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Hausdorff dimensions of self-similar sets and shortest path metrics

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Introduction.

Let (X, d) be a complete metric space and let $f_1, f_2, \cdots, f_N$ be contractions from X to itself, that is,

$$\sup_{x,y\in X}\frac{d(f_i(x), f_i(y))}{d(x, y)}<1.$$

Then it follows that

THEOREM 1.1 (Hutchinson [**Hu**]). *There exists a unique non-empty compact set K such that*

$$K=\bigcup_{i=1}^{N} f_i(K).$$

K is called a self-similar set with respect to $((X, d), \{f_i\}_{i=1}^{N})$.

This paper contains two main subjects. First in §2, we will study the Hausdorff dimension of a self-similar set. For the case that X is an Euclidian space $\boldsymbol{R}^n$ and f_i's are similitudes, there is a well-known result by Moran [**M**].

THEOREM 1.2. *Let $X=\boldsymbol{R}^n$ and let f_i be an r_i-similitude of $\boldsymbol{R}^n$ for $i=1, 2, \cdots, N$; that is, for all $x, y\in\boldsymbol{R}^n$,*

$$d(f_i(x), f_i(y))=r_i d(x, y),$$

where d is the ordinary Euclidean distance on $\boldsymbol{R}^n$. If there exists an open set $O\subset\boldsymbol{R}^n$ such that

$$\bigcup_{i=1}^{N} f_i(O)\subset O \quad \text{and} \quad f_i(O)\cap f_j(O)=\emptyset \qquad \text{for } i\neq j,$$

then the Hausdorff dimension of the self-similar set (K, d) with respect to $((\boldsymbol{R}^n, d), \{f_i\}_{i=1}^{N})$ is the unique number α that satisfies

$$\sum_{i=1}^{N} r_i^{\alpha}=1. \tag{1.1}$$

Furthermore, $0<\mathcal{H}^\alpha(K)<\infty$, *where* $\mathcal{H}^\alpha$ *is the* α*-dimensional Hausdorff measure.*

REMARK. It is not known whether the converse of the above theorem is true or not. Recently, Bandt and Graf [**BG**] obtained a necessary and sufficient condition for the consequence of the above theorem.

The condition on the open set O in the above theorem is called the open set condition by Hutchinson [**Hu**].

By this result we can calculate the Hausdorff dimensions of typical self-similar sets, for example, the Sierpinski gasket, the Sierpinski carpet and the Koch curve.

Recently, however, some metrics on self-similar sets that are different from the restriction of Euclidean metrics are introduced from the viewpoint of analysis on self-similar sets such as the interior distance by Bandt et al. [**BS, BK**] and the effective resistance metric by Kigami [**Ki3**]. In [**Ki3**], the self-similar sets are not embedded in Euclidean spaces and the contractions are no longer similitudes under effective resistance metrics in general. In these cases, the open set condition does not work, because in the proof of Theorem 1.2, we should use some special properties of the Euclidean spaces.

In the present paper we will give a result on the Hausdorff dimension of self-similar sets under general metric. For the simplest case, our result is

COROLLARY 1.3. *Let* K *be a self-similar set with respect to* $((X, d), \{f_i\}_{i=1}^N)$. *If there exist constants* $0<r<1$, $0<c_1$, c_2 *and* $M>0$ *such that*

(1) *for all* $w=w_1w_2\cdots w_m\in\{1, 2, \cdots, N\}^m$,

$$d(K_w)\leqq c_1r^m\,,$$

where $K_w=f_w(K)$, $f_w=f_{w_1}\circ f_{w_2}\circ\cdots\circ f_{w_m}$ *and* $d(A)=\sup_{x,y\in A}d(x, y)$ *for* $A\subset K$,

(2) *for all* $x\in K$ *and all* $m\geqq 0$,

$$\#\{w : w\in\{1, 2, \cdots, N\}^m,\ d(x, K_w)\leqq c_2r^m\}\leqq M\,,$$

where $d(x, K_w)=\inf_{y\in K_w}d(x, y)$, *then for* $\alpha=-\log N/\log r$,

$$0<\mathcal{H}^\alpha(K)<\infty\,,$$

where $\mathcal{H}^\alpha$ *is the* α*-dimensional Hausdorff measure. Especially, the Hausdorff dimension of the compact metric space* (K, d) *is* $-\log N/\log r$.

We will state the complete version of our main theorem in §2. Our main theorem, Theorem 2.4 is used to calculate Hausdorff dimensions of self-similar sets under the effective resistance metrics in [**Ki3**].

Of course, our result covers the case of ordinary self-similar sets with the Euclidean metrics. In general, it is shown in Proposition 2.8 that our main result, Theorem 2.4 includes Theorem 1.2 as a special case. Furthermore, it is easy to verify directly the conditions (1) and (2) in Corollary 1.3 for well-known self-similar sets as the Sierpinski gasket, the Sierpinski carpet and the Koch curve. Moreover, we can apply the above result to the Lévy curve defined by Lévy [**Le**], which is a self-similar set with respect to $(\boldsymbol{C}, \{f_1, f_2\})$ where

$$f_1(z) = \frac{1+i}{2}z \quad \text{and} \quad f_2(z) = \frac{1-i}{2}z + \frac{1+i}{2}.$$

See Figure 1. It is known that $0 < \mathcal{H}^2(K) < \infty$ and so $\dim_H K = 2$. In this case, however, it is quite difficult to find an open set that satisfies the open set condition. See the appendix for details.

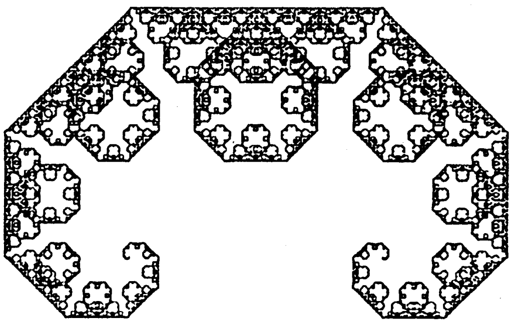


Figure 1. Lévy curve.

The second subject is shortest path metrics on p. c. f. self-similar sets. The notion of p. c. f. self-similar sets is a mathematical formulation of finitely ramified self-similar sets and it includes nested fractals introduced by Lindstrøm [**Li**]. ("p. c. f." is an abbreviation of "post critically finite".) In § 3, we give the definition of p. c. f. self-similar sets. See also the examples in § 4.

DEFINITION 1.4. Let (X, d) be a metric space. Then a continuous injection $g: [0, d(p, q)] \to X$ is called a geodesic between p and q if and only if, for all $0 \leqq s \leqq t \leqq d(p, q)$,

$$d(g(s), g(t)) = t - s.$$

A metric d is called a shortest path metric if and only if, for all $p, q \in X$, there exists a geodesic between p and q.

We will study a special kind of shortest path metrics on p. c. f. self-similar sets where the contraction mappings become similitudes. Precisely let K be a self-similar sets with respect to $((X, \rho), \{f_i\}_{i=1}^N)$, we focus on a metric d on K that satisfies

(B.1) $\qquad d$ is a shortest path metric

and there exist $0 < r_1, r_2, \cdots, r_N < 1$ such that

(B.2) $\qquad d(f_i(x), f_i(y)) = r_i d(x, y)$

for all $i=1, 2, \cdots, N$ and for all $x, y \in K$.

The interior distances introduced by Bandt et al. [**BK, BS**] are metrics that satisfies (B.1), (B.2) and some additional conditions. We will show in Theorem 3.2 that the Hausdorff dimension of a p. c. f. self-similar set under a metric which satisfies (B.1) and (B.2) is given by the unique positive number α such that

$$\sum_{i=1}^{N} r_i{}^{\alpha} = 1 .$$

In Theorem 4.3, we will give a necessary and sufficient condition for existence of a metric on p. c. f. self-similar sets which satisfy (B.1) and (B.2).

In the rest of this section we will recall the definitions of Hausdorff measures and Hausdorff dimensions.

DEFINITION 1.5. Let (X, d) be a metric space and A is a subset of X. Then we define

$$\mathcal{H}_{\delta}^{\alpha}(A) = \inf_{\{U_i\} \text{ is a } \delta\text{-covering of } A} \sum_i d(U_i)^{\alpha} ,$$

where $d(U)=\sup_{x, y \in U} d(x, y)$ is the diameter of the set U and $\{U_i\}$ is a δ-covering of A if and only if

$$\bigcup_i U_i \supset A \quad \text{and} \quad d(U_i) \leqq \delta .$$

Also the α-dimensional Hausdorff measure of A, $\mathcal{H}^{\alpha}(A)$ is defined by

$$\mathcal{H}^{\alpha}(A) = \lim_{\delta \to 0} \mathcal{H}_{\delta}^{\alpha}(A) .$$

It is well-known that $\mathcal{H}^{\alpha}$ becomes a Borel measure on X. See Rogers [**R**] or Falconer [**F**].

