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Normality of affine toric varieties associated with Hermitian symmetric spaces

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§ 0. Introduction.

Let an algebraic torus T of dimension n act on a vector space V of dimension N $(N>n)$ via N characters $\chi_1, \cdots, \chi_N$ of T. We assume the above characters to generate the character group $X(T)$ of T and to lie on one hyperplane of $\boldsymbol{R}\otimes_{\boldsymbol{Z}} X(T)$. Let A be the polynomial ring $\boldsymbol{Z}[\xi_1, \cdots, \xi_N]$, and let L be the subgroup of $\boldsymbol{Z}^N$ consisting of the elements $a=(a_j)_{1\leq j\leq N}$ such that $\sum_{j=1}^N a_j\chi_j=0$. We consider the ring

$$R = A/\sum_{a\in L} A\xi_a\,.$$

Here $\sum_{a\in L} A\xi_a$ denotes the ideal of A consisting of all sums $\sum_{a\in L} p_a\xi_a$ with $p_a\in A$ where $\xi_a=\prod_{a_j>0}\xi_j^{a_j}-\prod_{a_j<0}\xi_j^{-a_j}$, and only finitely many p_a are not zero. In this situation Gelfand and his collaborators studied generalized hypergeometric systems (cf. [**G**], [**GGZ**], [**GZK1**], [**GZK2**], [**GKZ**]). We notice that the idea of this kind of generalized hypergeometric systems goes back to [**H**] and [**KMM**]. We remark that Aomoto also defined and studied generalized hypergeometric functions by use of integral representations (cf. [**A1**]-[**A4**]). We can find in [**GZK2**] the computation of the characteristic cycles of generalized hypergeometric systems; we cannot follow this computation unless the Z-algebra R is normal, however. In [**S**] we defined the b-functions of generalized hypergeometric systems, and used the normality of the $\boldsymbol{Z}$-algebra R in order to determine those b-functions. Hence the normality of the $\boldsymbol{Z}$-algebra R is very important.

In this paper we assume V to be an open Schubert cell of a simple compact Hermitian symmetric space and T to be a maximal torus of its motion group. We remark that the generalized hypergeometric system corresponding to the Lauricella function F_C, and the one to the Lauricella function F_D are defined in this setup (cf. [**GZK2**]). Then we prove

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THEOREM. *In the above situation, the $\boldsymbol{Z}$-algebra R is normal.*

This theorem appears dividedly as Propositions 2.1, 3.1, 4.1, 5.1, 6.2, 7.2 and 8.2. We also determine a minimal system of generators of the ideal $\sum_{a\in L} A\xi_a$ except for type E_7 (Propositions 2.3, 3.2, 4.4, 5.2, 6.4 and 7.4). The generation by the elements of degree 2 is known at least for classical types (cf. [**H**]); we have not been able to find a proof in the literature, however.

In § 1 we prepare some lemmas for the normality of the $\boldsymbol{Z}$-algebra (Lemmas 1.1, 1.2, 1.3 and 1.7) or the generation of the ideal (Lemmas 1.4, 1.5 and 1.6). From § 2 through § 8 we work on the above problems type by type.

In the previous version of this paper, systems of generators of degree 2 were presented except for type E_7. It was the referee who suggested that minimal systems of generators should have been written down. The author would like to thank the referee for this suggestion.

§ 1. Preliminaries.

Suppose we are given N integral vectors $\chi_j=(\chi_{1j}, \cdots, \chi_{nj})\in\boldsymbol{Z}^n$ $(j=1, \cdots, N)$ satisfying two conditions:

(1) The vectors $\chi_1, \cdots, \chi_N$ generate the lattice $\boldsymbol{Z}^n$.

(2) All the vectors χ_j lie on one affine hyperplane $\sum_{i=1}^n c_i x_i=1$ in $\boldsymbol{R}^n$, where $c_i\in\boldsymbol{Z}$.

We denote by L the subgroup in $\boldsymbol{Z}^N$ consisting of the $a=(a_1, \cdots, a_N)$ such that $\sum_{j=1}^N a_j\chi_j=0$, by Q the Newton polyhedron, i.e., Q is the convex hull in $\boldsymbol{R}^n$ of the points $\chi_1, \cdots, \chi_N$, by $\mathcal{F}$ the set of faces of Q of codimension one, by Λ the semigroup $\boldsymbol{Z}_{\geq 0}\chi_1+\cdots+\boldsymbol{Z}_{\geq 0}\chi_N$, by A the polynomial ring $\boldsymbol{Z}[\xi_1, \cdots, \xi_N]$, and by R the semigroup ring

$$\boldsymbol{Z}[\Lambda]=A/\sum_{\alpha\in L} A\xi_a$$

where $\xi_a=\prod_{a_j>0}\xi_j^{a_j}-\prod_{a_j<0}\xi_j^{-a_j}$. The polynomial ring A has the gradation by degrees: $A=\bigoplus_{n=0}^\infty A_n$. Since all ξ_a $(a\in L)$ are homogeneous, the $\boldsymbol{Z}$-algebra R has the induced gradation: $R=\bigoplus_{n=0}^\infty R_n$. For $\Gamma\in\mathcal{F}$, we denote by F_Γ the linear form for the hyperplane spanned by Γ such that the coefficients of F_Γ are integers, that their greatest common divisor is one, and that $F_\Gamma(\chi)\geqq 0$ for all $\chi\in\Lambda$. It is clear that we have

$$\boldsymbol{R}_{\geq 0}\chi_1+\cdots+\boldsymbol{R}_{\geq 0}\chi_N=\bigcap_{\Gamma\in\mathcal{F}}\{\chi\in\boldsymbol{R}^n \mid F_\Gamma(\chi)\geqq 0\}.$$

LEMMA 1.1. *The $\boldsymbol{Z}$-algebra R is normal if and only if we have*

$$\Lambda=\boldsymbol{Z}^n\cap\bigcap_{\Gamma\in\mathcal{F}}\{\chi\in\boldsymbol{R}^n \mid F_\Gamma(\chi)\geqq 0\}.$$

PROOF. The semigroup Λ is said to be saturated when the condition $m\chi \in \Lambda$, where m is a positive integer and $\chi \in \boldsymbol{Z}^n$, implies $\chi \in \Lambda$. It is well known that the $\boldsymbol{Z}$-algebra R is normal if and only if Λ is saturated (cf. [TE]). Suppose that $\boldsymbol{Z}^n \cap (\boldsymbol{R}_{\geq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geq 0}\chi_N) = \Lambda$. Then it is clear that Λ is saturated. Conversely suppose that Λ is saturated. We have

$$\boldsymbol{Z}^n \cap (\boldsymbol{R}_{\geq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geq 0}\chi_N) = \boldsymbol{Z}^n \cap (\boldsymbol{Q}_{\geq 0}\chi_1 + \cdots + \boldsymbol{Q}_{\geq 0}\chi_N)$$

by Carathéodory's theorem (see, e.g., [Gr]) and Cramér's formula. For $\chi \in \boldsymbol{Z}^n \cap (\boldsymbol{Q}_{\geq 0}\chi_1 + \cdots + \boldsymbol{Q}_{\geq 0}\chi_N)$, there exists a positive integer m such that $m\chi \in \Lambda$. By the saturatedness of Λ, it implies $\chi \in \Lambda$. ■

For $i \geqq n$, we denote by C_i the cone generated by $\chi_1, \cdots, \chi_i$. Suppose that

$$C_i = \boldsymbol{R}_{\geq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geq 0}\chi_i = \bigcap_{f \in F_i} \{\chi \in \boldsymbol{R}^n \mid f(\chi) \geqq 0\},$$

where F_i is a finite set of linear forms on $\boldsymbol{R}^n$. We decompose the set F_i according to the sign at χ_{i+1}, i.e., $F_i^+ := \{f \in F_i \mid f(\chi_{i+1}) \geqq 0\}$ and $F_i^- := \{f \in F_i \mid f(\chi_{i+1}) < 0\}$. We then define a finite set F_{i+1} of linear forms by

$$F_{i+1} := F_i^+ \cup \{f(\chi_{i+1})f' - f'(\chi_{i+1})f \mid f \in F_i^+,\ f' \in F_i^-\}.$$

LEMMA 1.2. $C_{i+1} = \bigcap_{f \in F_{i+1}} \{\chi \in \boldsymbol{R}^n \mid f(\chi) \geqq 0\}$.

PROOF. It is clear that $C_{i+1} \subset \bigcap_{f \in F_{i+1}} \{\chi \in \boldsymbol{R}^n \mid f(\chi) \geqq 0\}$. Let $\chi \in \bigcap_{f \in F_{i+1}} \{\chi \in \boldsymbol{R}^n \mid f(\chi) \geqq 0\}$. If we have $f'(\chi) \geqq 0$ for all $f' \in F_i^-$, then $\chi \in C_i \subset C_{i+1}$. Hence we suppose that there exists $f_0' \in F_i^-$ such that $|f_0'(\chi_{i+1})|^{-1} f_0'(\chi) = \min_{f' \in F_i^-} |f'(\chi_{i+1})|^{-1} \cdot f'(\chi) < 0$. Put $\chi' := \chi + |f_0'(\chi_{i+1})|^{-1} f_0'(\chi) \chi_{i+1}$. For $f' \in F_i^-$, we have

$$\begin{aligned} f'(\chi') &= f'(\chi) + |f_0'(\chi_{i+1})|^{-1} f_0'(\chi) f'(\chi_{i+1}) \\ &= f'(\chi) - |f_0'(\chi_{i+1})|^{-1} |f'(\chi_{i+1})| f_0'(\chi) \geqq 0. \end{aligned}$$

For $f \in F_i^+$, we have

$$\begin{aligned} f(\chi') &= f(\chi) + |f_0'(\chi_{i+1})|^{-1} f_0'(\chi) f(\chi_{i+1}) \\ &= |f_0'(\chi_{i+1})|^{-1} [f(\chi_{i+1}) f_0' - f_0'(\chi_{i+1}) f](\chi) \geqq 0. \end{aligned}$$

Hence we have $\chi' \in C_i$ and $\chi = \chi' + |f_0'(\chi_{i+1})|^{-1} |f_0'(\chi)| \chi_{i+1}$ belongs to C_{i+1}. ■

For $i \geqq n$, we denote by Λ_i the semigroup generated by $\chi_1, \cdots, \chi_i$.

LEMMA 1.3. *Suppose that we have* $\Lambda_i = C_i \cap \boldsymbol{Z}^n$, $f(\boldsymbol{Z}^n) \subset \boldsymbol{Z}$ *for all* $f \in F_i^+$, *and* $f'(\chi_{i+1}) = -1$ *for all* $f' \in F_i^-$. *Then we obtain* $\Lambda_{i+1} = C_{i+1} \cap \boldsymbol{Z}^n$.

PROOF. It is clear from the proof of Lemma 1.2. ■

EXAMPLE. Let $n=2p-1$ and $N=2p$ for $p\geqq 1$. Let $e_1, \cdots, e_n$ be a basis of $\boldsymbol{Z}^n$, and $f_1, \cdots, f_n$ the dual basis to it. We suppose that N vectors $\chi_1=e, \cdots, \chi_n=e_n$, $\chi_{n+1}=e_1+\cdots+e_p-e_{p+1}-\cdots-e_n$ are given. Then we have $C_n=\boldsymbol{R}_{\geqq 0}\chi_1+\cdots+\boldsymbol{R}_{\geqq 0}\chi_n=\bigcap_{i=1}^n(f_i\geqq 0)$. Since $f_i(\chi_{n+1})=1$ for $1\leqq i\leqq p$ and -1 for $p+1\leqq i\leqq n$, we have $C_{n+1}\cap\boldsymbol{Z}^n=\Lambda_{n+1}=\Lambda$ and $F_{n+1}=\{f_1, \cdots, f_p, f_i+f_j\,(1\leqq i\leqq p,\ p+1\leqq j\leqq n)\}$.

