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Recurrence conditions for multidimensional processes of Ornstein-Uhlenbeck type

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1. Introduction and results.

A stochastic process of Ornstein-Uhlenbeck type on the Euclidean space is a Markov process obtained from a spatially homogeneous Markov process undergoing a linear drift force determined by a matrix $-Q$. We give a criterion of recurrence and transience for a process of this type under the assumption that Q is diagonalizable and its eigenvalues are positive. No restriction is imposed on the part of the spatially homogeneous Markov process.

Rigorous definition of our process is as follows. Let $\boldsymbol{G}$ be an operator defined by

$$\begin{aligned}(1.1)\qquad \boldsymbol{G}f(x)= &\sum_{j=1}^{d} a_j D_j f(x)+\frac{1}{2}\sum_{j,k=1}^{d} B_{jk}D_jD_kf(x)\\ &+\int_{\boldsymbol{R}^d}\left[f(x+y)-f(x)-\sum_{j=1}^{d}\frac{y_j}{1+|y|^2}D_jf(x)\right]\rho(dy)\\ &-\sum_{j,k=1}^{d} Q_{jk}x_kD_jf(x),\end{aligned}$$

where D_j stands for partial derivative in x_j. Here $a=(a_j)$ is a constant vector, $B=(B_{jk})$ is a symmetric nonnegative-definite constant matrix, ρ is a measure on $\boldsymbol{R}^d$ with $\rho(\{0\})=0$ and $\int|y|^2(1+|y|^2)^{-1}\rho(dy)<\infty$, and $Q=(Q_{jk})$ is a constant matrix. We consider the real Banach space $C_0(\boldsymbol{R}^d)$ of continuous functions vanishing at infinity with the norm of uniform convergence. The operator $\boldsymbol{G}$ is acting in this space and its domain is the class of C^2 functions with compact supports. It is proved in Sato and Yamazato [**10**] that the smallest closed extension $\bar{\boldsymbol{G}}$ of $\boldsymbol{G}$ is the infinitesimal generator of a strongly continuous nonnegative contraction semigroup on $C_0(\boldsymbol{R}^d)$. So a Markov process $\boldsymbol{X}$ on $\boldsymbol{R}^d$ is associated and it is represented, as usual (see [**1**]), by $(\Omega, \mathcal{F}, \mathcal{F}_t, \boldsymbol{P}^x, X_t)$ with $\boldsymbol{P}^x(X_0=x)=1$. The Markov process $\boldsymbol{X}$ is called in [**10**] the process of Ornstein-Uhlenbeck type associated with $\boldsymbol{G}$. The measure ρ is called the Lévy measure of $\boldsymbol{X}$. We consider $\boldsymbol{R}^d$ as the set of d-column vectors $x=(x_j)_{1\leq j\leq d}$, and denote the inner product and the norm by $\langle x, y\rangle=\sum_{j=1}^{d} x_j y_j$ and $|x|=\langle x, x\rangle^{1/2}$. Define $\boldsymbol{H}$ by

$$\boldsymbol{G}=\boldsymbol{H}-\sum_{j,k=1}^{d} Q_{jk}x_k D_j\,.$$

Then $\boldsymbol{H}$ gives the most general spatially homogeneous Markov process on $\boldsymbol{R}^d$. Fixing the starting point at the origin, let $\{Z_t : t\geqq 0\}$ be the Lévy process (process with stationary independent increments with $Z_0=0$ with paths being right-continuous and having left limits) determined by $\boldsymbol{H}$. The process $\boldsymbol{X}$ is sometimes called the process of Ornstein-Uhlenbeck type associated with $\{Z_t\}$ and Q. An equivalent definition of $\boldsymbol{X}$ for any specified starting point x is given by the unique solution of the equation

$$X_t = x+Z_t-\int_0^t QX_s ds\,. \tag{1.2}$$

The solution is expressed as

$$X_t = e^{-tQ}x+\int_0^t e^{(s-t)Q}dZ_s\,, \tag{1.3}$$

where the stochastic integral with respect to the Lévy process is defined by convergence in probability from integrals of simple functions.