DEFINITION 1.6. For $A \subset X$, the Hausdorff dimension of A with respect to the metric d denoted by $\dim_H(A, d)$ is defined by

$$\dim_H(A, d) = \sup\{\alpha : \mathcal{H}^{\alpha}(A)=\infty\} = \inf\{\alpha : \mathcal{H}^{\alpha}(A)=0\} .$$

In particular, if $0<\mathcal{H}^{\alpha}(A)<\infty$, then $\dim_H(A, d)=\alpha$.

§ 2. Hausdorff dimension of self-similar sets.

In this section, we will state and prove our main result on Hausdorff dimensions of self-similar sets.

First, we introduce the notion of self-similar structure, which is a purely topological formulation of self-similar sets defined by [**Ki2**].

DEFINITION 2.1. Let K be a compact metrizable topological space and S be a finite set[1]. Also, let F_i, for $i \in S$, be a continuous injection from K to itself. Then, $(K, S, \{F_i\}_{i\in S})$ is called a self-similar structure if there exists a continuous surjection $\pi: \Sigma \to K$, where $\Sigma = S^N$ is the one-sided shift space, such that $F_i \circ \pi = \pi \circ i$ for every $i \in S$, where $i: \Sigma \to \Sigma$ is defined by $i(w_1w_2w_3\cdots) = iw_1w_2w_3\cdots$.

NOTATION. $W_m = S^m$ is the collection of words with length m. For $w = w_1w_2\cdots w_m \in W_m$, we define $F_w: K \to K$ by $F_w = F_{w_1}\circ F_{w_2}\circ\cdots\circ F_{w_m}$ and $K_w = F_w(K)$. Also we define

$$W_* = \bigcup_{m\geq 0} W_m .$$

If $(K, S, \{F_i\}_{i=S})$ is a self-similar structure, then it was shown in [**Ki2**] that for all $\omega = \omega_1\omega_2\cdots \in \Sigma$,

$$\bigcap_{m\geq 1} K_{\omega_1\omega_2\cdots\omega_m} = \{\pi(\omega)\},$$

And so, the map π is uniquely determined. Conversely, it is easy to see that

PROPOSITION 2.2. *$(K, S, \{F_i\}_{i\in S})$ is a self-similar structure if and only if K is a compact metrizable space, F_i's are continuous, $K = \bigcup_{i=1}^N F_i(K)$ and $\bigcap_{m\geq 1} K_{\omega_1\omega_2\cdots\omega_m}$ consists of a single point for all $\omega = \omega_1\omega_2\cdots \in \Sigma$.*

By the above proposition, a self-similar set in the sense of Theorem 1.1 is a self-similar structure. A self-similar structure is, however, purely topological object and so without specifying a metric d on K that is compatible with the original topology of K, we cannot think about its Hausdorff dimension. For example, let f_1^β and f_2^β be contractions from $\boldsymbol{C}$ to itself defined by

$$f_1^\beta(z) = \frac{1+\beta i}{2}\bar{z} \quad\text{and}\quad f_2^\beta(z) = \frac{1-\beta i}{2}\bar{z} + \frac{1+\beta i}{2},$$

where $0 \leq \beta < 1$, then the corresponding self-similar sets $K(\beta)$ have different Hausdorff dimensions under the Euclidean metric on $\boldsymbol{C}$. Precisely $\dim_H K(\beta) = -\log 2/\log(\sqrt{1+\beta^2}/2)$. Especially $K(0) = [0, 1]$ and $K(\sqrt{3}/6)$ is the Koch curve. Now let $p_\beta: K(0) \to K(\beta)$ be the natural parametrization and let $\pi_\beta: \{1, 2\}^N \to K(\beta)$ be the natural map defined by

$$\{\pi_\beta(\omega)\} = \bigcap_{m\geq 0} K_w(\beta)$$

then p_β is obviously a homeomorphism and we have $p_\beta \circ \pi_0 = \pi_\beta$ and, for $i = 1, 2$, $f_i^\beta \circ p_\beta = p_\beta \circ f_i^0$ on $K(0)$. In this manner, we can identify $(K(\beta), \{1, 2\}, \{f_1^\beta, f_2^\beta\})$ with $(K(0), \{1, 2\}, \{f_1^0, f_2^0\})$ as a self-similar structure.

Hereafter, we will fix a self-similar structure $(K, \{1, 2, \cdots, N\}, \{F_i\}_{i=1}^N)$.

DEFINITION 2.3. For $\boldsymbol{r} = (r_1, r_2, \cdots, r_N)$ where $0 < r_i < 1$ and for $0 < a < 1$,

[1] In this paper, $S = \{1, 2, \cdots, N\}$ except for Appendix.

$$\Lambda(\boldsymbol{r}, a) = \{w : w = w_1 w_2 \cdots w_m \in W_*,\ r_{w_1 w_2 \cdots w_{m-1}} > a \geqq r_w\},$$

where $r_v = r_{v_1} r_{v_2} \cdots r_{v_k}$ for $v = v_1 v_2 \cdots v_k \in W_k$.

Now the following is our main theorem.

THEOREM 2.4. *Let d be a metric on K which is compatible with the original topology of K. If there exist $\boldsymbol{r} = (r_1, r_2, \cdots, r_N)$ where $0 < r_i < 1$ and positive constants c_1, c_2, c_* and M such that*

$$d(K_w) \leqq c_1 r_w \tag{A.1}$$

for all $w \in W_*$ and

$$\#\{w : w \in \Lambda(\boldsymbol{r}, a),\ d(x, K_w) \leqq c_2 a\} \leqq M \tag{A.2}$$

for any $x \in K$ and any $a \in (0, c_*)$, then, there exist constants $0 < c_3$, c_4 such that for all $w \in W_*$,

$$c_3 r_w{}^\alpha < \mathcal{H}^\alpha(K_w) < c_4 r_w{}^\alpha$$

where α is the unique positive number that satisfies

$$\sum_{i=1}^N r_i{}^\alpha = 1\,.$$

In particular, $0 < \mathcal{H}^\alpha(K) < \infty$ and $\dim_H(K, d) = \alpha$.

REMARK 1. If the conditions (A.1) and (A.2) in Theorem 2.4 hold, then it is easy to see that, for any $x \in K$,

$$\#(\pi^{-1}(x)) \leqq M\,.$$

REMARK 2. By the proofs of Lemma 2.5 and Theorem 2.4, we can easily see that the assumptions of the above theorem, (A.1) and (A.2), can be replaced by the following weaker assumptions (A.1a) and (A.2a). Let c and a be positive constants less than 1.

$$d(K_w) \leqq c_1 a^n \tag{A.1a}$$

for all $n \geqq 0$ and $w \in \Lambda(\boldsymbol{r}, ca^n)$,

$$\#\{w : w \in \Lambda(\boldsymbol{r}, ca^n),\ d(x, K_w) \leqq c_2 a^n\} \leqq M \tag{A.2a}$$

for any $x \in K$ and $n \geqq 0$.

Corollary 1.3 is the case when $r = r_1 = r_2 = \cdots = r_N$ and $a = r$. In several points, the proof of the above theorem depends on the same ideas as Moran's proof of Theorem 1.2. For convenience of the readers, however, we will give the whole proof. First we will recall the following well-known lemma about Hausdorff measures. See Moran [**M**] and also Hutchinson [**Hu**].

LEMMA 2.5. *Let (K, d) be a compact metric space. If $\mathcal{H}^{\alpha}(K)<\infty$ and there exists a probability measure μ on K such that, for a constant $c>0$,*

$$\mu(B(x, l)) \leqq cl^{\alpha}$$

for all $x\in K$ and sufficiently small $l>0$, where $B(x, l)=\{y : y\in K, d(x, y)\leqq l\}$, then for each Borel set $A\subset K$,

$$\mu(A) \leqq c\mathcal{H}^{\alpha}(A) .$$

In particular, $0<\mathcal{H}^{\alpha}(K)<\infty$.

REMARK. According to the discussion of Moran [**M**], the converse of the above lemma is true: If $0<\mathcal{H}^{\alpha}(K)<\infty$, then there exists a probability measure μ on K such that, for some $c>0$,

$$\mu(B(x, l)) \leqq cl^{\alpha}$$

for all $x\in K$ and $l>0$.

PROOF OF LEMMA 2.5. For $U\subset K$ and $x\in U$, note that $U\subset B(x, d(U))$. Let $\{U_i\}$ be a covering of a Borel set $A\subset K$, then

$$\sum_i \mu(B(x_i, d(U_i))) \leqq c \sum_i d(U_i)^{\alpha} ,$$

where $x_i\in U_i$. As $A\subset\bigcup_i B(x_i, d(U_i))$, we have

$$\mu(A) \leqq c \sum_i d(U_i)^{\alpha} .$$

Hence by Definition 1.5, $\mu(A)\leqq c\mathcal{H}^{\alpha}_l(A)$. Letting $l\to 0$, it follows that $\mu(A)\leqq c\mathcal{H}^{\alpha}(A)$.