For $a\in L$, we denote $a\chi:=\sum_{a_j\geqq 0}a_j\chi_j\in\Lambda$. By the homogeneity, we have $a\chi=(-a)\chi$ for any $a\in L$. For $\chi, \chi'\in\Lambda$, we denote $\chi>\chi'$ when $\chi-\chi'\in\Lambda-\{0\}$.

LEMMA 1.4. *Let a, b, $c\in L$ satisfy that $a=b+c$ and $a_j\geqq c_j$ if $c_j>0$.*
(1) *If there exists j such that $0>a_j$ and $0>c_j$, then we have $a\chi>b\chi$.*
(2) *If there exists no j such that $0>a_j$ and $0>c_j$, then $a\chi=b\chi$.*

PROOF. We define the subsets S_{++}, S_{+-}, S_{--} and B_+ of the set $\{1, \cdots, N\}$ by

$$S_{++}=\{i\,|\,a_i>0,\ c_i>0\}$$
$$S_{+-}=\{i\,|\,a_i>0,\ c_i\leqq 0\}$$
$$S_{--}=\{i\,|\,a_i\leqq 0,\ c_i\leqq 0\}$$
$$B_+=\{i\,|\,a_i\geqq c_i\}.$$

Then we have

$$\begin{aligned}b\chi&=\sum_{j\in S_{++}}(a_j-c_j)\chi_j+\sum_{j\in S_{+-}}(a_j-c_j)\chi_j+\sum_{j\in S_{--}\cap B_+}(a_j-c_j)\chi_j\\&=\sum_{j\in S_{++}}a_j\chi_j+\sum_{j\in S_{+-}}a_j\chi_j-\sum_{j\in S_{++}}c_j\chi_j+\sum_{j\in S_{+-}}(-c_j)\chi_j+\sum_{j\in S_{--}\cap B_+}(a_j-c_j)\chi_j\\&=a\chi-c\chi+\sum_{j\in S_{+-}}(-c_j)\chi_j+\sum_{j\in S_{--}\cap B_+}(a_j-c_j)\chi_j\\&=a\chi-\sum_{j\in S_{--}}(-c_j)\chi_j+\sum_{j\in S_{--}\cap B_+}(a_j-c_j)\chi_j\\&=a\chi+\sum_{j\in S_{--}-B_+}c_j\chi_j+\sum_{j\in S_{--}\cap B_+}a_j\chi_j\,.\end{aligned}$$

Hence we see that $a\chi\geqq b\chi$, and that $a\chi>b\chi$ if and only if there exists $1\leqq j\leqq N$ such that $a_j<0$ and $c_j<0$. ■

LEMMA 1.5. *Let a, b, $c\in L$ satisfy that $a=b+c$ and $a_j\geqq c_j$ if $c_j>0$. Then ξ_a is generated by ξ_b and ξ_c.*

PROOF. Let S_{++}, S_{+-}, S_{--} and B_+ be as in the proof of Lemma 1.4. We remark that $S_{++}=\{i\,|\,c_i>0\}$, $S_{--}=\{i\,|\,a_i\leqq 0\}$ and $S_{++}\cup S_{+-}\subset B_+$ by assumption. We have

$$\xi_a = \prod_{a_i>0} \xi_i^{a_i} - \prod_{a_i<0} \xi_i^{-a_i} = \prod_{i\in S_{++}} \xi_i^{a_i-c_i} \cdot \prod_{c_i>0} \xi_i^{c_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i} - \prod_{a_i<0} \xi_i^{-a_i}$$

$$= \prod_{i\in S_{++}} \xi_i^{a_i-c_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i} \cdot \xi_c + \prod_{i\in S_{++}} \xi_i^{a_i-c_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i} \cdot \prod_{c_i\leqq 0} \xi_i^{-c_i} - \prod_{a_i<0} \xi_i^{-a_i}.$$

Since $B_+ = S_{++} \cup S_{+-} \cup (S_{--} \cap B_+)$, we have

$$\prod_{i\in S_{++}} \xi_i^{a_i-c_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i} \cdot \prod_{c_i\leqq 0} \xi_i^{-c_i}$$

$$= \prod_{i\in S_{++}} \xi_i^{a_i-c_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{a_i-c_i} \prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i}$$

$$= \prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i} \cdot \xi_b + \prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i} \cdot \prod_{b_i<0} \xi_i^{-b_i}.$$

Hence we obtain

$$\xi_a = \prod_{i\in S_{++}} \xi_i^{b_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i} \cdot \xi_c + \prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i} \cdot \xi_b$$

$$+ \prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i} \cdot \prod_{b_i<0} \xi_i^{-b_i} - \prod_{a_i<0} \xi_i^{-a_i}.$$

Since $\prod_{b_i<0} \xi_i^{-b_i} = \prod_{i\in S_{--}-B_+} \xi_i^{-b_i} = \prod_{i\in S_{--}-B_+} \xi_i^{c_i-a_i}$, we have

$$\prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i} \cdot \prod_{b_i<0} \xi_i^{-b_i} = \prod_{i\in S_{--}} \xi_i^{-a_i} = \prod_{a_i<0} \xi_i^{-a_i}.$$

Therefore we obtain

$$\xi_a = \prod_{i\in S_{++}} \xi_i^{b_i} \cdot \prod_{i\in S_{+-}} \xi_i^{a_i} \cdot \xi_c + \prod_{i\in S_{--}-B_+} \xi_i^{-c_i} \cdot \prod_{i\in S_{--}\cap B_+} \xi_i^{-a_i} \cdot \xi_b.$$

■

LEMMA 1.6. *Let $a=(a_j)_{1\leqq j\leqq N} \in \boldsymbol{Z}^N$. Then we have $\sum_{i=1}^N a_j \chi_j = 0$ if and only if $\sum_{j=1}^N a_j F_\Gamma(\chi_j) = 0$ for all faces $\Gamma \in \mathcal{F}$.*

PROOF. By the conditions on $\chi_1, \cdots, \chi_N$, it is clear that the cone $\boldsymbol{R}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geqq 0}\chi_N$ has a vertex at $\{0\}$. Hence we see

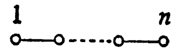
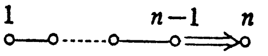
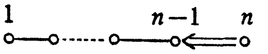
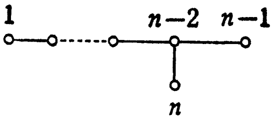
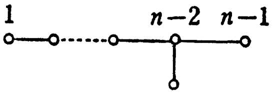
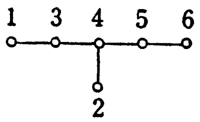
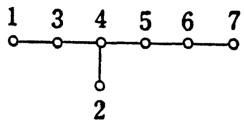
$$\{\chi \in X(T) \mid F_\Gamma(\chi) = 0 \text{ for all } \Gamma \in \mathcal{F}\} = \{0\}.$$

■

Let $\mathfrak{g}$ be a simple Lie algebra over $\boldsymbol{C}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{b}$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots determined by $\mathfrak{b}$ and $\mathfrak{h}$, and $\{f_1, \cdots, f_n\}$ be the basis of $(\boldsymbol{Z}^n)^* = (\boldsymbol{Z}\alpha_1 \oplus \cdots \oplus \boldsymbol{Z}\alpha_n)^*$ dual to $\{\alpha_1, \cdots, \alpha_n\}$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ with the abelian nilradical $\mathfrak{n}$. Then there exists a unique $1 \leqq i(\mathfrak{n}) \leqq n$ such that the root space corresponding to $\alpha_{i(\mathfrak{n})}$ lies in $\mathfrak{n}$. For $1 \leqq i \leqq n$, let s_i be the reflection corresponding to α_i. Let $W(\mathfrak{g}, \mathfrak{h})$ denote the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and W the subgroup of $W(\mathfrak{g}, \mathfrak{h})$ generated by all s_i $(i \neq i(\mathfrak{n}))$. We denote by $X = \{\chi_1, \cdots, \chi_N\}$ the set of roots whose root spaces lie in $\mathfrak{n}$, by Λ the semigroup generated by

X. Then the group W acts on X. There are five classical types of pairs $(\mathfrak{g}, \mathfrak{p})$ and two exceptional pairs, which are listed in Table 1 (cf. [**LSS**], the labelling of the simple roots is that of Bourbaki [**B**]). There exists a nondegenerate symmetric bilinear form $(\,,\,): \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ induced by the Killing form.

Table 1.

$\mathfrak{g}$	Dynkin diagram	$i(\mathfrak{n})$	W	$N=\dim \mathfrak{n}$
A_n $(n \geqq 1)$		$1 \leqq p \leqq n$	$A_{p-1} \times A_{n-p}$	$p(n-p+1)$
B_n $(n \geqq 2)$		1	B_{n-1}	$2n-1$
C_n $(n \geqq 2)$		n	A_{n-1}	$\frac{n(n+1)}{2}$
D_n $(n \geqq 4)$		1	D_{n-1}	$2n-2$
D_n $(n \geqq 4)$		n	A_{n-1}	$\frac{n(n-1)}{2}$
E_6		6	D_5	16
E_7		7	E_6	27

LEMMA 1.7. *If any element* $\chi = a_1\alpha_1 + \cdots + a_n\alpha_n$ *with* $a_i \in \boldsymbol{Z}_{\geqq 0}$ $(1 \leqq i \leqq n)$ *and* $(\chi, \alpha_i) \leqq 0$ $(i \neq i(\mathfrak{n}))$ *belongs to the semigroup* Λ, *then the* $\boldsymbol{Z}$*-algebra* R *is normal.*

PROOF. Let $\chi \in \boldsymbol{Z}^n \cap (\boldsymbol{R}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geqq 0}\chi_N)$. There exists an element $w \in W$ such that $(w \cdot \chi, \alpha_i) \leqq 0$ for all $i \neq i(\mathfrak{n})$. Since W acts on X and $\boldsymbol{Z}^n$, the conjugate $w \cdot \chi$ also belongs to $\boldsymbol{Z}^n \cap (\boldsymbol{R}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geqq 0}\chi_N)$. By the assumption, $w \cdot \chi$ belongs to Λ, accordingly χ belongs to Λ. ■

§2. Type A_n.

In this section we suppose that $\mathfrak{g}$ is of A_n-type and $i(\mathfrak{n}) = p$.