A point y in $\boldsymbol{R}^d$ is called a recurrent point of $\boldsymbol{X}$ if

$$\boldsymbol{P}^x(\liminf_{t\to\infty}|X_t-y|=0)=1 \qquad \text{for every } x.$$

The process $\boldsymbol{X}$ is called recurrent if it has a recurrent point. The process $\boldsymbol{X}$ is said to be transient if

$$\boldsymbol{P}^x(\lim_{t\to\infty}|X_t|=\infty)=1 \qquad \text{for every } x.$$

If all eigenvalues of Q have positive real parts, then, as is shown by Shiga [**12**], $\boldsymbol{X}$ is either recurrent or transient. The problem that we tackle is to give a criterion of recurrence and transience in terms of a, B, ρ, and Q (it will be seen that a and B do not affect transience and recurrence).

We will prove two theorems.

THEOREM A. *Assume that $Q=\alpha I$, where $\alpha>0$ and I is the identity matrix. Fix $c>0$ arbitrarily. Then $\boldsymbol{X}$ is recurrent if and only if*

$$\int_0^1 \frac{dv}{v}\exp\left[\int_0^1\frac{du}{u}\int_{|x|\geq c}(e^{-u^{\alpha}|x|}-1)\rho(dx)\right]=\infty\,. \tag{1.4}$$

To state the second theorem, assume that all eigenvalues of Q are real and positive and that the eigenvectors of Q span the whole space $\boldsymbol{R}^d$. Let n be the number of different eigenvalues of Q and let $\alpha_1, \cdots, \alpha_n$ be the eigenvalues of Q. Let V_j be the eigenspace of α_j for Q for each j. Thus

$$\boldsymbol{R}^d = V_1 \oplus \cdots \oplus V_n . \tag{1.5}$$

Denote the projectors associated with this direct sum decomposition by $T_1, \cdots, T_n$, so that

$$x = T_1 x + \cdots + T_n x, \quad T_j x \in V_j \qquad \text{for } j=1, \cdots, n. \tag{1.6}$$

THEOREM B. *Fix $c>0$ arbitrarily. Under the assumption stated above, $\boldsymbol{X}$ is recurrent if and only if*

$$\int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u} \int_{|x|\geq c} \left(\exp\left(-\sum_{j=1}^n u^{\alpha_j} |T_j x|\right) - 1\right) \rho(dx)\right] = \infty . \tag{1.7}$$

We emphasize that we do not impose any condition on the Lévy process $\{Z_t\}$. Theorem A is a special case of Theorem B with $n=1$. But it is the most important special case, being a direct generalization of the classical Ornstein-Uhlenbeck process. Its proof is not so much involved as that of Theorem B and common idea of the proof is more visible. So we prove Theorem A prior to Theorem B.

Results related to this paper are as follows. When the eigenvalues of Q have positive real parts, the limit distribution of $\boldsymbol{X}$ exists if and only if the Lévy measure ρ satisfies

$$\int_{|x|\geq 1} \log |x| \rho(dx) < \infty . \tag{1.8}$$

Sato and Yamazato [**10**] prove it and explicitly describe the characteristic function of the limit distribution, which is independent of the starting point. The class of limit distributions coincides with the class of operator-selfdecomposable distributions. These or similar results are obtained by [**4, 5, 6, 9, 10, 13, 14**] almost simultaneously. In [**10**] the continuity of the correspondence between $\{Z_t\}$ and the limit distribution is established. There an example (with $d=1$) is given which is recurrent but does not possess a limit distribution. The example shows that, in recurrence and transience, not only ρ but also Q is relevant, while the condition (1.8) involves only ρ. Shiga [**12**] attacks the problem of recurrence and transience. He finds their criterion in one dimension ($d=1$). Our theorems generalize Shiga's criterion to arbitrary dimensions. In general dimensions Shiga [**12**] gives criteria in the following three cases: (i) $Q=\alpha I$ and ρ is symmetric (i.e. $\rho(-E)=\rho(E)$ for all Borel sets E); (ii) Q is diagonal and ρ is symmetric and concentrated to the coordinate axes; (iii) Q symmetric (i.e. $Q=Q'$ where Q' is the transpose of Q) and ρ is rotation invariant. All of them are special cases of Theorem B but none of his criteria has a form that directly generalizes to (1.7). In finding the criterion (1.7), we encounter analytical subtlety of the interplay of the matrix Q and the Lévy measure ρ. We do not have any intuitive or probabilistic reasoning to obtain the criterion.

