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Harmonic functions of polynomial growth on complete manifolds II

Dedicated to Professor S. Kobayashi for his 60th birthday

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0. Introduction.

Let M be a complete noncompact Riemannian manifold of dimension m. For a nonnegative number d, we denote by $\mathcal{H}_d(M)$ the space of harmonic functions h of polynomial growth of order d, namely, $|h| \leqq Ar^d + B$ for some positive constants A and B, where r stands for the distance function to a fixed point, say o, of M. In this paper, we say that M has *the strong Liouville property* if the dimension of $\mathcal{H}_d(M)$ is finite for any d (cf. [**19**]). It is known that the strong Liouville property holds on M, for example, if M is so called an asymptotically (locally) Euclidean space (cf. [**2**], [**3**], [**19**], [**22**] and the references therein), or if $m=2$ and the total curvature is finite (cf. [**14**], [**21**]). The purpose of this paper is to give some geometric conditions under which M has the strong Liouville property.

Our first result is stated in the following

THEOREM I. *Let M be a Hadamard manifold of dimension m. Suppose that M has minimal volume growth, that is,*

$$V_m(B(t)) \leqq At^m$$

for some positive constant A, where $V_m(B(t))$ denotes the volume of the metric ball $B(t)$ of radius t around a point o of M. Then M has the strong Liouville property, if in addition, the sectional curvature K_M of M decays to zero at a quadratic rate, namely,

$$K_M \geqq -\frac{B}{r^2}$$

for some positive constant B.

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The condition on volume growth is sharp in this theorem (cf. Remark 1.4), but it is not clear whether the latter on curvature can be deleted or not.

We shall prove secondly the following

THEOREM II. *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. Then M has the strong Liouville property, if in addition,*

(i) *M has maximal volume growth, that is,*

$$V_m(B(t)) \geqq At^m$$

for some positive constant A, where $m=\dim M$;

(ii) *the sectional curvature K_M of M decays to zero quadratically.*

In these theorems, we impose the decay condition on curvature of the manifold in consideration. Although this condition might be deleted in the theorems, our method of this paper depends heavily on it, because our approach to the problem is based on the convergence theory of Riemannian manifolds of bounded curvature. It would be worth noting that a class of complete manifolds with curvature of quadratic decay may contain a variety of manifolds with respect to the problem discussed in this paper (see Examples in Section 1).

Complete flat manifolds are typical examples which enjoy the strong Liouville property in our sence. Unfortunately, Theorem II does not cover these important cases (except Euclidean space), because of the condition on volume growth. In this regard, we shall attempt to replace this condition with those which all complete flat spaces satisfy (see Theorems III and IV in Sections 4 and 5, respectively).

We would like to mention here some earlier results related to this paper. A theorem by Yau [**28**] says that on a complete manifold M of nonnegative Ricci curvature, there are no nonconstant positive harmonic functions. Moreover according to a result due to S. Y. Cheng [**6**], any harmonic function h of sublinear growth, $|h|=o(r)$, must be constant on such a manifold (see also [**26**]). More recently, under the same assumption, Li and Tam [**20**] showed that $\dim \mathcal{H}_1(M)$ is less than or equal to $n+1$ if the volume of the metric ball of radius t is bounded by Ct^n for some positive constant C and an integer n, $0<n\leqq m=\dim M$. We note that there exist complete manifolds of positive Ricci curvature which admit nonconstant harmonic functions of linear growth (cf. [**18**]). To the contrary, in case that a complete manifold has nonnegative sectional curvature of quadratic decay, any closed harmonic one form of bounded length must be parallel, and hence it vanishes if in addition, the Ricci curvature is positive somewhere (cf. [**14**]). Finally, for a complete Riemannian manifold M, if we denote by $\kappa(M)$ the infimum of the nonnegative number d

such that the space $\mathcal{H}_d(M)$ contains nonconstant harmonic functions, then we have a lower bound for this number $\kappa(M)$ in terms of a lower bound of the Ricci curvature and an upper bound for *the diameter at infinity of* M in a certain sense (cf. [**14**], [**15**]). Wu [**27**] on the other hand discussed closely the interactions between the growth rate of the volume of a geodesic ball and the growth rate of the maximum modulas of a nonconstant subharmonic function and its differential.

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1. Weighted Sobolev spaces.

In this section, we shall introduce weighted Sobolev spaces of a complete Riemannian manifold in relation with the problem to be discussed.

Let M be a complete Riemannian manifold of dimension m, and let r be the distance function to a fixed point o of M. For a smooth function f with compact support, $f\in C_o^\infty(M)$, the weighted L^p norm $\|f\|_{p,\delta}$, $1<p<+\infty$, with weight $\delta\in\boldsymbol{R}$ is defined by

$$\|f\|_{p,\delta}=\left(\int_M |f|^p(1+r)^{-p\delta-m}dvol\right)^{1/p}.$$

The weighted Sobolev norms $\|f\|_{n,p,\delta}$, $n\in\boldsymbol{Z}^+$, are defined in the usual manner:

$$\|f\|_{n,p,\delta}=\sum_{i=0}^{n}\|D^i f\|_{p,\delta-i}\,.$$

We denote by $W_{n,p,\delta}(M)$ the closure of $C_o^\infty(M)$ with respect to this norms $\|*\|_{n,p,\delta}$.

Given p, $1<p<\infty$, and a weight $\delta\in\boldsymbol{R}$, we want to know whether the following estimate holds or not:

$$\|f\|_{2,p,\delta}\leqq C_1\|\Delta_M f\|_{p,\delta-2}+C_2\left(\int_{B(R)}|f|^p dvol\right)^{1/p} \tag{1.1}$$

for some positive constants C_1, C_2 and R, and for all $f\in W_{2,p,\delta}(M)$, where $B(R)$ stands for the metric ball of radius R around o, and Δ_M denotes the Laplace operator of M. This estimate is equivalent to the following:

$$\|f\|_{2,p,\delta}\leqq C\|\Delta_M f\|_{p,\delta-2} \tag{1.2}$$

for some positive constants C and R, and for all $f\in W_{2,p,\delta}(M)$ with $\mathrm{supp}\, f\subset M\diagdown B(R)$. If (1.1) (or equivalently (1.2)) holds, then it is clear that the kernel of $\Delta_M: W_{2,p,\delta}\to W_{p,\delta-2}$ is of finite dimensional. See e.g., [**3**], [**22**] for details and further topics.

Now we restrict our attention first to a special class of Riemannian manifolds, for which estimate (1.1) (or equivalently (1.2)) holds except the case that the weight δ belongs to some discrete subset of $\boldsymbol{R}$.

Let M be a complete Riemannian manifold the end of which is isometric to that of a cone. Namely, the complement of a compact subset K in M is isometric to the product of a half line $[R, \infty)$ and a compact manifold N with cone metric $dt^2+t^2ds^2$, where ds^2 is a Riemannian metric of N. For simplicity, we identify $M\setminus K$ with $[R, \infty)\times N$ equipped with the metric $dt^2+t^2ds^2$. In this case, it is known that (1.2) holds for those weights δ which do not belong to some discrete subset of $\boldsymbol{R}$. More generally, let us consider an elliptic differential operator of second order $\mathcal{L}$ in the form:

$$\mathcal{L}f = \frac{1}{\chi} div(\chi \nabla f),$$

where χ is a positive smooth function on M. We would like to restrict ourselves to simple cases and assume that $\chi(t, x)=t^l\theta(x)$ on $[R, \infty)\times N$, where $l\in\boldsymbol{R}$ and θ is a positive smooth function on N. Then we have the following

LEMMA 1.1. *If δ is nonexceptional, namely, it does not belong to some discrete subset of $\boldsymbol{R}$, then*

$$\|f\|_{2,p,\delta} \leqq C\|\mathcal{L}f\|_{p,\delta-2}$$

for some constant $C>0$, and for all $f\in W_{2,p,\delta}(M)$ with supp $f\subset M\setminus K$.

See e.g., [22] for this lemma.

REMARK 1.2. Lemma 1.1 still holds for the case that the metric ds^2 and the density function χ are of class $C^{1,\alpha}$ $(0<\alpha<1)$.

EXAMPLES.

Simple manifolds considered as above are typical examples of complete manifolds having the strong Liouville property. We observe that they also have the sectional curvature of quadratic decay. This curvature condition itself, however, never implies the strong Liouville property. Let us show simple examples of complete manifolds whose sectional curvature goes to zero at a rate $O(r^{-2})$.

We choose a metric g on $\boldsymbol{R}^4$ by setting

$$g = dr^2+u^2(\sigma_1^2+\sigma_2^2)+v^2\sigma_3^2,$$

where σ_i $(i=1, 2, 3)$ are the standard left invariant coframing of S^3 with σ_3 tangent to the Hopf fibers. The warping functions u, v are chosen to be functions of the distance r and assume the initial conditions: $u(0)=v(0)=0$, $u'(0)=v'(0)=1$. Now we take u, v so that

$$u(r) = Ar^a\,, \qquad v(r) = Br^b$$

for large r. Then we have the following assertions.

(i) *For* $a<1$, $\kappa(M)=+\infty$, *namely, any harmonic function of polynomial growth must be constant.*

(ii) *In case* $a>1$ *and* $-2a+b\leqq-1$, *the sectional curvature of* M *decays to zero at a rate* $O(r^{-2})$.

(iii) *For* $a=1$, M *has the strong Liouville property. Moreover*

$$\kappa(M) = \frac{1}{2}\{-(1+b)+\sqrt{(1+b)^2+32A^{-2}}\}$$

if $b<1$, *and*

$$\kappa(M) = \frac{1}{2}\{-(1+b)+\sqrt{(1+b)^2+8A^{-2}}\}$$

if $b\geqq 1$.