PROPOSITION 2.1. *The* $\boldsymbol{Z}$*-algebra* R *is normal.*

PROOF. We number the elements of the set X by

$$\chi_i=\alpha_i+\cdots+\alpha_p \qquad (1\leqq i\leqq p),$$

$$\chi_i=\alpha_p+\cdots+\alpha_i \qquad (p\leqq i\leqq n),$$

$$\chi_{n+k}=\alpha_{k-(s-1)(p-1)}+\cdots+\alpha_{p+s} \quad ((s-1)(p-1)<k\leqq s(p-1),\ 1\leqq s\leqq n-p).$$

As in §1, we put $\Lambda_i=\boldsymbol{Z}_{\geqq 0}\chi_1+\cdots+\boldsymbol{Z}_{\geqq 0}\chi_i$ and $C_i=\boldsymbol{R}_{\geqq 0}\chi_1+\cdots+\boldsymbol{R}_{\geqq 0}\chi_i$ for $i\geqq n$. Then we have $C_n=\bigcap_{f\in F_n}\{\chi\in\boldsymbol{R}^n \mid f(\chi)\geqq 0\}$, where $F_n=\{f_1, f_2-f_1, \cdots, f_{p-1}-f_{p-2}, f_p-f_{p-1}-f_{p+1}, f_{p+1}-f_{p+2}, \cdots, f_{n-1}-f_n, f_n\}$. By using Lemmas 1.2 and 1.3, we can verify the normality of R, and we see $\boldsymbol{R}_{\geqq 0}\chi_1+\cdots+\boldsymbol{R}_{\geqq 0}\chi_N=\bigcap_{f\in F_N}\{\chi\in\boldsymbol{R}^n \mid f(\chi)\geqq 0\}$, where $F_N=\{f_1, f_2-f_1, \cdots, f_{p-1}-f_{p-2}, f_p-f_{p-1}, f_p-f_{p+1}, f_{p+1}-f_{p+2}, \cdots, f_{n-1}-f_n, f_n\}$. ■

We re-index the elements of the set X by $\chi_{ij}:=\sum_{i\leqq k\leqq j}\alpha_k$ for $1\leqq i\leqq p\leqq j\leqq n$. Then we have $X=\{\chi_{ij} \mid 1\leqq i\leqq p\leqq j\leqq n\}$. For $1\leqq i\neq i'\leqq p$ and $p\leqq j\neq j'\leqq n$, we define $a(ii', jj')\in L$ by

$$a(ii', jj')_{st}=\begin{cases} 1 & (s, t)=(i', j), (i, j') \\ -1 & (s, t)=(i, j), (i', j') \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.2. *Let A be the polynomial ring $\boldsymbol{Z}[\xi_{ij}\ (1\leqq i\leqq p\leqq j\leqq n)]$ Then we have*

$$R=A/\sum_{1\leqq i\neq i'\leqq p\leqq j\neq j'\leqq n} A(\xi_{i'j}\xi_{ij'}-\xi_{ij}\xi_{i'j'}).$$

PROOF. We have obtained the set F_N in the proof of Proposition 2.1. For $a=(a_{ij})_{i\leqq p\leqq j}\in\boldsymbol{Z}^N$, we have $a\in L$ if and only if $\sum_{p\leqq j\leqq n}a_{ij}=0$ for $1\leqq i\leqq p$ and $\sum_{1\leqq i\leqq p}a_{ij}=0$ for $p\leqq j\leqq n$ by Lemma 1.6. Hence for $a\in L-\{0\}$ there exist $1\leqq i\neq i'\leqq p$ and $p\leqq j\neq j'\leqq n$ such that $a_{ij'}>0$, $a_{i'j}>0$ and $a_{ij}<0$. By Lemmas 1.4 and 1.5, we see that ξ_a is generated by $\xi_{a(ii',jj')}$ and $\xi_{a-a(ii',jj')}$, and that $a\chi>(a-a(ii', jj'))\chi$. By recurrence, we see that ξ_a is generated by $\xi_{a(ii',jj')}$ $(1\leqq i\neq i'\leqq p\leqq j\neq j'\leqq n)$. Obviously we have $\xi_{a(ii',jj')}=\xi_{i'j}\xi_{ij'}-\xi_{ij}\xi_{i'j'}$. ■

PROPOSITION 2.3. *The set $\Xi:=\{\xi_{ij}\xi_{i'j'}-\xi_{ij'}\xi_{i'j} \mid 1\leqq i<i'\leqq p\leqq j<j'\leqq n\}$ is a minimal system of generators of the ideal $\sum_{a\in L}A\xi_a$.*

PROOF. Since we have $\xi_{a(ii',jj')}=-\xi_{a(i'i,jj')}$ and $\xi_{a(ii',jj')}=-\xi_{a(ii',j'j)}$, the set Ξ generates the ideal $\sum_{a\in L}A\xi_a$. Clearly we have

$$\begin{aligned}\dim R_2 &= \#\left\{\sum_{i=1}^{n}k_i\alpha_i \;\middle|\; \begin{matrix}0\leqq k_1\leqq k_2\leqq\cdots\leqq k_p=2\\ 0\leqq k_n\leqq k_{n-1}\leqq\cdots\leqq k_p=2\end{matrix}\right\}\\ &=\binom{p+2-1}{2}\times\binom{n-(p-1)+2-1}{2}\\ &=p(p+1)(n-p+1)(n-p+2)/4\,.\end{aligned}$$

On the other hand, we have

$$\dim A_2 = \binom{p(n-(p-1))+2-1}{2}, \qquad |\Xi| = \binom{p}{2}\binom{n-(p-1)}{2}.$$

Hence we have $\dim R_2 + |\Xi| = \dim A_2$, and thus we proved the minimality of Ξ. ■

PROPOSITION 2.4. *The ring R is the Segre product of two polynomial rings, $\boldsymbol{Z}[e(\varepsilon_1), \cdots, e(\varepsilon_p)]$ and $\boldsymbol{Z}[e(-\varepsilon_{p+1}), \cdots, e(-\varepsilon_n)]$.*

PROOF. This is clear from the standard realization of the root system A_n (cf. [**B**, planche I]). ■

§3. Type B_n.

In this section we suppose that $\mathfrak{g}$ is of B_n-type. We index the elements of the set X as follows:

$$\chi_j = \sum_{1\leq k\leq j} \alpha_k + 2\sum_{j<k\leq n} \alpha_k \qquad (1\leqq j\leqq n-1)$$

$$\chi_{-j} = \sum_{1\leq k\leq j} \alpha_k \qquad (1\leqq j\leqq n-1)$$

$$\chi_0 = \sum_{1\leq k\leq n} \alpha_k .$$

Then we have $X = \{\chi_0, \chi_{\pm j}\ (1\leqq j\leqq n-1)\}$.

PROPOSITION 3.1. *The $\boldsymbol{Z}$-algebra R is normal.*

PROOF. We denote by $\triangle$ the cone generated by $\chi_0, \chi_{-1}, \cdots, \chi_{-(n-1)}$. It is clear that $\triangle \cap \boldsymbol{Z}^n = \boldsymbol{Z}_{\geq 0}\chi_0 + \boldsymbol{Z}_{\geq 0}\chi_{-1} + \cdots + \boldsymbol{Z}_{\geq 0}\chi_{-(n-1)}$ and $\triangle = (f_1 - f_2 \geqq 0) \cap \cdots \cap (f_{n-1} - f_n \geqq 0) \cap (f_n \geqq 0)$. Let $\delta = a_1\alpha_1 + \cdots + a_n\alpha_n$ with $a_i \in \boldsymbol{Z}_{\geq 0}$ $(1\leqq i\leqq n)$ satisfy $(\delta, \alpha_i) \leqq 0$ for $2\leqq i\leqq n$. Then we have $-a_{i-1} + 2a_i - a_{i+1} \leqq 0$ for $2\leqq i\leqq n-1$ and $-a_{n-1} + 2a_n \leqq 0$. Hence we have $a_1 - a_2 \geqq a_2 - a_3 \geqq \cdots \geqq a_{n-1} - a_n \geqq a_n \geqq 0$, which implies $\delta \in \triangle$, accordingly $\delta \in \boldsymbol{Z}_{\geq 0}\chi_0 + \boldsymbol{Z}_{\geq 0}\chi_{-1} + \cdots + \boldsymbol{Z}_{\geq 0}\chi_{-(n-1)} \subset \Lambda$. By Lemma 1.7, we obtain the normality of R. ■

The proof of Proposition 3.1 shows that, for any $\Gamma \in \mathfrak{F}$, there exists $w \in W$ such that $\Gamma = (w \cdot f_n = 0)$. The number $|\mathfrak{F}|$ is equal to $|W|/|W(A_{n-2})| = \{2^{n-1}\cdot(n-1)!\}/(n-1)! = 2^{n-1}$ where $W(A_{n-2})$ is the subgroup of W generated by $s_2, s_3, \cdots, s_{n-1}$. The group W can be identified with the semidirect of the symmetric group S_{n-1} and the group of sign changes by $s_i = (i-1, i)$ for $2\leqq i\leqq n-1$ and s_n=the sign change of $(n-1)$. For $\sigma \in W$ we have $\sigma(\chi_0) = \chi_0$ and $\sigma(\chi_j) = \chi_{\sigma(j)}$ for any $j \in \{\pm 1, \cdots, \pm(n-1)\}$. For $1\leqq i\leqq n-1$ we define $b(i) \in L$ by

$$b(i)_s=\begin{cases}-2 & s=0\\ 1 & s=\pm i\\ 0 & \text{otherwise.}\end{cases}$$

For $1\leqq i\neq j\leqq n-1$ we define $c(ij)\in L$ by

$$c(ij)_s=\begin{cases}1 & s=\pm i\\ -1 & s=\pm j\\ 0 & \text{otherwise.}\end{cases}$$

We remark that $\xi_{c(ij)}=\xi_{b(i)}-\xi_{b(j)}$.

PROPOSITION 3.2. *Let A be the polynomial ring $\boldsymbol{Z}[\xi_i(-(n-1)\leqq i\leqq n-1)]$. Then the set $\Xi:=\{\xi_i\xi_{-i}-\xi_0^2 \mid 1\leqq i\leqq n-1\}$ is a minimal system of generators of the ideal $\sum_{a\in L} A\xi_a$.*

PROOF. Let $a=(a_i)_{-(n-1)\leqq i\leqq n-1}\in\boldsymbol{Z}^N$. By Lemma 1.6, we have $a\in L$ if and only if $a_0+2\sum_{i\in I}a_i+2\sum_{i\in I'}a_{-i}=0$ for any disjoint decomposition $I\amalg I'=\{1,\cdots,n-1\}$. Let $a\in L-\{0\}$. Then we have $a_0+2\sum_{k=1}^{n-1}a_k=0$ and $a_0+2\sum_{k\neq i}a_k+2a_{-i}=0$ for any $i\in\{1,\cdots,n-1\}$. Hence we obtain $a_i=a_{-i}$ for $i=1,\cdots,n-1$. Since $a\neq 0$, there exists $i\in\{1,\cdots,n-1\}$ such that $a\neq 0$. If necessary, replacing by $-a$, we may suppose that $a_i>0$. Then there exists $j\in\{1,\cdots,n-1\}$ such that $a_j<0$ or $a_0\leqq -2$. In the former case, ξ_a is generated by $\xi_{c(ij)}$ and $\xi_{a-c(ij)}$, and we have $a\chi>(a-c(ij))\chi$. In the latter case, ξ_a is generated by $\xi_{b(i)}$ and $\xi_{a-b(i)}$, and we have $a\chi>(a-b(i))\chi$. By recurrence, we see that ξ_a is generated by $\xi_{b(i)}$ $(1\leqq i\leqq n-1)$. Obviously we have $\xi_{b(i)}=\xi_i\xi_{-i}-\xi_0^2$. The minimality is obvious. ■

§4. Type C_n.

In this section, we suppose that $\mathfrak{g}$ is of C_n-type. For $1\leqq i\leqq j\leqq n$ we define $\chi_{ij}\in X$ by $\chi_{ij}:=\sum_{i\leqq k<j}\alpha_k+2\sum_{j\leqq k<n}\alpha_k+\alpha_n$. Then we have $X=\{\chi_{ij}\mid 1\leqq i\leqq j\leqq n\}$.