In the methodological aspect of the problem, Shiga's proof in one dimension that finiteness of the integral implies transience is Fourier-analytic; his proof that infiniteness of the integral implies recurrence uses probabilistic argument, which is peculiar to one dimension. His three cases (i)-(iii) are in the situation that reduction to one dimension is possible. In multi-dimensions we cannot find any useful probabilistic technique. We have to adopt purely analytical method. Thus Section 2 of this paper gives a simple new proof of Shiga's one-dimensional result if we let $d=1$. An important point is that we can reduce the case of non-symmetric Lévy measures to the case of symmetric ones by an analytical manipulation.

Some consequences of our results in special cases are discussed in another paper [**11**] of Sato and Yamazato. In the case of an Ornstein-Uhlenbeck type process for which the eigenvalues of Q have positive real parts but Q does not satisfy our assumption, it is still hard to conjecture a recurrence-transience criterion of the integral type in terms of Q and ρ. We add that some related problems in Gaussian case are studied by [**2**, **3**, **7**, **8**].

Organization of this paper is as follows. In Section 2 we will prove Theorem A. Two technical lemmas on boundedness of some integrals containing trigonometric functions, exponential function, and powers will be given in Section 3. We will establish Theorem B in Section 4, using these lemmas. An example will be illustrated in Section 5.

The first version of this paper proved Theorem A and Theorem B for $n=2$ and gave the conjecture for general n. It was written by Sato and Yamazato in August, 1991 (No. 7 of the 1991 Preprint Series from Department of Mathematics, College of General Education, Nagoya University). After that Watanabe found the way (given in Section 3) to handle the case of general n. Now we jointly present our results in a complete form.

2. Proof of Theorem A.

Let $\boldsymbol{X}$ be the Markov process on $\boldsymbol{R}^d$ given in the previous section. It is the process of Ornstein-Uhlenbeck type associated with the Lévy process $\{Z_t\}$ and the matrix Q. We assume that the eigenvalues of Q have positive real parts. For $c>0$ the restriction of the Lévy measure ρ to the set $\{x: |x|\geqq c\}$ is denoted by ρ^c, and the compound Poisson process with Lévy measure ρ^c is denoted by $\{Z_t^c: t\geqq 0\}$. Further $\boldsymbol{X}^c$ denotes the process of Ornstein-Uhlenbeck type associated with $\{Z_t^c\}$ and Q. The transition probabilities of $\boldsymbol{X}$ and $\boldsymbol{X}^c$

are written as $p_t(x, E)$ and $p_t^c(x, E)$, respectively. The characteristic functions of $p_t(x, \cdot)$ and $p_t^c(x, \cdot)$ are denoted by $\hat{p}_t(x, z)$ and $\hat{p}_t^c(x, z)$. The following facts are known.

Fact 1. Let $\psi(z)$ be the function such that

$$\boldsymbol{E}e^{i\langle z, Z_t\rangle} = e^{t\psi(z)},$$

that is

$$\psi(z) = i\langle a, z\rangle - \frac{1}{2}\langle Bz, z\rangle + \int\left[e^{i\langle z, y\rangle} - 1 - \frac{i\langle z, y\rangle}{1+|y|^2}\right]\rho(dy).$$

Then

$$\hat{p}_t(x, z) = \exp\left[i\langle x, e^{-tQ'}z\rangle + \int_0^t \psi(e^{-sQ'}z)ds\right], \tag{2.1}$$

where Q' is the transpose of Q.

Fact 2. The process $\boldsymbol{X}$ is either recurrent or transient.

Fact 3. The process $\boldsymbol{X}$ is recurrent if and only if there is a point y in $\boldsymbol{R}^d$ such that, for any x and any open neighborhood E of y,

$$\int_0^\infty p_t(x, E)dt = \infty. \tag{2.2}$$

It is transient if and only if, for any x and any compact set E,

$$\int_0^\infty p_t(x, E)dt < \infty. \tag{2.3}$$

Fact 4. The process $\boldsymbol{X}$ has a limit distribution if and only if (1.8) holds.

