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Quasi-umbilical, locally strongly convex homogeneous affine hypersurfaces

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0. Introduction.

In this paper, we continue our investigations on homogeneous, locally strongly convex affine hypersurfaces in $\boldsymbol{R}^{n+1}$, which we started in [**DV1**] and [**DV2**].

A nondegenerate hypersurface M of the equiaffine space $\boldsymbol{R}^{n+1}$ is called locally homogeneous if for all points p and q of M, there exists a neighborhood U_p of p in M, and an equiaffine transformation A of $\boldsymbol{R}^{n+1}$, i.e. $A\in SL(n+1, \boldsymbol{R})\ltimes \boldsymbol{R}^{n+1}$, such that $A(p)=q$ and $A(U_p)\subset M$. If $U_p=M$ for all p, then M is called homogeneous.

We denote the affine normal by ξ and the induced affine connection by ∇. We will always assume here that M is locally strongly convex. Let S denote the shape operator of the affine immersion. Since M is locally strongly convex, S is diagonalizable. If S is a multiple of the identity, M is called an affine sphere. Locally strongly convex homogeneous affine spheres have been studied in [**S**], see also the discussions in [**DV2**]. If the affine shape operator at each point has an eigenvalue λ with multiplicity exactly $n-1$, where n is the dimension of M, we call M proper quasi-umbilical. If $\lambda=0$ (so rank $(S)=1$), we recall the following result from [**DV1**].

THEOREM A [**DV1**]. *Let M be a locally strongly convex, locally homogeneous affine hypersurface with* rank $(S)=1$ *in $\boldsymbol{R}^{n+1}$. Then M is affine equivalent to the convex part of the hypersurface with equation*

$$\left(Z-\frac{1}{2}\sum_{i=1}^{r}X_i^2\right)^{r+2}\left(W-\frac{1}{2}\sum_{j=1}^{s}Y_j^2\right)^{s+2}=1,$$

where $r+s=n-1$ and $(X_1, \cdots, X_r, Y_1, \cdots, Y_s, Z, W)$ are the coordinates of $\boldsymbol{R}^{n+1}$.

Here, we will mainly consider the case that $\lambda\neq 0$. In Section 2, we will start to construct a special local tangent frame on a locally strongly convex,

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proper quasiumbilical, homogeneous affine hypersurface. As a consequence of that construction, we derive the following theorem.

THEOREM 1. *Let M^n be a locally strongly convex, locally homogeneous affine hypersurface in $\boldsymbol{R}^{n+1}$. Assume that M is also proper quasi-umbilical. Then* $\det(S)=0$.

If M is a surface ($n=2$), the above theorem remains true, if the condition that M is locally affine homogeneous is replaced by the weaker condition that the eigenvalues of the shape operator of S are constant ([V]). In general, if the affine shape operator has constant eigenvalues, it is not known how many different non-zero eigenvalues can occur (cf. for a Euclidean hypersurface of $\boldsymbol{R}^{n+1}$, if all the eigenvalues of the Euclidean shape operator are constant, there can be at most 1 non-zero eigenvalue).

In Section 3, we will then gradually improve our choice of h-orthonormal frame. By combining then the results obtained there, with Theorem A, we obtain the following classification theorems.

THEOREM 2. *Let M^3 be a locally strongly convex, locally homogeneous, proper quasiumbilical affine hypersurface in $\boldsymbol{R}^4$. Then M is affine equivalent to the convex part of one of the following hypersurfaces:*

$$\left(y-\frac{1}{2}(x^2+z^2)\right)^4 w^2=1,$$

$$\left(y-\frac{1}{2}x^2\right)^3\left(z-\frac{1}{2}w^2\right)^3=1,$$

$$\left(y-\frac{1}{2}x^2\right)^3 v^2w^2=1,$$

$$\left(y-\frac{1}{2}x^2-\frac{1}{2}w^2/z\right)^4 z^3=1,$$

where (x, y, z, w) are the coordinates of $\boldsymbol{R}^4$.

THEOREM 3. *Let M^4 be a locally strongly convex, locally homogeneous, proper quasiumbilical affine hypersurface in $\boldsymbol{R}^5$. Then M is affine equivalent to the convex part of one of the following hypersurfaces:*

$$\left(y-\frac{1}{2}(x^2+z^2+v^2)\right)^5 w^2=1,$$

$$\left(y-\frac{1}{2}(x^2+u^2)\right)^4\left(z-\frac{1}{2}w^2\right)^3=1,$$

$$\left(y-\frac{1}{2}x^2\right)^3 u^2v^2w^2=1,$$

$$\left(y-\frac{1}{2}x^2-\frac{1}{2}(w^2/z+u^2/z)\right)^5 z^4 = 1\,,$$

$$\left(y-\frac{1}{2}x^2-\frac{1}{2}w^2/z\right)^4 z^3u^2 = 1\,,$$

$$\left(y-\frac{1}{2}x^2\right)(z^2-(w^2+u^2)) = 1\,,$$

where (x, y, z, w, u) *are the coordinates of* $\boldsymbol{R}^5$.

Remark that, by the constructions of [**DV2**], all of the above examples can be viewed as compositions of affine spheres. Also from [**DV2**], we get that all the above examples are globally homogeneous.

1. Preliminaries.

Let M^n be a differentiable n-dimensional manifold in the affine space $\boldsymbol{R}^{n+1}$ equipped with its usual flat connection D and a parallel volume element ω and let ξ be an arbitrary local transversal vector field to M^n. For any vector fields $X, Y, X_1, \cdots, X_n$, we write

(1.1) $$D_XY = \nabla_XY + h(X, Y)\xi\,,$$

(1.2) $$\theta(X_1, \cdots, X_n) = \omega(X_1, \cdots, X_n, \xi)\,,$$

thus defining an affine connection ∇, a symmetric (0, 2)-type tensor h, called the second fundamental form, and a volume element θ. We say that M is nondegenerate if h is nondegenerate and this condition is independent of the choice of transversal vector field ξ. In this case, it is known (see [**N2**]) that there is a unique choice (up to sign) of transversal vector field such that the induced connection ∇, the induced second fundamental form h and the induced volume element θ satisfy the following conditions:

(i) $$\nabla\theta = 0\,,$$

(ii) $$\theta = \omega_h\,,$$

where ω_h is the metric volume element induced by h. We call ∇ the induced affine connection, ξ the affine normal and h the affine metric.

Let A be an equiaffine transformation of $\boldsymbol{R}^{n+1}$. Since both D and the volume form ω of $\boldsymbol{R}^{n+1}$ are preserved by equiaffine transformation of $\boldsymbol{R}^{n+1}$, we get that the affine normal $\tilde{\xi}$ to $A(M)$ is related to ξ by

$$\tilde{\xi}(A(p)) = A_*\xi(p)\,.$$

Moreover, we also have

$$D_{A_*X}A_*(Y)=\tilde{\nabla}_{A_*X}A_*Y+\tilde{h}(A_*X,\ A_*Y)\tilde{\xi}(A)$$
$$A_*(D_X)Y=A_*(\nabla_XY)+h(X,\ Y)A_*\xi\,,$$

where $\tilde{\nabla}$ and $\tilde{h}$ are respectively the induced affine connection and the affine metric on $A(M)$. So, since $D_{A_*X}A_*(Y)=A_*(D_XY)$, by comparing tangential and transversal parts of the above expressions, the affine metric and the induced connection are preserved as well.

By combining (i) and (ii), we obtain the apolarity condition which states that $\nabla\omega_h=0$. A nondegenerate hypersurface equipped with this special transversal vector field is called a Blaschke hypersurface. Throughout this paper, we will always assume that M is a Blaschke hypersurface. If h is positive (or negative) definite, the hypersurface is called locally strongly convex. Notice that if h is negative definite, we can always replace ξ by $-\xi$, thus making the new affine metric positive definite. Therefore, if we say that M is locally strongly convex, we will always assume that ξ is chosen so that h is positive definite.

Condition (i) implies that $D_X\xi$ is tangent to M^n for any tangent vector X to M. Hence, we can define a (1, 1)-tensor field S, called the affine shape operator by

$$D_X\xi=-SX\,. \tag{1.3}$$

M is called an affine sphere if $S=\lambda I$. We define the affine mean curvature H by $H=(1/n)\,\mathrm{trace}\,(S)$. Again, if A is an equiaffine transformation of $\boldsymbol{R}^{n+1}$, we can relate the shape operators on $A(M)$ and M by

$$\tilde{S}A_*(X)=-D_{A_*(X)}\tilde{\xi}(A)=-D_{A_*(X)}A_*\xi=-A_*(D_X\xi)=-A_*SX\,.$$

Hence the shape operator is affine invariant.

The following fundamental equations of Gauss, Codazzi and Ricci are given by

(1.4) $R(X,\ Y)Z=h(Y,\ Z)SX-h(X,\ Z)SY$ (Equation of Gauss)

(1.5) $(\nabla h)(X,\ Y,\ Z)=(\nabla h)(Y,\ X,\ Z)$ (Equation of Codazzi for h)

(1.6) $(\nabla_XS)Y=(\nabla_YS)X$ (Equation of Codazzi for S)

(1.7) $h(X,\ SY)=h(SX,\ Y)$ (Equation of Ricci).

Since $\dim(M)\geqq 2$, if M is an affine sphere, it follows from (2.6) that λ is constant. If $\lambda\neq 0$, we say that M is a proper affine sphere and if $\lambda=0$, we call M an improper affine sphere. From (1.5) it follows that the cubic form $C(X,Y,Z)=(\nabla h)(X,\ Y,\ Z)$ is symmetric in $X,\ Y,\ Z$. The theorem of Berwald states that C vanishes identically if and only if M is an open part of a nondegenerate quadric.