PROPOSITION 4.1. *The $\boldsymbol{Z}$-algebra R is normal.*

PROOF. We put $\chi_i:=\chi_{ii+1}$ for $i=1,2,\cdots,n-1$, $\chi_n:=\chi_{nn}$, $\gamma_{n-i}:=\sum_{k=0}^{(i-1)/2}\chi_{n-2k-1}$ for odd $i<n$, and $\gamma_{n-i}:=2\sum_{k=0}^{i/2}\chi_{n-2k}+\chi_n$ for even $i<n$. We denote by $\triangle$ (resp. $\triangle'$) the cone generated by $\chi_1,\chi_2,\cdots,\chi_n$ (resp. $\gamma_1,\gamma_2,\cdots,\gamma_n$). It is clear that $\triangle'\subset\triangle$ and $\boldsymbol{Z}^n\cap\triangle=\boldsymbol{Z}_{\geqq 0}\chi_1+\cdots+\boldsymbol{Z}_{\geqq 0}\chi_n$. We can verify that

$$\triangle'=(f_1\geqq 0)\cap(f_2-2f_1\geqq 0)$$
$$\cap\bigcap_{i=3}^{n-1}(f_{i-2}-2f_{i-1}+f_i\geqq 0)\cap(f_{n-2}-2f_{n-1}+2f_n\geqq 0).$$

Let $\delta=a_1\alpha_1+\cdots+a_n\alpha_n$ with $a_i\in \boldsymbol{Z}_{\geqq 0}$ $(1\leqq i\leqq n)$ satisfy $(\delta, \alpha_j)\leqq 0$ for $j=1, \cdots, n-1$. Then we have

$$\begin{aligned} 2a_1-a_2 &\leqq 0 \\ -a_{i-1}+2a_i-a_{i+1} &\leqq 0 \qquad (i=2, \cdots, n-2) \\ -a_{n-2}+2a_{n-1}-2a_n &\leqq 0\,. \end{aligned}$$

Hence we have $\delta\in\triangle'\subset\triangle$, which implies $\delta\in \boldsymbol{Z}_{\geqq 0}\chi_1+\cdots+\boldsymbol{Z}_{\geqq 0}\chi_n\subset\Lambda$. By Lemma 1.7, we obtain the normality of R. ■

The proof of Proposition 4.1 shows that, for any $\Gamma\in\mathfrak{F}$, there exists $w\in W$ such that $\Gamma=(w\cdot f_1=0)$. The number $|\mathfrak{F}|$ is equal to $|W|/|W(A_{n-2})|=n!/(n-1)! =n$ where $W(A_{n-2})$ is the subgroup of W generated by $s_2, s_3, \cdots, s_{n-1}$. Set $\chi_{ij}:=\chi_{ji}$ for $i>j$. The group W can be identified with the symmetric group S_n by $s_i=(i, i+1)$ for $i=1, \cdots, n-1$. Then for $\sigma\in W$ we have $\sigma(\chi_{ij})=\chi_{\sigma(i)\sigma(j)}$ for all $1\leqq i, j\leqq n$. For $a=(a_{ij})_{1\leqq i\leqq j\leqq n}\in \boldsymbol{Z}^N$, we set $a_{ij}:=a_{ji}$ for $i>j$, and thus we identify $\boldsymbol{Z}^N$ with the space of $n\times n$ symmetric matrices with integer coefficients. For four distinct numbers i, j, k, l (j may coincide with k), we define $a(ij, kl)\in L$ by

$$a(ij, kl)_{st}=\begin{cases} 1 & (s, t)=(i, k), (j, l) \\ -1 & (s, t)=(i, l), (j, k) \\ 0 & \text{otherwise.} \end{cases}$$

For $1\leqq i\neq j\leqq n$ we define $b(ij)\in L$ by

$$b(ij)_{st}=\begin{cases} 2 & (s, t)=(i, j) \\ -1 & (s, t)=(i, i), (j, j) \\ 0 & \text{otherwise.} \end{cases}$$

For three distinct numbers i, j, k, we define $c(ijk)\in L$ by

$$c(ijk)_{st}=\begin{cases} 1 & (s, t)=(i, j), (i, k) \\ -1 & (s, t)=(i, i), (j, k) \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.2. *Let A be the polynomial ring $\boldsymbol{Z}[\xi_{ij}\ (1\leqq i\leqq j\leqq n)]$. We identify A with the ring $A'/\sum_{i<j}A'(\xi_{ij}-\xi_{ji})$ where A' is the polynomial ring $\boldsymbol{Z}[\xi_{ij}\ (1\leqq i, j\leqq n)]$. Then we have*

$$R=A/\Big(\sum_{1\leq i,j,k,l\leq n}A\xi_{a(ij,kl)}+\sum_{1\leq i,j\leq n}A\xi_{b(ij)}+\sum_{1\leq i,j,k\leq n}A\xi_{c(ijk)}\Big).$$

PROOF. Let $a=(a_{ij})_{1\leq i\leq j\leq n}\in \boldsymbol{Z}^N$. By Lemma 1.6, we have $a\in L$ if and only if $\sum_{k=1}^{i} a_{ki}+\sum_{k=i}^{n} a_{ik}=0$ for $i=1, \cdots, n$. Suppose that $a\in L-\{0\}$. Then there exist $i\neq j$ such that $a_{ij}\neq 0$. Without loss of generality, we may suppose that $a_{ij}<0$. Since we have $2a_{ii}+\sum_{k\neq i} a_{ik}=0$ and $2a_{jj}+\sum_{k\neq j} a_{jk}=0$, there exist the following cases:

(1) There exist $k\neq j$ and $l\neq i$ such that $a_{ik}>0$ and $a_{jl}>0$.

(2) We have $a_{ii}>0$ and $a_{is}\leqq 0$ for all $s\neq i$. Moreover there exists $k\neq j$ such that $a_{ik}<0$.

(3) We have $a_{ii}>0$, $a_{ij}\leqq -2$ and $a_{is}=0$ for all $s\neq i, j$.

In the case (1), ξ_a is generated by $\xi_{a(il,kj)}$ and $\xi_{a-a(il,kj)}$, and we have $a\chi>(a-a(il, kj))\chi$ by Lemmas 1.4 and 1.5. In the case (2), ξ_a is generated by $\xi_{c(ijk)}$ and $\xi_{-a-c(ijk)}$, and we have $a\chi>(a+c(ijk))\chi$ by Lemmas 1.4 and 1.5. In the case (3), ξ_a is generated by $\xi_{b(ij)}$ and $\xi_{-a-b(ij)}$, and we have $a\chi>(a+b(ij))\chi$ by Lemmas 1.4 and 1.5. By recurrence, we see that ξ_a is generated by $\xi_{a(ij,kl)}$, $\xi_{b(ij)}$ and $\xi_{c(ijk)}$. ∎

PROPOSITION 4.3. *The ring R is the Veronese subring of degree 2 of the polynomial ring $\boldsymbol{Z}[e(\varepsilon_1), \cdots, e(\varepsilon_n)]$.*

PROOF. This is clear because we have $\chi_{ij}=\varepsilon_i+\varepsilon_j$ in the standard realization of the root system C_n (cf. [**B**, planche III]). ∎

PROPOSITION 4.4. *Put*

$$\Xi_1 := \{\xi_{ik}\xi_{jl}-\xi_{il}\xi_{jk} \mid i<j<k<l\}$$

$$\Xi_2 := \{\xi_{ij}\xi_{kl}-\xi_{ik}\xi_{jl} \mid i<j<k<l\}$$

$$\Xi_3 := \{\xi_{ij}\xi_{ik}-\xi_{ii}\xi_{jk} \mid i<j<k\}$$

$$\Xi_4 := \{\xi_{ij}\xi_{jk}-\xi_{jj}\xi_{ik} \mid i<j<k\}$$

$$\Xi_5 := \{\xi_{ik}\xi_{jk}-\xi_{ij}\xi_{kk} \mid i<j<k\}$$

$$\Xi_6 := \{\xi_{ij}^2-\xi_{ii}\xi_{jj} \mid i<j\},$$

and $\Xi := \bigcup_{i=1}^{6} \Xi_i$. Then Ξ is a minimal system of generators of the ideal $\sum_{a\in L} A\xi_a$.

PROOF. First we verify that all $\xi_{a(ij,kl)}$ are generated by Ξ. Since $\xi_{a(ij,kl)} =-\xi_{a(ji,kl)}=-\xi_{a(ij,lk)}=\xi_{a(ji,lk)}$, we may assume $i<j$ and $k<l$. For distinct i, j, k, l, there are six cases: (1) $i<j<k<l$, (2) $i<k<j<l$, (3) $i<k<l<j$, (4) $k<i<j<l$, (5) $k<i<l<j$, (6) $k<l<i<j$. In the case of (1), we have $\xi_{a(ij,kl)}= \xi_{ik}\xi_{jl}-\xi_{il}\xi_{jk}\in\Xi_1$. In the case of (2), we have $\xi_{a(ij,kl)}=\xi_{ik}\xi_{jl}-\xi_{il}\xi_{kj}=(\xi_{ik}\xi_{jl}- \xi_{ij}\xi_{kl})+(\xi_{ij}\xi_{kl}-\xi_{il}\xi_{kj})\in\Xi_2+\Xi_1$. In the case of (3), we have $\xi_{a(ij,kl)}=\xi_{ik}\xi_{lj}- \xi_{il}\xi_{kj}\in\Xi_2$. In the case of (4), we have $\xi_{a(ij,kl)}=\xi_{ki}\xi_{jl}-\xi_{il}\xi_{kj}\in\Xi_2$. In the case of (5), we have $\xi_{a(ij,kl)}=\xi_{ki}\xi_{lj}-\xi_{il}\xi_{kj}\in\Xi_2+\Xi_1$. In the case of (6), we have

$\xi_{a(ij,kl)}=\xi_{ki}\xi_{lj}-\xi_{li}\xi_{kj}\in\Xi_1$. In the case of $i<j=k<l$ we have $\xi_{a(ij,kl)}=\xi_{ij}\xi_{jl}-\xi_{il}\xi_{jj}\in\Xi_4$.

Second we verify that all $\xi_{b(ij)}$ are generated by Ξ. Since $b(ij)=b(ji)$, we may assume $i<j$. Then we have $\xi_{b(ij)}=\xi_{ij}^2-\xi_{ii}\xi_{jj}\in\Xi_6$.

Third we verify that all $\xi_{c(ijk)}$ are generated by Ξ. There are three cases: (1) $i<j<k$, (2) $j<i<k$, (3) $j<k<i$. In the case of (1), we have $\xi_{c(ijk)}=\xi_{ij}\xi_{ik}-\xi_{ii}\xi_{jk}\in\Xi_3$. In the case of (2), we have $\xi_{c(ijk)}=\xi_{ji}\xi_{ik}-\xi_{ii}\xi_{jk}\in\Xi_4$. In the case of (3), we have $\xi_{c(ijk)}=\xi_{ji}\xi_{ki}-\xi_{ii}\xi_{jk}\in\Xi_5$. Hence we have proved that Ξ generates the ideal $\sum_{a\in L}A\xi_a$.

Finally we verify that $\dim A_2-|\Xi|=\dim R_2$ for the minimality. Clearly we have

$$|\Xi|=2\binom{n}{4}+3\binom{n}{3}+\binom{n}{2}$$
$$=n^2(n-1)(n+1)/12,$$

and

$$\dim A_2=\binom{n(n+1)/2+2-1}{2}.$$

By Proposition 4.3, we have

$$\dim R_2=\binom{n+4-1}{4}.$$

We can check $\dim A_2-\dim R_2=n^2(n-1)(n+1)/12$, and thus we see the minimality of Ξ. ■

REMARK 4.5. In [S, Example 4], the subset Ξ_2 was missed.

§ 5. Type D_n with $i(\mathfrak{n})=1$.

In this section, we suppose that $\mathfrak{g}$ is of D_n-type, and that $i(\mathfrak{n})=1$. Let $\gamma=2\alpha_1+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\in\Lambda$. We index the elements of the set X as follows:

$$\chi_i:=\sum_{1\leq k\leq i}\alpha_k \qquad (i=1,\cdots,n-1)$$

$$\chi_{-i}:=\gamma-\chi_i \qquad (i=1,\cdots,n-1).$$

Then we have $X=\{\chi_i,\chi_{-i}\ (i=1,\cdots,n-1)\}$.