Facts 1 and 4 are proved by Sato and Yamazato [**10**], while Facts 2 and 3 are shown by Shiga [**12**].

We prepare two lemmas. The first one is essentially by Shiga [**12**], p. 439.

LEMMA 2.1. *If $\boldsymbol{X}^c$ is recurrent for some $c>0$, then $\boldsymbol{X}$ is recurrent.*

PROOF. Let $\{W_t\}$ be a Lévy process independent of $\{Z_t^c\}$ such that $\{W_t+Z_t^c\}$ is equivalent with $\{Z_t\}$. The process $\boldsymbol{X}^c$ under the condition $X_0^c=0$ can be considered as the solution of

$$X_t^c = Z_t^c - \int_0^t QX_s^c ds.$$

Let $\{Y_t\}$ be the solution of

$$Y_t = W_t - \int_0^t QY_s ds.$$

Then $\{Y_t\}$ is the process of Ornstein-Uhlenbeck type associated with $\{W_t\}$ and Q. The processes $\{X_t^c\}$ and $\{Y_t\}$ are independent, and the process $\{X_t^c+Y_t\}$ is equivalent with $\boldsymbol{X}$ starting at 0. Since the Lévy measure of $\{Y_t\}$ is supported by the set $\{|x|\leqq c\}$, the process $\{Y_t\}$ has a limit distribution μ by Fact 4. Choose a compact set E_1 and a compact continuity set E_2 for μ such that $\int_0^\infty \boldsymbol{P}(X_t^c\in E_1)dt=\infty$ and $\mu(E_2)>0$. Let $E=E_1+E_2$. Then E is compact and

$$\begin{aligned}\boldsymbol{P}^0(X_t\in E)&=\boldsymbol{P}(X_t^c+Y_t\in E)\geqq \boldsymbol{P}(X_t^c\in E_1,\ Y_t\in E_2)\\&=\boldsymbol{P}(X_t^c\in E_1)\boldsymbol{P}(Y_t\in E_2)\geqq \boldsymbol{P}(X_t^c\in E_1)\mu(E_2)/2\end{aligned}$$

for large t. Hence $\int_0^\infty \boldsymbol{P}^0(X_t\in E)dt=\infty$, which shows that X is recurrent by Facts 2 and 3. □

LEMMA 2.2. *If ν is a measure on $\boldsymbol{R}^d$ such that $\nu(\{0\})=0$ and $\nu(\boldsymbol{R}^d)<1$, then*

$$\int_{|z|<1}dz\exp\left[\int\log\frac{|x|}{|\langle z,\ x\rangle|}\nu(dx)\right]<\infty\,. \tag{2.4}$$

PROOF. If $d=1$, then $|x|/|\langle z,\ x\rangle|=1/|z|$ and the assertion is trivial. Let $d\geqq 2$ and let $S=\{\xi\in\boldsymbol{R}^d:\ |\xi|=1\}$, the unit sphere. Let $A=\nu(\boldsymbol{R}^d)$. Disintegrate ν as

$$\nu(E)=\int_S\sigma(d\xi)\int_{(0,\infty)}1_E(r\xi)\tau_\xi(dr)$$

for any Borel set E, where σ is a probability measure on S and τ_ξ is a measure on $(0,\infty)$ with total mass A such that $\tau_\xi(F)$ is measurable in ξ for each Borel set F in $(0,\infty)$. Let σ_0 be the Euclidean surface measure on S. Then

$$\begin{aligned}&\int_{|z|<1}dz\exp\left[\int\log\frac{|x|}{|\langle z,\ x\rangle|}\nu(dx)\right]\\&=\int_{|z|<1}dz\exp\left[A\int_S\log\frac{1}{|\langle z,\ \xi\rangle|}\sigma(d\xi)\right]\\&=\int_S\sigma_0(d\zeta)\int_0^1 s^{d-1}ds\exp\left[A\int_S\log\frac{1}{s|\langle\zeta,\ \xi\rangle|}\sigma(d\xi)\right]\\&=\frac{1}{d-A}\int_S\sigma_0(d\zeta)\exp\left[A\int_S\log\frac{1}{|\langle\zeta,\xi\rangle|}\sigma(d\xi)\right]\\&\leqq\frac{1}{d-A}\int_S\sigma_0(d\zeta)\int_S\sigma(d\xi)\exp\left(A\log\frac{1}{|\langle\zeta,\xi\rangle|}\right)\\&=\frac{1}{d-A}\int_S\sigma(d\xi)\int_S|\langle\zeta,\ \xi\rangle|^{-A}\sigma_0(d\zeta)\end{aligned}$$