(iv) *In case* $a>1$, *the strong Liouville property does not hold for* M. *More precisely, setting* $c=-2a-b+1$, *we have the following assertions*:

(1) $\dim \mathcal{H}_0(M)=+\infty$ *and* $V_m(B(r))\sim r^{2-c}$ *if* $c<0$;

(2) $\dim \mathcal{H}_0(M)=1$, $\dim\{h:\Delta_M h=0,\ |h|\sim\log r\}=+\infty$, *and* $V_m(B(r))\sim r^2$ *if* $c=0$;

(3) $\kappa(M)=c$, $\dim \mathcal{H}_d(M)=+\infty$ *for any* $d>c$, *and* $V_m(B(r))\sim r^{2-c}$ *if* $0<c<2$;

(4) $\kappa(M)=c$, $\dim \mathcal{H}_d(M)=+\infty$ *for any* $d>c$, *and* $V_m(B(r))\sim\log r$ *if* $c=2$;

(5) $\kappa(M)=c$, $\dim \mathcal{H}_d(M)=+\infty$ *for any* $d>c$, *and* M *has finite volume if* $c>2$.

These assertions also show that some results by Wu [**27**] are sharp in a sense.

REMARK 1.3. Let us take u, v in such a way that

$$u(r) = Ar^a\,, \qquad v(r) = Be^{br}$$

for large r. Then $\kappa(M)=+\infty$ if $b<0$, and $\dim \mathcal{H}_0(M)=1$ if $a=0$ and $b>0$. In the former case, the intrinsic diameter of the geodesic sphere $S(t)$ grows like t^a when $a>0$. On the other hand, in the latter case, the volume $V_m(B(t))$ of the geodesic ball $B(t)$ grows like e^{bt} if $b>0$.

REMARK 1.4. Let us take $u=v$ and assume that $u'\geqq 1$ and $u''\geqq 0$. Then the sectional curvature K_M is nonpositive. Moreover if $u(r)=Ar^a$ for large r and $a>1$, then $\dim \mathcal{H}_0(M)=+\infty$. In this case, $V_m(B(t))\sim t^{3a+1}$ and K_M decays to zero at a quadratic rate. On the other hand, if $u(r)=Ar\log r$ for large r, then $\dim \mathcal{H}_0(M)=1$ and $\dim \mathcal{H}_d(M)=+\infty$ for any $d>0$. In this case, $V_m(B(t))\sim t^4(\log t)^3$ and K_M decays at a rate $O(\log r/r^2)$.

See [**15**] for the proofs of these assertions and remarks.

Now we shall prove the following

LEMMA 1.5. *Let M be a complete Riemannian manifold of dimension m. Suppose the sectional curvature decays in its absolute values at a rate $O(r^{-2})$. Then there is a positive constant β such that for any $d\geqq 0$, the space $\mathcal{H}_d(M)$ is included in the space $W_{2,p,\delta}(M)$, if $\delta>d-(m-1-\beta)/p$. In particular, M has the strong Liouville property, provided that estimate* (1.1) *or* (1.2) *holds for every nonexceptional weight δ.*

To prove this lemma, we recall here the following fact on harmonic coordinates.

LEMMA 1.6. *Let N be a Riemannian manifold of dimension n. Suppose the sectional curvature of N is bounded in its absolute values by a positive constant Λ, and the injectivity radius at a point x is bounded from below by a positive constant ι. Then given α, $0<\alpha<1$, there exist positive constants $a=a(n, \Lambda, \iota)$ $(<\iota)$, $b=b(n, \Lambda, \iota)$, $c=c(n, \Lambda, \iota, \alpha)$ and a coordinate system $H=(h_1, \cdots, h_n)$ defined on the metric ball $B(x, a)$ around x with radius a such that each component h_i $(i=1, \cdots, n)$ is a harmonic function and the metric tensor*

$$g=\sum_{i,j} g_{i,j}dh_i\otimes dh_j$$

in terms of this coordinates satisfies

$$e^{-b}|\xi|^2\leqq \sum_{i,j} g_{ij}\xi^i\xi^j\leqq e^b|\xi|^2$$

for all $\xi=(\xi_1, \cdots, \xi_n)$, and

$$\|g_{ij}\|_{C^{1,\alpha}}\leqq c\,.$$

See e.g., [**11**] for details.

PROOF OF LEMMA 1.5. Let f be a harmonic function of $\mathcal{H}_d(M)$. We want to show that $|df|$ (resp. $|Ddf|$) grows at a rate $O(r^{d-1})$ (resp. $O(r^{d-2})$). For a point x of M, we scale the metric g_M of M and consider the metric g_t defined by $g_t=g_M/t^2$ with $t=r(x)$. Then we take a sufficiently small positive constant a in such a way that the exponential mapping at x with respect to the scaled metric g_t has maximal rank on the ball $B^*(a)$ of radius a around the origin in the tangent space T_xM. Let us denote by g_t^* the pull-back of the metric on $B^*(a)$ via the exponential mapping. Then the sectional curvature of g_t^* is bounded uniformly in x by a positive constant. Hence by taking smaller a if necessarily, we have a harmonic coordinate system around the origin with the properties discribed in Lemma 1.6. Therefore we can apply the standard elliptic regularity estimates to the pull-back f^* of f (cf. e.g., [8]). In fact, f^* satisfies

$$\sum_{i,j=1,\cdots,n} g_t^{*ij} \frac{\partial^2 f^*}{\partial h_i \partial h_j} = 0\,,$$

and in particular

$$\|f^*\|_{C^2(B^*(a/2))} \leqq C \sup_{B^*(a)} |f^*|$$

for some positive constant C. Since

$$\|f^*\|_{C^2(B^*(a/2))} = \|f\|_{C^0(B(x,at/2))} + t\|df\|_{C^0(B(x,at/2))} + t^2\|Ddf\|_{C^0(B(x,at/2))}\,,$$

we see that $|df| = O(r^{d-1})$ and $|Ddf| = O(r^{d-2})$.

Now it follows from the Rauch's comparison theorem and the (lower) bound for the sectional curvature that, in terms of the polar coordinate around a reference point o, the volume element of M grows at most like r^γ for some positive constant γ (depending on A). Thus the assertion of the lemma is clear.

The scaling argument as above together with Lemma 1.6 will be often used in the subsequent sections.

2. Proof of Theorem I.

Let M be a Hadamard manifold of dimension m and g_M the metric of M. We shall consider the equivalence classes of rays on M, denoted by $M(\infty)$, equipped with the Tits metric Td (cf. [1]). For a point $z=(\tau, q)$ of the product $[0, \infty) \times M(\infty)$ and a positive number t, it is convenient to write tz for $(t\tau, q)$ and also we shall identify $M(\infty)$ with $\{1\} \times M(\infty)$ if there is no confusion. Let us fix a point, say o, of M and denote by $S(t)$ and $B(t)$, respectively, the metric sphere and the metric ball around o of radius t. We write r for the distance function to the point o. We first observe from the Rauch's comparison theorem that

$$V_{m-1}\Big(\frac{1}{t} S(t)\Big) \geqq \omega_{m-1} > 0\,, \tag{2.1}$$

where ω_{m-1} stands for the volume of Euclidean unit sphere of dimension $m-1$. For a point x of $M \setminus \{o\}$, we take a unique ray σ_x of M starting at o and going through x, and we set

$$\Phi_0(x) = r(x)[\sigma_x], \quad \text{or} \quad (r(x), [\sigma_x]),$$

where $[\sigma]$ denotes the equivalence class of a ray σ.

We suppose first that the diameter of $M(\infty)$ is finite, namely,

$$diam(M(\infty)) \leqq D \tag{2.2}$$

for some constant $D>0$. Then as t goes to infinity, the scaled metric sphere $S(t)/t$ converges to $M(\infty)$ with respect to the Gromov-Hausdorff distance (cf. [**1**], [**10**]). In fact, in this case, we see that

$$(2.3)\qquad Td\Big(\frac{1}{t}\Phi_0(x),\ \frac{1}{t}\Phi_0(y)\Big) \geqq d_t(x,\ y) \geqq Td\Big(\frac{1}{t}\Phi_0(x),\ \frac{1}{t}\Phi_0(y)\Big)-\varepsilon(t)$$

for all x, y of $S(t)/t$, where d_t denotes the intrinsic distance on $S(t)/t$ and $\varepsilon(t)$ is a positive constant which tends to zero as t goes to infinity.

We suppose next the sectional curvature K_M of M satisfies

$$(2.4)\qquad K_M \geqq -\frac{B}{r^2}$$

for some constant $B>0$. Then for any $t>0$, the second fundamental form Π_t of the metric sphere $S(t)$ satisfies

$$1 \leqq t\Pi_t \leqq C_1$$

for some positive constant C_1. Hence if $m\geqq 3$, the sectional curvature K_t of $S(t)/t$ is pinched as follows:

$$(2.5)\qquad 1-B \leqq K_t \leqq B'$$

for all t and some constant $B'>0$. Therefore we can apply the $C^{1,\alpha}$ convergence theorem to this family $\{S(t)/t\}$ and the limit $M(\infty)$ (cf. [**13**], [**9**], [**25**]), and we can assert that $M(\infty)$ is a smooth manifold of dimension $m-1$ and there is a Riemannian metric ds_∞^2 of class $C^{1,\alpha}$ (for any $\alpha\in(0,1)$) which induces the distance Td on $M(\infty)$. We write $CM(\infty)$ for the cone over $M(\infty)$, namely,

$$CM(\infty) = ([0,\ \infty)\times M(\infty),\ g_\infty = dt^2+t^2ds_\infty^2).$$

We remark here that *under the assumption* (2.4), *M has minimal volume growth, $V_m(B(t))\leqq At^m$, if and only if* (2.2) *holds*. In fact, it follows from (2.4) that for some positive constants L and C_2, and for any t and every smooth curve γ in $S(t)$ joining two points x and y,

$$Length(\gamma) \leqq C_2\, dis_M(x,\ y)$$

if γ is a geodesic with respect to the intrinsic metric of $S(t)$ and the length is bounded by Lt. We can also deduce from (2.2) and (2.4) that if $m=2$, then M has finite total curvature, and actually,

$$1 = \frac{1}{2\pi}\int_M K_M + \frac{1}{2\pi} Length(M(\infty))$$

(cf. e.g., the arguments in [**13**: § 3]). For this reason, we may assume that $\dim M \geqq 3$.