PROPOSITION 5.1. *The $\boldsymbol{Z}$-algebra R is normal.*

PROOF. We denote by $\triangle_1$ (resp. $\triangle_2$) the cone generated by $\chi_1,\cdots,\chi_{n-1}$ and γ (resp. $\chi_1,\cdots,\chi_{n-2},\chi_{-(n-1)}$ and γ). It is clear that

$$\boldsymbol{Z}^n\cap\triangle_1=\boldsymbol{Z}_{\geq0}\chi_1+\cdots+\boldsymbol{Z}_{\geq0}\chi_{n-1}+\boldsymbol{Z}_{\geq0}\gamma$$

and

$$\boldsymbol{Z}^n \cap \triangle_2 = \boldsymbol{Z}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{Z}_{\geqq 0}\chi_{n-2} + \boldsymbol{Z}_{\geqq 0}\chi_{-(n-1)} + \boldsymbol{Z}_{\geqq 0}\gamma .$$

We can verify that

$$\triangle_1 = (f_n \geqq 0) \cap (f_{n-1} - f_n \geqq 0) \cap (f_{n-2} - f_{n-1} - f_n \geqq 0) \cap \bigcap_{i=1}^{n-3} (f_i - f_{i+1} \geqq 0)$$

and

$$\triangle_2 = (f_{n-1} \geqq 0) \cap (f_n - f_{n-1} \geqq 0) \cap (f_{n-2} - f_{n-1} - f_n \geqq 0) \cap \bigcap_{i=1}^{n-3} (f_i - f_{i+1} \geqq 0) .$$

Hence we have

$$\triangle_1 \cup \triangle_2 = (f_n \geqq 0) \cap (f_{n-1} \geqq 0) \cap (f_{n-2} - f_{n-1} - f_n \geqq 0) \cap \bigcap_{i=1}^{n-3} (f_i - f_{i+1} \geqq 0)$$

and

$$\boldsymbol{Z}^n \cap (\triangle_1 \cup \triangle_2) = \boldsymbol{Z}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{Z}_{\geqq 0}\chi_{n-1} + \boldsymbol{Z}_{\geqq 0}\chi_{-(n-1)} + \boldsymbol{Z}_{\geqq 0}\gamma .$$

Let $\delta = a_1\alpha_1 + \cdots + a_n\alpha_n$ with $a_i \in \boldsymbol{Z}_{\geqq 0}$ $(1 \leqq i \leqq n)$ satisfy $(\delta, \alpha_i) \leqq 0$ for $i = 2, 3, \cdots, n$. Then we have

$$-a_{i-1} + 2a_i - a_{i+1} \leqq 0 \qquad (i = 2, 3, \cdots, n-3)$$

$$-a_{n-3} + 2a_{n-2} - a_{n-1} - a_n \leqq 0$$

$$-a_{n-2} + 2a_{n-1} \leqq 0$$

$$-a_{n-1} + 2a_n \leqq 0 .$$

From these inequalities, we obtain

$$a_1 - a_2 \geqq a_2 - a_3 \geqq \cdots \geqq a_{n-3} - a_{n-2} \geqq a_{n-2} - a_{n-1} - a_n \geqq 0 ,$$

which implies $\delta \in \triangle_1 \cup \triangle_2$. Hence we see that $\delta \in \boldsymbol{Z}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{Z}_{\geqq 0}\chi_{n-1} + \boldsymbol{Z}_{\geqq 0}\chi_{-(n-1)} + \boldsymbol{Z}_{\geqq 0}\gamma \subset \Lambda$. By Lemma 1.7, we obtain the normality of R. ∎

The proof of Proposition 5.1 shows that, for any $\Gamma \in \mathfrak{F}$, there exists $w \in W$ such that $\Gamma = (w \cdot f_{n-1} = 0)$ or $(w \cdot f_n = 0)$. The number $|\mathfrak{F}|$ is equal to $|W|/|W(A_{n-2})| + |W|/|W(A_{n-2})'| = 2\{2^{n-2} \cdot (n-1)!\}/(n-1)! = 2^{n-1}$ where $W(A_{n-2})$ (resp. $W(A_{n-2})'$) is the subgroup of W generated by $s_2, s_3, \cdots, s_{n-1}$ (resp. $s_2, s_3, \cdots, s_{n-2}$ and s_n). The group W can be identified with the semidirect product of the symmetric group S_{n-1} and the group of even number of sign changes, by $s_i = (i-1, i)$ for $i = 2, 3, \cdots, n-1$, and by s_n = the sign change of $n-1$ and $n-2$ following the transposition $(n-2, n-1)$. Then we have $\sigma(\chi_j) = \chi_{\sigma(j)}$ for any $\sigma \in W$ and any $j \in \{\pm 1, \cdots, \pm(n-1)\}$. For distinct $i, j \in \{1, \cdots, (n-1)\}$, we define $c(ij) \in L$ by

$$c(ij)_s = \begin{cases} 1 & s = \pm j \\ -1 & s = \pm i \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 5.2. *Let A be the polynomial ring $\boldsymbol{Z}[\xi_{\pm i}\ (1 \leqq i \leqq n-1)]$. Then the set $\Xi := \{\xi_i \xi_{-i} - \xi_{i+1} \xi_{-(i+1)} \mid 1 \leqq i \leqq n-2\}$ is a minimal system of generators of the ideal $\sum_{a \in L} A\xi_a$.*

PROOF. Let $a = (a_i)_{i=\pm 1, \cdots, \pm(n-1)} \in \boldsymbol{Z}^N$. By Lemma 1.6, we have $a \in L$ if and only if $\sum_{i \in I} a_i + \sum_{i \in I'} a_{-i} = 0$ for any disjoint decomposition $I \amalg I' = \{1, 2, \cdots, n-1\}$. Suppose that $a \in L - \{0\}$. We see that $a_k = a_{-k}$ for any $1 \leqq k \leqq n-1$, and that there exist $i \neq j$ such that $a_i < 0$ and $a_j > 0$ since $\sum_{j=1}^{n-1} a_j = 0$ and $a_{-k} + \sum_{j>0, j \neq k} a_j = 0$ for any $1 \leqq k \leqq n-1$. Hence ξ_a is generated by $\xi_{c(ij)}$ and $\xi_{a-c(ij)}$, and we have $a\chi > (a - c(ij))\chi$ by Lemmas 1.4 and 1.5. By recurrence, we see that ξ_a is generated by $\xi_{c(ij)}$ $(1 \leqq i \neq j \leqq n-1)$. Clearly all $\xi_{c(ij)}$ are generated by Ξ. The minimality of Ξ is also clear. ∎

§6. Type D_n with $i(\mathfrak{n}) = n$.

In this section, we suppose that $\mathfrak{g}$ is of D_n-type, and that $i(\mathfrak{n}) = n$. We index the elements of the set X as follows:

$$\chi_{ij} := \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq n-2} \alpha_k + \alpha_{n-1} + \alpha_n \qquad (1 \leqq i < j \leqq n-1)$$

$$\chi_{in} := \sum_{i \leq k \leq n-2} \alpha_k + \alpha_n \qquad (1 \leqq i \leqq n-1).$$

Then we have $X = \{\chi_{ij} \mid 1 \leqq i < j \leqq n\}$.

We define χ_i and χ'_i $(1 \leqq i \leqq n)$ by

$$\chi_n := \chi'_n := \chi_{n-1n}$$

$$\chi_{n-1} := \chi'_{n-1} := \chi_{n-2n-1}$$

$$\chi_{n-2} := \chi'_{n-2} := \chi_{n-2n}$$

$$\chi_i := \chi_{ii+1} \qquad (1 \leqq i \leqq n-3)$$

$$\chi'_i := \chi_{in} \qquad (1 \leqq i \leqq n-3).$$

We also define γ_i $(1 \leqq i \leqq n)$ by

$$\gamma_n := \chi_n$$

$$\gamma_{n-1} := \chi_{n-1}$$

$$\gamma_{n-2} := \chi_{n-2}$$

$$\gamma_{n-i} := \sum_{k=1}^{(i-1)/2} \chi_{n-i-2+2k} + \chi_n \qquad (i \text{ is odd and } i \geqq 3)$$

$$\gamma_{n-i} := 2\sum_{k=1}^{(i-2)/2} \chi_{n-i-2+2k} + \chi_{n-2} + \chi_{n-1} + \chi_n \qquad (i \text{ is even and } i > 3).$$

We denote by C_{n+i} (resp. Λ_{n+i}) the cone (resp. the semigroup) generated by $\chi'_1, \chi'_2, \cdots, \chi'_n, \chi_1, \chi_2, \cdots, \chi_i$ for $0 \leqq i \leqq n-3$.

LEMMA 6.1.

$$\boldsymbol{Z}^n \cap C_{2n-3} = \Lambda_{2n-3}\,.$$

PROOF. It is clear that $\boldsymbol{Z}^n \cap C_n = \Lambda_n$. We can verify that

$$C_n = (f_1 \geqq 0) \cap \bigcap_{i=2}^{n-3} (f_i - f_{i-1} \geqq 0)$$
$$\cap (f_{n-2} - f_{n-3} - f_{n-1} \geqq 0) \cap (f_{n-1} \geqq 0) \cap (f_n - f_{n-2} \geqq 0)\,.$$

Then we have

$$f_1(\chi_i) \geqq 0 \qquad (1 \leqq i \leqq n-3)$$
$$f_{n-1}(\chi_i) \geqq 0 \qquad (1 \leqq i \leqq n-3)$$
$$(f_n - f_{n-2})(\chi_i) = -1 \qquad (1 \leqq i \leqq n-3)$$
$$(f_{n-2} - f_{n-3} - f_{n-1})(\chi_i) = -1 \qquad (1 \leqq i < n-3)$$
$$(f_{n-2} - f_{n-3} - f_{n-1})(\chi_{n-3}) = 0$$
$$(f_j - f_{j-1})(\chi_i) \geqq 0 \qquad (1 \leqq i \leqq n-3,\ 2 \leqq j \leqq n-3)\,.$$

Hence we obtain the assertion by Lemmas 1.4 and 1.5. ■

PROPOSITION 6.2. *The* $\boldsymbol{Z}$*-algebra* R *is normal.*

PROOF. We denote by $\triangle_\gamma$ the cone generated by $\gamma_1, \gamma_2, \cdots, \gamma_n$, and by $\triangle_i$ the one generated by $\gamma_1, \gamma_2, \cdots, \gamma_i, \chi'_i, \chi'_{i+1}, \cdots, \chi'_{n-2}$ and χ'_n for $1 \leqq i \leqq n-3$. We can verify that

$$\triangle_\gamma = (f_1 \geqq 0) \cap (-2f_1 + f_2 \geqq 0) \cap \bigcap_{i=3}^{n-3} (f_{i-2} - 2f_{i-1} + f_i \geqq 0)$$
$$\cap (f_{n-4} - 2f_{n-3} + 2f_{n-2} - f_{n-1} \geqq 0)$$
$$\cap (f_{n-4} - 2f_{n-3} + 2f_{n-1} \geqq 0) \cap (f_{n-4} - 2f_{n-2} + 2f_n \geqq 0)\,,$$

and that, for $1 \leqq i \leqq n-3$,

$$\triangle_i = \bigcap_{i=1}^{i}(F_j^i \geqq 0) \cap \bigcap_{i \leqq j \leqq n, j \neq n-1} (G_j^i \geqq 0)$$

where

$$F_j^i = \begin{cases} f_{j-2} - 2f_{j-1} + f_j & (1 \leqq j \leqq i-1) \\ (n-i-1)f_{i-2} - (n-i)f_{i-1} + 2f_{n-1} & (j=i), \end{cases}$$

and

$$G_j^i = \begin{cases} -(n-i-2)f_{i-1} + (n-i-1)f_i - 2f_{n-1} & (j=i) \\ f_{i-1} - (n-i-1)f_{j-1} + (n-i-1)f_j - 2f_{n-1} & (i<j \leqq n-2) \\ f_{i-1} - (n-i-1)f_{n-2} + (n-i-3)f_{n-1} + (n-i-1)f_n & (j=n). \end{cases}$$

We remark that

$$G_i^i = -F_{i+1}^{i+1} \qquad (1 \leqq i \leqq n-3)$$

$$G_{i+1}^i = (f_{i-1} - 2f_i + f_{i+1}) - F_{i+2}^{i+2} \qquad (1 \leqq i \leqq n-3)$$

$$G_{j+1}^i = G_j^i + (n-i-1)(f_{j-1} - 2f_j + f_{j+1}) \qquad (1 \leqq i < j \leqq n-3)$$

$$G_n^i = G_{n-2}^i + (n-i-1)(f_{n-3} - 2f_{n-2} + f_{n-1} + f_n) \qquad (1 \leqq i \leqq n-3)$$

where we put $F_{n-2}^{n-2} := f_{n-4} - 2f_{n-3} + 2f_{n-1}$, $F_{n-1}^{n-1} := 2f_{n-1} - f_{n-2}$, and $f_i = 0$ for $i \leqq 0$.