by the use of Jensen's inequality. Since $\int_S|\langle\zeta,\ \xi\rangle|^{-A}\sigma_0(d\zeta)$ does not depend on ξ,

$$\int_S \sigma(d\xi)\int_S |\langle\zeta, \xi\rangle|^{-A}\sigma_0(d\zeta) = \int_S |\zeta_1|^{-A}\sigma_0(d\zeta),$$

where ζ_1 is the first coordinate of ζ. Using the polar coordinates in $\boldsymbol{R}^d$, we get

$$\begin{aligned}\int_S |\zeta_1|^{-A}\sigma_0(d\zeta) &= 2^d\int_0^{\pi/2}\cdots\int_0^{\pi/2}(\cos\theta_1)^{-A}(\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3} \\ &\qquad\cdots(\sin\theta_{d-2})d\theta_1 d\theta_2\cdots d\theta_{d-1} \\ &= \text{const}\int_0^{\pi/2}(\sin\theta_1)^{d-2}(\cos\theta_1)^{-A}d\theta_1 < \infty\,,\end{aligned}$$

since A is less than 1. $\square$

PROOF OF THEOREM A. Assume that $Q=\alpha I$, $\alpha>0$. First we note that, if (1.4) holds for some $c>0$, then it holds for any $c'>0$ in place of c. In fact, this is obvious for $c'>c$ and, for $c'<c$, it suffices to note that

$$\begin{aligned}\left|\int_v^1\frac{du}{u}\int_{c'\leq|x|<c}(e^{-u^\alpha|x|}-1)\rho(dx)\right| &\leq \int_0^1\frac{du}{u}\int_{c'\leq|x|<c}(1-e^{-u^\alpha|x|})\rho(dx) \\ &= \int_{c'\leq|x|<c}\rho(dx)\int_0^{|x|}(1-e^{-u})\frac{du}{\alpha u} < \infty\,.\end{aligned}$$

Suppose that X is transient. Let us prove that

$$\int_0^1\frac{dv}{v}\exp\left[\int_v^1\frac{du}{u}\int_{|x|\geq c}(e^{-u^\alpha|x|}-1)\rho(dx)\right] < \infty \tag{2.5}$$

for some $c>0$. By Lemma 2.1 the process X^c is transient for every $c>0$. Hence

$$\int_0^\infty p_t^c(0, E)dt < \infty \tag{2.6}$$

for every c and every compact set E. Let

$$h(x) = \prod_{j=1}^d((1-|x_j|)\vee 0)\,. \tag{2.7}$$

Then

$$\hat{h}(z) = \int e^{i\langle z,x\rangle}h(x)dx = \prod_{j=1}^d 4z_j^{-2}\sin^2(2^{-1}z_j),$$

$$h(x) = (2\pi)^{-d}\int e^{-i\langle z,x\rangle}\hat{h}(z)dz\,.$$

It follows from (2.6) that

$$\begin{aligned}\infty > \int_0^\infty dt\int p_t^c(0, dx)h(x) &= (2\pi)^{-d}\int_0^\infty dt\int\hat{h}(z)\hat{p}_t^c(0, z)dz \\ &= (2\pi)^{-d}\int_0^\infty dt\int\hat{h}(z)\,\mathrm{Re}\,\hat{p}_t^c(0, z)dz\,.\end{aligned}$$

Since

$$\hat{p}_t^c(0, z) = \exp\left[\int_0^t ds \int (e^{ie^{-\alpha s}\langle z, x\rangle} - 1)\rho^c(dx)\right]$$

by Fact 1, we have

$$\mathrm{Re}\ \hat{p}_t^c(0, z) = (\cos F_c(t, z))(\exp G_c(t, z)),$$

where

$$F_c(t, z) = \int_0^t ds \int \sin(e^{-\alpha s}\langle z, x\rangle)\rho^c(dx),$$

$$G_c(t, z) = \int_0^t ds \int (\cos(e^{-\alpha s}\langle z, x\rangle) - 1)\rho^c(dx).$$