For the proof of Theorem 2.4, we also need some facts on the one-sided shift space Σ.

DEFINITION 2.6. A subset $\Lambda\subset W_*$ is called a partition if and only if

$$\bigcup_{w\in\Lambda} \Sigma_w = \Sigma \quad \text{and} \quad \Sigma_w\cap\Sigma_v = \emptyset \qquad \text{for } w\neq v\in\Lambda ,$$

where, for $w=w_1w_2\cdots w_m\in W_m$,

$$\Sigma_w = \{\omega : \omega=\omega_1\omega_2\cdots\in\Sigma,\ \omega_1\omega_2\cdots\omega_m=w_1w_2\cdots w_m\}$$

LEMMA 2.7. *For $a_1, a_2, \cdots, a_N\geqq 0$ satisfying $\sum_{i=1}^N a_i=1$, if Λ be a partition, then*

$$\sum_{w_1w_2\cdots w_m\in\Lambda} a_{w_1}a_{w_2}\cdots a_{w_m} = 1 .$$

The proof of the above lemma is exactly the same as that of Lemma 2.7 of [**KL**].

PROOF OF THEOREM 2.4. We write $\Lambda_a=\Lambda(\boldsymbol{r}, a)$ and $R=\min\{r_1, r_2, \cdots, r_N\}$. First we will show that $\mathcal{H}^{\alpha}(K_w)\leqq(c_1/R)^{\alpha}r_w{}^{\alpha}$ for all $w\in W_*$.

For $w=w_1w_2\cdots w_m\in W_*$, we define

$$\Lambda_a(w)=\{v=v_1v_2\cdots v_k : wv\in\Lambda_a\},$$

where $wv=w_1w_2\cdots w_mv_1v_2\cdots v_k$. Then we can see that $\Lambda_a(w)$ is a partition for sufficiently small a. Hence by Lemma 2.7,

$$r_w{}^{\alpha}=\sum_{v\in\Lambda_a(w)} r_{wv}{}^{\alpha}. \tag{2.1}$$

As $\{K_{wv}\}_{v\in\Lambda_a(w)}$ is a c_1a-covering of K_w, we have

$$\mathcal{H}^{\alpha}_{c_1a}(K_w)\leqq c_1{}^{\alpha}\sum_{v\in\Lambda_a(w)} a^{\alpha}.$$

Note that $r_{wv}>aR$ for $v\in\Lambda_a$, it follows

$$\mathcal{H}^{\alpha}_{c_1a}(K)\leqq(c_1/R)^{\alpha}\sum_{v\in\Lambda_a(w)} r_{wv}{}^{\alpha}.$$

Using (2.1) and letting $a\to 0$, we obtain

$$\mathcal{H}^{\alpha}(K_w)\leqq(c_1/R)^{\alpha}r_w{}^{\alpha}.$$

Next we show that $r_w^{\alpha}\leqq Mc_2^{-\alpha}\mathcal{H}^{\alpha}(K_w)$. Let $\tilde{\mu}$ be the unique probability Borel measure on Σ satisfying

$$\tilde{\mu}(\Sigma_w)=r_w{}^{\alpha}.$$

Then we define a probability Borel measure μ on K by, for any Borel set $A\subset K$,

$$\mu(A)=\tilde{\mu}(\pi^{-1}(A)).$$

Now for every $x\in K$,

$$\pi^{-1}(B(x, c_2a))\subset\bigcup_{w\in\Lambda_{a,x}}\Sigma_w,$$

where $\Lambda_{a,x}=\{w : w\in\Lambda_a,\ d(x, K_w)\leqq c_2a\}$. Hence

$$\mu(B(x, c_2a))\leqq\sum_{w\in\Lambda_{a,x}}\tilde{\mu}(\Sigma_w).$$

Note that $\tilde{\mu}(\Sigma_w)=r_w^{\alpha}\leqq a^{\alpha}$ and $\#(\Lambda_{a,x})\leqq M$, we have

$$\mu(B(x, c_2a))\leqq Mc_2{}^{-\alpha}(c_2a)^{\alpha}.$$

Thus using Lemma 2.5,

$$r_w^{\alpha}\leqq\mu(K_w)\leqq Mc_2^{-\alpha}\mathcal{H}^{\alpha}(K_w).$$

Hence we have completed the proof of Theorem 2.4.

In the rest of this section, we show that the open set condition implies (A.1) and (A.2) of Theorem 2.4.

PROPOSITION 2.8. *Let $f_i: \boldsymbol{R}^k \to \boldsymbol{R}^k$ be r_i-similitude for $i=1, 2, \cdots, N$. And let K be the self-similar set with respect to $(\boldsymbol{R}^k, \{f_i\}_{i=1}^N)$. If the open set condition holds; there exists an open set $O \subset \boldsymbol{R}^k$ such that*

$$\bigcup_{i=1}^{N} f_i(O) \subset O \quad \text{and} \quad f_i(O) \cap f_j(O) = \emptyset \qquad \text{for } i \neq j\,,$$

then there exist constants $c_1, c_2, M>0$ such that

$$\text{(A.1)} \qquad d(K_w) \leqq c_1 r_w$$

for all $w \in W_*$

$$\text{(A.2)} \qquad \#\{w : w \in \Lambda(\boldsymbol{r}, a),\ d(x, K_w) \leqq c_2 a\} \leqq M$$

for all $0<a<1$ and $x \in K$.

PROOF. We can see that $K_w \subset \bar{O}_w$ where $O_w = F_w(O)$. Without loss of generality, we may assume that $d(O) \leqq 1$. Then obviously, for all $w \in W_*$,

$$d(K_w) \leqq d(\bar{O}_w) \leqq r_w\,.$$

Now let $\mathfrak{m}$ be the k-dimensional Lesbegue measure and let

$$\Lambda_{a,x} = \{w : w \in \Lambda(\boldsymbol{r}, a),\ d(x, K_w) \leqq a\}.$$

Then $\bigcup_{w \in \Lambda_{a,x}} O_w \subset B(x, 2a)$. Since O_w's are mutually disjoint, we have

$$\sum_{w \in \Lambda_{a,x}} \mathfrak{m}(O_w) \leqq \mathfrak{m}(B(x, 2a)).$$

Hence we have

$$\#(\Lambda_{a,x}) r_w{}^k \mathfrak{m}(O) \leqq 2^k C a^k$$

where $C = \mathfrak{m}$ (unit ball). Since $r_w \geqq aR$ where $R = \min\{r_1, r_2, \cdots, r_N\}$,

$$\#(\Lambda_{a,x}) \leqq 2^k C R^{-k} \mathfrak{m}(O)^{-1}\,.$$

§ 3. Shortest path metrics on p.c.f. self-similar sets.

In this section, we will apply Theorem 2.4 to shortest path metrics on p. c. f. self-similar sets. The notion of p. c. f. self-similar sets introduced by [**Ki2**] is a mathematical justification of the "finitely ramified" self-similar sets. Roughly speaking, a self-similar set K is finitely ramified if $\#(\bigcup_{i \neq j} (K_i \cap K_j))$ is finite.

DEFINITION 3.1. Let $(K, \{1, 2, \cdots, N\}, \{F_i\}_{i=1}^N)$ be a self-similar structure. We define the critical set $\mathcal{C}\subset\Sigma$ and the post critical set $\mathcal{P}\subset\Sigma$ by

$$\mathcal{C} = \pi^{-1}\big(\bigcup_{i\neq j}(K_i\cap K_j)\big) \quad\text{and}\quad \mathcal{P}=\bigcup_{n\geq 1}\sigma^n(\mathcal{C}),$$

where σ is the shift map from Σ to itself defined by $\sigma(\omega_1\omega_2\omega_3\cdots)=\omega_2\omega_3\omega_4\cdots$. A self-similar structure is called post critically finite (p. c. f. for short) if and only if $\#(\mathcal{P})$ is finite. Moreover if $(K, \{1, 2, \cdots, N\}, \{F_i\}_{i=1}^N)$ is p. c. f., then K is called a p. c. f. self-similar set.

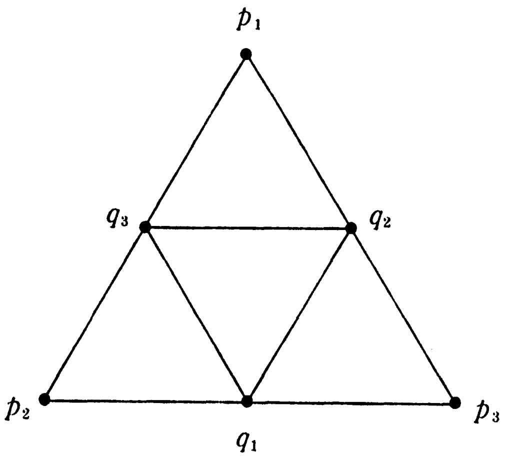
EXAMPLE (Sierpinski Gasket): Figure 2. Let p_1, p_2, p_3 be the vertices of a regular triangle in $\boldsymbol{C}$. Then we define, for $i=1, 2, 3$,

$$F_i(z) = \frac{1}{2}(z-p_i)+p_i\,.$$

The Sierpinski gasket K is the self-similar set with respect to $(\boldsymbol{C}, \{F_1, F_2, F_3\})$. The self-similar structure associated with the Sierpinski gasket is post critically finite. In fact,

$$\mathcal{C} = \{1\dot{2},\ 2\dot{1},\ 1\dot{3},\ 3\dot{1},\ 2\dot{3},\ 3\dot{2}\} \quad\text{and}\quad \mathcal{P} = \{\dot{1},\ \dot{2},\ \dot{3}\},$$

where $\dot{k}=kkkk\cdots$.