Let $\delta = a_1\alpha_1 + \cdots + a_n\alpha_n$ with $a_i \in \boldsymbol{Z}_{\geqq 0}$ $(1 \leqq i \leqq n)$ satisfy $(\delta, \alpha_j) \leqq 0$ for $j = 1, 2, \cdots, n-1$. Then we have

$$-a_{i-1} + 2a_i - a_{i+1} \leqq 0 \qquad (i = 1, 2, \cdots, n-3)$$

$$-a_{n-3} + 2a_{n-2} - a_{n-1} - a_n \leqq 0$$

$$-a_{n-2} + 2a_{n-1} \leqq 0 .$$

If $F_{n-2}^{n-2}(\delta) = a_{n-4} - 2a_{n-3} + 2a_{n-1} \geqq 0$, then we have

$$a_{n-4} - 2a_{n-3} + 2a_{n-2} - 2a_{n-1} = a_{n-4} - 2a_{n-3} + 2a_{n-1} + 2(a_{n-2} - 2a_{n-1}) \geqq 0 ,$$

$$a_{n-4} - 2a_{n-2} + 2a_n = (a_{n-4} - 2a_{n-3} + 2a_{n-2} - 2a_{n-1}) + 2(a_{n-3} - 2a_{n-2} + a_{n-1} + a_n) \geqq 0 ,$$

and hence $\delta \in \triangle_\gamma$. If $F_{n-2}^{n-2}(\delta) \leqq 0$ and $F_{n-3}^{n-3}(\delta) \geqq 0$, then we obtain $\delta \in \triangle_{n-3}$. For $1 \leqq i \leqq n-3$, if $F_j^j(\delta) \leqq 0$ $(i+1 \leqq j \leqq n-2)$ and $F_i^i(\delta) \geqq 0$, then we obtain $\delta \in \triangle_i$. Since we have $F_1^1(\delta) = a_{n-1} \geqq 0$, we obtain $\delta \in \triangle_\gamma \cup \bigcup_{i=1}^{n-3} \triangle_i$. It is clear that $\triangle_\gamma \cup \bigcup_{i=1}^{n-3} \triangle_i \subset C_{2n-3}$. Hence we obtain the normality of R by Lemmas 1.7 and 6.1. ■

We put

$$\triangle := \{\chi \in \boldsymbol{R}^n \mid f_i(\chi) \geqq 0 \ (1 \leqq i \leqq n), \text{ and } (\chi, \alpha_j) \leqq 0 \ (1 \leqq j \leqq n-1)\}.$$

Let $\chi = \sum_{i=1}^n a_i \chi_i \in \boldsymbol{R}^n$ satisfy $a_1 \geqq 0$, $a_{n-1} \geqq 0$ and $(\chi, \alpha_j) \leqq 0$ for $1 \leqq j \leqq n-1$. Then we have

$$a_{n-2} - a_{n-3} \geqq a_{n-3} - a_{n-4} \geqq \cdots \geqq a_2 - a_1 \geqq a_1 \geqq 0,$$

$$a_{n-2} - a_{n-1} \geqq a_{n-1} \geqq 0,$$

$$a_n \geqq (a_{n-2} - a_{n-3}) + (a_{n-2} - a_{n-1}) \geqq 0.$$

Hence we have $\chi \in \triangle$ and

$$\triangle = \{\chi \in \boldsymbol{R}^n \mid f_1(\chi) \geqq 0, \ f_{n-1}(\chi) \geqq 0, \text{ and } (\chi, \alpha_j) \leqq 0 \ (1 \leqq j \leqq n-1)\}.$$

Since $(\chi'_j, \alpha_j) > 0$ for $1 \leqq j \leqq n-1$, the linear form $(\cdot, -\alpha_j)$ can not define a face of Q of codimension one. On the other hand, it is easy to check that the linear forms f_1 and f_{n-1} actually define faces of Q of codimension one. For any $\Gamma \in \mathfrak{F}$, there exists $w \in W$ such that $\Gamma = (w \cdot f_1 = 0)$ or $(w \cdot f_{n-1} = 0)$. The number $|\mathfrak{F}|$ is equal to $|W|/|W(A_{n-2})| + |W|/|W(A_{n-2})'| = 2n!/(n-1)! = 2n$ where $W(A_{n-2})$ (resp. $W(A_{n-2})'$) denotes the subgroup of W generated by $s_2, s_3, \cdots, s_{n-1}$ (resp. $s_1, s_2, \cdots, s_{n-2}$).

The group W can be identified with the symmetric group S_n by $s_i = (i, i+1)$ for $i = 1, 2, \cdots, n-1$. We set $\chi_{ij} := \chi_{ji}$ for $j < i$. We have $\sigma(\chi_{ij}) = \chi_{\sigma(i)\sigma(j)}$ for $\sigma \in W$ and $1 \leqq i \neq j \leqq n$. For all $a = (a_{ij})_{1 \leqq i < j \leqq n} \in \boldsymbol{Z}^N$ we set $a_{ij} := a_{ji}$ for $i > j$, and thus we consider them as elements of $\boldsymbol{Z}^{n(n-1)/2} = \{(a_{ij})_{1 \leqq i \neq j \leqq n} \mid a_{ij} \in \boldsymbol{Z}\}$. For four distinct $i, j, k, l \in \{1, \cdots, n\}$, we define $a(ij, kl) \in L$ by

$$a(ij, kl)_{st} = \begin{cases} 1 & (s, t) = (i, j), (k, l) \\ -1 & (s, t) = (i, k), (j, l) \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 6.3. *Let A be the polynomial ring $\boldsymbol{Z}[\xi_{ij} \ (1 \leqq i < j \leqq n)]$. We identify A with the ring $A'/\sum_{i<j} A'(\xi_{ij} - \xi_{ji})$ where A' is the polynomial ring $\boldsymbol{Z}[\xi_{ij} \ (1 \leqq i \neq j \leqq n)]$. Then we have*

$$R = A / \sum_{i,j,k,l \text{ are distinct}} A \xi_{a(ij,kl)}.$$

PROOF. Let $a = (a_{ij})_{1 \leqq i < j \leqq n} \in \boldsymbol{Z}^N$. By Lemma 1.6, we have $a \in L$ if and only if $\sum_{k=1}^{i-1} a_{ki} + \sum_{k=i+1}^n a_{ik} = 0$ for $1 \leqq i \leqq n$. For convenience, we set $a_{ij} := a_{ji}$ for $i > j$. Suppose that $a \in L - \{0\}$. Then there exist $i \neq k$ such that $a_{ik} < 0$. There also exist $j \neq i, k$ and $l \neq i, k$ such that $a_{ij} > 0$ and $a_{kl} > 0$. If $j \neq l$, then ξ_a is generated by $\xi_{a(ij,kl)}$ and $\xi_{a-a(ij,kl)}$, and we have $a\chi > (a - a(ij, kl))\chi$ by Lemmas 1.4

and 1.5. If $j=l$, then there exists $m\neq i, j, k$ such that $a_{jm}<0$, which implies that ξ_a is generated by $\xi_{a(ik,jm)}$ and $\xi_{a+a(ik,jm)}$, and that $a\chi>(a+a(ik, jm))\chi$ by Lemmas 1.4 and 1.5. By recurrence, we see that ξ_a is generated by $\xi_{a(ij,kl)}$ (i, j, k, l are distinct). ■

PROPOSITION 6.4. *Put*

$$\Xi_1 := \{\xi_{ij}\xi_{kl}-\xi_{ik}\xi_{jl} \mid i<j<k<l\},$$

$$\Xi_2 := \{\xi_{ik}\xi_{jl}-\xi_{il}\xi_{jk} \mid i<j<k<l\},$$

and $\Xi := \Xi_1 \cup \Xi_2$. *Then* Ξ *is a minimal system of generators of the ideal* $\sum_{a\in L} A\xi_a$.

PROOF. First we prove that all $\xi_{a(ij,kl)}$ are generated by Ξ. Since $\xi_{a(ij,kl)} = -\xi_{a(ik,jl)} = -\xi_{a(lj,ki)} = \xi_{a(lk,ji)}$, we may assume $i<l$ and $j<k$. There are six cases: (1) $i<l<j<k$, (2) $i<j<l<k$, (3) $i<j<k<l$, (4) $j<k<i<l$, (5) $j<i<k<l$, (6) $j<i<l<k$. Since $a(ij, kl)=a(ji, lk)$, we do not have to consider the cases (4), (5), (6). In the case of (1), we have $\xi_{a(ij,kl)}=\xi_{ij}\xi_{lk}-\xi_{ik}\xi_{lj}\in\Xi_2$. In the case of (2), we have $\xi_{a(ij,kl)}=\xi_{ij}\xi_{lk}-\xi_{ik}\xi_{jl}(\xi_{ij}\xi_{lk}-\xi_{il}\xi_{jk})+(\xi_{il}\xi_{jk}-\xi_{ik}\xi_{jl})\in\Xi_1+\Xi_2$. In the case of (3), we have $\xi_{a(ij,kl)}=\xi_{ij}\xi_{kl}-\xi_{ik}\xi_{jl}\in\Xi_1$. Hence Ξ generates the ideal $\sum_{a\in L} A\xi_a$.

Next we prove the minimality. We have

$$\dim A_2 = \binom{n(n-1)/2+2-1}{2}$$

and

$$|\Xi| = 2\binom{n}{4}.$$

In the standard realization of the root system of type D_n (cf. [**B**, planche IV]), we have $\chi_{ij}=\varepsilon_i+\varepsilon_j$ $(1\leqq i<j\leqq n)$. Hence we have

$$\dim R_2 = \binom{n}{4}+3\binom{n}{3}+\binom{n}{2}.$$

Consequently we obtain $\dim R_2+|\Xi|=\dim A_2$, and the minimality of Ξ. ■

§7. Type E_6.

In this section, we suppose that $\mathfrak{g}$ is of E_6-type. We consider the following elements of the set X:

$$\chi_1 := \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6$$

$$\chi_2 := \alpha_2+\alpha_4+\alpha_5+\alpha_6$$

$$\chi_i := \sum_{k=i}^{6} \alpha_k \qquad (3 \leqq i \leqq 6)$$

$$\chi_{16} := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 .$$

We also consider $\gamma := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 = \chi_1 + \chi_2 + \chi_3 + \chi_{16} \in \Lambda$ and $\gamma' := \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 = \chi_2 + \chi_3 \in \Lambda$. We denote by C the cone generated by χ_1, $\chi_2, \cdots, \chi_6$ and χ_{16}.