Let

$$\text{(2.8)} \qquad \sup\left|\int_M^N \sin u \frac{du}{u}\right| = K_1, \quad \sup\left|\int_M^N (\cos u - e^{-u})\frac{du}{u}\right| = K_2,$$

where supremums are taken over $M, N \in (0, \infty)$. Then K_1 and K_2 are finite, since $\int_0^M (\sin u)u^{-1}du$ and $\int_0^M (\cos u - e^{-u})u^{-1}du$ are convergent as $M \to \infty$. Now we have

$$|F_c(t, z)| = \left|\int \rho^c(dx)\int_{e^{-\alpha t}\langle z, x\rangle}^{\langle z, x\rangle} \sin u \frac{du}{\alpha u}\right| \leqq \frac{K_1}{\alpha}\rho(\{|x| \geqq c\}).$$

Choose c so large that $\rho(\{|x| \geqq c\}) \leqq \alpha\pi/(4K_1)$. Then $\cos F_c(t, z) \geqq 1/\sqrt{2}$. Therefore

$$\int \hat{h}(z)dz \int_0^\infty \exp G_c(t, z)dt < \infty.$$

Hence, for some z with $0 < |z| \leqq 1$,

$$\text{(2.9)} \qquad \int_0^\infty \exp G_c(t, z)dt < \infty.$$

Rewrite $G_c(t, z)$ as

$$G_c(t, z) = \int_0^t ds \int (e^{-e^{-\alpha s}|\langle z, x\rangle|} - 1)\rho^c(dx) + H_c(t, z),$$

$$H_c(t, z) = \int_0^t ds \int (\cos(e^{-\alpha s}\langle z, x\rangle) - e^{-e^{-\alpha s}|\langle z, x\rangle|})\rho^c(dx).$$

Then

$$\text{(2.10)} \qquad |H_c(t, z)| = \left|\int \rho^c(dx)\int_{e^{-\alpha t}|\langle z, x\rangle|}^{|\langle z, x\rangle|} (\cos u - e^{-u})\frac{du}{\alpha u}\right| \leqq \frac{K_2}{\alpha}\rho(\{|x| \geqq c\}).$$

Hence, by (2.9),

$$\int_0^\infty dt \exp\left[\int_0^t ds \int (e^{-e^{-\alpha s}|\langle z, x\rangle|}-1)\rho^c(dx)\right] < \infty .$$

This holds with $|\langle z, x\rangle|$ replaced by $|x|$, since $|z| \leqq 1$. Change of variables $u=e^{-s}$ and $v=e^{-t}$ gives (2.5).

Conversely, suppose that (2.5) holds for some (hence all) $c>0$. We will show transience of $\boldsymbol{X}$. For $a>0$ let

$$h_a(x) = \prod_{j=1}^d ((a-|x_j|)\vee 0) .$$

We have

$$\hat{h}_a(z) = \prod_{j=1}^d 4z_j^{-2} \sin^2(2^{-1}az_j) .$$

Notice that $\hat{h}_a(z)$ is bounded from below by a positive constant on the set $\{z : |z_j| \leqq \pi/a \text{ for } j=1, \cdots, d\}$. Hence, in order to prove transience, it suffices to show that

$$\int_0^\infty dt \int p_t(0, dx)\hat{h}_a(x) < \infty$$

for all small a (see Facts 2 and 3). Since

$$\int p_t(0, dx)\hat{h}_a(x) = \int \hat{p}_t(0, z)h_a(z)dz \leqq a^d \int_{|z|<a\sqrt{d}} |\hat{p}_t(0, z)|dz$$

and $|\hat{p}_t(0, z)| \leqq |\hat{p}_t^c(0, z)|$, it is enough to show that

$$\int_0^\infty dt \int_{|z|<1} |\hat{p}_t^c(0, z)|dz < \infty \tag{2.11}$$

for some $c>0$. We have

$$\int_{|z|<1} |\hat{p}_t^c(0, z)|dz = \int_{|z|<1} \exp G_c(t, z)dz$$

$$= \int_{|z|<1} dz \exp\left[\int_0^t ds \int (e^{-e^{-\alpha s}|x|}-1)\rho^c(dx) + H_c(t, z) + I_c(t, z)\right],$$

where

$$I_c(t, z) = \int_0^t ds \int (e^{-e^{-\alpha s}|\langle z, x\rangle|} - e^{-e^{-\alpha s}|x|})\rho^c(dx) .$$