Now we want to construct an asymptotically isometry between the end of M and that of the tangent cone at infinity $CM(\infty)$, which approximates the canonical map $\Phi_0: M\to CM(\infty)$ at infinity. For this, we first fix a large number R and a small one c, in such a way that for any point x of $M\setminus B(R)$, we can take a harmonic coordinate system $H_x: B(x, 10c)/t\to \boldsymbol{R}^m$ defined on the scaled metric ball $B(x, 10c)/t$ around x of radius $10c$ with respect to the scaled metric $g_t=g_M/t$ with $t=r(x)$. Here the coordinates H_x are assumed to have the same properties as discribed in Lemma 1.6. Let $B_\infty(q_x, 10c)$ be the metric ball in $CM(\infty)$ around $q_x=\Phi_0(x)/t$ of radius $10c$. We shall now consider the following Dirichlet problem in the metric ball $B_\infty(q_x, 2c)$ of $CM(\infty)$:

$$\begin{aligned} \Delta_\infty F &= 0 && \text{in } B_\infty(q_x, 2c) \\ F &= H_x\circ\left(\frac{1}{t}\Psi_0\right) && \text{on } \partial B_\infty(q_x, 2c). \end{aligned}$$

Here we set

$$\frac{1}{t}\Psi_0(\tau, q)=\Phi_0^{-1}(t\tau, q).$$

Let $F_x=(F_x^1, \cdots, F_x^m)$ be the solution of the above equation. Then since Ψ_0/t is a Lipschitz map with dilatation $\leqq 1$ because of (2.3), it turns out from the standard elliptic regularity estimates that for any $\alpha\in(0, 1)$,

$$|F_x|_{C^{3,\alpha}(B_\infty(q_x, c))}\leqq C_3, \tag{2.6}$$

where C_3 is a positive constant independent of x. If we set

$$\Psi_x(\tau, q)=H_x^{-1}\left(F_x\left(\frac{\tau}{t}, q\right)\right) \qquad (t=r(x)),$$

then we obtain a map Ψ_x of $B_\infty(\Phi_0(x), tc)$ into $B(x, tc)$ which coincides with Φ_0^{-1} on the boundary $\partial B_\infty(x, tc)$. It is not hard to deduce from (2.3) and (2.6) that for some positive constant $\delta(t)$ independent of x with $\lim_{t\to\infty}\delta(t)=0$,

$$\max\left\{\frac{1}{t}dis_M(\Psi_x(z), \Phi_0^{-1}(z)): z\in B_\infty(\Phi_0(x), tc)\right\}\leqq\delta(t),$$

$$\max\left\{\frac{1}{t^2}|\Psi_x^*g_M-g_\infty|(z): z\in B_\infty(\Phi_0(x), tc)\right\}\leqq\delta(t),$$

$$\max\{t|Dd\Psi_x|(z): z\in B_\infty(\Phi_0(x), tc)\}\leqq\delta(t),$$

where $Dd\Psi_x$ denotes the second fundamental form of the map Ψ_x.

Now taking a sufficiently large R and a positive constant $c'<c$, we may assume that for any $x\in M\setminus B(R)$, Ψ_x is a diffeomorphism of $B_\infty(\Phi_0(x), tc)$ into $B(x, tc)$ and the image $\Psi_x(B_\infty(\Phi_0(x), tc))$ contains the metric ball $B(x, tc')$. Hence we have a diffeomorphism Φ_x of $B(x, tc')$ into $B_\infty(\Phi_0(x), tc)$ by setting $\Phi_x=\Psi_x^{-1}$.

Then Φ_x enjoys the following properties:

$$\max\left\{\frac{1}{t}\, dis_{CM(\infty)}(\Phi_x(y),\ \Phi_0(y)):\ y\in B(x,\ tc')\right\} \leqq \delta'(t)$$

$$\max\left\{\frac{1}{t^2}\,|\Phi_x^* g_\infty - g_M|(y):\ y\in B(x,\ tc')\right\} \leqq \delta'(t) \tag{2.7}$$

$$\max\ \{t|Dd\Phi_x|(y):\ y\in B(x,\ tc')\} \leqq \delta'(t),$$

where $t=r(x)$ and $\delta'(t)$ is as before a positive constant independent of the choice of x with $\lim_{t\to\infty}\delta'(t)=0$. Thus we have obtained local approximations $\{\Phi_x\}$ $(x\in M\setminus B(R))$ as above for the canonical map Φ_0.

Now let us choose a finite number of points of $M(\infty)$, $q_1, \cdots, q_\nu$, and a sufficiently small constant ε in such a way that if we put $d=1+\varepsilon$, $q_{j\mu}=(d^j,\ q_\mu)$, and $x_{j,\mu}=\Phi_0^{-1}(q_{j,\mu})$, then

$$B(d^{j+1})\setminus B(d^{j-1}) \subset \bigcup_{1\leq\mu\leq\nu} B(x_{j,\mu},\ c'd^j) \subset B(d^{j+2})\setminus B(d^{j-2}).$$

Taking a sufficient large j_0, we may assume that $x_{j,\mu}\in M\setminus B(R)$ for any $j\geqq j_0$, and

$$M\setminus B(2R) \subset \bigcup_{j\geq j_0,\ 1\leq\mu\leq\nu} B(x_{j,\mu},\ c'd^j).$$

Let us write $\Phi_{j,\mu}$ for the maps $\Phi_{x_{j,\mu}}$ of $B(x_{j,\mu},\ c'd^j)$ into $B_\infty(q_{j,\mu},\ cd^j)$ satisfying (2.7) with $t=d^j$. Making use of these maps $\{\Phi_{j,\mu}\}$, we can take a partition of unity $\{\xi_{j,\mu}\}$ on $M\setminus B(2R)$ subordinate to the covering $\{B(x_{j,\mu},\ c'd^j)\}$ so that

$$|\xi_{j,\mu}| + d^j|d\xi_{j,\mu}| + d^{2j}|Dd\xi_{j,\mu}| \leqq C_4$$

for some positive constant C_4 which is independent of j and μ. We shall assume here that the limit metric ds_∞^2 on $M(\infty)$ is smooth, and hence so is the metric g_∞. Then we have uniquely a smooth map Φ of $M\setminus B(2R)$ into $CM(\infty)$, called a center of mass of $\{\Phi_{j,\mu}\}$ with weights $\{\xi_{j,\mu}\}$, which is given by

$$\sum_{j\geq j_0,\ 1\leq\mu\leq\nu} \xi_{j,\mu}(x)\exp_{\Phi(x)}{}^{-1}\Phi_{j,\mu}(x) = 0$$

for each x of $M\setminus B(2R)$. In view of (2.7), it is not hard to see that for sufficiently large R, this map Φ induces a diffeomorphism of $M\setminus B(2R)$ into the cone $CM(\infty)$ such that

$$\lim_{x\to\infty}\frac{1}{r(x)}\, dis_{CM(\infty)}(\Phi(x),\ \Phi_0(x)) = 0;$$

$$\lim_{x\to\infty}\frac{1}{r(x)^2}\,|\Phi^* g_\infty - g_M|(x) = 0; \tag{2.8}$$

$$\lim_{x\to\infty} r(x)|Dd\Phi|(x) = 0$$

(cf. [**16**: Appendix]). Obviously the image $\Phi(M\setminus B(2R))$ is contained in $[R', \infty)\times M(\infty)$ for some R' with $\lim_{R\to\infty} R'/R=1$.

For the construction of the above map Φ, we have assumed that the metric ds_∞^2 is smooth. When ds_∞^2 is not smooth, we can employ smooth approximations of the metric in the above argument. Namely, we have a sequence of smooth metrics, ds_n^2, which converges to ds_∞^2 in the $C^{1,\alpha}$ topology for any $\alpha\in(0, 1)$. Using the exponential maps with respect to these smooth approximations step by step as j goes to infinity, we can obtain the map Φ as above. We refer the reader to [**16**] for similar arguments.

We are now in a position to complete the proof of Theorem I. In what follows, we denote by $\varepsilon_*(R)$'s positive constants which tend to zero as R goes to infinity. For any smooth function f on M whose support is compact and lies in $M\setminus B(2R)$, we set $f^*=f\circ\Phi^{-1}$. Then in view of (2.8), we can compare the weighted Sobolev norms of f and f^*. To be precise, given $p>1$ and $\delta\in\boldsymbol{R}$, it follows from (2.8) that

$$e^{-\varepsilon_1(R)}\|f\|_{2,p,\delta}\leqq\|f^*\|_{2,p,\delta}\leqq e^{\varepsilon_1(R)}\|f\|_{2,p,\delta}\,. \tag{2.9}$$

We note also that

$$e^{-\varepsilon_2(R)}\|\Delta_\infty f^*\|_{p,\delta-2}\leqq\|(\Delta_\infty f^*)\circ\Phi\|_{p,\delta-2}\leqq e^{\varepsilon_2(R)}\|\Delta_\infty f^*\|_{p,\delta-2}\,. \tag{2.10}$$

Moreover we observe from (2.8) that

$$\begin{aligned}|\Delta_M f-(\Delta_\infty f^*)\circ\Phi| &= |df^*(tr.\,Dd\Phi)+tr.(Ddf^*(d\Phi,\, d\Phi))-(\Delta_\infty f^*)\circ\Phi| \\ &\leqq \varepsilon_3(R)\Big(\frac{1}{r}|df^*|\circ\Phi+|Ddf^*|\circ\Phi\Big).\end{aligned}$$

This implies that

$$\begin{aligned}\|\Delta_M f-(\Delta_\infty f^*)\circ\Phi\|_{p,\delta-2} &\leqq \varepsilon_4(\|df^*\|_{p,\delta-1}+\|Ddf^*\|_{p,\delta-2}) \\ &\leqq \varepsilon_4(R)\|f^*\|_{2,p,\delta}\,.\end{aligned}$$

Hence we have by (2.9)

$$\|\Delta_M f-(\Delta_\infty f^*)\circ\Phi\|_{p,\delta-2}\leqq\varepsilon_5(R)\|f\|_{2,p,\delta}\,. \tag{2.11}$$

Here we recall that (1.2) holds for $CM(\infty)$ if δ is nonexceptional. Therefore for such a weight δ, we can derive that

$$\begin{aligned}\|f\|_{2,p,\delta} &\leqq C_5\|f^*\|_{2,p,\delta} && \text{by (2.9)} \\ &\leqq C_6\|\Delta_\infty f^*\|_{p,\delta-2} && \text{by (1.2)} \\ &\leqq C_7\|(\Delta_\infty f^*)\circ\Phi\|_{p,\delta-2} && \text{by (2.10)} \\ &\leqq C_8(\|\Delta_M f-(\Delta_\infty f^*)\circ\Phi\|_{p,\delta-2}+\|\Delta_M f\|_{p,\delta-2}) \\ &\leqq \varepsilon_6(R)\|f\|_{2,p,\delta}+C_9\|\Delta_M f\|_{p,\delta-2} && \text{by (2.11).}\end{aligned}$$

Here C_*'s are some positive constants. Thus if we take R so large that $\varepsilon_6(R) \leqq 1/2$, then we obtain

$$\|f\|_{2,p,\delta} \leqq 2C_{10}\|\Delta_M f\|_{p,\delta-2}$$

for all $f \in C_0^\infty(M \setminus B(2R))$. This proves that M has the strong Liouville property.