Let $\hat{\nabla}$ denote the Levi Civita connection of the affine metric h. The difference tensor K is defined by

$$K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y,$$

for vector fields X and Y on M. Notice that, since both connections are torsion free, K is symmetric in X and Y. We also write $K_X Y = K(X, Y)$. From [**N2**], we have that

$$h(K_X Y, Z) = -\frac{1}{2} C(X, Y, Z). \tag{1.8}$$

Notice also that the apolarity condition together with (1.8) implies that $\operatorname{Tr} K_X = 0$ for every tangent vector X.

2. Proof of Theorem 1.

In the remainder of this paper, M^n will always denote a locally strongly convex, locally homogeneous, proper quasi-umbilical hypersurface of $\boldsymbol{R}^{n+1}$. Since M is locally strongly convex, it follows from the Ricci equation that the affine shape operator is diagonalizable. The fact that M is locally homogeneous then implies that the eigenvalues of the shape operator are constant on M. This can be seen in the following way.

Let $p, q \in M$. Then, by the definition of local homogeneity, there exists an equiaffine transformation A of $\boldsymbol{R}^{n+1}$ which maps p to q and which maps a neighbourhood U of p to a neighbourhood V of q in M. Since the affine shape operator is affine invariant, we know that if $e_1, \cdots, e_n$ are eigenvectors of the affine shape operator at the point p with eigenvalues λ_i, then $A_* e_1, \cdots, A_* e_n$ are eigenvectors of S at the point q with eigenvalues λ_i.

Therefore, if M is also proper quasi-umbilical, S has two different constant eigenvalues λ and μ, where the multiplicity of λ is $n-1$ and the multiplicity of μ is equal to 1. In view of Theorem A, we can restrict ourselves to the case that $\lambda \neq 0$. Then we have the following basic lemma.

LEMMA 2.1. *Let $p \in M$. Then there exists a local orthonormal frame $\{E_1, U_1, \cdots, U_{n-1}\}$ defined on a neighborhood of p, such that*

$$SE_1 = \mu E_1, \qquad SU_i = \lambda U_i, \qquad \lambda \neq 0$$

$$\nabla_{U_1} E_1 = bE_1, \qquad \nabla_{U_i} E_1 = 0, \qquad i > 1$$

$$\nabla_{E_1} E_1 = cE_1 + 2bU_1, \qquad \nabla_{E_1} U_i = \sum_{j=1}^{n-1} a_{ij} U_j,$$

$$\nabla_{U_i} U_j = (a_{ij} + a_{ji}) E_1 + \sum_{k=1}^{n-1} d_{ij}^k U_k,$$

where $c = -\sum_{i=1}^{n-1} a_{ii}$ and b are constants.

PROOF. We take $p\in M$. We construct a tangent basis $e_1, u_1, \cdots, u_{n-1}$ at the point p such that $Se_1=\mu e_1$ and $Su_i=\lambda u_i$. Then, since μ and λ are different numbers, and the eigenvalues of the shape operator are constant it follows from [**N1**] (Lemma 1, pp. 48-49 with $A=S-\mu I$) that we can extend these vectors to local vector fields $E_1, U_1, \cdots, U_{n-1}$, such that $SE_1=\mu E_1$ and $SU_i=\lambda U_i$. Notice that, up to sign, the vector field E_1 is uniquely determined. This implies that by taking 2 points p and q and the equiaffine transformation of $\boldsymbol{R}^{n+1}$ which maps p to q and a neighbourhood U of p to a neighbourhood of q, we have that

$$A_*(E_1(x)) = \pm E_1(A(x)).$$

Since A is an equiaffine transformation (which from Section 1 preserves the affine metric and the induced affine connection), this implies that

$$h(\nabla_{A_*(E_1)}A_*E_1,\ A_*(E_1)) = h(A_*(\nabla_{E_1}E_1),\ A_*(E_1)) = h(\nabla_{E_1}E_1,\ E_1).$$

So $c(q)=\pm c(p)$ and thus c is a constant on M. Similar arguments will be used throughout the paper, without further mentioning them.

We define functions d_{ij}^k, a_{ij}, h_{ij}, b_i, g_i, c, f_i, c_{ij}, α_{ij} such that the connection ∇ is given by

$$\nabla_{E_1}E_1 = cE_1+\sum_{i=1}^{n-1} g_iU_i$$

$$\nabla_{E_1}U_i = f_iE_1+\sum_{j=1}^{n-1} a_{ij}U_j$$

$$\nabla_{U_i}E_1 = b_iE_1+\sum_{j=1}^{n-1} h_{ij}U_j$$

$$\nabla_{U_i}U_j = \alpha_{ij}E_1+\sum_{k=1}^{n-1} d_{ij}^kU_k.$$

We first apply the Codazzi equation for S. We obtain that

$$\begin{aligned}0 &= (\nabla_{E_1}S)U_i-(\nabla_{U_i}S)E_1\\ &= \lambda\nabla_{E_1}U_i-S(\nabla_{E_1}U_i)-\mu\nabla_{U_i}E_1+S(\nabla_{U_i}E_1)\\ &= (\lambda-\mu)f_iE_1-(\mu-\lambda)\sum_{j=1}^{n-1} h_{ij}U_j.\end{aligned}$$

Hence $f_i=h_{ij}=0$.

Next it is clear that we can pick U_1 such that $b_2=b_3=\cdots=b_{n-1}=0$. We call $b_1=b$. Hence $\nabla_{U_1}E_1=bE_1$ and $\nabla_{U_j}E_1=0$ for $j>1$. We now show that b is constant. Let us assume that at some point q, $b(q)\neq 0$. Then, U_1 is also nniquely determined around q. Since M is locally homogeneous, we immediately get that b is a constant on a neighbourhood of q. Since b varies differentiably, b is a constant.

Next, we will apply the Codazzi equation for h. From

$$(\nabla h)(E_1, U_i, E_1) = (\nabla h)(U_i, E_1, E_1),$$

we deduce that $f_i + g_i = 2b_i$. Hence $g_1 = 2b$ and $g_j = 0$, for $j > 1$. The Codazzi equation

$$(\nabla h)(E_1, U_i, U_j) = (\nabla h)(U_i, U_j, E_1),$$

gives us that

$$\alpha_{ij} = a_{ij} + a_{ji}.$$

Finally we find from the apolarity condition

$$(\nabla h)(E_1, E_1, E_1) + \sum_{i=1}^{n-1} (\nabla h)(E_1, U_i, U_i) = 0,$$

that $c = -\sum_{i=1}^{n-1} a_{ii}$. This completes the proof of the lemma. □

In the next lemmas, we will then gradually obtain more information about the other coefficients, using also the Gauss equation.

LEMMA 2.2. *We have* $b \neq 0$.

PROOF. Let us suppose that $b=0$. Then we have from the Gauss equation that

$$\begin{aligned} -\lambda U_1 &= R(E_1, U_1)E_1 \\ &= \nabla_{E_1}\nabla_{U_1}E_1 - \nabla_{U_1}\nabla_{E_1}E_1 - \nabla_{[E_1, U_1]}E_1 \\ &= -c\nabla_{U_1}E_1 - \sum_{j=1}^{n-1} a_{1j}\nabla_{U_j}E_1 = 0. \end{aligned}$$

Since we assumed that $\lambda \neq 0$, this gives us a contradiction. □

The following lemma then describes the derivatives of the vector field U_1, which is, since E_1 is uniquely determined and $b \neq 0$, also uniquely determined and therefore affine invariant (i.e. applying an equiaffine transformation, which locally preserves the hypersurface, maps the vector field U_1 into itself).

LEMMA 2.3. *We can choose* U_2 *in such a way that we have that*

$$\begin{aligned} \nabla_{U_1}U_1 &= \left(\frac{c}{2} - \frac{a_{11}}{2}\right)E_1 + \left(2b + \frac{\lambda}{2b}\right)U_1, \\ \nabla_{U_2}U_1 &= -\frac{a_{21}}{2}E_1 + \frac{\lambda}{2b}U_2, \\ \nabla_{U_j}U_1 &= \frac{\lambda}{2b}U_j, \qquad j > 2, \end{aligned}$$

where a_{21} *is constant.*

PROOF. From the Gauss equation, using the fact c and b are both constants, we have that

$$\begin{aligned}
-\lambda U_j &= R(E_1, U_j)E_1 \\
&= \nabla_{E_1}\nabla_{U_j}E_1 - \nabla_{U_j}\nabla_{E_1}E_1 - \nabla_{[E_1, U_j]}E_1 \\
&= \nabla_{E_1}(\delta_{j1}bE_1) - \nabla_{U_j}(cE_1 + 2bU_1) - \nabla_{\Sigma_{k=1}^{n-1} a_{jk}U_k - \delta_{j1}bE_1}E_1 \\
&= 2\delta_{j1}b(cE_1 + 2bU_1) - c\delta_{j1}bE_1 - a_{j1}bE_1 - 2b\nabla_{E_j}U_1 .
\end{aligned}$$

Taking $j=1$ then gives us that

$$\nabla_{U_1}U_1 = \left(\frac{c}{2} - \frac{a_{11}}{2}\right)E_1 + \left(2b + \frac{\lambda}{2b}\right)U_1 .$$

If $j>1$, we obtain that

$$\nabla_{U_j}U_1 = -\frac{a_{j1}}{2}E_1 + \frac{\lambda}{2b}U_j . \tag{2.1}$$

From this last formula it is clear that we can choose $U_2, U_3, \cdots, U_{n-1}$ in such a way that

$$\begin{aligned}
\nabla_{U_2}U_1 &= -\frac{a_{21}}{2}E_1 + \frac{\lambda}{2b}U_2 , \\
\nabla_{U_j}U_1 &= \frac{\lambda}{2b}U_j, \qquad j > 2 .
\end{aligned}$$

Since E_1 and U_1 are uniquely determined, we get from these last equations that a_{21} is constant. If this constant is non-zero, we also see that U_2 is uniquely determined. □

By combining the formulas of Lemma 2.3 and (2.1) with those of Lemma 2.1, we also see that

$$\begin{aligned}
c &= 5a_{11} , \\
a_{12} &= -\frac{3}{2}a_{21} , \\
a_{1j} &= a_{j1} = 0, \qquad j > 2 .
\end{aligned}$$

About the d's, we obtain already some information in the following lemma.