Notice that, for any $0<a\leqq b$,

$$\int_0^1 (e^{-au^\alpha} - e^{-bu^\alpha})\frac{du}{u} = \int_0^1 (e^{-au} - e^{-bu})\frac{du}{\alpha u} \tag{2.12}$$

$$= \int_0^1 \frac{du}{\alpha} \int_a^b e^{-su} ds = \int_a^b (1-e^{-s})\frac{ds}{\alpha s} \leqq \frac{1}{\alpha} \log \frac{b}{a} .$$

Thus

$$|I_c(t, z)| = \left|\int \rho^c(dx)\int_{e^{-t}}^1 (e^{-u^\alpha|\langle z,x\rangle|}-e^{-u^\alpha|x|})\frac{du}{u}\right|$$

$$\leqq \frac{1}{\alpha}\int \log\frac{|x|}{|\langle z, x\rangle|}\rho^c(dx).$$

Choose c so large that $\alpha^{-1}\rho(\{|x|\geqq c\})<1$. Use Lemma 2.2, (2.10), and the assumption (2.5). Then we get (2.11), which completes the proof. □

3. Boundedness of some integrals.

In preparation of the proof of Theorem B we will give two lemmas of analytic nature. Let n be a positive integer. Fix n distinct positive reals $\alpha_1, \cdots, \alpha_n$. Let $a_1, \cdots, a_n$ be real numbers and let $p_1, \cdots, p_n$ be real numbers satisfying $0<|p_j|\leqq 1$ for $1\leqq j\leqq n$. Define

$$F(u)=\sum_{j=1}^n a_j u^{\alpha_j},\quad G(u)=\sum_{j=1}^n p_j a_j u^{\alpha_j},\quad H(u)=\sum_{j=1}^n |a_j| u^{\alpha_j}$$

for $u>0$. Let $0<M<N$. Our lemmas are as follows.

LEMMA 3.1. *There are positive constants K_1 and K_2 independent of M, N, $a_1, \cdots, a_n$ such that*

$$\left|\int_M^N \sin F(u)\frac{du}{u}\right| \leqq K_1, \tag{3.1}$$

$$\left|\int_M^N [\cos F(u)-e^{-|F(u)|}]\frac{du}{u}\right| \leqq K_2. \tag{3.2}$$

LEMMA 3.2. *There are positive constants C_j $(0\leqq j\leqq n)$ independent of M, N, $a_1, \cdots, a_n, p_1, \cdots, p_n$ such that*

$$\int_M^N (e^{-|G(u)|}-e^{-H(u)})\frac{du}{u} \leqq C_0+\sum_{j=1}^n C_j\log\frac{1}{|p_j|}. \tag{3.3}$$

In proving Lemma 3.1 we may assume that $a_j\neq 0$ $(1\leqq j\leqq n)$, because the integrals in (3.1) and (3.2) are continuous in a_j. Further we assume that

$$a_j>0 \quad\text{for } 1\leqq j\leqq l \qquad\text{and}\qquad a_j<0\quad\text{for } l+1\leqq j\leqq n, \tag{3.4}$$

where l is an integer with $0\leqq l\leqq n$. This does not harm generality, as we can rearrange $a_1, \cdots, a_n$. Denote

$$m=n-l \quad\text{and}\quad b_k=-a_{l+k},\quad \beta_k=\alpha_{l+k}\qquad\text{for } 1\leqq k\leqq m. \tag{3.5}$$

Thus

$$F(u)=\sum_{j=1}^{l} a_j u^{\alpha_j}-\sum_{k=1}^{m} b_k u^{\beta_k}.$$

If $l=0$, then understand $\sum_{j=1}^{l}$ to be zero; similarly if $m=0$.