REMARKS. Let M be a Hadamard manifold of dimension m.

(i) The strong Liouville property does not follow from condition (2.2) only. For example, the Riemannian product of hyperbolic space form and Euclidean space satisfies (2.2) (with $D=\pi$).

(ii) Suppose $m \geqq 3$ and condition (2.4) is satisfied with $B<1$. Then by (2.5), the (intrinsic) diameter of the scaled metric sphere $S(t)/t$ is bounded from above by $\pi/\sqrt{1-B}$, and hence so is the diameter of $M(\infty)$. Thus in this case, M has the strong Liouville property.

3. Proof of Theorem II.

In this section, we shall first recall the counter part to the Tits' geometry for complete manifold of nonnegative sectional curvature and then carry out the proof of Theorem II, using the same idea as in the preceding one for Theorem I.

Let M be a complete noncompact manifold with a Riemannian metric g_M of nonnegative sectional curvature. To begin with, we shall recall the counter part to the Tits' geometry for M introduced in [**1**, pp. 58-59]. Let us first fix a point, say o, of M as before, and consider the set $\mathcal{R}_0$ of all rays starting at o. Given two elements γ, σ of $\mathcal{R}_o$, let $\Delta(s, t)$, $0<s<t$, be the triangle sketched on $\boldsymbol{R}^2$ whose edge length are s, t and $dis_M(\gamma(s), \sigma(t))$, and denote by $\tilde{\angle}\gamma(s)o\sigma(t)$ the angle of $\Delta(s, t)$ opposite to the edge of length $dis_M(\gamma(s), \sigma(t))$. Then the Toponogov's theorem says that $(s, t) \to \tilde{\angle}\gamma(s)o\sigma(t)$ is monotone nonincreasing in the sense that $\tilde{\angle}\gamma(s')o\sigma(t') \leqq \tilde{\angle}\gamma(s)o\sigma(t)$ if $s \leqq s'$ and $t \leqq t'$. Hence we have the limit

$$\lim_{s,t\to\infty} \tilde{\angle}\gamma(s)o\sigma(t)$$

for all γ, σ of $\mathcal{R}_o$. Let us introduce an equivalent relation $\sim$ on $\mathcal{R}_o$ by setting $\gamma \sim \sigma$ when the above limit is equal to zero. Now on the set of all equivalence classes of such rays is defined naturally a distance $\angle_\infty$ by

$$\angle_\infty([\gamma], [\sigma]) = \lim_{t\to\infty} \tilde{\angle}\gamma(t)o\sigma(t),$$

where $[\gamma]$ stands for the equivalence class of a ray γ. Then we have a metric space $M(\infty)=(\mathcal{R}_o/\sim, \angle_\infty)$. The tangent cone at infinity $CM(\infty)$ of M is given by the product of $[0, \infty)$ and $M(\infty)$ equipped with the distance

$$dis_{CM(\infty)}((s, q), (t, p)) = \sqrt{s^2+t^2+2st\cos\angle(q, p)}.$$

Moreover if we set $M_0=\{\gamma(t): \gamma\in\mathcal{R}_0, t\geqq 0\}$, we have a natural map Φ_0 of M_0 onto $CM(\infty)$ defined by

$$\Phi_0(\gamma(t)) = (t, [\gamma]).$$

When $M(\infty)$ consists of only one point, we know that $\kappa(M)=+\infty$, namely, there are no nonconstant harmonic functions of polynomial growth (cf. [**14**], [**15**]). For this reason, we assume that the diameter of $M(\infty)$ is positive.

Now we suppose that the sectional curvature K_M of M satisfies

$$K_M \leqq \frac{B}{r^2} \tag{3.1}$$

for some positive constant B. Then for fixed numbers a, b with $0<a<b$, we have a family of compact Riemannian manifolds with boundary $\{A(at, bt)/t\}$ of uniformly bounded curvature and diameter. Here and after $A(s, t)$ denotes the annular domain $B(t)\setminus\overline{B(s)}$ in M, and also we write $A_\infty(s, t)$ for $(s, t)\times M(\infty)$. Via the natural map Φ_0/t, we know that as t goes to infinity, $A(at, bt)/t$ converges to $A_\infty(a, b)$ with respect to the Gromov-Hausdorff distance. It should be noted that the limit space $A_\infty(a, b)$ is the quotient space of a smooth manifold with $C^{1,\alpha}$ metric by an isometric action of $O(m)$ (see [**7**]) and hence it may not be a smooth manifold. In order to construct a smooth approximation of the map Φ_0, we suppose that *$A_\infty(a, b)$ is a smooth manifold, or equivallently, $M(\infty)$ is a smooth manifold.* Let us denote by ds^2_∞ the metric of class $C^{1,\alpha}$, $0<\alpha<1$, which induces the distance $\angle_\infty$ of $M(\infty)$. In addition, we write as before g_∞ for the metric of the tangent cone at infinity $CM(\infty)$, namely, $g_\infty= dt^2+t^2ds^2_\infty$.

We would like to employ the same method as in the preceding section. However in the present case, the map Φ_0 is defined only on the subset M_0 of M. For this reason, we need first to approximate the map Φ_0 by a smooth one defined on an end of M. In fact, by virtue of the assumption of $M(\infty)$ being smooth, we can apply the method of smoothing Hausdorff approximations developed in [**4**: § 2], and we see that for large R, there is a smooth submersion Φ_1 of $M\setminus B(R)$ mapping over the outside of a compact set of $CM(\infty)$ which satisfies

$$\begin{gathered}\lim_{M_0\ni x\to\infty}\frac{1}{r(x)}dis_{CM(\infty)}(\Phi_0(x), \Phi_1(x)) = 0\,;\\ \lim_{x\to\infty}\frac{1}{r(x)^2}|\Phi_1^*g_{\infty|\mathcal{H}_x}-g_{M|\mathcal{H}_x}| = 0\,;\\ \limsup_{x\to\infty} r(x)|Dd\Phi_1|(x) < +\infty\,,\end{gathered} \tag{3.2}$$

where $\mathcal{H}_x$ denotes the horizontal subspace of T_xM, namely the orthogonal complement of the vertical subspace consisting of vectors tangent to the fibre $\Phi_1^{-1}(\Phi_1(x))$ over the point $\Phi_1(x)$ of $CM(\infty)$.

PROOF OF THEOREM II. Let M be a complete noncompact manifold of nonnegative sectional curvature which satisfies (3.1) and further which has maximal volume growth, namely,

$$V_m(B(t)) \geqq At^m \tag{3.3}$$

for some positive constant A, where $m=\dim M$. Then for some positive constants C and D, the injectivity radius of M at a point x is bounded from below by $Cr(x)+D$ (cf. [**6**]). Hence as in the case of Theorem I, we can apply the $C^{1,\alpha}$ convergence theorem and deduce that $A_\infty(a, b)$ is a smooth manifold of dimension m, to which $A(at, bt)/t$ converges in the $C^{1,\alpha}$ topology as t goes to infinity. Therefore taking R sufficiently large, we may assume that the map Φ_1 is actually a diffeomorphism between $M\setminus B(R)$ and $CM(\infty)\setminus K$. Moreover repeating the same arguments as in the proof of Theorem I, we are able to obtain a diffeomorphism Φ of $M\setminus B(R)$ onto the outside of a compact set in $CM(\infty)$ such that

$$\begin{gathered}\lim_{M_0\ni x\to\infty}\frac{1}{r(x)}dis_{CM(\infty)}(\Phi_0(x),\ \Phi(x))=0\,;\\ \lim_{x\to\infty}\frac{1}{r(x)^2}|\Phi^*g_\infty-g_M|=0\,;\\ \lim_{x\to\infty}r(x)|Dd\Phi|=0\,.\end{gathered}\tag{3.4}$$

Thus by the same reasons as in the proof of Theorem I, we can conclude that M has the strong Liouville property. This completes the proof of Theorem II.

4. Measured Hausdorff convergence at infinity.

Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. Throughout this section, we assume that *the sectional curvature of M decays at a rate $O(r^{-2})$ and further the points at infinity $M(\infty)$ is smooth*. Our concern is the case M does not have maximal volume growth, that is, $V_m(B(t))/t^m$ tends to zero as t goes to infinity, where $m=\dim M$. At this stage, it is unclear whether M satisfies the strong Liouville property, even if $M(\infty)$ is assumed to be smooth. The purpose of this section is to introduce a notion of measured Hausdorff convergence at infinity and show that the strong Liouville property holds if M satisfies such an additional condition.