$p_1=\pi(\dot{1})$, $p_2=\pi(\dot{2})$, $p_3=\pi(\dot{3})$
$q_1=\pi(2\dot{3})=\pi(3\dot{2})$
$q_2=\pi(1\dot{3})=\pi(3\dot{1})$
$q_3=\pi(1\dot{2})=\pi(2\dot{1})$

Figure 2. a. Sierpinski Gasket.

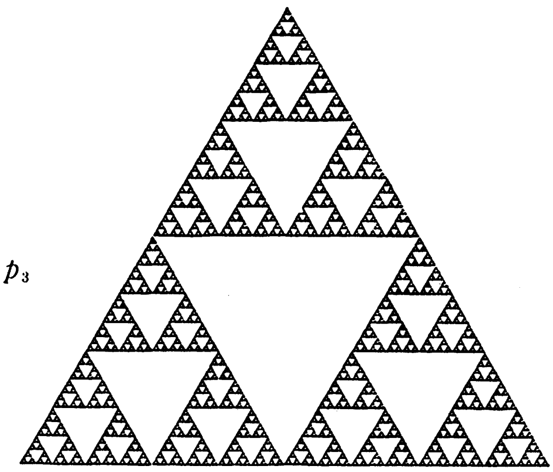


Figure 2. b. Sierpinski Gasket.

One can find other examples of p. c. f. self-similar sets in § 4. Moreover, the nested fractals introduced by Lindstrøm [**Li**] are p. c. f. self-similar sets.

The following is the main result of this section.

THEOREM 3.2. *Let $(K, \{1, 2, \cdots, N\}, \{F_i\}_{i=1}^N)$ be a p.c.f. self-similar structure. If a metric d on K satisfies:*

(B.1) *d is a shortest path metric.*

(B.2) *There exist $0<r_1, r_2, \cdots, r_N<1$ such that, for all $x, y \in K$,*

$$d(F_i(x), F_i(y)) = r_i d(x, y),$$

then there exists a constant $c>0$ such that

$$\mathcal{H}^{\alpha}(K_w) = c r_w{}^{\alpha}$$

for all $w \in W_*$, where α is the unique constant that satisfies

$$\sum_{i=1}^{N} r_i{}^{\alpha} = 1 .$$

In particular, $\dim_H(K, d)=\alpha$.

In the next section, we will establish a necessary and sufficient condition for the existence of metrics satisfying the above assumptions (B.1) and (B.2) and give some examples.

For the proof of Theorem 3.2, we will use

LEMMA 3.3. *Let $\Lambda \subset W_*$ be a partition of Σ. Then, for $w \neq v \in \Lambda$,*

$$K_w \cap K_v = F_w(V_0) \cap F_v(V_0), \tag{1}$$

where $V_0=\pi(\mathcal{P})$. Also for all $w \in \Lambda$,

$$\#\{v : v \neq w \in \Lambda,\ K_w \cap K_v \neq \emptyset\} \leqq \#(C)\#(V_0). \tag{2}$$

PROOF. (1) is obvious by the definition of $\mathcal{P}$ and C. For (2), it is enough to show that, for all $p \in K$,

$$\#(\pi^{-1}(p)) \leqq \#(C). \tag{3.1}$$

Let $k=\#(C)$ and suppose $\pi^{-1}(p)$ contains $k+1$ elements, that is,

$$\pi^{-1}(p) \supset \{\omega^1, \omega^2, \cdots, \omega^{k+1}\},$$

where $\omega^n=\omega_1^n\omega_2^n \cdots$. Then, there exists $m \geqq 1$ such that

$$\omega_1^1\omega_2^1 \cdots \omega_{m-1}^1 = \omega_1^2\omega_2^2 \cdots \omega_{m-1}^2 = \cdots = \omega_1^{k+1}\omega_2^{k+1} \cdots \omega_{m-1}^{k+1}$$

and $\omega_m^i \neq \omega_m^j$ for some $i \neq j$. Let $q=F^{-1}_{\omega_1^1\omega_2^1\cdots\omega_{m-1}^1}(p)$, then $q \in K_i \cap K_j$ and

$$\pi^{-1}(q) \supset \{\omega_*^1, \omega_*^2, \cdots, \omega_*^{k+1}\},$$

where $\omega_*^n=\omega_m^n\omega_{m+1}^n \cdots$.

On the other hand, $\pi^{-1}(q) \subset C$, hence $\#(\pi^{-1}(q)) \leqq k$. Thus we have a contradiction.

PROOF OF THEOREM 3.2. Obviously by (B.2), we have, for all $w\in W_*$,

$$d(K_w)=r_w d(K)\,. \tag{A.1}$$

Now for $x\in K$, there exists $w\in\Lambda(\boldsymbol{r}, a)$ such that $x\in K_w$. If $K_w\cap K_v=\emptyset$, then for $y\in K_v$, a geodesic between x and y must intersect some K_s for some $s\in\Lambda(\boldsymbol{r}, a)\setminus\{w, v\}$. Using Lemma 3.3-(1), we have

$$d(x, y)\geqq \min_{p\neq q\in F_s(V_0)} d(p, q)\,.$$

By (B.2), we have $d(x, y)\geqq r_s b\geqq aRb$, where $R=\min\{r_1, r_2, \cdots, r_N\}$ and $b=\min\{d(p, q): p\neq q\in V_0\}$. Hence let $c_2=Rb/2$, then

$$\{t: t\in\Lambda(\boldsymbol{r}, a),\ d(x, K_t)\leqq c_2 a\}\subset\{t: t\in\Lambda(\boldsymbol{r}, a),\ K_t\cap K_w\neq\emptyset\}\,.$$

Using Lemma 3.3-(2), we have

$$\#\{t: t\in\Lambda(\boldsymbol{r}, a),\ d(x, K_t)\leqq c_2 a\}\leqq M\,, \tag{A.2}$$

where $M=\#(C)\#(V_0)+1$. Thus applying Theorem 2.4 we have $0<\mathcal{H}^\alpha(K)<\infty$. By (B.2), we can see that, for all $w\in W_*$,

$$\mathcal{H}^\alpha(K_w)=r_w{}^\alpha\mathcal{H}^\alpha(K)\,.$$

Thus we have completed the proof of Theorem 3.2.

§ 4. Existence of shortest path metric.

In this section, we will give a necessary and sufficient condition for existence of metrics on p.c.f. self-similar sets that satisfy the assumptions of Theorem 3.2, (B.1) and (B.2). First we introduce several notions.

DEFINITION 4.1. Let V be a finite set. A family of $\#(V)\times\#(V)$-matrices, $\mathcal{D}=\{D_1, D_2, \cdots, D_n\}$ is called a family of paths on V if and only if, for every $i=1, 2, \cdots, n$,

(1) D_i is symmetric,

(2) $D_i(p, q)\geqq 0$ for all $p, q\in V$ and $D_i(p, p)=0$ for all $p\in V$, where $D_i=(D_i(p, q))_{p,q\in V}$.

A sequence $\{(p_k, p_{k+1}: i_k)\}_{k=1}^m$ where $p_k, p_{k+1}\in V$ and $i_k\in\{1, 2, \cdots, n\}$ is called a $\mathcal{D}$-path between p_1 and p_{m+1} if and only if, for all $k=1, 2, \cdots, m$, $D_{i_k}(p_k, p_{k+1})>0$. Further a family of paths on V, $\mathcal{D}$ is said to be irreducible if there exists a $\mathcal{D}$-path between p and q for all $p, q\in V$.

NOTATION. For two $\mathcal{D}$-paths $\boldsymbol{p}=\{(p_k, p_{k+1}: i_k)\}_{k=1}^{m_1}$ and $\boldsymbol{q}=\{(q_k, q_{k+1}: j_k)\}_{k=1}^{m_2}$, if $p_{m_1+1}=q_1$, we define a $\mathcal{D}$-path between p_1 and q_{m_2+1}, $\boldsymbol{h}=\{(h_k, h_{k+1}: l_k)\}_{k=1}^{m_1+m_2}$, by

$$h_k = \begin{cases} p_k & \text{for } 1 \leqq k \leqq m_1 \\ q_{k-m_1} & \text{for } m_1+1 \leqq k \leqq m_1+m_2+1 \end{cases}$$

$$l_k = \begin{cases} i_k & \text{for } 1 \leqq k \leqq m_1 \\ j_{k-m_1} & \text{for } m_1+1 \leqq k \leqq m_1+m_2 \end{cases}$$

we will denote $\boldsymbol{h}=\boldsymbol{p}\vee\boldsymbol{q}$.