LEMMA 7.1.

$$\boldsymbol{Z}^6 \cap C = \boldsymbol{Z}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{Z}_{\geqq 0}\chi_6 + \boldsymbol{Z}_{\geqq 0}\chi_{16} .$$

PROOF. Let C' be the cone generated by $\chi_1, \chi_2, \cdots, \chi_6$. It is clear that

$$\boldsymbol{Z}^6 \cap C' = \boldsymbol{Z}_{\geqq 0}\chi_1 + \cdots + \boldsymbol{Z}_{\geqq 0}\chi_6 ,$$

and

$$C' = (f_1 \geqq 0) \cap (f_2 \geqq 0) \cap (f_3 - f_1 \geqq 0) \cap (f_4 - f_2 - f_3 \geqq 0) \cap (f_5 - f_4 \geqq 0) \cap (f_6 - f_5 \geqq 0).$$

Since $f_2(\chi_{16}) = 2$, $f_1(\chi_{16}) = (f_3 - f_1)(\chi_{16}) = 1$ and $(f_4 - f_2 - f_3)(\chi_{16}) = (f_5 - f_4)(\chi_{16}) = (f_6 - f_5)(\chi_{16}) = -1$, we obtain the assertion by Lemma 1.3. ■

PROPOSITION 7.2. *The $\boldsymbol{Z}$-algebra R is normal.*

PROOF. We denote by $\triangle_1$ the cone generated by $\chi_2, \chi_4, \chi_5, \chi_6, \gamma'$ and γ, by $\triangle_2$ the one generated by $\chi_3, \chi_4, \chi_5, \chi_6, \gamma'$ and γ, and by $\triangle_3$ the one generated by $\chi_1, \chi_3, \chi_4, \chi_5, \chi_6$ and γ. Then we can verify that

$$\triangle_1 = (f_1 + 2f_2 - 2f_3 \geqq 0) \cap (f_1 - 2f_2 - 2f_3 + 2f_4 \geqq 0) \cap (f_1 - 2f_4 + 2f_5 \geqq 0) \cap (f_1 - 2f_5 + 2f_6 \geqq 0) \cap (-2f_1 + f_3 \geqq 0) \cap (f_1 \geqq 0),$$

$$\triangle_2 = (-f_1 - 2f_2 + 2f_3 \geqq 0) \cap (f_1 - 2f_2 - 2f_3 + 2f_4 \geqq 0) \cap (f_1 - 2f_4 + 2f_5 \geqq 0) \cap (f_1 - 2f_5 + 2f_6 \geqq 0) \cap (-3f_1 + 2f_2 \geqq 0) \cap (f_1 \geqq 0),$$

and

$$\triangle_3 = (3f_1 - 2f_2 \geqq 0) \cap (-3f_1 - 2f_2 + 3f_3 \geqq 0) \cap (-2f_2 - 3f_3 + 3f_4 \geqq 0) \cap (f_2 - 3f_4 + 3f_5 \geqq 0) \cap (f_2 - 3f_5 + 3f_6 \geqq 0) \cap (f_2 \geqq 0).$$

Let $\delta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_6\alpha_6$ with $a_i \in \boldsymbol{Z}_{\geqq 0} (1 \leqq i \leqq 6)$ satisfy $(\delta, \alpha_j) \leqq 0$ for $j = 1, 2, \cdots, 5$. Then we have

$$2a_1-a_3 \leqq 0$$
$$2a_2-a_4 \leqq 0$$
$$-a_1+2a_3-a_4 \leqq 0$$
$$-a_2-a_3+2a_4-a_5 \leqq 0$$
$$-a_4+2a_5-a_6 \leqq 0.$$

If we have $a_1+2a_2-2a_3 \geqq 0$, then $\delta \in \triangle_1$. If we have $a_1+2a_2-2a_3<0$ and $-3a_1+2a_2 \geqq 0$, then $\delta \in \triangle_2$. If we have $a_1+2a_2-2a_3<0$ and $-3a_1+2a_2<0$, then $\delta \in \triangle_3$. Hence we have $\delta \in \triangle_1 \cup \triangle_2 \cup \triangle_3 \subset C$. By Lemma 7.1, we see that $\delta \in \Lambda$. We obtain the normality of R by Lemma 1.7. ■

We put $\triangle = \{\chi \in \boldsymbol{R}^6 \mid f_i(\chi) \geqq 0\ (1 \leqq i \leqq 6),\ (\chi, \alpha_j) \leqq 0\ (1 \leqq j \leqq 5)\}$. Let $\chi = \sum_{i=1}^6 a_i \alpha_i \in \boldsymbol{R}^6$ satisfy $a_1 \geqq 0$, $a_2 \geqq 0$ and $(\chi, \alpha_j) \leqq 0\ (1 \leqq j \leqq 5)$. Then we have

$$\begin{aligned}
a_3 &\geqq 2a_1 \geqq 0 \\
a_4 &\geqq 2a_2 \geqq 0 \\
a_5 &\geqq (a_4-a_2)+(a_4-a_3) \geqq (a_4-a_2)+(a_3-a_1) \geqq 0 \\
a_6 &\geqq 2a_5-a_4 \\
&\geqq 2(-a_2-a_3+2a_4)-a_4 = 3a_4-2a_2-2a_3 \\
&= (a_4-2a_2)+2(a_4-a_3) \\
&\geqq (a_4-2a_2)+2(a_3-a_1) \geqq 0.
\end{aligned}$$

Hence we have $\chi \in \triangle$ and

$$\triangle = \{\chi \in \boldsymbol{R}^6 \mid f_1(\chi) \geqq 0,\ f_2(\chi) \geqq 0 \text{ and } (\chi, \alpha_j) \leqq 0\ (1 \leqq j \leqq 5)\}.$$

Since $(\chi_j, \alpha_j)>0$ for $1 \leqq j \leqq 5$, the linear form $(\cdot, -\alpha_j)$ can not define a face of Q of codimension one. On the other hand, it is easy to check that the linear forms f_1 and f_2 actually define faces of Q of codimension one. For any $\Gamma \in \mathcal{F}$, there exists $w \in W$ such that $\Gamma = (w.f_1=0)$ or $(w.f_2=0)$. The number $|\mathcal{F}|$ is equal to $|W|/|W(D_4)|+|W|/|W(A_4)| = (2^4 \cdot 5!)/(2^3 \cdot 4!)+(2^4 \cdot 5!)/5! = 10+16=26$ where $W(D_4)$ (resp. $W(A_4)$) denotes the subgroup of W generated by s_2, s_3, s_4 and s_5 (resp. s_1, s_3, s_4 and s_5).

We take the new basis $e_1, e_2, \cdots, e_6$ of $\boldsymbol{R}^6 = \boldsymbol{R}\alpha_1 + \cdots + \boldsymbol{R}\alpha_6$ such that $\alpha_1 = e_1-e_2$, $\alpha_2 = e_4+e_5$, $\alpha_3 = e_2-e_3$, $\alpha_4 = e_3-e_4$, $\alpha_5 = e_4-e_5$ and $\alpha_6 = (-e_1-e_2-e_3-e_4+e_5)/2 + e_6$. Let $\mathcal{J}$ denote the set $\{I \subset \{1, 2, 3, 4, 5\} \mid |I| = \text{odd}\}$. For $I \in \mathcal{J}$, we define $\chi_I \in \boldsymbol{R}^6$ by

$$\chi_I := \frac{1}{2} \sum_{i \in I} e_i - \frac{1}{2} \sum_{i \notin I,\, i \leqq 5} e_i + e_6.$$

Then we have $X=\{\chi_I \mid I\in\mathcal{I}\}$. The group W can be identified with the semi-direct product of the symmetric group S_5 and the group of even number of sign changes by $s_1=(1, 2)$, $s_3=(2, 3)$, $s_4=(3, 4)$, $s_5=(4, 5)$ and $s_2=$ the sign change of 4 and 5 following the transposition (4, 5). For $I\in\mathcal{I}$ and $\sigma\in W$, we define $\sigma(I)\in\mathcal{I}$ by

$$\sigma(I):=\{\sigma(i) \mid i\in I,\ \sigma(i)>0\}\cup\{-\sigma(i) \mid i\notin I,\ \sigma(i)<0\}.$$

Then we have $\sigma(\chi_I)=\chi_{\sigma(I)}$ for all $\sigma\in W$ and all $I\in\mathcal{I}$. Supposing that $\{1,2,3,4,5\}=\{i, j, k, l, m\}$, we define $a(ij, k)=a(lm, k)\in L$ by

$$a(ij, k)_I=\begin{cases} 1 & I=\{i, j, k\},\ \{l, m, k\} \\ -1 & I=\{k\},\ \{1, 2, 3, 4, 5\} \\ 0 & \text{otherwise,}\end{cases}$$

$b(ijklm)\in L$ by

$$b(ijklm)_I=\begin{cases} 1 & I=\{i, j, k\},\ \{i, l, m\} \\ -1 & I=\{i, j, l\},\ \{i, k, m\} \\ 0 & \text{otherwise,}\end{cases}$$

and $c(ijkl)\in L$ by

$$c(ijkl)_I=\begin{cases} 1 & I=\{i, j, k\},\ \{l\} \\ -1 & I=\{i, j, l\},\ \{k\} \\ 0 & \text{otherwise.}\end{cases}$$

PROPOSITION 7.3. *Let A be the polynomial ring $\boldsymbol{Z}[\xi_I(I\in\mathcal{I})]$. Then we have*

$$R=A/\Big(\sum_{i,j,k\text{ are distinct}} A\xi_{a(ij,k)}+\sum_{i,j,k,l\text{ are distinct}} A\xi_{c(ijkl)}\Big).$$

PROOF. Let $a=(a_I)_{I\in\mathcal{I}}\in\boldsymbol{Z}^{16}$. The conditions $f_1(\sum_{I\in\mathcal{I}} a_I\chi_I)=0$ and $f_2(\sum_{I\in\mathcal{I}} a_I\chi_I)=0$ induce $\sum_{I\ni 1} a_I=0$ and $\sum_{|I|=1} a_I=a_{\{1,2,3,4,5\}}$. By Lemma 1.6, we have $a\in L$ if and only if

$$\sum_{I\ni i} a_I=0 \qquad (1\leqq i\leqq 5)$$

$$\sum_{|I|=1} a_I=a_{\{1,2,3,4,5\}}$$

$$a_{\{i\}}=a_{\{1,2,3,4,5\}}+\sum_{|I|=3,\, i\notin I} a_I \qquad (1\leqq i\leqq 5)$$

$$a_{\{1,2,3,4,5\}-\{i,j\}}=a_{\{i\}}+a_{\{j\}}+\sum_{k\neq i,j} a_{\{i,j,k\}} \qquad (i\neq j).$$

Suppose that $a\in L-\{0\}$. Then there exists $I\in\mathcal{I}$ such that $|I|=3$ and $a_I\neq 0$. We may assume $a_{\{i,j,k\}}>0$ for some distinct i, j, k. Let l and m be the other two elements of $\{1, 2, 3, 4, 5\}$. Since we have

$$a_{\{i,j,k\}} = a_{\{l\}} + a_{\{m\}} + a_{\{i,l,m\}} + a_{\{j,l,m\}} + a_{\{k,l,m\}},$$

there are the following two cases:

(1) $a_{\{i,j,k\}} > 0$ and $a_{\{i,l,m\}} > 0$.

(2) $a_{\{i,j,k\}} > 0$ and $a_{\{l\}} > 0$.

In the case (1), considering the equation $\sum_{I \ni i} a_I = 0$, we see that $a_I < 0$ for some $I \ni i$. If $a_{\{i\}} < 0$ or $a_{\{1,2,3,4,5\}} < 0$, ξ_a is generated by $\xi_{a(jk,i)}$ and $\xi_{a-a(jk,i)}$, and we have $a\chi > (a - a(jk, i))\chi$ by Lemmas 1.4 and 1.5. Otherwise we may suppose that $a_{\{i,j,l\}} < 0$. Then ξ_a is generated by $\xi_{b(ijklm)}$ and $\xi_{a-b(ijklm)}$, and we have $a\chi > (a - b(ijklm))\chi$ by Lemmas 1.4 and 1.5. In the case (2), considering the equation $\sum_{m \notin I} a_I = 0$, we see that $a_I < 0$ for some I not containing m. We may suppose that $a_{\{k\}} < 0$ or $a_{\{i,j,l\}} < 0$. Then ξ_a is generated by $\xi_{c(ijkl)}$ and $\xi_{a-c(ijkl)}$, and we have $a\chi > (a - c(ijkl))\chi$ by Lemmas 1.4 and 1.5. By recurrence, we see that ξ_a is generated by $\xi_{a(ij,k)}$'s, $\xi_{b(ijklm)}$'s and $\xi_{c(ijkl)}$'s. Since we have $\xi_{b(ijklm)} = \xi_{a(jk,i)} - \xi_{a(jl,i)}$, we obtain the assertion. ■

PROPOSITION 7.4. *Put*

$$\Xi_1 := \{\xi_{a(ij,k)} = \xi_{\{i,j,k\}}\xi_{\{l,m,k\}} - \xi_{\{k\}}\xi_{\{1,2,3,4,5\}} \mid \{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}\},$$

$$\Xi_2 := \{\xi_{c(ijkl)} = \xi_{\{i,j,k\}}\xi_{\{l\}} - \xi_{\{i,j,l\}}\xi_{\{k\}} \mid \max\{i, j, k, l\} = l\},$$

and $\Xi := \Xi_1 \cup \Xi_2$. *Then the set* Ξ *is a minimal system of generators of the ideal* $\sum_{a \in L} A\xi_a$.