Let M be as above, and let Φ_0 be the canonical map of M_0 into the tangent cone at infinity $CM(\infty)$ defined in Section 3. Then by virtue of $M(\infty)$ being smooth, we have shown that for large R, there is a smooth submersion Φ_1 of $M\setminus B(R)$ over the outside of a compact set in $CM(\infty)$, which satisfies the properties described in (3.2). We fix two positive numbers a and b with $a<b$ as before. Let μ_t be the push-forward of the normarized Riemannian measure $dvol/V_m(A(at, bt))$ by the map Φ_1/t. The density of μ_t with respect to the Riemannian measure $dvol_\infty$ of $A_\infty(a, b)$ is given by

$$\chi_t(\tau, q)=\frac{t^k V_{m-k}(\Phi_1^{-1}(t\tau, q))}{V_m(A(at, bt))}$$

where $k=\dim M(\infty)+1$, and t is assumed to be sufficiently large. Then χ_t converges almost everywhere on $A_\infty(a, b)$ as t goes to infinity if and only if $A(at, bt)/t$ converges to A_∞ with respect to the measure Hausdorff topology via the map Φ_1/t of $A(at, bt)/t$ into the tangent cone $CM(\infty)$ at infinity, namely,

$$\mu_t := \frac{1}{t}\Phi_{1*}(dvol/V_m(A(at, bt))) \longrightarrow \mu_\infty \tag{4.1}$$

in the weak* topology. We note that (4.1) does not depend on the choice of an approximation Φ_1 as above of Φ_0 nor the choice of positive constants a and b (cf. the proof of Lemma 4.1). For this reason, we shall say that M *is tangent at infinity to the cone* $CM(\infty)$ *with respect to the measured Hausdorff topology,* if the condition (4.1) holds. It is not clear that this is true for the manifold M in consideration.

Now our result is stated in the following

THEOREM III. *A complete noncompact Riemannian manifold M of nonnegative sectional curvature has the strong Liouville property, if in addition,*

(i) *the sectional curvature of M decays at a quadratic rate;*

(ii) *the points at infinity $M(\infty)$ of M is a smooth manifold;*

(iii) *M is tangent at infinity to the cone $CM(\infty)$ over $M(\infty)$ with respect to the measured Hausdorff topology.*

REMARK. This theorem is a generalization of Theorem II, since under the condition (i), the others (ii) and (iii) are satisfied automatically if M has maximal volume growth.

PROOF OF THEOREM III.

STEP 1. We begin with recalling that the limit measure μ_∞ has the positive density χ_∞ of class $C^{1,\alpha}$, $0<\alpha<1$ (cf. [**16**: §1]). Since the length of the gradient of χ_t is uniformly bounded in t by (3.2), χ_t converges to χ_∞ uniformly in the annular domain $A_\infty(a, b)$. Moreover we have the following

LEMMA 4.1. *The density $\chi_\infty(\tau, q)$ of the limit measure μ_∞ can be written as*

$$\chi_\infty(\tau, q) = \tau^l \theta(q)$$

for a constant l and a positive function θ on $M(\infty)$ of class $C^{1,\alpha}$.

The proof of this lemma will be given after Lemma 4.3 in the next step.

Now we define an elliptic differential operator $\mathcal{L}_\infty$ of divergent form on $A_\infty(a, b)$ by

$$\mathcal{L}_\infty h = \frac{1}{\chi_\infty} div(\chi_\infty \nabla h) = \Delta_\infty h + \nabla \log \chi_\infty \cdot h .$$

By virtue of Lemma 4.1, this operator $\mathcal{L}_\infty$ is defined on $C^*M(\infty)$ $(:= (0, +\infty) \times M(\infty))$, which can be written as

$$\mathcal{L}_\infty = \frac{\partial^2}{\partial \tau^2} + \frac{k-1+l}{\tau}\frac{\partial}{\partial \tau} + \frac{1}{\tau^2}\hat{\mathcal{L}}_\infty ,$$

where $\hat{\mathcal{L}}_\infty$ is the operator on $M(\infty)$ given by

$$\hat{\mathcal{L}}_\infty u = \frac{1}{\theta} div(\theta \nabla u) .$$

Therefore as mentioned in Lemma 1.1, we have the following

LEMMA 4.2. *Given $p>1$, a nonexceptional weight δ, and $R>0$, there is a constant C such that*

$$\|h\|_{2,p,\delta} \leqq C \|\mathcal{L}_\infty h\|_{p,\delta-2}$$

for any $h \in C_0^\infty(C^*M(\infty))$ with supp $h \subset A_\infty(R, \infty)$.

STEP 2. In this stage, we want to construct a smooth approximation Φ for Φ_1 and hence for Φ_0, so that we can investigate more closely the Laplace operator of M.

We first fix sufficiently small positive constants c, c' and c'' with $c''<c'<c<1$. Let $q=(1, q)$ be a point of $CM(\infty)$, and $H=(h_1, \cdots, h_k)$ be a coordinate system around q defined on the metric ball $B_\infty(q, c)$ in $CM(\infty)$ such that

$$\mathcal{L}_\infty H = 0 .$$

For large t, we take a point x of M so that $\Phi_1(x)=tq$. We may assume that

$$B_\infty(q, c'') \subset \frac{1}{t}\Phi_1(B(x, tc')) \subset B_\infty(q, c) ,$$

because of (3.2). We write $\tilde{B}(x, tc'')$ for the inverse image of $B_\infty(tq, c''t)$ by Φ_1. Let $F=(f_1, \cdots, f_k)$ be the solution of the Dirichlet problem:

$$\Delta_M F = 0 \qquad \text{in } \tilde{B}(x, tc'')$$

$$F = H\circ\frac{1}{t}\Phi_1 \qquad \text{on } \partial\tilde{B}(x, tc'').$$

Then we have a smooth map Φ_x of $\tilde{B}(x, tc'')$ into $B^\infty(tq, ct)$ defined by

$$\Phi_x = t(H^{-1}\circ F)\,.$$

Now we choose a finite number of points $\{q_\mu\}$ of $M(\infty)$ and a sufficiently small positive constant ε in such a way that if we put $d=1+\varepsilon$, and if we take points $x_{j,\mu}$ of M with $\Phi_1(x_{j,\mu})=d^j q_\mu$, then

$$A(d^{j-1}, d^{j+1}) \subset \bigcup_\mu \tilde{B}(x_{j,\mu}, c''d^j) \subset A(d^{j-2}, d^{j+2})$$

for all large j. Let us write $\Phi_{j,\mu}$ for the map $\Phi_{x_{j,\mu}}$ of $\tilde{B}(x_{j,\mu}, c''d^j)$ into $B_\infty(d^j q_\mu, cd^j)$. Then using the center-of-mass technique and repeating the same arguments as in the proof of Theorem I, we can obtain a smooth submersion Φ of an end of M, say $M\setminus B(R)$, over an end of $CM(\infty)$, say $CM(\infty)\setminus K$. Following [**16**], we want to describe some important properties of this submersion $\Phi: M\setminus B(R)\to CM(\infty)\setminus K$. For this, it is convenient to fix several notations (cf. [**24**]). For a point x of $M\setminus B(R)$ and a tangent vector $E\in T_xM$, we denote by $\mathcal{V}_x$, $\mathcal{H}_x$, $\mathcal{V}E$ and $\mathcal{H}E$, respectively, the subspace of T_xM which consists of vectors tangent to the fiber $\Phi^{-1}(\Phi(x))$ through x, the orthogonal complement of $\mathcal{V}_x$ in T_xM, the $\mathcal{V}$-component of E and the $\mathcal{H}$-component of E. Also $\mathcal{V}$ and $\mathcal{H}$ denote the projections onto them. For a vector field X on an open subset U of $CM(\infty)\setminus K$, we write $\hat{X}$ for the vector field on $\Phi^{-1}(U)$ satisfying

$$\hat{X}_x \in \mathcal{H}_x, \qquad d\Phi(\hat{X}_x) = X_{\Phi(x)}$$

for $x\in\Phi^{-1}(U)$. We define (2, 1) tensor fields $A(\Phi)$ and $T(\Phi)$ on $M\setminus B(R)$ whose values on vector fields E_1 and E_2 are respectively given by

$$A(\Phi)(E_1, E_2) = \mathcal{H}D_{\mathcal{H}E_1}\mathcal{V}E_2+\mathcal{V}D_{\mathcal{H}E_1}\mathcal{H}E_2\,,$$

$$T(\Phi)(E_1, E_2) = \mathcal{H}D_{\mathcal{V}E_1}\mathcal{V}E_2+\mathcal{V}D_{\mathcal{V}E_1}\mathcal{H}E_2\,.$$

Moreover let us introduce a (2, 1) tensor field $B(\Phi)$ on $M\setminus B(R)$ by setting

$$B(\Phi)(E_1, E_2) = \mathcal{H}D_{E_1}E_2-(D_{d\Phi(E_1)}d\Phi(E_2))^\wedge,$$

so that

$$d\Phi(B(\Phi)(E_1, E_2)) = -Dd\Phi(E_1, E_2).$$

Then by the construction of Φ and the same arguments as in [**16**: § 2], we can deduce the following

LEMMA 4.3. (i) *Φ approximates Φ_0 in the sense that*

$$\lim_{M_0 \ni x \to \infty} \frac{1}{r(x)} dis_{CM(\infty)}(\Phi_0(x), \Phi(x)) = 0.$$

(ii) *Φ is an asymptotically Riemannian submersion, that is,*

$$\lim_{M \ni x \to \infty} \min_{X \in \mathcal{H}_x} \frac{|d\Phi(X)|}{|X|} = \lim_{M \ni x \to \infty} \max_{X \in \mathcal{H}_x} \frac{|d\Phi(X)|}{|X|} = 1.$$

(iii) *If η stands for the mean curvature vector field along the fibers of Φ, then*

$$\lim_{M \ni x \to \infty} r(x) | d\Phi_x(\eta) + \nabla \log \chi_\infty(\Phi(x)) | = 0.$$

(iv) *The second fundamental form $Dd\Phi$ of Φ satisfies*

$$\lim \sup_{M \ni x \to \infty} r(x) | Dd\Phi | (x) < +\infty ,$$

and hence we have in particular

$$\lim \sup_{M \ni x \to \infty} r(x) | A(\Phi) | (x) < +\infty ,$$

$$\lim \sup_{M \ni x \to \infty} r(x) | T(\Phi) | (x) < +\infty .$$

(v) *Finally, given constants $p>1$, and a, b with $0<a<b$,*

$$\lim \sup_{t \to \infty} \int_{A(at, bt)} r^p | D\eta |^p \frac{dvol}{V_m(A(at, bt))} < +\infty ,$$

$$\lim \sup_{t \to \infty} \int_{A(at, bt)} r^p | DDd\Phi |^p \frac{dvol}{V_m(A(at, bt))} < +\infty .$$

Making use of this lemma, we shall verify Lemma 4.1.