LEMMA 2.4. *$d_{1i}^1=0$ and $d_{ik}^1 = d_{1i}^k + d_{1k}^i - \delta_{ik}(\lambda/2b)$ for $i, k>1$.*

PROOF. From the Gauss equation and Lemma 2.3 we have for $i>1$ that

$$\begin{aligned}
0 &= h(R(U_i, U_1)E_1, E_1) \\
&= h(\nabla_{U_i}\nabla_{U_1}E_1 - \nabla_{U_1}\nabla_{U_i}E_1 - \nabla_{\nabla_{U_i}U_1 - \nabla_{U_1}U_i}E_1, E_1) \\
&= h(b\nabla_{U_i}E_1, E_1) - h(\nabla_{U_i}U_1 - \nabla_{U_1}U_i, U_1)b = bd_{1i}^1 .
\end{aligned}$$

To obtain the second claim, we use the Codazzi equation. Then we find for $i, k>1$ that

$$d^1_{ik} = -d^k_{i1}+d^k_{1i}+d^i_{1k} = -\delta_{ik}\frac{\lambda}{2b}+d^k_{1i}+d^i_{1k}. \qquad \square$$

LEMMA 2.5. *We also have that* $\mu=0$, *hence* $\det(S)=0$.

PROOF. By applying the Gauss equation, using also the fact that a_{11} and a_{12} are both constants, we obtain that

$$\begin{aligned} -\mu E_1 &= R(U_1, E_1)U_1 \\ &= \nabla_{U_1}\nabla_{E_1}U_1-\nabla_{E_1}\nabla_{U_1}U_1-\nabla_{[U_1,E_1]}U_1 \\ &= \nabla_{U_1}(a_{11}U_1+a_{12}U_2)-\nabla_{E_1}\Big(2a_{11}E_1+\Big(2b+\frac{\lambda}{2b}\Big)U_1\Big) \\ &\quad -b\nabla_{E_1}U_1+a_{11}\nabla_{U_1}U_1+a_{12}\nabla_{U_2}U_1 \\ &= a_{11}\Big(2a_{11}E_1+\Big(2b+\frac{\lambda}{2b}\Big)U_1\Big)+a_{12}\nabla_{U_1}U_2 \\ &\quad -2a_{11}(5a_{11}E_1+2bU_1)-\Big(2b+\frac{\lambda}{2b}\Big)(a_{11}U_1+a_{12}U_2) \\ &\quad -b(a_{11}U_1+a_{12}U_2)+a_{11}\Big(2a_{11}E_1+\Big(2b+\frac{\lambda}{2b}\Big)U_1\Big)+a_{12}\Big(-\frac{a_{21}}{2}E_1+\frac{\lambda}{2b}U_2\Big). \end{aligned}$$

By taking different components, using also Lemma 2.4, we deduce that

$$-\mu = -6a_{11}^2+a_{12}(a_{12}+a_{21})+a_{12}\Big(-\frac{a_{21}}{2}\Big) = -6a_{11}^2-a_{12}a_{21}, \tag{2.2}$$

$$a_{11}\Big(-3b+\frac{\lambda}{2b}\Big) = 0, \tag{2.3}$$

$$a_{12}h(\nabla_{U_1}U_2, U_j) = 0, \qquad j>2, \tag{2.4}$$

$$a_{12}(h(\nabla_{U_1}U_2, U_2)-3b) = 0. \tag{2.5}$$

First, we assume that $a_{21}=0$. Then we apply the Gauss equation and find, using amongst others that $\nabla_{E_1}U_j$ is orthogonal to U_1 for $j>1$, that

$$\begin{aligned} 0 &= R(U_j, E_1)U_1 \\ &= \nabla_{U_j}\nabla_{E_1}U_1-\nabla_{E_1}\nabla_{U_j}U_1-\nabla_{[U_j,E_1]}U_1 \\ &= a_{11}\frac{\lambda}{2b}U_j-\frac{\lambda}{2b}\nabla_{E_1}U_j+\nabla_{\nabla_{E_1}U_j}U_1 = a_{11}\frac{\lambda}{2b}U_j. \end{aligned}$$

It follows that $a_{11}=0$. From (2.2), we then get that $\mu=0$. This completes the proof of the lemma in this case.

Therefore, for the remainder of the proof, we shall assume that $a_{21}\neq 0$.

So U_2 is determined uniquely. Since M is homogeneous this implies that a_{22} is also constant. We will derive a contradiction in this case. First notice that it then follows from (2.4) and (2.5) that

$$\nabla_{U_1} U_2 = -\frac{a_{21}}{2} E_1 + 3bU_2. \tag{2.6}$$

Applying then the Gauss equation once more gives us that

$$\begin{aligned}
-\lambda U_2 &= R(U_1, U_2)U_1 \\
&= \nabla_{U_1}\nabla_{U_2}U_1 - \nabla_{U_2}\nabla_{U_1}U_1 - \nabla_{[U_1, U_2]}U_1 \\
&= \nabla_{U_1}\Big(-\frac{a_{21}}{2}E_1 + \frac{\lambda}{2b}U_2\Big) - \nabla_{U_2}\Big(2a_{11}E_1 + \Big(2b + \frac{\lambda}{2b}\Big)U_1\Big) \\
&\quad - \Big(3b - \frac{\lambda}{2b}\Big)\nabla_{U_2}U_1 \\
&= -\frac{a_{21}}{2}bE_1 + \frac{\lambda}{2b}\Big(-\frac{a_{21}}{2}E_1 + 3bU_2\Big) \\
&\quad - \Big(2b + \frac{\lambda}{2b}\Big)\Big(-\frac{a_{21}}{2}E_1 + \frac{\lambda}{2b}U_2\Big) - \Big(3b - \frac{\lambda}{2b}\Big)\Big(-\frac{a_{21}}{2}E_1 + \frac{\lambda}{2b}U_2\Big).
\end{aligned}$$

So, by comparing components, we get that $a_{21}(-4b + \lambda/(2b)) = 0$. Since we assumed that $a_{21} \neq 0$, this gives us that

$$\lambda = 8b^2. \tag{2.7}$$

By substituting this in (2.3), we see that $a_{11} = c = 0$.

Remark that in case M is 3-dimensional, the apolarity condition for U_1 implies $b + (2b + \lambda/(2b)) + 3b = 0$. Since $\lambda = 8b^2$, this yields a contradiction. Therefore, we may assume that the dimension of M is at least 4.

Next, we again apply the Gauss equation to obtain that

$$\begin{aligned}
0 &= R(U_2, E_1)U_1 \\
&= \nabla_{U_2}\nabla_{E_1}U_1 - \nabla_{E_1}\nabla_{U_2}U_1 - \nabla_{[U_2, E_1]}U_1 \\
&= \nabla_{U_2}a_{12}U_2 - \nabla_{E_1}\Big(-\frac{a_{21}}{2}E_1 + \frac{\lambda}{2b}U_2\Big) + \sum_{j=1}^{n-1} h(\nabla_{E_1}U_2, U_j)\nabla_{U_j}U_1 \\
&= a_{12}\nabla_{U_2}U_2 + \frac{a_{21}}{2}2bU_1 - \frac{\lambda}{2b}(\nabla_{E_1}U_2) \\
&\quad + \frac{\lambda}{2b}\nabla_{E_1}U_2 + a_{21}(2bU_1) + a_{22}\Big(-\frac{a_{21}}{2}E_1\Big).
\end{aligned} \tag{2.8}$$

Taking the E_1-component of this expression shows that $2a_{12}a_{22} - (a_{21}/2)a_{22} = 0$. Hence $a_{22} = 0$. Therefore (2.8) reduces to $a_{12}\nabla_{U_2}U_2 + 3a_{21}bU_1 = 0$, from which we deduce that

$$\nabla_{U_2} U_2 = 2bU_1 . \tag{2.9}$$

We then have to apply the Gauss equation several times more. First, using (2.7), we obtain that for $j>2$

$$\begin{aligned} 0 &= R(U_j, E_1)U_1 \\ &= \nabla_{U_j} a_{12} U_2 - \nabla_{E_1} 4bU_j + 4b\nabla_{E_1} U_j - h(\nabla_{E_1} U_j, U_2) \frac{a_{21}}{2} E_1 \\ &= -\frac{3}{2} a_{21} \nabla_{U_j} U_2 - \frac{1}{2} a_{21} a_{j2} E_1 . \end{aligned}$$

From this it follows that we can choose $U_3, \cdots, U_{n-1}$ such that

$$\begin{aligned} \nabla_{U_3} U_2 &= -\frac{1}{3} a_{32} E_1 , \\ \nabla_{U_j} U_2 &= 0, \qquad j > 3 . \end{aligned}$$

Moreover, we get that, for $j>3$, $a_{j2}=a_{2j}=0$ and $a_{23}=-(4/3)a_{32}$, where, because M is locally homogeneous and E_1, U_1 and U_2 are uniquely determined, a_{23} is a constant.