The next proposition follows immediately by Definition 4.1.

PROPOSITION 4.2. *Let V be a finite set and let $\mathcal{D}$ be a irreducible family of paths on V, then $d_{\mathcal{D}}$ is a metric on V where $d_{\mathcal{D}}$ is defined by*

$$d_{\mathcal{D}}(p, q) = \min_{\substack{\{(p_k, p_{k+1}: i_k)\}_{k=1}^m \text{ is a} \\ \mathcal{D}\text{-path between } p \text{ and } q}} \sum_{i=1}^{m} D_{i_k}(p_k, p_{k+1}).$$

A $\mathcal{D}$-path between p and q that attains the minimum of the above definition is called a minimal $\mathcal{D}$-path between p and q.

Now the main result of this section is

THEOREM 4.3. *Let $(K, \{1, 2, \cdots, N\}, \{F_i\}_{i=1}^N)$ be a p.c.f. self-similar structure where K is connected. For given $0<r_1, r_2, \cdots, r_N<1$, there exists a metric on K such that*

(B.1) $\qquad d$ *is a shortest path metric,*

(B.2) $$d(F_i(x), F_i(y)) = r_i d(x, y),$$

for all $x, y \in K$ if and only if there exists a metric d_0 on $V_0 = \pi(\mathcal{P})$ such that, for all $p, q \in V_0$,

(C.1) $$d_{\mathcal{D}}(p, q) = d_0(p, q),$$

where $\mathcal{D} = \{D_1, D_2, \cdots, D_N\}$ is a family of paths[2] on $V_1 = \bigcup_{i=1}^N F_i(V_0)$ defined by

$$D_i(p, q) = \begin{cases} r_i d_0(F_i^{-1}(p), F_i^{-1}(q)) & \text{if } p, q \in F_i(V_0), \\ 0 & \text{otherwise} \end{cases}$$

and

(C.2) $$d_{\mathcal{D}}(p, q) = D_i(p, q),$$

for all $p, q \in F_i(V_0)$.

REMARK 1. In [**BS**], they studied the existence problem of interior distances on p.c.f. self-similar sets where the contractions are similitudes of a

[2] If K is connected, we can easily see that $\mathcal{D}$ is irreducible.

Euclidean space $\boldsymbol{R}^n$. The interior distances satisfy (B.1) and (B.2). They get a condition which corresponds to (C.1) of the above theorem. In their restricted situation, however, the condition (C.2) will not appear. We will give an example where (C.2) becomes really a constraint. See Example 2 below.

REMARK 2. By the proof of Theorem 4.3, we can see that if (C.1) and (C.2) is satisfied then we can find a set of geodesics $\{g_{pq}\}_{p,q\in V_0}$ such that

$$g_{pq}(t) = g_{pq}(d(p, q)-t).$$

Moreover let $[t_1, t_2]=\{t: g_{pq}(t)\in K_i\}$, then

$$F_i(g_{\bar{p}\bar{q}}(t)) = g_{pq}(r_i t+d(p, p_1)),$$

where $p_1=g_{pq}(t_1)$, $q_1=g_{pq}(t_2)$, $\bar{p}=F_i^{-1}(p_1)$ and $\bar{q}=F_i^{-1}(q_1)$. Hence $\{g_{pq}\}_{p,q\in V_0}$ forms a frame of this self-similar sets. The concept of frames of self-similar sets was introduced by Kameyama [**Ka**].

Before proving our theorem, we apply it to several examples.

EXAMPLE 1 (Hata's tree-like set): Figure 3. This tree-like set was introduced by Hata [**Ha**]. For $\beta\in\boldsymbol{C}$ that satisfies

$$|\beta|<1,\ |\beta-1|<1 \quad\text{and}\quad \operatorname{Im}\beta\neq 0,$$

we define contractions F_1 and F_2 from $\boldsymbol{C}$ to itself by

$$F_1(z)=\beta\bar{z} \quad\text{and}\quad F_2(z)=(1-|\beta|^2)\bar{z}+|\beta|^2.$$

The Hata's tree-like set is the self-similar set associated with $(\boldsymbol{C}, \{F_1, F_2\})$. The corresponding self-similar structure is independent of β and it is post critically finite. In fact

$$\mathcal{C}=\{11\dot{2},\ 2\dot{1}\}, \qquad q=\pi(11\dot{2})=\pi(2\dot{1}),$$
$$\mathcal{P}=\{\dot{1},\ \dot{2},\ 1\dot{2}\}, \quad p_1=\pi(\dot{1}), \quad p_2=\pi(\dot{2}), \quad p_3=\pi(1\dot{2}).$$

Now let d_0 be a metric on $V_0=\{p_1, p_2, p_3\}$ and let

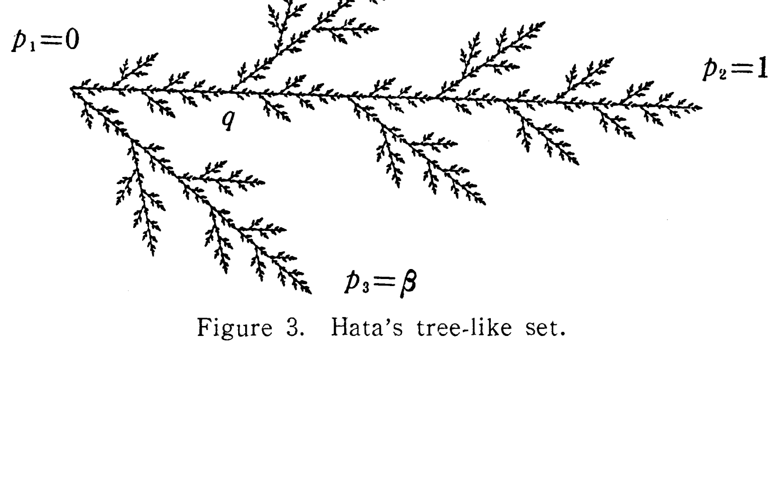


Figure 3. Hata's tree-like set.

$$a = d_0(p_1, p_3), \quad b = d_0(p_1, p_2) \quad \text{and} \quad c = d_0(p_2, p_3).$$

Then the condition (C.1) becomes

$$br_1 = a, \quad ar_1 + br_2 = b, \quad br_1 + ar_1 + br_2 = c. \tag{4.1}$$

In this case, (C.2) is satisfied under (4.1). By (4.1), we have, for $0<r<1$,

$$r_1 = r, \quad r_2 = 1-r^2, \quad a = br \quad \text{and} \quad c = a+b.$$

Hence by Theorem 4.3, we can construct a shortest path metric on the Hata's tree-like set for each choice of r. By Theorem 3.2, the corresponding Hausdorff dimension is the unique number α that satisfies

$$r^\alpha + (1-r^2)^\alpha = 1.$$

For the Hata's tree-like set, the contraction ratio $|\beta|$ naturally corresponds to the ratio of the shortest path metric r.

EXAMPLE 2: Figure 4. Let $\{p_1, p_2, p_3\}$ be the vertices of a regular triangle in $\boldsymbol{C}$ and let

$$p_4 = \frac{1}{2}(p_2+p_3), \quad p_5 = \frac{1}{2}(p_1+p_3), \quad p_6 = \frac{1}{2}(p_1+p_2), \quad p_7 = \frac{1}{3}(p_1+p_2+p_3).$$

Further, for $i=1, 2, \cdots, 7$, let F_i be a contraction from $\boldsymbol{C}$ to itself defined by

$$F_i(z) = \beta_i(z-p_i)+p_i,$$

where $\beta_1=\beta_2=\beta_3=\beta$, $\beta_4=\beta_5=\beta_6=1-2\beta$, $\beta_7=1-3\beta$ for $1/3<\beta<1/2$.

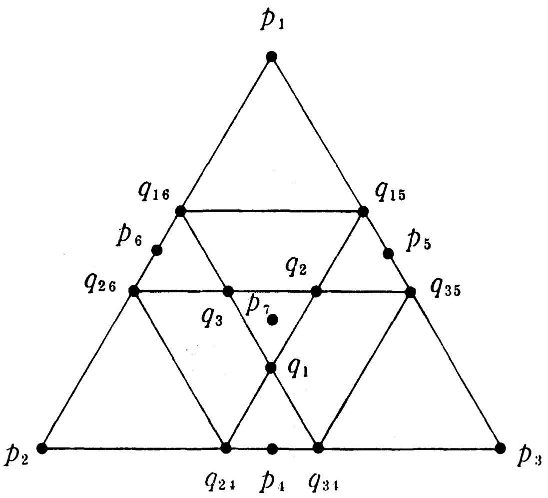


Figure 4. a.