PROOF OF LEMMA 4.1. We shall first define a positive smooth function $\hat{\chi}_t$ on $A_\infty(a, b)$ by

$$\hat{\chi}_t(\tau q) = \frac{t^k V_{m-k}(\Phi^{-1}(t\tau q))}{V_m(A(at, bt))} .$$

Since Φ approximates Φ_0 as in Lemma 4.3 (i), $\hat{\chi}_t(\tau q)$ converges uniformly to χ_∞. In view of the identity

$$\hat{\chi}_t(\tau q) = \tau^{-k} \frac{V_m(A(at\tau, bt\tau))}{V_m(A(at, bt))} \hat{\chi}_{t\tau}(q) ,$$

we see that

$$\lim_{t \to \infty} \tau^{-k} \frac{V_m(A(at\tau, bt\tau))}{V_m(A(at, bt))} = \frac{\chi_\infty(\tau q)}{\chi_\infty(q)} .$$

The left-hand side is clearly independent of points q of $M(\infty)$, and hence we may put

$$\xi(\tau) = \frac{\chi_\infty(\tau q)}{\chi_\infty(q)},$$

$$\theta'(q) = \chi_\infty(q),$$

so that $\chi_\infty(\tau q) = \xi(\tau)\theta'(q)$. We claim that $\xi(\tau) = \tau^l$ for some $l \in \boldsymbol{R}$. To show this, we set

$$r_\infty(tq) = t$$

and

$$\tilde{S}(t) = \{x \in M \setminus B(R) : r_\infty \circ \Phi(x) = t\}.$$

Then it is not hard to derive from Lemma 4.3 that

$$\lim_{t\to\infty} \frac{d}{d\tau} \log \frac{V_{m-1}(\tilde{S}(t\tau))}{V_{m-1}(\tilde{S}(t))} = \frac{k-1}{\tau} + \frac{\xi'(\tau)}{\xi(\tau)}.$$

The convergence is uniform on $[a, b]$. This shows that

$$\lim_{t\to\infty} \frac{V_{m-1}(\tilde{S}(t\tau))}{V_{m-1}(\tilde{S}(t))} = \tau^{k-1}\xi(\tau).$$

In particular, $\xi(\tau)$ does not depend on a and b. Moreover on account of Lemma 4.3 (i) and (ii), we have

$$\lim_{t\to\infty} \frac{V_m(A(at, bt))}{tV_{m-1}(\tilde{S}(t))} = \lim_{t\to\infty} \int_a^b \frac{V_{m-1}(\tilde{S}(t\sigma))}{V_{m-1}(\tilde{S}(t))} d\sigma$$

$$= \int_a^b \sigma^{k-1}\xi(\sigma) d\sigma .$$

Hence it follows that

$$\xi(\tau) = \lim_{t\to\infty} \tau^{-k} \frac{V_m(A(at\tau, bt\tau))}{V_m(A(at, bt))}$$

$$= \tau^{-k} \frac{\int_{a\tau}^{b\tau} \sigma^{k-1}\xi(\sigma) d\sigma}{\int_a^b \sigma^{k-1}\xi(\sigma) d\sigma}$$

$$= \frac{\int_a^b \sigma^{k-1}\xi(\tau\sigma) d\sigma}{\int_a^b \sigma^{k-1}\xi(\sigma) d\sigma}.$$

Now it is easy to see that $\xi(\tau) = \tau^l$ for some l.

STEP 3. In order to complete the proof of Theorem III, we shall introduce modified weighed Sobolev norms. For a point τq of the image $\Phi(M \setminus B(R))$, we denote by $w'(\tau q)$ the volume of the fibre $\Phi^{-1}(\tau q)$ over τq. We set $w = w' \circ \Phi$ and extend this to a positive function on M for our convenience. As noted in the proof of Lemma 4.1, by the assumption (iii) of the theorem, we have

$$\lim_{t\to\infty} \frac{t^k w'(t\tau q)}{V_m(A(at, bt))} = \tau^l \theta(q) \tag{4.2}$$

for all $\tau\in[a, b]$ and $q\in M(\infty)$. The convergence is uniform.

Let f be a smooth function on M. Given $p>1$ and a weight $\delta\in\boldsymbol{R}$, we put

$$\|f\|'_{p,\delta} = \left(\int_M |f|(1+r)^{-p\delta-k}\frac{dvol}{w}\right)^{1/p},$$

$$\|f\|'_{n,p,\delta} = \sum_{j=0}^{n} \|D^j f\|'_{p,\delta-j}$$

(if they are finite). Moreover for a smooth function f with $\operatorname{supp} f\subset M\setminus B(R)$, we define a function Θf on $C^*M(\infty)$ by

$$\Theta f(\tau q) = \frac{1}{w'(\tau q)}\int_{\Phi^{-1}(\tau q)} f .$$

Recall here that for $h\in C_0^\infty(C^*M(\infty))$, the weighted Sobolev norms are given by

$$\|h\|_{n,p,\delta} = \sum_{j=0}^{n}\left(\int |D^j h|^p (1+r_\infty)^{-p(\delta-j)-k} dvol_\infty\right)^{1/p}.$$

Then we have the following

LEMMA 4.4. *For some positive constant $\varepsilon(R)$ with $\lim_{R\to\infty}\varepsilon(R)=0$, the following estimates hold for all smooth functions f on M such that $\operatorname{supp} f\subset M\setminus B(R)$ and $\|f\|'_{p,\delta}+\|\Delta_M f\|'_{p,\delta-2}$ is finite:*

(i) $e^{-\varepsilon(R)}\|\Theta f\|_{1,p,\delta} \leqq \|\Theta f\circ\Phi\|'_{1,p,\delta} \leqq e^{\varepsilon(R)}\|\Theta f\|_{1,p,\delta}$;

(ii) $\|\Theta f\|_{p,\delta} \leqq e^{\varepsilon(R)}\|f\|'_{p,\delta}$;

(iii) $\|f-\Theta f\circ\Phi\|'_{1,p,\delta} \leqq \varepsilon(R)(\|f\|'_{p,\delta}+\|\Delta_M f\|'_{p,\delta-2})$;

(iv) $\|\mathcal{L}_\infty\Theta f-\Theta\Delta_M f\|_{p,\delta-2} \leqq \varepsilon(R)(\|f\|'_{p,\delta}+\|\Delta_M f\|'_{p,\delta-2})$.

Here it is assumed that $p>m$ for the last two estimates.

PROOF. The first two estimates are direct consequences from Lemma 4.3 (ii). To prove (iii), we shall fix two positive constants a' and b' with $a<a'<b'<b$. We observe that for large t and all $x\in A(a't, b't)$,

$$|f-\Theta f\circ\Phi|^p(x)+t^p|d(f-\Theta f\circ\Phi)|^p(x)$$
$$\leqq \varepsilon_1(t)\int_{A(at,bt)}(|f|^p+t^{2p}|\Delta_M f|^p)\frac{dvol}{V_m(A(at, bt))}.$$

Here and after $\varepsilon_*(t)$'s stand as before for some positive constants depending on t in such a way that $\lim_{t\to\infty}\varepsilon_*(t)=0$. The above estimate can be verified by the standard elliptic regularity estimates (cf. [8] and Lemma 1.3 in [**16**]) and further the properties of the submersion Φ described in Lemma 4.3 (cf. Theorem B (iv) in [**16**]). Hence we have

$$\int_{A(a't,b't)}(|f-\Theta f\circ\Phi|^p+r^p|d(f-\Theta f\circ\Phi)|^p)r^{-p\delta-k}\frac{dvol}{w}$$

$$\leqq \varepsilon_2(t)\int_{A(at,bt)}(|f|^p+t^{2p}|\Delta_M f|^p)\frac{dvol}{V_m(A(at,\ bt))}\times\int_{A(a't,b't)}r^{-p\delta-k}\frac{dvol}{w}$$

$$\leqq \varepsilon_3(t)\Big(\int_{A(at,bt)}|f|^p r^{-p\delta-k}\frac{dvol}{w}+\int_{A(at,bt)}|\Delta_M f|^p r^{-p(\delta-2)-k}\frac{dvol}{w}\Big),$$

where we have used the fact that $t^k w'(t\tau q)/V_m(A(at,\ bt))$ is uniformly bounded from above and also away from zero by (4.2). Clearly the above inequality leads to the estimate (iii).

It remains to show the estimate (iv). From Lemma 4.3 and the same arguments as in the proof of Theorem B (iv) in [16], we can derive that

$$\int_{A_\infty(at,bt)}|\mathcal{L}_\infty\Theta f-\Theta\Delta_M f|^p r_\infty{}^{-p(\delta-2)-k}dvol_\infty$$

$$\leqq \varepsilon_4(t)\int_{A(at,bt)}(|f|^p+t^{2p}|\Delta_M f|^p)r^{-p\delta+k}\frac{dvol}{V_m(A(at,\ bt))}.$$

Hence by (4.2), we have

$$\int_{A_\infty(at,bt)}|\mathcal{L}_\infty\Theta f-\Theta\Delta_M f|^p r_\infty{}^{-p(\delta-2)-k}dvol_\infty$$

$$\leqq \varepsilon_5(t)\Big(\int_{A(at,bt)}|f|^p r^{-p\delta-k}\frac{dvol}{w}+\int_{A(at,bt)}|\Delta_M f|^p r^{-p(\delta-2)-k}\frac{dvol}{w}\Big).$$

This implies (iv). The proof of Lemma 4.4 is now completed.

By Lemmas 4.2 and 4.4, we have the following

LEMMA 4.5. *Given $p>m$ and a nonexceptional weight δ, there are positive constants R and C such that*

$$\|f\|'_{1,p,\delta}\leqq C\|\Delta_M f\|'_{p,\delta-2}$$

for all smooth functions f on M with $\operatorname{supp} f\subset M\setminus B(R)$ *and* $\|f\|'_{p,\delta}+\|\Delta_M f\|'_{p,\delta-2}<+\infty$.