Next, we will use another Gauss equation, which will show us amongst others that $a_{32}\neq 0$. We have that

$$\begin{aligned} \mu E_1 &= R(E_1, U_2)U_2 \\ &= 2b\nabla_{E_1} U_1 - \nabla_{U_2}(a_{21}U_1 + a_{23}U_3) - a_{21}\nabla_{U_1} U_2 - a_{23}\nabla_{U_3} U_2 \\ &= 2ba_{12}U_2 - a_{21}\Big(-\frac{a_{21}}{2} E_1 + 4bU_2\Big) - a_{23}\nabla_{U_2} U_3 \\ &\quad - a_{21}\Big(-\frac{a_{21}}{2} E_1 + 3bU_2\Big) + a_{23}\Big(\frac{1}{3} a_{32} E_1\Big). \end{aligned}$$

Taking the E_1 component, using also (2.2), we get that

$$-\frac{3}{2} a_{21}^2 = a_{21}^2 - \frac{4}{9} a_{32}^2 - \frac{4}{9} a_{32}^2 .$$

From this we get that $a_{32}^2=(45/16)a_{21}^2\neq 0$. From the Gauss equation (2.10), we then deduce that

$$\nabla_{U_2} U_3 = -\frac{1}{3} a_{32} E_1 - 10b \frac{a_{21}}{a_{23}} U_2 .$$

Finally, we use the Gauss equation once more. We compute

$$\begin{aligned} 0 &= R(U_2, U_3)U_1 \\ &= 4b\nabla_{U_2} U_3 - 4b\nabla_{U_3} U_2 + \frac{a_{21}}{2} \nabla_{U_3} E_1 + 10b \frac{a_{21}}{a_{23}} \nabla_{U_2} U_1 \\ &= \Big(-10b \frac{a_{21}}{a_{23}}\Big)\Big(4bU_2 + \frac{a_{21}}{2} E_1 - 4bU_2\Big), \end{aligned}$$

in order to deduce that $a_{21}=0$. This completes the proof of the lemma. □

From this lemma, the proof of Theorem 1 follows immediately. Looking once more at the proof, we immediately obtain the following corollary.

COROLLARY 2.1. *$a_{11}=c=a_{1j}=a_{j1}=0$, where $j>1$.*

3. Proofs of Theorem 2 and Theorem 3.

We start by defining a matrix $A=(a_{ij})$. Since

$$h(K_{E_1}U_i, U_j) = -\frac{1}{2}(\nabla h)(E_1, U_i, U_j) = \frac{1}{2}(a_{ij}+a_{ji}), \tag{3.1}$$

we see that, although the matrix A itself depends on the choice of h-orthonormal $U_1, \cdots, U_{n-1}$, the fact that A is skew-symmetric or not, is independent of that choice. In the following lemma, we prove that this matrix A is a skew-symmetric matrix.

LEMMA 3.1. *${}^tA=-A$. In particular, for h-orthonormal $U_1, \cdots, U_{n-1}$ as defined in Section 2, we have*

$$\nabla_{U_i}U_j = \sum_{k=1}^{n-1} d_{ij}^k U_k .$$

PROOF. Clearly, we already know that for all j, $a_{j1}=a_{1j}=0$. Therefore, we may assume that $i, j>1$. Next, we consider the restriction K of K_{E_1} to $\{E_1, U_1\}^{\perp}$. By restriction, we always mean that for $U\in\{E_1, U_1\}^{\perp}$,

$$KU = K_{E_1}U - h(K_{E_1}U, E_1)E_1 - h(K_{E_1}U, U_1)U_1 .$$

We then choose $U_2, \cdots, U_{n-1}$ as eigenvectors of this symmetric linear operator. Remark that since $\{E_1, U_1\}^{\perp}$ is uniquely determined, the eigenvalues of this operator must be constant. This together with (3.1) implies that there exist constants μ_i such that $\mu_i\delta_{ij}=(a_{ij}+a_{ji})/2$ and $2\mu_i$ is an eigenvalue. Since $\mu=0$, we have from the Gauss equation that

$$\begin{aligned} 0 &= h(R(E_1, U_i)U_j, E_1) \\ &= h(\nabla_{E_1}\nabla_{U_i}U_j - \nabla_{U_i}\nabla_{E_1}U_j - \nabla_{[E_1, U_i]}U_j, E_1). \end{aligned}$$

Since $\nabla_{E_1}E_1$ and $\nabla_{E_1}U_k$ are orthogonal to E_1 and the numbers μ_i are constants, using also Corollary 2.1, we get that

$$\begin{aligned} 0 &= -\sum_{k=2}^{n-1} a_{jk}h(\nabla_{U_i}U_k, E_1) - \sum_{k=2}^{n-1} a_{ik}h(\nabla_{U_k}U_j, E_1) \\ &= -\sum_{k=2}^{n-1}(a_{jk}(a_{ik}+a_{ki}) + a_{ik}(a_{kj}+a_{jk})) \end{aligned}$$

$$= -2\sum_{k=2}^{n-1} a_{jk}\delta_{ik}\mu_k + a_{ik}\delta_{jk}\mu_j$$

$$= -2\mu_i a_{ji} - 2\mu_j a_{ij}.$$

So taking $i=j$ gives us $\mu_i^2=0$. Hence A is a skew-symmetric matrix. □

LEMMA 3.2. *We define a matrix D by $D_{ij}=d_{1i}^j$, $2\leqq i,\ j\leqq n-1$. Then $(D+{}^tD)=2\tilde{K}$, where $\tilde{K}$ is the restriction of K_{U_1} to $\{E_1, U_1\}^{\perp}$. Then there exist numbers r and s, $r+s=n-2$, such that the symmetric linear operator $2K$ has two eigenvalues $\lambda_2=\lambda/b$ (multiplicity s) and $\lambda_1=\lambda/(2b)+2b$ (multiplicity r). Moreover, we have that $\lambda=-2b^2(r+3)/(1+s+r/2)$. Finally, if j and k correspond to different eigenspaces then $d_{1j}^k=0$.*

PROOF. We choose $U_2,\cdots,U_{n-1}$ as eigenvectors of K_{U_1} restricted to $\{E_1, U_1\}^{\perp}$. Remark that since this space is affine invariant, the eigenvalues must be constant on M. However, since

$$h(K_{U_1}U_i, U_j) = -\frac{1}{2}(\nabla h)(U_1, U_i, U_j) = \frac{1}{2}(d_{1i}^j + d_{1j}^i),$$

there exist constants λ_i such that

$$\lambda_i\delta_{ij} = d_{1i}^j + d_{1j}^i. \tag{3.2}$$

Hence Lemma 2.4 implies that d_{ij}^1 is constant. Using Lemma 2.3, Lemma 2.4 and Lemma 3.1, we find from the Gauss equation for $i,\ j>1$ that

$$\begin{aligned}
\lambda\delta_{ij} &= h(R(U_1, U_i)U_j, U_1)\\
&= h(\nabla_{U_1}\nabla_{U_i}U_j - \nabla_{U_i}\nabla_{U_1}U_j - \nabla_{[U_1,U_i]}U_j, U_1)\\
&= d_{ij}^1 d_{11}^1 - \sum_{k=2}^{n-1} d_{1j}^k d_{ik}^1 - \sum_{k=2}^{n-1} d_{1i}^k d_{kj}^1 + \frac{\lambda}{2b} d_{ij}^1\\
&= \left(d_{11}^1 + \frac{\lambda}{2b}\right)\left(-\delta_{ij}\frac{\lambda}{2b} + d_{1i}^j + d_{1j}^i\right) - \sum_{k=2}^{n-1} d_{1j}^k\left(-\delta_{ki}\frac{\lambda}{2b} + d_{1k}^i + d_{1i}^k\right)\\
&\quad - \sum_{k=2}^{n-1} d_{1i}^k\left(-\delta_{kj}\frac{\lambda}{2b} + d_{1k}^j + d_{1j}^k\right)\\
&= \left(d_{11}^1 + \frac{\lambda}{2b}\right)\left(\lambda_i - \frac{\lambda}{2b}\right)\delta_{ij} - \sum_{k=2}^{n-1} d_{1j}^k\delta_{ik}\left(\lambda_i - \frac{\lambda}{2b}\right) - \sum_{k=2}^{n-1} d_{1i}^k\delta_{jk}\left(\lambda_j - \frac{\lambda}{2b}\right)\\
&= \left(d_{11}^1 + \frac{\lambda}{2b}\right)\left(\lambda_i - \frac{\lambda}{2b}\right)\delta_{ij} - d_{1j}^i\left(\lambda_i - \frac{\lambda}{2b}\right) - d_{1i}^j\left(\lambda_j - \frac{\lambda}{2b}\right).
\end{aligned} \tag{3.3}$$

Hence, by taking $i=j$, we get that

$$\lambda + d_{11}^1\frac{\lambda}{2b} + \left(\frac{\lambda}{2b}\right)^2 = \left(d_{11}^1 + \frac{\lambda}{b}\right)\lambda_i - \lambda_i^2. \tag{3.4}$$

This proves that $2\tilde{K}$ has at most two eigenvalues, namely the two solutions of the above equation.

Moreover, it follows also from (3.2) and (3.3) that if U_i and U_j belong to different eigenspaces, then

$$\lambda_i d^i_{1j}+\lambda_j d^j_{1i}=0. \tag{3.5}$$

Since in this case (3.2) becomes $d^j_{1i}+d^i_{1j}=0$, (3.5) implies that $d^i_{1j}=d^j_{1i}=0$.

So in order to obtain the proof of the lemma, we only have to relate λ, b, λ_1, λ_2. First remark that it follows from Lemma 2.3 that $d^1_{11}=(2b+\lambda/(2b))$. Then, by solving λ_i from (3.4), we get that

$$\lambda_2=\frac{\lambda}{b}, \tag{3.6}$$

$$\lambda_1=\frac{\lambda}{2b}+2b. \tag{3.7}$$

Let us denote by r (resp. s) the multiplicity of λ_1 (resp. λ_2). From the apolarity condition for the vector field U_1, we get that

$$\sum_{i=1}^{n-1} d^i_{1i}+h(\nabla_{U_1}E_1,\ E_1)=0.$$

This implies that $3b+\lambda/(2b)+(s/2)(\lambda/b)+r(\lambda/(2b)+2b)/2=0$, such that

$$\lambda=-2b^2\frac{r+3}{(1+s+r/2)}. \tag{3.8}$$

□

From now on, we will assume that $U_2, \cdots, U_{n-1}$ are chosen in such a way that

(1) $U_2, \cdots, U_{r+1}$ are eigenvectors of $\tilde{K}$ with eigenvalue $\lambda_1/2$;

(2) $U_{r+2}, \cdots, U_{r+s+1}$ are eigenvectors of $\tilde{K}$ with eigenvalue $\lambda_2/2$.