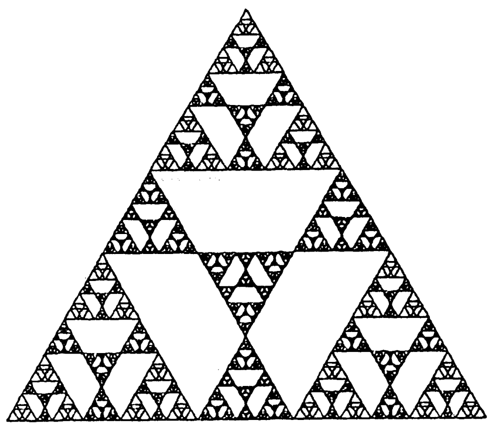


Figure 4. b.

The self-similar structure associated with the self-similar set with respect to $(\boldsymbol{C}, \{F_i\}_{i=1}^7)$ is independent of the value of β and it is post critically finite.

In fact

$$C=\bigcup_{k=1}^{3}\{7\dot{k},(k+3)\dot{k}\}\cup\bigcup_{\substack{k<l<m\\ k+l+m=9}}\{k\dot{l},m\dot{k},m\dot{l},l\dot{k}\},$$

$$q_k=\pi(7\dot{k})=\pi((k+3)\dot{k})\qquad\text{for } k=1,2,3,$$

$$q_{lm}=\pi(m\dot{l})=\pi(l\dot{k}),\qquad q_{km}=\pi(k\dot{l})=\pi(m\dot{k}),$$

for (k, l, m) such that $k<l<m$ and $k+l+m=9$. Also

$$\mathcal{P}=\{\dot{1},\dot{2},\dot{3}\}\quad\text{and}\quad p_k=\pi(\dot{k})$$

for $k=1, 2, 3$.

In this case, taking symmetries into account, we assume that $d_0(p_i, p_j)=1$ for all $i\neq j\in\{1, 2, 3\}$ where d_0 is a metric on $V_0=\{p_1, p_2, p_3\}$ and that $r_1=r_2=r_3=r$, $r_4=r_5=r_6=s$ and $r_7=t$. Then the condition (C.1) and (C.2) becomes

$$2r+s=1\quad\text{and}\quad 2s+t\geqq r$$

respectively. Here the condition (C.2) is really a constraint. For example, for $r=3/7$ and $s=1/7$, we have $t\geqq 1/7$ by (C.2).

EXAMPLE 3 (Pentakun): Figure 5. Let $\{p_1, p_2, \cdots, p_5\}$ be the vertices of a regular pentagon in $\boldsymbol{C}$. Then for $i=1, 2, \cdots, 5$, we define a contraction F_i by

$$F_i(z)=\frac{3-\sqrt{5}}{2}(z-p_i)+p_i\,.$$

The pentakun[3] is the self-similar set with respect to $(\boldsymbol{C}, \{F_i\}_{i=1}^{5})$. The self-

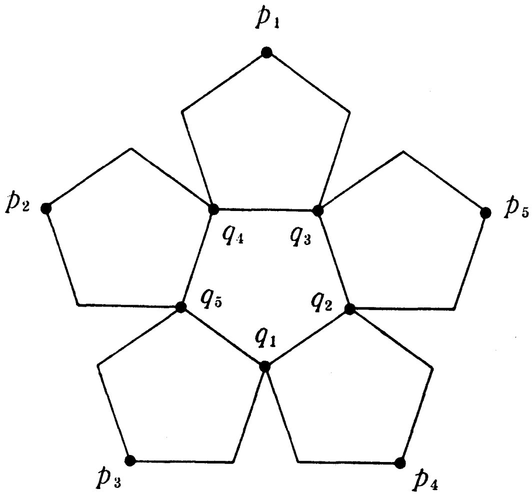


Figure 5.a. Pentakun.

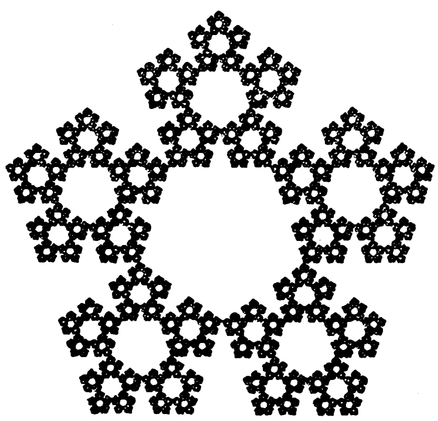


Figure 5.b. Pentakun.

[3] In the same way, we can also define hexakun, heptakun, octakun and so on. 'kun' is a Japanese which means 'Mr.'.

similar structure that corresponds to the pentakun is post critically finite. In fact

$$C = \bigcup_{k=1}^{5} \{[k-2][k\dot{+}1],\ [k+2][k\dot{-}1]\},$$

$$q_k = \pi([k-2][k\dot{+}1]) = \pi([k+2][k\dot{-}1]),$$

$$\mathcal{P} = \{\dot{1}, \dot{2}, \cdots, \dot{5}\} \quad \text{and} \quad p_k = \pi(\dot{k})$$

for $k=1, 2, \cdots, 5$, where $[i] \in \{1, 2, \cdots, 5\}$ is defined by $[i] \equiv i \bmod 5$.

The pentakun has a strong symmetry and it is a nested fractal. Here we will focus on shortest path metrics that have the same symmetry as the shape of the pentakun. Therefore, we assume that a metric d_0 on $V_0 = \{p_1, p_2, \cdots, p_5\}$ satisfies

$$d_0(p_i, p_j) = \begin{cases} a & \text{if } |i-j| = \pm 1 \bmod 5 \\ b & \text{if } |i-j| = \pm 2 \bmod 5 \end{cases}$$

and also $r = r_i$ for $i=1, 2, \cdots, 5$. Then the condition (C.1) becomes

$$2br = a \quad \text{and} \quad 2br + ar = b\,. \tag{4.2}$$

(C.2) is satisfied if (4.2) holds. By (4.2), we have

$$r = \frac{\sqrt{3}-1}{2} \quad \text{and} \quad a = (\sqrt{3}-1)b\,.$$

Hence by Theorem 4.3, the shortest path metrics with the above symmetry are essentially unique up to constant multiple. The Hausdorff dimension under this shortest path metric is $-\log 5/\log r$.

In the rest of this section, we will prove Theorem 4.3 in several steps. First we show that (B.1) and (B.2) imply (C.1) and (C.2).

LEMMA 4.4. *If* (B.1) *and* (B.2) *hold, then we have* (C.1) *and* (C.2).

PROOF. Let $d_0 = d|_{V_0 \times V_0}$. By virtue of (B.2), $d(p, q) = D_i(p, q)$ if $p, q \in F_i(V_0)$. Hence by the triangle inequality, we have (C.2) and, for $p, q \in V_0$,

$$d_{\mathcal{D}}(p, q) \geqq d_0(p, q)\,.$$

By (B.1), there is a geodesic $g: [0, d(p, q)] \to K$ between p and q for $p, q \in V_0$. Let $\{t_1, t_2, \cdots, t_{m+1}\} = \{t : g(t) \in V_1\}$ where $0 = t_1 < t_2 < \cdots < t_{m+1} = d(p, q)$, then $\{(g(t_k), g(t_{k+1}) : i_k)\}_{k=1}^{m}$ is a $\mathcal{D}$-path between p and q for some $i_1, i_2, \cdots, i_m$. As g is a geodesic,

$$d(p, q) = \sum_{i=1}^{m} d(g(t_k), g(t_{k+1}))\,.$$

Hence we have

$$d_{\mathcal{D}}(p, q) = d_0(p, q).$$

Next we prove the counter direction. From now on, we assume (C.1) and (C.2).

DEFINITION 4.5. Let $V_m = \bigcup_{w \in W_m} F_w(V_0)$. $\mathcal{D}_m = \{D_w\}_{w \in W_m}$ is a family of paths on V_m defined by

$$D_w(p, q) = \begin{cases} r_w d_0(F_w^{-1}(p), F_w^{-1}(q)) & \text{if } p, q \in F_w(V_0), \\ 0 & \text{otherwise.} \end{cases}$$

We write $d_m = d_{\mathcal{D}_m}$. Note that $\mathcal{D} = \mathcal{D}_1$.

As K is connected we can see that $\mathcal{D}_m$ is irreducible. Now, for each pair $(p, q) \in V_0 \times V_0$, we fix a minimal $\mathcal{D}$-path between p and q, $\boldsymbol{p}(p, q) = \{(p_k(p, q), p_{k+1}(p, q) : i_k(p, q))\}_{k=1}^{m(p,q)}$. Then using (C.2) inductively, we can see that

LEMMA 4.6. *For $w \in W_m$ and $p, q \in F_w(V_0)$, we define a $\mathcal{D}_{m+1}$-path between p and q, $\boldsymbol{p}_w(p, q)$ by*

$$\boldsymbol{p}_w(p, q) = \{(F_w(p_k(\bar{p}, \bar{q})), F_w(p_{k+1}(\bar{p}, \bar{q})) : wi_k(\bar{p}, \bar{q}))\}_{k=1}^{m(\bar{p}, \bar{q})},$$

where $\bar{p} = F_w^{-1}(p)$ and $\bar{q} = F_w^{-1}(q)$. Then $\boldsymbol{p}_w(p, q)$ is a minimal $\mathcal{D}_{m+1}$-path between p and q.