PROOF. We have by Lemma 4.4 (i) and (iii)

$$\|f\|'_{1,p,\delta}\leqq\|f-\Theta f\circ\Phi\|'_{1,p,\delta}+\|\Theta f\circ\Phi\|'_{1,p,\delta}$$
$$\leqq \varepsilon(R)(\|f\|'_{p,\delta}+\|\Delta_M f\|'_{p,\delta-2})+e^{\varepsilon(R)}\|\Theta f\|_{1,p,\delta}.$$

In view of Lemma 4.2, we see that

$$\|\Theta f\|_{1,p,\delta}\leqq C_1\|\mathcal{L}_\infty\Theta f\|_{p,\delta-2}$$

for some positive constant C_1, if δ is nonexceptional. Moreover it follows from

Lemma 4.4 (ii) and (iv) that

$$\|\mathcal{L}_\infty\Theta f\|_{p,\delta-2} \leqq \|\mathcal{L}_\infty\Theta f-\Theta\Delta_M f\|_{p,\delta-2}+\|\Theta\Delta_M f\|_{p,\delta-2}$$
$$\leqq \varepsilon(R)(\|f\|'_{p,\delta}+\|\Delta_M f\|'_{p,\delta-2})+e^{\varepsilon(R)}\|\Delta_M f\|'_{p,\delta-2}\,.$$

Therefore we obtain

$$\|f\|'_{1,p,\delta} \leqq \varepsilon_6(R)\|f\|'_{p,\delta}+C_2\|\Delta_M f\|'_{p,\delta-2}$$

for some positive constants $\varepsilon_6(R)$ as before and C_2. Thus by taking R so large that $\varepsilon_6(R)<1/2$, we get the required estimate. This completes the proof of Lemma 4.5.

STEP 4. We are now in a position to complete the proof of Theorem III. We fix large R as in Lemma 4.5 and choose a smooth function η on M in such a way that $|\eta|\leqq 1$, $\eta\equiv 1$ outside $B(2R)$, and $\eta\equiv 0$ in $B(R)$. Let h be a harmonic function on M with $\|h\|'_{p,\delta}<+\infty$. Suppose δ is nonexceptional. Then it follows from Lemma 4.5 that

$$\|h\|'_{p,\delta} \leqq \|\eta h\|'_{p,\delta}+\|(1-\eta)h\|'_{p,\delta}$$
$$\leqq C\|\Delta_M(\eta h)\|_{p,\delta-2}+\|(1-\eta)h\|'_{p,\delta}\,.$$

Hence applying the elliptic regularity estimates to h on $B(3R)$, we obtain

$$\|h\|'_{p,\delta} \leqq C'\Big(\int_{B(3R)}|h|^p dvol\Big)^{1/p} \tag{4.3}$$

for some positive constant C'. This implies that the space of harmonic functions h with $\|h\|'_{p,\delta}<+\infty$ is of finite dimension. In fact, let $\{h_i\}$ be a sequence of harmonic functions with $\|h_i\|'_{p,\delta}=1$. Then there is a subsequence $\{h_j\}$ which forms a Cauchy sequence in $L^p(B(3R))$. Hence by (4.3), it is also a Cauchy sequence with respect to the norm $\|*\|'_{p,\delta}$. This shows that h_j converges to a function h on M with respect to the norm. Obviously h is harmonic.

Now since $V_m(A(at, bt))\geqq C''t$ for some positive constant C'', we observe from (4.2) that for any $f\in L_{p,\delta}(M)$, and for some positive constant C''',

$$\|f\|'_{p,\delta+(m-1)/p} \leqq C'''\|f\|_{p,\delta}\,.$$

Hence for any $h\in\mathcal{H}_d(M)$, $\|h\|'_{p,\delta+(m-1)/p}$ is finite if δ is greater than $d-(m-1-\beta)/p$, where β is as in Lemma 1.4. Therefore $\mathcal{H}_d(M)$ is of finite dimension for any d. This completes the proof of Theorem III.

5. Further discussions.

As mentioned in Introduction, complete flat spaces are typical examples of manifolds possessing the strong Liouville property. This class is unfortunately not coverd by the results in the preceding sections. For example, in Theorem III, we imposed a regularity condition at infinity on the manifolds in consideration. In this section, we shall study a class of complete Riemannian manifolds which satisfy certain additional conditions other than that on curvature decay. The conditions are rather restrictive, but all of complete flat spaces satisfy them.

Let $M=(M, g_M)$ be a complete noncompact Riemannian manifold of dimension m. We suppose first that the sectional curvature K_M of M satisfies:

$$-\frac{B}{r^{2+\varepsilon}} \leqq K_M \leqq \frac{B'}{r^2} \tag{5.1}$$

for some positive constants B, B' and ε. Let D be a bounded domain with smooth boundary Σ and ν^+ the outer unit normal to the boundary Σ. Secondly we suppose that the normal exponential map of Σ induces a diffeomorphism between $[0, \infty)\times\Sigma$ and the outside of D, that is, if we put

$$E(t, x) = \exp t\nu^+(x)$$

for $(t, x)\in[0, \infty)\times\Sigma$, then

$$E: [0, \infty)\times\Sigma \longrightarrow M\setminus D \text{ is diffeomorphic.} \tag{5.2}$$

For each $t>0$, we set $\Sigma_t=E(\{t\}\times\Sigma)$ and define a map $E_t: \Sigma\to\Sigma_t$ by $E_t(x)=E(t, x)$. Let us consider a one-parameter family of Riemannian metrics ds_t^2 on Σ which are given by

$$ds_t^2 = \frac{1}{(1+t)^2} E_t^*(g_{M|\Sigma_t}).$$

We write $d_t(x, y)$ for the distance between two points x, y of Σ measured by the metric ds_t^2. Then it is not hard to see that d_t converges to a pseudo-distance d_∞ on Σ, namely, for all $x, y\in\Sigma$, the limit

$$\lim_{t\to\infty} d_t(x, y) = d_\infty(x, y)$$

exists. In fact, we may assume that the sectional curvature K_M satisfies

$$K_M \geqq -\frac{B''}{(1+\rho)^{2+\varepsilon}}$$

for some positive constant B'', where ρ stands for the distance to D. Let ξ be a unique solution of the classical Jacobi equation:

$$\xi''(t)-\frac{B''}{(1+t)^{2+\varepsilon}}\xi(t) = 0,$$

subject to the initial conditions: $\xi(0)=1$, $\xi'(0)=\lambda$, where a constant λ is chosen in such a way that the maximal eigenvalue of the second fundamental form of Σ

is not greater than λ. Then $\{\xi(t)^{-2}E_t^*(g_{M|\Sigma_t})\}$ forms a monotone nonincreasing family of metrics on Σ (cf. e.g., [**12**]). Since $\xi(t)/(1+t)$ converges to a positive constant as t goes to infinity, we see that d_t converges to a pseudo-distance d_∞. Now we introduce an equivalence relation $\sim$ on Σ by putting $x\sim y$ if $d_\infty(x, y)=0$. We write $M(\infty)$ for the metric space of all equivalence classes equipped with the induced distance, denoted by the same letter d_∞. Moreover we denote by ϕ the natural projection of Σ onto $M(\infty)$. The metric space (Σ, d_t) converges to $M(\infty)$ with respect to the Gromov-Hausdorff distance via the map ϕ. In fact we have

$$d_\infty(\phi(x), \phi(y)) \leqq e^{\varepsilon(t)} d_t(x, y)$$

for all $x, y\in\Sigma$, where a positive constant $\varepsilon(t)$ goes to zero as t tends to infinity.

We observe here that the sectional curvature of ds_t^2 is bounded uniformly in t, because of the assumption (5.1). Hence according to a result by Fukaya [7], there is a smooth manifold N of $C^{1,\alpha}$ Riemannian metric ds_N^2 on which the orthogonal group $O(m-1)$ acts as isometries, and $M(\infty)$ is isometric to the quotient space $N/O(m-1)$. Therefore we have the cone $CM(\infty)$ over $M(\infty)$ as the quotient space of the cone $CN=([0, \infty)\times N,\ dt^2+t^2ds_N^2)$ over N by the action $O(m-1)$.

To give a condition which we need, we shall define a canonical map Φ_0 of $M\setminus D$ onto the cone $CM(\infty)$ by

$$\Phi_0(\exp t\nu^+(x)) = t\phi(x)(= (t, \phi(x))) .$$

Given two positive constants a and b with $a<b$, we set $A_\rho(at, bt)=\{x\in M: at<\rho(x)<bt\}$ and consider the push-forward μ_t of the normalized Riemannian measure on $A_\rho(at, bt)$ by the scaled map $\Phi_0/t: A_\rho(at, bt)/t\to A_\infty(a, b)$, namely,

$$\mu_t := \frac{1}{t}\Phi_{0*}\Big(\frac{dvol}{V_m(A_\rho(at, bt))}\Big) .$$

The last condition states now

(5.3) $\qquad \mu_t$ converges to a measure μ_∞ in the weak* topology.