Notice that we allow both r and s to be equal to zero. It immediately follows from the previous lemma and (3.6), (3.7) and (3.8) that $\lambda_1\neq\lambda_2$. Remark also that because E_1 and U_1 are uniquely determined, the above vector spaces are uniquely determined. So we can define several differentiable distributions. We first define as I_1 and I_2 the following sets of indices:

$$I_1=\{2, \cdots, r+1\},$$
$$I_2=\{r+2, \cdots, r+s+1=n-1\}.$$

Then, we define distributions T_0, T_1 and T_2 by

$$T_0=\operatorname{span}\{E_1,\ U_1\}$$
$$T_\alpha=\operatorname{span}\{U_j \mid j\in I_\alpha\},$$

where $\alpha=1, 2$. Also, all these distributions are determined in an equiaffine invariant way, i.e. if A is an equiaffine transformation which locally preserves the surface and if $X\in T_{\beta}$, $\beta=0, 1, 2$, then also $A_{*}X\in T_{\beta}$. So, we can define constants a_1, a_2, a_3 and a_4 on M by

$$a_1=\sum_{i,j,k\in I_1}(\nabla h)(U_i, U_j, U_k)^2,$$

$$a_2=\sum_{i,j,k\in I_2}(\nabla h)(U_i, U_j, U_k)^2,$$

$$a_3=\sum_{i,j\in I_1,k\in I_2}(\nabla h)(U_i, U_j, U_k)^2,$$

$$a_4=\sum_{i\in I_1,j,k\in I_2}(\nabla h)(U_i, U_j, U_k)^2.$$

The following lemmas will then be useful to prove integrability properties for different combinations of the above distributions.

First, we have:

LEMMA 3.3. *We have for $i, j\in I_1$ and for $k, p\in I_2$ that $d_{ij}^k=0=d_{kp}^i$, i.e.*

$$h(\nabla_{U_i}U_j, U_k)=h(\nabla_{U_k}U_p, U_i)=0.$$

PROOF. We compute, using the fact that by Lemma 2.4 and our choice of orthonormal frame d_{ij}^1 is constant for all indices i, j, that

$$\begin{aligned}0&=h(R(U_i, U_j)U_k, U_1)\\&=h(\nabla_{U_i}\nabla_{U_j}U_k-\nabla_{U_j}\nabla_{U_i}U_k-\nabla_{[U_i,U_j]}U_k, U_1)\\&=\sum_{p=2}^{n-1}d_{jk}^p d_{ip}^1-d_{ik}^p d_{jp}^1-(d_{ij}^p-d_{ji}^p)d_{pk}^1.\end{aligned}$$

Applying Lemma 2.4, we can rewrite this as

$$\begin{aligned}0=&\sum_{p=2}^{n-1}\Big(d_{jk}^p\Big(-\delta_{ip}\frac{\lambda}{2b}+d_{1i}^p+d_{1p}^i\Big)-d_{ik}^p\Big(-\delta_{jp}\frac{\lambda}{2b}+d_{1j}^p+d_{1p}^j\Big)\Big)\\&-\sum_{p=2}^{n-1}(d_{ij}^p-d_{ji}^p)\Big(-\delta_{pk}\frac{\lambda}{2b}+d_{1p}^k+d_{1k}^p\Big).\end{aligned}$$

Since the vector fields U_i are eigenvectors of $\tilde{K}$, this equation reduces to

$$0=d_{jk}^i\Big(\frac{\lambda}{2b}+2d_{1i}^i\Big)-d_{ik}^j\Big(\frac{\lambda}{2b}+2d_{1j}^j\Big)-(d_{ij}^k-d_{ji}^k)\Big(\frac{\lambda}{2b}+2d_{1k}^k\Big).$$

Applying then the Codazzi equation $d_{ji}^k=-d_{jk}^i+d_{ij}^k+d_{ik}^j$ gives us that

$$0=d_{jk}^i(2d_{1i}^i-2d_{1k}^k)-d_{ik}^j(2d_{1j}^j-2d_{1k}^k).$$

Hence if $i, k\in I_1$ and $j\in I_2$, we see that $0=d_{ik}^j$. Similarly, we get $d_{ik}^j=0$ for $i, k\in I_2$ and $j\in I_1$. This completes the proof of the lemma. □

LEMMA 3.4. *The distributions T_α, $\alpha=1, 2$ are integrable.*

PROOF. Let $U_2, \cdots, U_{r+1}$ be the local basis of T_1 which we constructed earlier. Then Lemma 3.3 implies that $\nabla_{U_i}U_j$, $i, j\in I_1$ is orthogonal to T_2. So, since ∇ is torsion free, we also have that $[U_i, U_j]$ is orthogonal to T_2. From Lemma 2.4, we then see that $d_{ij}^1=d_{ji}^1$, which implies that $[U_i, U_j]$ is orthogonal to U_1. Finally, from Lemma 2.1, we see that $[U_i, U_j]$ is also orthogonal to E_1. This completes the proof that T_1 is integrable. The proof that T_2 is integrable is similar. □

LEMMA 3.5. *We also have that $a_{ij}=0$, if $i\in I_1$ and $j\in I_2$, i.e. $\nabla_{E_1}T_\alpha\subset T_\alpha$, $\alpha=1, 2$.*

PROOF. We use the Gauss equation to obtain for $j>1$ that

$$\begin{aligned}0 &= R(E_1, U_1)U_j\\ &= \nabla_{E_1}\nabla_{U_1}U_j-\nabla_{U_1}\nabla_{E_1}U_j-\nabla_{[E_1,U_1]}U_j\\ &= \sum_{k,m=2}^{n-1}(d_{1j}^k a_{km}-a_{jk}d_{1k}^m)U_m+b\sum_{m=2}^{n-1}a_{jm}U_m+\sum_{m=2}^{n-1}(E_1(d_{1j}^m)-U_1(a_{jm}))U_m.\end{aligned}$$

Hence from this, we get that

$$0=\sum_{k=2}^{n-1}(d_{1j}^k a_{km}-a_{jk}d_{1k}^m)+ba_{jm}+E_1(d_{1j}^m)-U_1(a_{jm}), \tag{3.9}$$

and since A is skew-symmetric matrix, we also get that

$$0=-\sum_{k=2}^{n-1}(d_{1j}^k a_{mk}-a_{kj}d_{1k}^m)-ba_{mj}+E_1(d_{1j}^m)+U_1(a_{mj}).$$

Interchanging m and j then yields

$$0=-\sum_{k=2}^{n-1}(d_{1m}^k a_{jk}-a_{km}d_{1k}^j)-ba_{jm}+E_1(d_{1m}^j)+U_1(a_{jm}). \tag{3.10}$$

Adding (3.9) and (3.10) then gives us that

$$0=\sum_{k=2}^{n-1}(d_{1j}^k+d_{1k}^j)a_{mk}-a_{kj}(d_{1k}^m+d_{1m}^k).$$

So for $j\in I_1$ and $m\in I_2$, we get $(\lambda_1-\lambda_2)a_{jm}=0$. Since $\lambda_1\neq\lambda_2$, this completes the proof. □

LEMMA 3.6. *The distributions T_0, T_α, $T_0\oplus T_\alpha$, and $T_1\oplus T_2$, where $\alpha=1, 2$ are integrable.*

PROOF. From Lemma 3.4, we already know that the distributions T_α, $\alpha=1, 2$ are integrable. The integrability of T_0 is an immediate consequence of Lemma 2.1 and Corollary 2.1. In the same way, the integrability of $T_1\oplus T_2$ follows from Lemma 2.1 and Lemma 2.4.

So, we only have to proof that $T_0\oplus T_\alpha$ is integrable. By the above, it is sufficient to show that $[E_1, U_i]$ and $[U_1, U_i]$ belong to T_α for $i\in I_\alpha$. The first assumption follows immediately from Lemma 2.1 and Lemma 3.5, while the second one is an immediate consequence of Lemma 2.3 and Lemma 3.2. □

In the following lemma, we will then obtain a new simplification of our orthonormal frame.

LEMMA 3.7. *There are local h-orthonormal frames* $\{V_1, \cdots, V_r\}$ *of* T_1 *and* $\{W_1, \cdots, W_s\}$ *of* T_2 *such that*

(1) $\nabla_{U_1}E_1 = bE_1, \qquad \nabla_{E_1}U_1 = 0,$

(2) $\nabla_{V_i}E_1 = 0, \qquad \nabla_{E_1}V_i = 0,$

(3) $\nabla_{W_i}E_1 = 0, \qquad \nabla_{E_1}W_i = 0,$

(4) $\nabla_{E_1}E_1 = 2bU_1, \qquad \nabla_{U_1}U_1 = \left(2b+\frac{\lambda}{2b}\right)U_1,$

(5) $\nabla_{V_i}U_1 = \frac{\lambda}{2b}V_i, \qquad \nabla_{U_1}V_i = \frac{1}{2}\left(2b+\frac{\lambda}{2b}\right)V_i,$

(6) $\nabla_{W_i}U_1 = \frac{\lambda}{2b}W_i, \qquad \nabla_{U_1}W_i = \frac{\lambda}{2b}W_i.$

PROOF. We consider the integrable distributions T_0, T_1 and T_2 defined earlier. Since also $T_i\oplus T_j$ is integrable, we know that there exist coordinates $x_1, x_2, y_1, \cdots, y_r, z_1, \cdots, z_s$ such that $\partial/\partial x_i\in T_0$, $\partial/\partial y_j\in T_1$ and $\partial/\partial z_k\in T_2$. In order to simplify the notation, we denote $\partial/\partial x_i$, $\partial/\partial y_j$ and $\partial/\partial z_k$ respectively by ∂x_i, ∂y_j and ∂z_k.