LEMMA 4.7. *For $p, q \in V_m$, let $\boldsymbol{p} = \{(p_k, p_{k+1} : w^{(k)})\}_{k=1}^{m}$ be a minimal $\mathcal{D}_m$-path between p and q. Then*

$$\mathcal{F}(\boldsymbol{p}) = \boldsymbol{p}_{w^{(1)}}(p_1, p_2) \vee \boldsymbol{p}_{w^{(2)}}(p_2, p_3) \vee \cdots \boldsymbol{p}_{w^{(m)}}(p_m, p_{m+1})$$

is a minimal $\mathcal{D}_{m+1}$-path between p and q.

By the above lemma, we can easily see that

$$d_{m+1}(p, q) = d_m(p, q)$$

for all $p, q \in V_m$. Hence we can define a metric d on $V_* = \bigcup_{m \geq 0} V_m$ by $d(p, q) = d_m(p, q)$ for $p, q \in V_m$. Next we will extend this metric d to a metric on K.

LEMMA 4.8. *Define $d(p, q)$ for $p, q \in K$ by*

$$d(p, q) = \lim_{n \to \infty} d(p_n, q_n),$$

where $p_n, q_n \in V_*$ and $p_n \to p$, $q_n \to q$ as $n \to \infty$. Then d is well-defined and it is a metric on K that satisfies (B.2).

REMARK. The notion of the convergence of a sequence in K is equivalent to the following. A sequence in K, $\{p_n\}_{n=1}^{\infty}$, converges to $p\in K$ if and only if, for each m, there exists $n(m)$ such that, for all $n>n(m)$, $p_n\in U_{m,p}$ where

$$U_{m,p}=\bigcup_{w:p\in K_w,w\in W_m} K_w .$$

PROOF OF LEMMA 4.8. Let $p_n\to p$, $p'_n\to p$ as $n\to\infty$ where $\{p_n\}$, $\{p'_n\}\subset V_*$. Note that, for $w\in W_m$,

$$d(K_w)=\max_{x,y\in K_w\cap V_*} d(x, y)\leqq R^m M$$

where $R=\max\{r_1, r_2, \cdots, r_N\}$ and $M=\max_{p,q\in V_0} d_0(p, q)$. Hence by (3.1), we have, for sufficient large n,

$$d(p_n, p'_n)\leqq \sum_{w\in U_{m,p}} d(K_w)\leqq \#(C)R^m M .$$

Therefore $d(p_n, p'_n)\to 0$ as $n\to\infty$.

Now by the triangle inequality,

$$|d(p_n, q_n)-d(p_m, q_m)|\leqq d(p_n, p_m)+d(q_n, q_m) .$$

By the above fact, we have $d(p_n, p_m)$, $d(q_n, q_m)\to 0$ as $m, n\to\infty$. Hence there exists a limit of $d(p_n, q_n)$ as $n\to\infty$. We can show that this d is a well-defined on K by the same discussion. It is obvious that d is a metric on K and it satisfies (B.2).

The final step of the proof of Theorem 4.3 is to construct geodesics for d. First we construct a geodesic between p and q for $p, q\in V_*$. For $p, q\in V_*$, there is some m such that $p, q\in V_m$. Let $\boldsymbol{p}$ be a minimal $\mathcal{D}_m$-path between p and q. We define $\boldsymbol{p}^n=\mathcal{F}^n(\boldsymbol{p})$, then by Lemma 4.7, $\boldsymbol{p}^n$ is a minimal $\mathcal{D}_{m+n}$-path between p and q. Let

$$\boldsymbol{p}^n=\{(p_k^n, p_{k+1}^n : w^n(k))\}_{k=1}^{m(n)} ,$$

then define $T_n=\{t_1^n, t_2^n, \cdots, t_{m(n)+1}^n\}\subset[0, d(p, q)]$ by $t_k^n=d_{m+n}(p, p_k^n)$, where $t_1^n=0$ and $t_{m(n)+1}^n=d(p, q)$. We can see that $T_n\subset T_{n+1}$ and $T_*=\bigcup_{n\geqq 0} T_n$ is a dense subset of $[0, d(p, q)]$. Now define $g_{pq}: T_*\to[0, d(p, q)]$ by, for $t_k^n\in T_n$, $g_{pq}(t_k^n)=p_k^n$, then this is well-defined and we can extend g_{pq} to a continuous function $g_{pq}: [0, d(p, q)]\to K$. It is obvious by the method of construction that g_{pq} is a geodesic between p and q.

Next, we construct a geodesic between $p\in V_0$ and $q\in K\setminus V_*$. Let $n=\#(V_0)$, $\pi^{-1}(q)=\omega_1\omega_2\omega_3\cdots$ and $F_{\omega_1\omega_2\cdots\omega_m}(V_0)=\{g_1^m, g_2^m, \cdots, g_n^m\}$. By Lemma 3.3-(1), every geodesic between p and q_i^m intersects $F_{\omega_1\omega_2\cdots\omega_{m-1}}(V_0)$. Hence we can choose a family of geodesics $g(p, q_i^m : \cdot)$ such that $g(p, q_i^m : \cdot)$ is a geodesic between p and q_i^m and, for some j, $g(p, q_i^m : t)=g(p, q_j^{m-1} : t)$ on $[0, d(p, q_j^{m-1})]$.

LEMMA 4.9. *There exists a sequence* $\{i_m\}_{m=1,2,\dots}$ *where* $1\leqq i_k\leqq n$ *such that*

$$g(p,\, q^m_{i_m} : t) = g(p,\, q^k_{i_k} : t)$$

on $[0,\, d(p,\, q^k_{i_k})]$ *for all* $m>k$.

PROOF. We can choose i_1 such that for all $k>1$, there exists j_k that satisfies

$$g(p,\, q^k_{j_k} : t) = g(p,\, q^1_{i_1} : t)$$

on $[0,\, d(p,\, q^1_{i_1})]$. Now suppose that we can choose $i_1, i_2, \cdots, i_m$ such that for all $k>m$, there exists j_k satisfying

$$g(p,\, q^k_{j_k} : t) = g(p,\, q^l_{i_l} : t)$$

on $[0,\, d(p,\, q^l_{i_l})]$ for $l=1, 2, \cdots, m$. Then let

$$U_i = \{k : k\geqq m+1,\, g(p,\, q^k_{j_k} : d(p,\, q^{m+1}_i))=q^{m+1}_i\}.$$

Choosing i_{m+1} so that $\#(U_{i_{m+1}})=\infty$, we can find $j'_{m+2}, j'_{m+3}, \cdots$ satisfying

$$g(p,\, q^k_{j'_k} : t) = g(p,\, q^l_{i_l} : t)$$

on $[0,\, d(p,\, q^l_{i_l})]$ for $l=1, 2, \cdots, m+1$. Hence we can inductively construct a sequence $\{i_m\}_{m=1,2,\dots}$ that satisfies Lemma 4.9.

Since $q^m_{i_m}\to q$ as $m\to\infty$, we can define $g_{pq}\colon [0,\, d(p,\, q)]\to K$ by

$$g_{pq}(t) = \begin{cases} g(p,\, q^m_{i_m} : t) & \text{on } [0,\, d(p,\, q^m_{i_m})] \\ q & \text{for } t=d(p,\, q)\,. \end{cases}$$

Then $g_{p,q}$ is a geodesic between p and q.

Finally we construct a geodesic between p and q for $p, q\in K$. For sufficiently large m, we can choose $w\neq v\in W_m$ so that $p\in K_w$ and $q\in K_v$. Then there exist $p_1\in F_w(V_0)$ and $q_1\in F_v(V_0)$ such that

$$d(p,\, q) = d(p,\, p_1)+d(p_1,\, q_1)+d(q_1,\, q)\,.$$

Let $\bar{p}=F_w^{-1}(p)$, $\bar{p}_1=F_w^{-1}(p_1)$, $\bar{q}=F_v^{-1}(q)$ and $\bar{q}_1=F_v^{-1}(q_1)$ and let $g_{\bar{p}_1\bar{p}}$, g_{pq} and $g_{\bar{q}_1\bar{q}}$ be geodesics constructed in the previous steps. We define

$$g_{pq}(t) = \begin{cases} g_{\bar{p}_1\bar{p}}((d(p,\, p_1)-t)/r_w) & \text{on } [0,\, d(p,\, p_1)] \\ g_{p_1q_1}(t-d(p,\, p_1)) & \text{on } [d(p,\, p_1),\, d(p,\, q_1)] \\ g_{\bar{q}_1\bar{q}}((t-d(p,\, q_1))/r_v) & \text{on } [d(p,\, q_1),\, d(p,\, q)]\,. \end{cases}$$

It follows easily that g_{pq} is a geodesic between p and q. Thus we have shown that d is a shortest path metric.