As mentioned at the begining of this section, complete flat spaces enjoy all of these conditions (5.1), (5.2) and (5.3). In fact, let K be a compact flat manifold of dimension s and σ an orthogonal representation on $\boldsymbol{R}^{n+1}$ of the fundamental group Γ of K. Then Γ acts diagonally on the product space $\boldsymbol{R}^s\times\boldsymbol{R}^{n+1}$ and we obtain a complete flat manifold $M=\boldsymbol{R}^s\times_\Gamma\boldsymbol{R}^{n+1}$. That is, M is a flat vector bundle over K with fibre $\boldsymbol{R}^{n+1}$. Set $D=\boldsymbol{R}^s\times_\Gamma B^{n+1}$, where B^{n+1} denotes the unit ball around the origin in $\boldsymbol{R}^{n+1}$. Then D satisfies condition (5.2). In this case, $M(\infty)$ is just the quotient space of the unit sphere S^n by the action of the closure $\overline{\sigma(\Gamma)}$ of the subgroup $\sigma(\Gamma)$ in $O(n+1)$. Then the projection

$\phi : \Sigma(=\partial D) \to M(\infty)$ is given by

$$\phi([x, v]) = \phi'(v),$$

where $[x, v]$ stands for the equivalence class of a point (x, v) of $\boldsymbol{R}^s \times \boldsymbol{R}^{n+1}$ and also ϕ' denotes the projection of S^n onto $M(\infty) = S^n/\overline{\sigma(\Gamma)}$. Since the measure μ_t as above is given by

$$\mu_t = \frac{(\tau + 1/t)^{n+1-k}}{b^{n+1} - a^{n+1}} \phi'_*(dvol_{S^n}),$$

where $k = \dim M(\infty) + 1$ and $dvol_{S^n}$ stands for the Riemannian measure of the unit sphere S^n, condition (5.3) is clearly satisfied. Actually, μ_t converges weakly to a measure μ_∞ defined by

$$\mu_\infty = \frac{\tau^{n+1-k}}{b^{n+1} - a^{n+1}} \phi'_*(dvol_{S^n}).$$

Now we shall prove the following

THEOREM IV. *Let M be a complete Riemannian manifold of dimension m satisfying* (5.1). *Then M has the strong Liouville property, if in addition, there exists a bounded domain D for which* (5.2) *holds, and further* (5.3) *is satisfied.*

Before going into the proof of the theorem, we note that the same arguments as in the proof of Theorem III are valid for the case $M(\infty)$ is smooth. However if this is not the case, we can not expect to have an appropriate smooth approximation for the map Φ_0 as before. To overcome such difficulty, we shall make use of the frame bundle FM over M endowed with a suitable $O(m)$ invariant metric.

PROOF OF THEOREM IV. Let us denote by g_{FM} a canonical metric of FM such that the orthogonal group $O(m)$ acts on (FM, g_{FM}) as isometries and the projection $\pi : FM \to M$ is a Riemannian submersion with totally geodesic fibres. For our purpose, we shall modify the metric as follows. We choose first an auxiliary positive function ζ on $[0, \infty)$ with $\zeta(t) = t$ for $t \geqq 1$. Set $\bar{\rho} = \pi(\zeta(\rho))$ and define a metric $\bar{g}$ on FM by

$$\bar{g}(\bar{X}, \bar{Y}) = g_{FM}(\mathcal{H}\bar{X}, \mathcal{H}\bar{Y}) + \bar{\rho}^2 g_{FM}(\mathcal{V}\bar{X}, \mathcal{V}\bar{Y})$$

for tangent vectors $\bar{X}, \bar{Y}$ of FM. Here as before, $\mathcal{H}\bar{X}$ (resp., $\mathcal{V}\bar{X}$) denotes the horizontal (resp., vertical) component of $\bar{X}$ with respect to the metric g_{FM}. In what follows, FM is assumed to be endowed with this metric $\bar{g}$, unless otherwise stated. Clearly the action of $O(m)$ on FM is still isometric and the projection π is a Riemannian submersion. Moreover if we put $\bar{D} = \pi^{-1}(D)$ and $\bar{\Sigma}_t = \pi^{-1}(\Sigma_t)$, and define a map $\bar{E}_t$ of $\bar{\Sigma}$ onto $\bar{\Sigma}_t$ by

$$\bar{E}_t(\bar{x}) = \exp t\bar{\nu}^+(\bar{x}),$$

where $\bar{\nu}^+$ stands for the outer unit normal to $\bar{\Sigma}$, the boundary of $\bar{D}$, then $\bar{E}_t$ induces a diffeomorphism between $\bar{\Sigma}$ and $\bar{\Sigma}_t$. Hence we have a one-parameter family of metrics $d\bar{s}_t^2$ on $\bar{\Sigma}$ given by

$$d\bar{s}_t^2 = \frac{1}{(1+t)^2}\bar{E}_t{}^*(\bar{g}_{|\bar{\Sigma}_t}).$$

We observe that $O(m)$ acts on $(\bar{\Sigma}, d\bar{s}_t^2)$ as isometries and (Σ, ds_t^2) is just the quotient space (for large t). Let $\bar{d}_t$ be the distance measured by the metric $d\bar{s}_t^2$. Then $\bar{d}_t$ converges to a pseudo-distance $\bar{d}_\infty$, and we obtain as before a compact metric space $\bar{M}(\infty)$ with the distance, denoted by the same letter $\bar{d}_\infty$. The metric space $(\bar{\Sigma}, \bar{d}_t)$ converges to $\bar{M}(\infty)$ with respect to the Gromov-Hausdorff distance and moreover we have the $O(m)$ action on $\bar{M}(\infty)$ in such a way that the quotient space $\bar{M}(\infty)/O(m)$ coincides with $M(\infty)$. We write $\pi(\infty)$ for the projection of $\bar{M}(\infty)$ onto $M(\infty)$ and set $id.\times\pi(\infty)(\tau\bar{q})=\tau\pi(\infty)(\bar{q})$.

Now by the definition of the metric $\bar{g}$ and the assumption (5.1), we see that the sectional curvature of FM decays at a rate $O(\bar{\rho}^{-2})$. Hence it turns out from the same reasons as in [7] that *$\bar{M}(\infty)$ is a smooth manifold of $C^{1,\alpha}$ metric $d\bar{s}_\infty^2$.* In fact, the frame bundle $FA_\rho(at, bt)$ over $A_\rho(at, bt)$ with the scaled metric $\bar{g}/t$, say as before $FA_\rho(at, bt)/t$, converges to a smooth manifold with $C^{1,\alpha}$ metric. Thus for FM itself, we are able to repeat the arguments in Section 4. However we note that the pull-back of a harmonic function on M by the projection π is not harmonic on FM. Hence we need some modification, which will be explained below.

Let us denote as before by $C\bar{M}(\infty)$ the cone over $\bar{M}(\infty)$ and define a canonical map $\bar{\Phi}_0\colon FM\to C\bar{M}(\infty)$ by

$$\bar{\Phi}_0(\exp t\bar{\nu}^+(\bar{x})) = t\bar{\phi}(\bar{x}),$$

where $\bar{\phi}:\bar{\Sigma}\to\bar{M}(\infty)$ stands for the projection. Then $(id.\times\pi(\infty))\circ\bar{\Phi}_0=\Phi_0\circ\pi$.

The measure Ω on FM which corresponds to the Riemannian measure on M via the projection π is given by

$$\Omega = \frac{1}{v_0\bar{\rho}^{m'}}dvol,$$

where v_0 stands for the volume of the fibres of FM with respect to the canonical metric g_{FM} and $m'=m(m-1)/2$. We observe that the operator $\mathcal{L}$ associated with the Dirichlet form $\int|df|^2\Omega$ is given by

$$\mathcal{L}f = v_0\bar{\rho}^{m'}div\Big(\frac{1}{v_0\bar{\rho}^{m'}}\nabla f\Big) = \Delta_{FM}f - m'\nabla\log\bar{\rho}\cdot f.$$

We set

$$\bar{\mu}_t := \frac{1}{t}\bar{\Phi}_{0*}\Big(\frac{\Omega}{\Omega(FA_\rho(at, bt))}\Big).$$

Then we have

$$\mu_t = (id.\times\pi(\infty))_*(\bar{\mu}_t),$$

and hence by (5.3), there is a measure $\bar{\mu}_\infty$ on $(a, b)\times\bar{M}(\infty)$ such that $\bar{\mu}_t$ converges to $\bar{\mu}_\infty$ as $t\to\infty$ in the weak* topology. Moreover replacing the Laplace operator in Section 3 with the operator $\bar{\mathcal{L}}$ and repeating the same arguments there, we can deduce that Lemmas 4.1 and 4.3 are valid. (In this case, the density $\bar{\chi}_\infty(\tau, \bar{q})$ of the limit measure $\bar{\mu}_\infty$ can be written as

$$\bar{\chi}_\infty(\tau, \bar{q}) = \tau^l\bar{\theta}(\bar{q})$$

for a constant l and a positive function $\bar{\theta}$ on $\bar{M}(\infty)$ of class $C^{1,\alpha}$ which is $O(m)$ *invariant*.) Therefore we can assert that for every d, the space

$$\mathcal{H}_d(FM, \Omega) = \{h\in C^\infty(FM):\ \bar{\mathcal{L}}h=0, |h|\leqq A\bar{\rho}^d+B \text{ for some constants } A, B>0\}$$

is of finite dimension. Since π^* maps the space $\mathcal{H}_d(M)$ into the space $\mathcal{H}_d(FM, \Omega)$, we can conclude that M has the strong Liouville property. This completes the proof of Theorem IV.

REMARK. Let M be a complete Riemannian manifold of dimension m and assume there is a domain D of M satisfying condition (5.2). Then we have a smooth family of metrics ds_t^2 on Σ as we mentioned at the begining of this section. When condition (5.1) is satisfied, we have a compact metric space $M(\infty)$ and a map $\phi: \Sigma\to M(\infty)$ such that (Σ, ds_t^2) converges to $M(\infty)$ with respect to the Gromov-Hausdorff distance via the map ϕ. Moreover if condition (5.3) holds, and it we write ξ_t for the normalized Riemannian measure $dvol/V_{m-1}((\Sigma, ds_t^2))$ of (Σ, ds_t^2), then the push-forward measure $\phi_*\xi_t$ converges weakly to a measure ξ_∞ on $M(\infty)$. Since the sectional curvature of ds_t^2 is uniformly bounded in t, we can assert that (Σ, ds_t^2) converges to $(M(\infty), ds_t^2)$ with respect to the spectral distance in the sense of [**17**]. In particular, we have the convergence of the eigenvalues and eigenfunctions of (Σ, ds_t^2) (cf. [**17**] for details).

Now we suppose instead of (5.1) and (5.3) that the metric ds_t^2 converges to a positive semidefinite symmetric tensor on the manifold Σ with respect to the C^0 topology. Then we may ask when M has the strong Liouville property. In general, this condition does not imply the convergence of (Σ, ds_t^2) with respect to the spectral distance, or even the Gromov-Hausdorff distance. See [**23**] for a related topic.

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