From (3.1), Lemma 3.1 and Lemma 3.2, we get that $K_{T_0}T_\alpha\subset T_\alpha$ and from Lemma 3.1, Lemma 3.2 and Lemma 3.4, we also have that $\nabla_{T_0}T_\alpha\subset T_\alpha$. Hence, since $K=\nabla-\hat{\nabla}$, it follows that $\hat{\nabla}_{T_0}T_\alpha\subset T_\alpha$, where $\alpha=1, 2$ and $\hat{\nabla}$ is the Levi Civita connection of the affine metric. Therefore, we get that $\hat{\nabla}_{\partial x_i}\partial y_j\in T_1$. On the other hand using Lemma 2.1, Lemma 2.3 and the fact that ∂x_i is a combination of E_1 and U_1, we have $(\hat{\nabla}_{\partial y_j}\partial x_i)^{T_0^\perp}=\alpha_i\partial y_j$, where α_i is a local function on M. We now proceed in the following way. We define V_1 as the unit length vector field in the direction of ∂y_1. Then it follows from the previous formulas that $\hat{\nabla}_{E_1}V_1=\hat{\nabla}_{U_1}V_1=0$. So, if we put $\tilde{V}_2=\partial y_2-h(\partial y_2, V_1)V_1$, we get that

$$\hat{\nabla}_{\partial x_i}\tilde{V}_2=\alpha_i\partial y_2-\alpha_i h(\partial y_2, V_1)V_1 = \alpha_i\tilde{V}_2.$$

Therefore, taking V_2 to be of unit length in the direction of $\tilde{V}_2$, we see that also $\hat{\nabla}_{E_1}V_2=\hat{\nabla}_{U_1}V_2=0$. Completing the Gramm-Schmidt orthogonalization procedure in this way, we get a local h-orthonormal frame $V_1, \cdots, V_r$ of T_1 such that $\hat{\nabla}_{E_1}V_i=\hat{\nabla}_{U_1}V_i=0$. Similarly, we get an h-orthonormal frame $W_1, \cdots, W_s$ of T_2 such that

$$\hat{\nabla}_{E_1}W_j = \hat{\nabla}_{U_1}W_j = 0.$$

Hence, the frame $E_1, U_1, V_1, \cdots, V_r, W_1, \cdots, W_s$ satisfies conditions (1) up to

(6) of the lemma. □

Next, we put $U_2=V_1, \cdots, U_{r+1}=V_r, U_{r+2}=W_1, \cdots, U_{n-1}=W_s$ and define $\mu_1=\mu_2=\cdots=\mu_r=\lambda_1$ and $\mu_{r+1}=\mu_{r+2}=\cdots=\mu_{r+s}=\lambda_2$. Then, we obtain from the Gauss equation for $i, j, k>1$ that

$$\begin{aligned} 0 &= h(R(E_1, U_i)U_j, U_k) \\ &= h(\nabla_{E_1}\nabla_{U_i}U_j-\nabla_{U_i}\nabla_{E_1}U_j-\nabla_{[E_1,U_i]}U_j, U_k) \\ &= E_1(d_{ij}^k), \end{aligned} \tag{3.11}$$

$$\begin{aligned} 0 &= h(R(U_1, U_i)U_j, U_k) \\ &= h(\nabla_{U_1}\nabla_{U_i}U_j-\nabla_{U_i}\nabla_{U_1}U_j-\nabla_{[U_1,U_i]}U_j, U_k) \\ &= U_1(d_{ij}^k)+\left(\frac{\mu_k-\mu_j-\mu_i}{2}+\frac{\lambda}{2b}\right)d_{ij}^k. \end{aligned} \tag{3.12}$$

Then we have the following lemma:

LEMMA 3.8. *Let $X_1, X_2, X_3\in T_1$ and let $Y_1, Y_2\in T_2$. Then*

$$(\nabla h)(X_1, X_2, X_3)=(\nabla h)(X_1, Y_1, Y_2)=0.$$

PROOF. First, we take $i, j, k\in I_1$. Then we get from (3.12) that

$$U_1(d_{ij}^k)=\left(\frac{\lambda_1}{2}-\frac{\lambda}{2b}\right)d_{ij}^k=\frac{1}{2}\left(2b-\frac{\lambda}{2b}\right)d_{ij}^k.$$

We put $c=(2b-\lambda/(2b))$. Since $\lambda<0$, from Lemma 3.2, we see that $c\neq 0$. So, since a_1 is constant we get

$$0=U_1(a_1)=U_1\sum_{i,j,k\in I_1}(d_{ij}^k+d_{ik}^j)^2=c\sum_{i,j,k\in I_1}(d_{ij}^k+d_{ik}^j)^2=ca_1.$$

Hence $a_1=0$. This implies for all $i, j, k\in I_1$ that $(\nabla h)(V_i, V_j, V_k)=0$.

Next, we take $i\in I_1$, $j, k\in I_2$. From (3.9) it follows in this case that

$$U_1(d_{ij}^k)=\left(\lambda_1-\frac{\lambda}{2b}\right)d_{ij}^k=\frac{1}{2}\left(2b-\frac{\lambda}{2b}\right)d_{ij}^k.$$

We recall that $c=(2b-\lambda/(2b))\neq 0$. So we again get that

$$0=U_1(a_4)=U_1\sum_{i\in I_1, j,k\in I_2}(d_{ij}^k+d_{ik}^j)^2=c\sum_{i\in I_1, j,k\in I_2}(d_{ij}^k+d_{ik}^j)^2=ca_4.$$

Hence $a_4=0$. This implies for all $i\in I_1$, $j, k\in I_2$ that $(\nabla h)(V_i, V_j, V_k)=0$. □

Applying the previous lemma, using also the fact that $\nabla_{T_2}T_2$ has no component in the T_1 direction, we also see that $h(\nabla_{T_2}T_1, T_2)=h(\hat{\nabla}_{T_2}T_1, T_2)=0$. Using this, going again through the proof of Lemma 3.5, we find that

$$h(\hat{\nabla}_{\partial z_p}\partial y_j, \partial z_q)=0,$$

and hence also $h(\hat{\nabla}_{\partial y_j}\partial z_p, \partial z_q)=0$. So

$$\frac{\partial}{\partial y_j}(h(\partial z_p, \partial z_q)) = 0 .$$

Since $W_1, \cdots, W_s$ were constructed out of $\partial z_1, \cdots, \partial z_s$ using Gramm-Schmidt orthonormalization, the above formulas imply that $h(\hat{\nabla}_V W_p, W_q)=0$, where V is an arbitrary vector field belonging to T_1. Combining this with Lemma 3.8 and the Codazzi equation, we get that

$$\nabla_{V_i}V_j = \delta_{ij}2bU_1 + \sum_{k=1}^{r} v_{ij}^k V_k , \tag{3.13}$$

$$\nabla_{V_i}W_p = \sum_{k=1}^{r}(c_{pi}^k + c_{pk}^i)V_k, \qquad \nabla_{W_p}V_i = \sum_{k=1}^{r} c_{pi}^k V_k , \tag{3.14}$$

$$\nabla_{W_p}W_q = \left(\frac{\lambda}{2b}\right)\delta_{ij}U_1 + \sum_{l=1}^{s} w_{pq}^l W_l ,$$

where v_{ij}^k, w_{pq}^l and c_{pi}^k are local functions on M. In the following lemma, we shall see how we can still improve our choice of h-orthonormal $V_1, \cdots, V_r$ which span T_1.

LEMMA 3.9. *We can choose h-orthonormal $V_1, \cdots, V_r$ which span T_1 in such a way that all previous equations remain valid and such that* (3.13) *reduces to*

$$\nabla_{V_i}V_j = 2b\delta_{ij}U_1 .$$

PROOF. First, from the Gauss equation, we obtain that

$$\begin{aligned} 0 &= -\lambda(\delta_{jk}V_i - \delta_{ji}V_k) + R(V_i, V_k)V_j \\ &= \sum_{t=1}^{r}(V_i(v_{kj}^t) - V_k(v_{ij}^t))V_t + \sum_{t,u=1}^{r}(v_{kj}^t v_{it}^u - v_{ij}^t v_{kt}^u)V_u - \sum_{t,u=1}^{r}(v_{ik}^t - v_{ki}^t)v_{tj}^u V_u . \end{aligned}$$

From this, we obtain the following matrix equation.

$$V_i(v_k) - V_k(v_i) - [v_i, v_k] - \sum_{j=1}^{r}(v_{ik}^j - v_{ki}^j)v_j = 0 , \tag{3.15}$$

where v_k is the (r, r)-matrix with components v_{ki}^j. Similarly, we obtain from the Gauss equations $0=-\lambda\delta_{ij}W_p + R(V_i, W_p)V_j$ and $0=R(W_p, W_q)V_i$ that

$$V_i(c_p) - W_p(v_i) - [v_i, c_p] - \sum_{k=1}^{r} c_{pk}^i v_k = 0 , \tag{3.16}$$

$$W_p(c_q) - W_q(c_p) - [c_p, c_q] - \sum_{l=1}^{s}(w_{pq}^l - w_{qp}^l)c_l = 0 , \tag{3.17}$$

where c_k is the (r, r)-matrix with components c_{ki}^j.