Appendix. The Hausdorff dimension of the Lévy curve.

At the beginning, we recall the definition of the Lévy curve.

Let F_0 and F_1 be contractions[4] from $\boldsymbol{C}$ to itself defined by, for $b=(1+i)/2$,

$$F_0(z)=\beta z \quad \text{and} \quad F_1(z)=\bar{\beta}z+\beta\,,$$

The Lévy curve K is the self-similar set with respect to $(\boldsymbol{C}, \{F_0, F_1\})$.

First, we explain the difficulty in applying Theorem 1.2 (the open set condition) to the Lévy curve. The following fact was pointed out by Hata in a private communication.

PROPOSITION A.1. *If there is an open set O that satisfies the open set condition for the Lévy curve K, then $O \subset K$.*

This means that if we try to find an open set that satisfies the open set condition, then we should prove that the Lévy curve K contains an open ball of $\boldsymbol{C}$. However, if we could find an open ball which contained in K, it proved that $0<\mathcal{H}^2(K)<\infty$, because K is a bounded subset of $\boldsymbol{C}$. As a consequence, it seems quite difficult to show the open set condition before proving that $\dim_H K=2$. In fact, we can show that the Lévy curve contains an open ball of $\boldsymbol{C}$ and then $\mathrm{int}(K)$ is the unique open set that satisfies the open set condition.

Now we will show that Corollary 1.3 can be applied to the Lévy curve.

PROPOSITION A.2. *For the Lévy curve K, let $r=|\beta|=\sqrt{1/2}$, then*

$$d(K_w) \leqq r^m d(K) \tag{1}$$

for all $m \geqq 1$ and all $w \in W_m$. Also there exists a positive constant M such that

$$\#\{w : w \in W_m,\ d(x, K_w) \leqq r^m d(K)\} \leqq M \tag{2}$$

for all $m \geqq 1$ and $x \in K$.

By Corollary 1.3, the above proposition implies

$$0<\mathcal{H}^2(K)<\infty$$

and so $\dim_H K=2$.

We will prove Proposition A.2 hereafter. (1) is obvious because $d(K_w)=r^m d(K)$ for all $w \in \{0, 1\}^m$. For (2), we will construct a series of broken lines which approximates the Lévy curve as in [**Kil**].

DEFINITION. For $n \geqq 0$, $\boldsymbol{a}_n=\{a_n(i)\}_{i=1}^{2^n}$ is defined inductively by

[4] We change the notation of contractions for convenience of the following discussion. In §1, we used f_1 and f_2 where $f_1=F_0$ and $f_2=F_1$.

$$a_0(1) = 1 \quad \text{and} \quad \begin{cases} a_{n+1}(2k-1) = \beta a_n(k) \\ a_{n+1}(2k) = \beta a_n(k). \end{cases}$$

We think of $\boldsymbol{a}_n$ as a broken line whose turning points are $\boldsymbol{z}_n = \{z_n(k)\}_{k=0}^{2^n}$ where $z(k)$'s are defined inductively by

$$z_n(0) = 0 \quad \text{and} \quad z_n(k) = z_n(k-1) + a_n(k).$$

See Figure 6. It is easy to see that

LEMMA A.3.

(a) $$z_{n+1}(2k) = z_n(k). \quad \textit{In particular, } z_n(2^n) = 1.$$

(b) $$\boldsymbol{z}_{n+1} = F_0(\boldsymbol{z}_n) \cup F_1(\boldsymbol{z}_n).$$

(c) $$\boldsymbol{z}_n \subset \beta^n \boldsymbol{Z}^2.$$

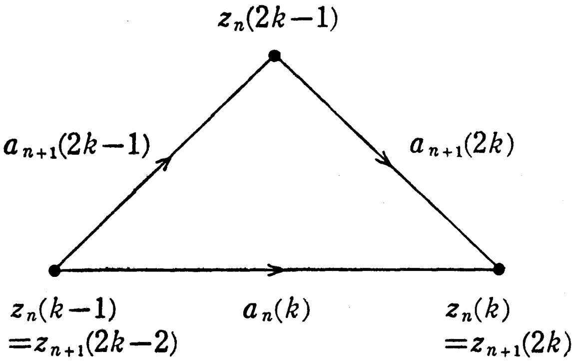


Figure 6. a.

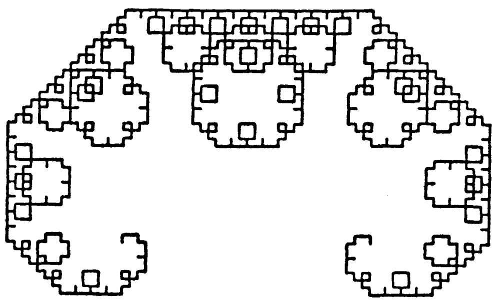


Figure 6. b. Z_{10}.

By (b) of the above lemma, it follows that the Lévy curve is the closure of $\bigcup_{n\geq 0} \boldsymbol{z}_n$. For each $(z_n(k), z_n(k+1))$, there exists $w \in \{0, 1\}^n$ such that $F_w(0) = z_n(k)$ and $F_w(1) = z_n(k+1)$. We denote this w by $w(n, k)$. Precisely, $w(n, k) = w_1 w_2 \cdots w_n \in \{0, 1\}^n$ where

$$k = \sum_{i=1}^{n} w_i 2^{n-i}.$$

LEMMA A.4. *If $k \neq l$ then $(z_n(k), z_n(k+1)) \neq (z_n(l), z_n(l+1))$.*

PROOF. For $n=0$, Lemma A.4 holds obviously. Now let's suppose that Lemma A.4 is true for n. Let $x, y \in \beta^{n+1} \boldsymbol{Z}^2$ satisfy $|x-y| = |\beta|^{n+1}$. Then we define $(x, y)^{-1} = (X, Y)$ where $X, Y \in \beta^n \boldsymbol{Z}^2$ and $|X-Y| = |\beta|^n$ by

$$(X, Y) = \begin{cases} (x, x + \beta^{-1}(y-x)) & \text{if } x \in \beta^n \boldsymbol{Z}^2, \\ (y - \bar{\beta}^{-1}(x-y), y) & \text{if } y \in \beta^n \boldsymbol{Z}^2. \end{cases}$$

For $(z_{n+1}(k), z_{n+1}(k+1))$, there exists m such that $(z_{n+1}(k), z_{n+1}(k+1))^{-1} = (z_n(m),$

$z_n(m+1))$. Precisely, $m=k/2$ if k is even and $m=(k-1)/2$ if k is odd. Now if $(z_{n+1}(k), z_{n+1}(k+1))=(z_{n+1}(l), z_{n+1}(l+1))$, then $(z_{n+1}(k), z_{n+1}(k+1))^{-1}=(z_{n+1}(l), z_{n+1}(l+1))^{-1}$. This is impossible because Lemma A.4 holds for n. Hence we have shown that Lemma A.4 is true for $n+1$. This completes the proof of Lemma A.4.

NOTATION. For $x, y\in\beta^n\boldsymbol{Z}^2$ satisfying $|x-y|=|\beta|^n$, we define the edge of the square lattice $\beta^n\boldsymbol{Z}^2$ whose vertices are x and y by

$$e(x, y)=\{x+t(y-x): 0\leqq t\leqq 1\}.$$

Further we denote the collection of the edges of the square lattice $\beta^n\boldsymbol{Z}^2$ by E_n.

PROOF OF (2) OF PROPOSITION A.2. As $d(K_w)=r^m d(K)$ for $w\in W_m$, we have

$$\Lambda_{m,x}\subset A_{m,x},$$

where $\Lambda_{m,x}=\{w: w\in W_m,\ d(x, K_w)\leqq d(K)r^m\}$ and

$$A_{m,x}=\{w(m, k): B(x, 2d(K)r^m)\cap e(z_m(k), z_m(k+1))\neq\emptyset\}.$$

By Lemma A.4, each edge $e(x, y)$ of $\beta^n\boldsymbol{Z}^2$ corresponds at most two $w(m, k)$'s such as $(x, y)=(z_m(k), z_m(k+1))$ and $(y, x)=(z_m(l), z_m(l+1))$. Hence we have

$$\begin{aligned}\#(A_{m,x}) &\leqq 2\#(\{e: e\in E_m,\ B(x, 2d(K)r^m)\cap e\neq\emptyset\})\\ &\leqq 2\#(\{e: e\in E_0,\ B(\beta^{-m}x, 2d(K))\cap e\neq\emptyset\}).\end{aligned}$$

Obviously there exists $M>0$ such that the last value of the above inequality is not larger than M for all x and m. Hence we have

$$\#(\Lambda_{m,x})\leqq M$$

for all $x\in K$ and all $m\geqq 1$.

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