proof for minimizers in  $\mathcal{C}(\Gamma, S)$  does not guarantee that minimizers lie on the same side of  $S$  as  $\Gamma$ , i.e., that they are physically realistic. Only if  $S$  is convex towards  $\Gamma$ , it can easily be seen by means of the maximum principle that no stationary minimal surface in  $\langle \Gamma, S \rangle$  can penetrate  $S$ .

We can use Theorem 1 to derive a new existence result. For this purpose we formulate

CONDITION (A). *The normal lines  $\mathcal{L}(s_1)$  and  $\mathcal{L}(s_2)$  of the curve  $\Sigma_0$  at  $p_1$  and  $p_2$  respectively do not meet  $\mathcal{G}$ .*

THEOREM 2. *Let  $\langle \Gamma, S \rangle$  be a projecting boundary configuration as described in Section 1 satisfying also Condition (A). Then there exists a solution  $z = \zeta(x, y)$  of the minimal surface equation (5.1) which is of class  $C^0(\bar{\mathcal{G}}) \cap C^2(\mathcal{G} \cup \overset{\circ}{\Sigma})$  and satisfies the mixed boundary condition (5.2).*

PROOF. Let  $\Sigma_1$  and  $\Sigma_2$  be the two components of  $\Sigma_0 - \Sigma$ . Then Condition (A) implies that we can find two arcs  $\tilde{\Sigma}_1$  and  $\tilde{\Sigma}_2$  such that  $\tilde{\Sigma}_0 := \tilde{\Sigma}_1 \cup \Sigma \cup \tilde{\Sigma}_2$  has the same behaviour as  $\Sigma_0$  and that the pair  $\underline{\Gamma}, \tilde{\Sigma}_0$  satisfies Condition (B). By applying Theorem 1 to the boundary configuration  $\langle \Gamma, \tilde{S} \rangle$  with the support

surface  $\tilde{S} := \tilde{\Sigma}_0 \times \mathbf{R}$  having the new curve  $\tilde{\Sigma}_0$  as directrix we obtain the desired result. □

REMARK 2. This result is much stronger than Theorem 3 of [6] since we do not need Condition (B2) required in that theorem, but only the fairly weak assumption (A). Moreover, a standard approximation device shows that we only need to require continuity of the boundary values  $\gamma: \underline{\Gamma} \rightarrow \mathbf{R}$ .

Let us also mention that by entirely different methods (cf. reference [7] in our paper [6]) E. Giusti has solved the mixed boundary value problem (5.2) for nonparametric minimal surfaces (and, more generally, for  $H$ -surfaces) provided that the domain  $\mathcal{Q}$  is convex. In contrast, we have only to assume that the part  $\underline{\Gamma}$  of  $\mathcal{Q}$  is convex where Dirichlet data of the solution are prescribed, while  $\Sigma$  may be nonconvex.

REMARK 3. In general the uniqueness result of Theorem 1 becomes false if we drop the assumption that  $\underline{\Gamma}$  and  $\Sigma_0$  satisfy Condition (B). This can be seen by means of the following example. Let  $\langle \Gamma, S \rangle$  be a boundary configuration whose orthogonal projection into the  $x^1, x^2$ -plane looks as in Fig. 1, while the orthogonal projection  $\Gamma^*$  of  $\Gamma$  into the  $x^2, x^3$ -plane has a shape as depicted in Fig. 2. Inspecting Fig. 1 it is evident that the pair  $\underline{\Gamma}, \Sigma_0$  does satisfy Condition (A) but not (B). If we choose  $a, h$  and  $\varepsilon$  in an appropriate way, then there is a disk-type surface spanning  $\langle \Gamma, S \rangle$  whose area is less than  $2ah$  and therefore also less than the area of the domain  $\mathcal{Q}^*$  in the  $x^2, x^3$ -plane whose boundary consists of the arc  $\Gamma^*$  and the segment of the  $x^3$ -axis between  $P_1^*$  and  $P_2^*$ . On the other hand  $\text{meas } \mathcal{Q}^*$  is certainly a lower bound for the area of the nonparametric minimal surface which, according to Theorem 2, exists as graph of a solution  $\zeta$  of the boundary value problem (5.1), (5.2) above the domain  $\mathcal{Q}$ . In our case it is not difficult to see that this surface is a freely stable minimal surface which is stationary in  $\langle \Gamma, S \rangle$ , and so there exist at least two freely stable solutions in  $\langle \Gamma, S \rangle$ . By assuming that  $\langle \Gamma, S \rangle$  is symmetric with respect to the  $x^2$ -axis we even obtain that there exist at least three freely stable minimal surfaces stationary in  $\langle \Gamma, S \rangle$ .

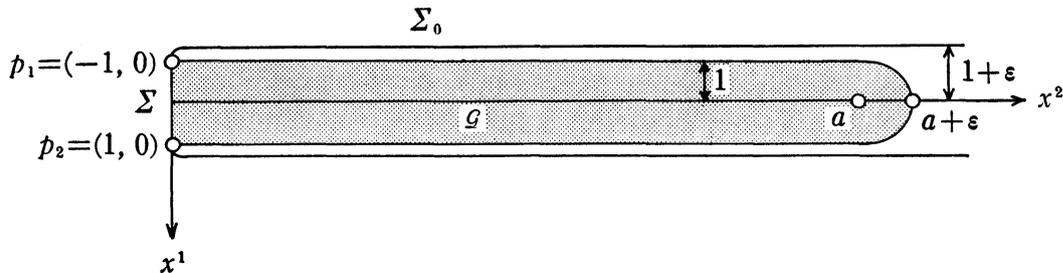


Fig. 1. The projection of  $\langle \Gamma, S \rangle$  on the  $x^1, x^2$ -plane.

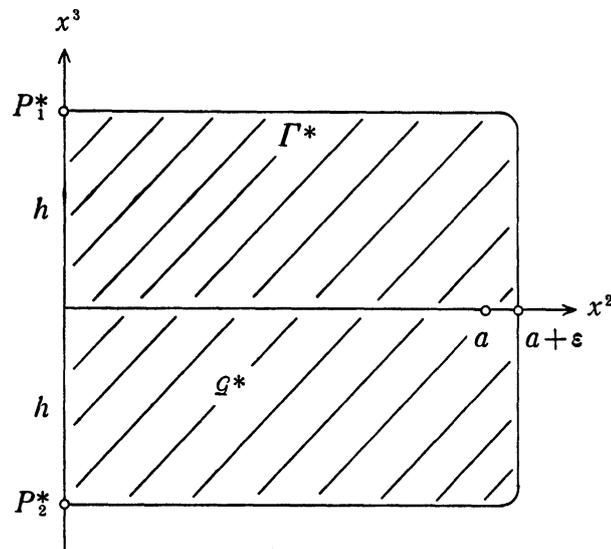


Fig. 2. The projection  $\Gamma^*$  of  $\Gamma$  on the  $x^2, x^3$ -plane.

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