Now, we look at the following system of differential equations for an (r, r)-matrix α.

$$
(3.18)\qquad \begin{cases} U_1(\alpha)=0\,, \\ E_1(\alpha)=0\,, \\ V_k(\alpha)=-\alpha v_k\,, \\ W_p(\alpha)=-\alpha c_p\,. \end{cases}
$$

Using (3.11), (3.12), (3.15), (3.16) and (3.17) we get that

$$
\begin{aligned}
&(E_1U_1-U_1E_1-[E_1, U_1])(\alpha)=0\,,\\
&(E_1V_k-V_kE_1-[E_1, V_k])(\alpha)=-\alpha E_1(v_k)=0\,,\\
&(E_1W_p-W_pE_1-[E_1, W_p])(\alpha)=-\alpha E_1(c_p)=0\,,\\
&(U_1V_k-V_kU_1-[U_1, V_k])(\alpha)=-\alpha U_1(v_k)+\frac{1}{2}\Big(2b-\frac{\lambda}{2b}\Big)\alpha v_k\\
&\qquad=\Big(-\frac{1}{2}\Big(2b-\frac{\lambda}{2b}\Big)+\frac{1}{2}\Big(2b-\frac{\lambda}{2b}\Big)\Big)\alpha v_k=0,\\
&(U_1W_p-W_pU_1-[U_1, W_p])(\alpha)=-\alpha U_1(w_p)=0\,,\\
&(V_iV_k-V_kV_i-[V_i, V_k])(\alpha)\\
&\qquad=-\alpha\Big(V_i(v_k)-V_k(v_i)-[v_i, v_k]-\sum_{j=1}^{r}(v_{ik}^j-v_{ki}^j)v_j\Big)=0\,,\\
&(V_iW_p-W_pV_i-[V_i, W_p])(\alpha)\\
&\qquad=-\alpha\Big(V_i(c_p)-W_p(v_i)-[v_i, c_p]-\sum_{k=1}^{r}c_{pk}^i v_k\Big)=0\,,\\
&(W_pW_q-W_qW_p-[W_p, W_q])(\alpha)\\
&\qquad=-\alpha\Big(W_p(c_q)-W_q(c_p)-[c_p, c_q]-\sum_{l=1}^{s}(w_{pq}^l-w_{qp}^l)c_l\Big)=0\,.
\end{aligned}
$$

The above equations now imply that for all vector fields X and Y tangent to M, we have

$$(XY-YX-[X, Y])\alpha=0\,.$$

Hence, for instance by introducing coordinates, it is then clear that the system of differential equations has a unique solution $\alpha=(\alpha_{ij})$ with initial conditions $\alpha(p)=I$. We define local vector fields V_i^* by

$$V_i^*=\sum_{j=1}^{r}\alpha_{ij}V_j\,.$$

Then, it follows from the definition of α that

$$
\begin{aligned}
&\nabla_{U_1}V_i^*=\frac{1}{2}\Big(2b+\frac{\lambda}{2b}\Big)V_i^*\,,\\
&\nabla_{E_1}V_i^*=\nabla_{W_p}V_i^*=0\,,\\
&\nabla_{V_k}V_i^*=2bh(V_k, V_i^*)U_1\,.
\end{aligned}
$$

We denote by $\bar{V}_i^*$ a Gramm-Schmidt orthogonalization of the V_i^*. Since by Lemma 3.5,

$$E_1(h(V_i^*, V_j^*)) = E_1\Big(\sum_{k=1}^{r} \alpha_{ik}\alpha_{jk}\Big) = 0,$$

$$U_1(h(V_i^*, V_j^*)) = U_1\Big(\sum_{k=1}^{r} \alpha_{ik}\alpha_{jk}\Big) = 0,$$

$$V_k(h(V_i^*, V_j^*)) = (\nabla h)(V_k, V_i^*, V_j^*) + h(\nabla_{V_k} V_i^*, V_j^*) + h(V_i^*, \nabla_{V_k} V_j^*) = 0,$$

we get that $\bar{V}_i^*$ form an h-orthonormal frame of T_1 which satisfies

$$\nabla_{U_1}\bar{V}_i^* = \frac{1}{2}\Big(2b + \frac{\lambda}{2b}\Big)\bar{V}_i^*,$$

$$\nabla_{E_1}\bar{V}_i^* = 0,$$

$$\nabla_{\bar{V}_k^*}\bar{V}_i^* = \delta_{ik} 2bU_1.$$

This completes the proof of the lemma. □

From now on, we will always work with the special h-orthonormal frame that we constructed in the previous lemmas. Its properties can be summarized in the following lemma.

LEMMA 3.10. *Let M^n be a locally strongly convex, affine homogeneous, proper quasiumbilical hypersurface in $\boldsymbol{R}^{n+1}$. Assume that* rank$(S)>1$. *Then there exists a local h-orthonormal basis $\{E_1, U_1, V_1, \cdots, V_r, W_1, \cdots, W_s\}$, where $r+s=n-2$, such that*

$$\nabla_{U_1}E_1 = bE_1 \qquad \nabla_{E_1}U_1 = 0$$

$$\nabla_{V_i}E_1 = 0 \qquad \nabla_{E_1}V_i = 0$$

$$\nabla_{W_i}E_1 = 0 \qquad \nabla_{E_1}W_i = 0$$

$$\nabla_{E_1}E_1 = 2bU_1 \qquad \nabla_{U_1}U_1 = \Big(2b + \frac{\lambda}{2b}\Big)U_1$$

$$\nabla_{V_i}U_1 = \frac{\lambda}{2b}V_i \qquad \nabla_{U_1}V_i = \frac{1}{2}\Big(2b + \frac{\lambda}{2b}\Big)V_i$$

$$\nabla_{W_i}U_1 = \frac{\lambda}{2b}W_i \qquad \nabla_{U_1}W_i = \frac{\lambda}{2b}W_i$$

$$\nabla_{V_i}V_j = \delta_{ij}2bU_1$$

$$\nabla_{V_i}W_j = \sum_{k=1}^{r}(c_{ji}^k + c_{jk}^i)V_k \qquad \nabla_{W_j}V_i = \sum_{k=1}^{r} c_{ji}^k V_k$$

$$\nabla_{W_i}W_j = \Big(\frac{\lambda}{2b}\Big)\delta_{ij}U_1 + \sum_{k=1}^{s} w_{ij}^k W_k,$$

where λ and b are constants on M related by $\lambda = -2b^2(r+3)/(1+s+r/2)$ and w_{ij}^k

and c_{ij}^k are local functions.

We consider now different cases depending on the dimension of M.

LEMMA 3.11. *Let M^3 be a locally strongly convex, affine homogeneous, proper quasiumbilical hypersurface in $\boldsymbol{R}^4$. Assume that* rank$(S)>1$. *Then either*

(1) *there exists a local h-orthonormal frame E_1, U_1, V_1 on M such that*

$$\begin{aligned} &\nabla_{U_1}E_1 = bE_1, && \nabla_{E_1}E_1 = 2bU_1, && \nabla_{V_1}V_1 = 2bU_1, \\ &\nabla_{V_1}E_1 = 0, && \nabla_{E_1}V_1 = 0, && \nabla_{E_1}U_1 = 0, \\ &\nabla_{U_1}U_1 = \left(2b+\frac{\lambda}{2b}\right)U_1, && \nabla_{U_1}V_1 = \frac{1}{2}\left(2b+\frac{\lambda}{2b}\right)V_1, && \nabla_{V_1}U_1 = \frac{\lambda}{2b}V_1, \end{aligned}$$

where $\lambda=-(16/3)b^2$, or

(2) *there exists a local h-orthonormal frame E_1, U_1, W_1 on M such that*

$$\begin{aligned} &\nabla_{U_1}E_1 = bE_1, && \nabla_{E_1}E_1 = 2bU_1, && \nabla_{W_1}W_1 = \frac{\lambda}{2b}U_1, \\ &\nabla_{W_1}E_1 = 0, && \nabla_{E_1}W_1 = 0, && \nabla_{E_1}U_1 = 0, \\ &\nabla_{U_1}U_1 = \left(2b+\frac{\lambda}{2b}\right)U_1, && \nabla_{U_1}W_1 = \frac{\lambda}{2b}W_1, && \nabla_{W_1}U_1 = \frac{\lambda}{2b}W_1, \end{aligned}$$

where $\lambda=-3b^2$.

PROOF. Since M is 3-dimensional, we have either $r=1$ and $s=0$ or $r=0$ and $s=1$. In the first case, the result follows immediately from Lemma 3.10. In the second case, we use the apolarity condition for W_1 to obtain that $h(\nabla_{W_1}W_1, W_1)=0$. □

LEMMA 3.12. *Let M^4 be a locally strongly convex, locally homogeneous, proper quasiumbilical hypersurface in $\boldsymbol{R}^5$. Assume that* rank$(S)>1$. *Then either*

(1) *there exists a local h-orthonormal frame E_1, U_1, V_1, V_2 on M such that*

$$\begin{aligned} &\nabla_{U_1}E_1 = bE_1, && \nabla_{E_1}E_1 = 2bU_1, && \nabla_{V_i}V_j = 2b\delta_{ij}U_1, \\ &\nabla_{V_i}E_1 = 0, && \nabla_{E_1}V_i = 0, && \nabla_{E_1}U_1 = 0, \\ &\nabla_{U_1}U_1 = \left(2b+\frac{\lambda}{2b}\right)U_1, && \nabla_{U_1}V_i = \frac{1}{2}\left(2b+\frac{\lambda}{2b}\right)V_i, && \nabla_{V_i}U_1 = \frac{\lambda}{2b}V_i, \end{aligned}$$

where $\lambda=-5b^2$,

(2) *there exists a local h-orthonormal frame E_1, U_1, V_1, W_1 on M such that*

$$\begin{aligned} &\nabla_{U_1}E_1 = bE_1, && \nabla_{E_1}E_1 = 2bU_1, && \nabla_{V_1}V_1 = 2bU_1, \\ &\nabla_{W_1}E_1 = \nabla_{E_1}W_1 = 0, && \nabla_{V_1}E_1 = \nabla_{E_1}V_1 = 0, && \nabla_{E_1}U_1 = 0, \\ &\nabla_{U_1}U_1 = \left(2b+\frac{\lambda}{2b}\right)U_1, && \nabla_{U_1}V_1 = \frac{1}{2}\left(2b+\frac{\lambda}{2b}\right)V_1, && \nabla_{V_1}U_1 = \frac{\lambda}{2b}V_1, \end{aligned}$$

$$\nabla_{U_1}W_1 = \nabla_{W_1}U_1 = \frac{\lambda}{2b}W_1,$$

$$\nabla_{W_1}W_1 = \frac{\lambda}{2b}U_1 - cW_1, \qquad \nabla_{W_1}V_1 = cV_1, \qquad \nabla_{V_1}W_1 = 2cV_1,$$

where $\lambda = -(16/5)b^2$ *and* c *is a positive number with* $c^2 = (24/25)b^2$,