PROOF. If $\max\{i, j, k, l\} = i$, then we have $\xi_{c(ijkl)} = \xi_{\{i,j,k\}}\xi_{\{l\}} - \xi_{\{i,j,l\}}\xi_{\{k\}} = (\xi_{\{j,k,l\}}\xi_{\{i\}} - \xi_{\{i,j,l\}}\xi_{\{k\}}) - (\xi_{\{j,k,l\}}\xi_{\{i\}} - \xi_{\{i,j,k\}}\xi_{\{l\}}) = \xi_{c(jlki)} - \xi_{c(jkli)}$. Since we have $\xi_{c(ijkl)} = \xi_{c(jikl)} = -\xi_{c(ijlk)}$, all $\xi_{c(ijkl)}$ are generated by Ξ_2. By Proposition 7.3, we see that Ξ generates the ideal $\sum_{a \in L} A\xi_a$.

Next we want to prove the minimality. Since we have $\xi_{a(ij,k)} = \xi_{a(lm,k)}$, we see $|\Xi_1| = \binom{5}{1} \times \binom{4}{2}/2 = 15$. Since we have $\xi_{c(ijkl)} = \xi_{c(jikl)}$, we see $|\Xi_2| = \binom{5}{1} \times \binom{3}{1} = 15$. Clearly we have $|\mathcal{J}| = \binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 16$. Consequently we have $\dim A_2 = \binom{16+2-1}{2} = 17 \cdot 16/2 = 136$. For $I, J \in \mathcal{J}$, we put $I' := [1, 5] - I$, and $J' := [1, 5] - J$. Then we have $\chi_I + \chi_J = \sum_{i \in I \cap J} e_i - \sum_{i \in I' \cap J'} e_i + 2e_6$. We can check

$$\begin{aligned} &\{(|I \cap J|, |I' \cap J'|) \mid I, J \in \mathcal{J}\} \\ &= \{(0, 1), (0, 3), (1, 0), (1, 2), (1, 4), (2, 1), (3, 0), (3, 2), (5, 0)\} \\ &=: S. \end{aligned}$$

For $(s_1, s_2)\in S$, we put

$$X^2_{(s_1,s_2)} := \{\chi_I+\chi_J \mid |I\cap J| = s_1,\ |I'\cap J'| = s_2\},$$

and $X^2=\bigcup_{(s_1,s_2)\in S}X^2_{(s_1,s_2)}$. Then we can check

$$|X^2_{(0,1)}| = \binom{5}{3}\binom{2}{1} = 20, \qquad |X^2_{(0,3)}| = \binom{5}{1}\binom{4}{1} = 20,$$

$$|X^2_{(1,0)}| = \binom{5}{1} = 5, \qquad |X^2_{(1,2)}| = \binom{5}{1}\binom{4}{2} = 30,$$

$$|X^2_{(1,4)}| = \binom{5}{1} = 5, \qquad |X^2_{(2,1)}| = \binom{5}{1} = 5,$$

$$|X^2_{(3,0)}| = \binom{5}{3} = 10, \qquad |X^2_{(3,2)}| = \binom{5}{3} = 10,$$

$$|X^2_{(5,0)}| = \binom{5}{5} = 1.$$

Hence we have $|X^2|=106=\dim A_2-|\Xi|$. Therefore the set Ξ is a minimal system of generators of the ideal $\sum_{\alpha\in L}A\xi_\alpha$. ∎

§8. Type E_7.

In this section, we suppose that $\mathfrak{g}$ is of E_7-type. We consider the following elements of the set X:

$$\chi_1 := \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$$

$$\chi_2 := \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7$$

$$\chi_i := \sum_{i\leq k\leq 7}\alpha_k \qquad (i = 3, 4, \cdots, 7)$$

$$\chi_{27} := 2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7.$$

We also consider $\gamma:=2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7=\chi_2+\chi_3+\chi_{27}\in\Lambda$ and $\gamma':=\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7=\chi_2+\chi_3\in\Lambda$. We denote by $\triangle$ the cone generated by $\chi_1, \chi_2, \cdots, \chi_7$ and χ_{27}.

LEMMA 8.1.

$$\boldsymbol{Z}^7\cap\triangle = \boldsymbol{Z}_{\geq 0}\chi_1+\boldsymbol{Z}_{\geq 0}\chi_2+\cdots+\boldsymbol{Z}_{\geq 0}\chi_7+\boldsymbol{Z}_{\geq 0}\chi_{27}.$$

PROOF. Let $\triangle'$ denote the cone generated by $\chi_1, \chi_2, \cdots, \chi_7$. It is clear that

$$\boldsymbol{Z}^7\cap\triangle' = \boldsymbol{Z}_{\geq 0}\chi_1+\boldsymbol{Z}_{\geq 0}\chi_2+\cdots+\boldsymbol{Z}_{\geq 0}\chi_7$$

and

$$\triangle' = (f_1 \geqq 0) \cap (f_2 \geqq 0) \cap (f_3 - f_1 \geqq 0) \cap (f_4 - f_2 - f_3 \geqq 0)$$
$$\cap (f_5 - f_4 \geqq 0) \cap (f_6 - f_5 \geqq 0) \cap (f_7 - f_6 \geqq 0).$$

Since we have $f_1(\chi_{27}) = f_2(\chi_{27}) = 2$, $(f_3 - f_1)(\chi_{27}) = 1$ and $(f_4 - f_2 - f_3)(\chi_{27}) = (f_5 - f_4)(\chi_{27}) = (f_6 - f_5)(\chi_{27}) = (f_7 - f_6)(\chi_{27}) = -1$, we obtain the assertion by Lemma 1.3. ■

PROPOSITION 8.2. *The $\boldsymbol{Z}$-algebra R is normal.*

PROOF. We denote by $\triangle_1$ the cone generated by $\chi_2, \chi_4, \chi_5, \chi_6, \chi_7, \gamma'$ and γ, by $\triangle_2$ the one generated by $\chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \gamma'$ and γ, and by $\triangle_3$ the one generated by $\chi_1, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$ and γ. Then we can verify that

$$\triangle_1 = (f_1 \geqq 0) \cap (-2f_1 + f_3 \geqq 0) \cap (f_1 - 2f_2 - 2f_3 + 2f_4 \geqq 0)$$
$$\cap \bigcap_{i=5}^{7} (f_1 - 2f_{i-1} + 2f_i \geqq 0) \cap (f_1 + 2f_2 - 2f_3 \geqq 0),$$
$$\triangle_2 = (f_1 \geqq 0) \cap (-3f_1 + 2f_2 \geqq 0) \cap (f_1 - 2f_2 - 2f_3 + 2f_4 \geqq 0)$$
$$\cap \bigcap_{i=5}^{7} (f_1 - 2f_{i-1} + 2f_i \geqq 0) \cap (-f_1 - 2f_2 + 2f_3 \geqq 0)$$

and

$$\triangle_3 = (f_2 \geqq 0) \cap (3f_1 - 2f_2 \geqq 0) \cap (-2f_2 - 3f_3 + 3f_4 \geqq 0)$$
$$\cap \bigcap_{i=5}^{7} (f_2 - 3f_{i-1} + 3f_i \geqq 0) \cap (-3f_1 - 2f_2 + 3f_3 \geqq 0).$$

Let $\delta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_7\alpha_7$ with $a_i \in \boldsymbol{Z}_{\geq 0} (1 \leqq i \leqq 7)$ satisfy $(\delta, \alpha_i) \leqq 0$ for $i = 1, 2, \cdots, 6$. Then we have

$$2a_1 - a_3 \leqq 0$$
$$2a_2 - a_4 \leqq 0$$
$$-a_1 + 2a_3 - a_4 \leqq 0$$
$$-a_2 - a_3 + 2a_4 - a_5 \leqq 0$$
$$-a_4 + 2a_5 - a_6 \leqq 0$$
$$-a_5 + 2a_6 - a_7 \leqq 0.$$

If we have $a_1 + 2a_2 - a_3 \geqq 0$, then δ belongs to $\triangle_1$. If we have $a_1 + 2a_2 - a_3 \leqq 0$ and $-3a_1 + 2a_2 \geqq 0$, then δ belongs to $\triangle_2$. Finally if we have $a_1 + 2a_2 - a_3 \leqq 0$ and $-3a_1 + 2a_2 \leqq 0$, then δ belongs to $\triangle_3$. Hence we have $\delta \in \triangle_1 \cup \triangle_2 \cup \triangle_3 \subset \triangle$, which implies $\delta \in \Lambda$ by Lemma 8.1. We obtain the normality of R by Lemma 1.7. ■

Denote

$$\triangle'' = \{\chi \in \boldsymbol{R}^7 \mid f_i(\chi) \geqq 0 \ (1 \leqq i \leqq 7), \quad (\chi, \alpha_j) \leqq 0 \ (1 \leqq j \leqq 6)\}.$$

Suppose that $\chi=\sum_{i=1}^{7}a_i\chi_i\in\boldsymbol{R}^7$ satisfies $a_1\geqq 0$, $a_2\geqq 0$ and $(\chi, \alpha_j)\leqq 0$ for $1\leqq j\leqq 6$. Then we have that $a_i\geqq 0\,(1\leqq i\leqq 6)$ as in §7 and

$$\begin{aligned} a_7 &\geqq 2a_6-a_5 \geqq 2(-a_4+2a_5)-a_5 = 3a_5-2a_4 \\ &\geqq 3(-a_2-a_3+2a_4)-2a_4 = 4a_4-3a_2-3a_3 \\ &= \frac{1}{2}a_4+\frac{3}{2}(a_4-2a_2)+(2a_4-3a_3) \geqq 0. \end{aligned}$$

Hence we have $\chi\in\triangle''$ and

$$\triangle'' = \{\chi\in\boldsymbol{R}^7\,|\,f_i(\chi)\geqq 0\ (i=1,2),\qquad (\chi,\alpha_j)\leqq 0\ (1\leqq j\leqq 6)\}.$$

Since $(\chi_j, \alpha_j)>0$ for $1\leqq j\leqq 6$, the linear form $(\cdot, -\alpha_j)$ can not define a face of Q of codimension one. On the other hand, it is easy to check that the linear forms f_1 and f_2 actually define faces of Q of codimension one. For any $\Gamma\in\mathcal{F}$, there exists $w\in W$ such that $\Gamma=(w.f_1=0)$ or $(w.f_2=0)$. The number $|\mathcal{F}|$ is equal to $|W|/|W(D_5)|+|W|/|W(A_5)|=(2^7\cdot 3^4\cdot 5)/(2^4\cdot 5!)+(2^7\cdot 3^4\cdot 5)/6!=27+72=99$ where $W(D_5)$ (resp. $W(A_5)$) denotes the subgroup of W generated by s_2, s_3, s_4, s_5 and s_6 (resp. s_1, s_3, s_4, s_5 and s_6).

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