(3) *with respect to the affine metric* h, M *is a product manifold* $M_0 \times M_2$, *where* M_2 *is a space of constant sectional curvature* $-3b^2$. *Moreover* E_1 *and* U_1 *locally span* $TM_0(=T_0)$ *and for* W_1 *and* W_2 *tangent to* M_2, *we have*

$$\hat{\nabla}_{U_1}E_1 = \hat{\nabla}_{U_1}U_1 = 0, \qquad \hat{\nabla}_{E_1}E_1 = bU_1, \qquad \hat{\nabla}_{E_1}U_1 = -bE_1$$

$$K_{E_1}E_1 = bU_1, \qquad K_{U_1}U_1 = \left(2b + \frac{\lambda}{2b}\right)U_1, \qquad K_{E_1}U_1 = bE_1,$$

$$K_{E_1}W_i = 0, \qquad K_{U_1}W_i = \frac{\lambda}{2b}W_i, \qquad K_{W_i}W_j = \delta_{ij}\frac{\lambda}{2b}U_1,$$

where $\lambda = -2b^2$, *or*

(4) *there exists a local h-orthonormal frame* E_1, U_1, W_1, W_2 *on* M *such that*

$$\nabla_{U_1}E_1 = bE_1, \qquad \nabla_{E_1}E_1 = 2bU_1, \qquad \nabla_{U_1}U_1 = \left(2b + \frac{\lambda}{2b}\right)U_1,$$

$$\nabla_{W_1}W_1 = \frac{\lambda}{2b}U_1 + \alpha W_1, \qquad \nabla_{W_2}W_2 = \frac{\lambda}{2b}U_1 - \alpha W_1, \qquad \nabla_{E_1}U_1 = 0,$$

$$\nabla_{W_1}W_2 = \nabla_{W_2}W_1 = -\alpha W_2,\ \nabla_{W_i}E_1 = \nabla_{E_1}W_i = 0,\ \nabla_{U_1}W_i = \nabla_{W_i}U_1 = \frac{\lambda}{2b}W_i,$$

where $\lambda = -2b^2$ *and* α *is a positive number with* $\alpha^2 = (3/2)b^2$.

PROOF. First, we consider the case that $r=2$ and $s=0$. Applying Lemma 3.10, immediately completes the proof in this case.

Next we consider the case that $r=1=s$. From Lemma 3.10, we obtain that there exist local functions w and c such that

$$\nabla_{V_1}V_1 = 2bU_1, \qquad \nabla_{W_1}W_1 = \frac{\lambda}{2b}U_1 + wW_1$$

$$\nabla_{W_1}V_1 = cV_1, \qquad \nabla_{V_1}W_1 = 2cV_1.$$

The apolarity condition for W_1 then implies that $w=-c$ and since a_3 is constant on M, we get that c is a constant. Also, if necessary by changing the sign of W_1, we may assume that c is positive. The Gauss equation then gives us that

$$\lambda V_1 = R(V_1, W_1)W_1 = \left(\frac{\lambda}{2b}\right)^2 V_1 - 6c^2V_1.$$

Since $\lambda = -(16/5)b^2$, Lemma 3.10 now completes the proof in this case as well.

Finally, we consider the case that $r=0$ and $s=2$. In this case, we get from

Lemma 3.10 that

$$\hat{\nabla}_{W_i} U_1 = \hat{\nabla}_{U_1} W_i = \hat{\nabla}_{W_i} E_1 = \hat{\nabla}_{E_1} W_i = 0 .$$

Let $p \in M$. Denote by M_α the integral manifold through p corresponding to the distribution T_α, $\alpha=0, 2$. The above formula implies that, as a Riemannian manifold, M is the product of M_0 and M_2. It also follows that W_1 and W_2 can be interpreted as tangent vector fields to M_2. Further,

$$\begin{aligned} K_{E_1} E_1 &= b E_1 , \\ K_{E_1} U_1 &= K_{E_1} W_i = 0 , \\ K_{U_1} U_1 &= \left(2b + \frac{\lambda}{2b}\right) U_1 , \\ K_{U_1} W_i &= \frac{\lambda}{2b} W_i . \end{aligned}$$

We now restrict ourselves to the integral manifold M_2. Clearly, we have that

$$\begin{aligned} D_{W_i} W_j &= \sum_{k=1}^{2} w_{ij}^k W_k + \delta_{ij} \left(\frac{\lambda}{2b} U_1 + \xi\right) \\ D_{W_i} \left(\frac{\lambda}{2b} U_1 + \xi\right) &= \left(\left(\frac{\lambda}{2b}\right)^2 - \lambda\right) W_i . \end{aligned}$$

By [**NP**] we get that the image of M_2 lies in a 3-dimensional affine space, which we will denote by $\boldsymbol{R}^3$. By the apolarity condition of M, the affine normal is given by $(\lambda/2b)U_1 + \xi$. Since the distribution T_2 is determined in an affine invariant way, the local homogeneity of M implies that also the leaves of T_2 are homogeneous hypersurfaces. So M_2 is a locally strongly convex, locally homogeneous 2-dimensional hyperbolic affine sphere. Applying then the classification theorem of [**NS**], we then see that either M_2 is a hyperboloid, with curvature $-3b^2$ and K^* vanishes identically or we can choose h-orthonormal W_1 and W_2 on M_2 in such a way that $\hat{\nabla}^*_{W_i} W_j = 0$, i, $j=1, 2$ and moreover

$$\begin{aligned} K^*_{W_1} W_1 &= \alpha W_1, \qquad K^*_{W_2} W_2 = -\alpha W_1 , \\ K^*_{W_1} W_2 &= K^*_{W_2} W_1 = -\alpha W_2 , \end{aligned}$$

where α is a positive number satisfying $2\alpha^2 = \lambda - (\lambda/2b)^2$ and ∇^* (resp. K^*) is the restriction of ∇ (resp. K) to the T_2-component.

We now extend in a parallel way W_1 and W_2 to local vector fields on $M_0 \times M_2$. Since, from (3.11) and (3.12) it follows that

$$U_1(h(K_{W_i} W_j, W_k)) = E_1(h(K_{W_i} W_j, W_k)) = 0 ,$$

we obtain (3) and (4). □

Using the two previous lemmas by introducing suitable coordinates and integrating explicitly the following theorems follow immediately.

THEOREM 3.1. *Let M^3 be a locally strongly convex, locally homogeneous, proper quasiumbilical hypersurface in $\boldsymbol{R}^4$. Assume also that* rank $(S)>1$. *Then M is affine equivalent to the convex part of one of the following hypersurfaces:*

$$\left(y-\frac{1}{2}x^2\right)^3 v^2 w^2 = 1\,,$$

$$\left(y-\frac{1}{2}x^2-\frac{1}{2}\frac{w^2}{z}\right)^4 z^3 = 1\,,$$

where (x, y, z, w) are the coordinates of $\boldsymbol{R}^4$.

THEOREM 3.2. *Let M^4 be a locally strongly convex, locally affine homogeneous, proper quasi-umbilical hypersurface in $\boldsymbol{R}^5$. Assume also that* rank $(S)>1$. *Then M is affine equivalent to the convex part of one of the following hypersurfaces:*

$$\left(y-\frac{1}{2}x^2\right)^3 u^2 v^2 w^3 = 1\,,$$

$$\left(y-\frac{1}{2}x^2-\frac{1}{2}\left(\frac{w^2}{z}+\frac{u^2}{z}\right)\right)^5 z^4 = 1\,,$$

$$\left(y-\frac{1}{2}x^2-\frac{1}{2}\frac{w^2}{z}\right)^4 z^3 u^2 = 1\,,$$

$$\left(y-\frac{1}{2}x^2\right)(z^2-(w^2+u^2)) = 1\,,$$

where (x, y, z, w, u) are the coordinates of $\boldsymbol{R}^5$.

Combining the above two theorems with the main theorem of [**DV1**] then completes the proof of Theorems 2 and 3.

REMARK. To obtain a complete classification of quasi-umbilical homogeneous hypersurfaces seems to be much more complicated. One of the reasons is technical. In dimensions 3 and 4, we immediately get from Lemma 3.7 that $[K_W, K_{\overline{W}}]T_1=0$, where $W, \overline{W}\in T_2$. For higher dimensions this condition is not satisfied automatically.

Also, the number of different examples increases rapidly with the dimension of M. For example

$$\left(v-\frac{1}{2}w^2-\frac{1}{2}\sum_{i=1}^{q}\left(\sum_{j=1}^{r_i}\left(\frac{x_j^i}{y_i}\right)^2 y_i\right)\right)^{3+\Sigma_{i=1}^q r_i}\prod_{i=1}^{q} y_i^{r_i+s_i+2}$$

$$\cdot\left(u-\frac{1}{2}\sum_{i=1}^{q}\left(\sum_{j=1}^{s_i}\left(\frac{z_j^i}{y_i}\right)^2 y_i\right)\right)^{2+\Sigma_{i=1}^q s_i} F(z_1, \cdots, z_p)^{2(p+2)/d} = 1\,,$$

where F is the homogeneous function of degree d associated with a homogeneous hyperbolic affine sphere (see [**DV2**]), is an example of a proper quasi-umbilical affine homogeneous hypersurface. Its dimension is equal to n, where

$$n=p+q+1+\sum_{i=1}^{q}(s_i+r_i).$$

It can be checked that all of the above examples satisfy the technical condition mentioned earlier. One could ask whether these are all the quasi-umbilical homogeneous hypersurfaces which satisfy that condition.

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