PROOF OF (3.1) IN LEMMA 3.1. Proof is by induction in n. If $n=1$, then (3.1) is evident by (2.8), since

$$\left|\int_M^N \sin a_1 u^{\alpha_1}\frac{du}{u}\right| = \alpha_1^{-1}\left|\int_{M'}^{N'} \sin u\,\frac{du}{u}\right| \quad \text{with } M'=|a_1|M^{\alpha_1},\ N'=|a_1|N^{\alpha_1}.$$

Now let $n\geqq 2$. Assume that (3.1) is true with n replaced by $n-1$. We divide the proof for n into three steps. Define

$$I(u)=F'(u)u=\sum_{j=1}^{l} a_j\alpha_j u^{\alpha_j}-\sum_{k=1}^{m} b_k\beta_k u^{\beta_k},$$

$$J(u)=I'(u)u=\sum_{j=1}^{l} a_j\alpha_j^2 u^{\alpha_j}-\sum_{k=1}^{m} b_k\beta_k^2 u^{\beta_k},$$

$$f_j(u)=a_j\alpha_j u^{\alpha_j} \qquad \text{for } 1\leqq j\leqq l,$$

$$g_k(u)=b_k\beta_k u^{\beta_k} \qquad \text{for } 1\leqq k\leqq m.$$

Let $C=\max_{1\leq j\leq n}|a_j|^{-1/\alpha_j}$. In this proof and in the proof of (3.2) we will denote by $K_3, K_4, \cdots$ positive constants independent of $M, N, a_1, \cdots, a_n$.

First step. Assume that $M<C$. We show that

$$\left|\int_M^C \sin F(u)\frac{du}{u}\right| \leqq K_3. \tag{3.6}$$

Without loss of generality, we can assume that $l\geqq 1$ and $C=a_1^{-1/\alpha_1}$. Define

$$F_*(u)=\sum_{j=2}^{l} a_j u^{\alpha_j}-\sum_{k=1}^{m} b_k u^{\beta_k}.$$

The induction hypothesis says that

$$\left|\int_M^N \sin F_*(u)\frac{du}{u}\right| \leqq K_4.$$

Since $F(u)-F_*(u)=a_1u^{\alpha_1}$,

$$\left|\int_M^C [\sin F(u)-\sin F_*(u)]\frac{du}{u}\right| \leqq \int_0^C a_1 u^{\alpha_1-1}du=\alpha_1^{-1}=K_5,$$

where we use $|\sin x-\sin y|\leqq|x-y|$. Hence we obtain (3.6).

Second step. We prove (3.1) when $l=0$ or $m=0$. We can assume $m=0$ since the other case can be reduced to this case. By virtue of the first step it is enough to prove (3.1) under the assumption that

$$M\geqq C. \tag{3.7}$$

Our basic observation is that integration by parts yields

$$\int_{u_1}^{u_2}\sin F(u)\frac{du}{u}=\left[-(\cos F(u))\frac{1}{I(u)}\right]_{u_1}^{u_2}-\int_{u_1}^{u_2}(\cos F(u))\frac{J(u)}{I(u)^2}\frac{du}{u} \tag{3.8}$$

for $0<u_1<u_2$. Let $\alpha_*=\max_{1\leqq j\leqq n}\alpha_j$. Since $I(u)\geqq f_1(u)$ and $J(u)/I(u)^2\leqq\alpha_*/f_1(u)$, we obtain from (3.7) and (3.8) that

$$\left|\int_M^N\sin F(u)\frac{du}{u}\right|\leqq 2f_1(C)^{-1}+\alpha_*(\alpha_1 f_1(C))^{-1}\leqq 2\alpha_1^{-1}+\alpha_*\alpha_1^{-2}\,.$$

Third step. We prove (3.1) when $l\geqq 1$ and $m\geqq 1$. Again we may assume (3.7). Let

$$\sigma=\begin{pmatrix}1 & 2 & \cdots & l\\ \sigma(1) & \sigma(2) & \cdots & \sigma(l)\end{pmatrix}\quad\text{and}\quad\tau=\begin{pmatrix}1 & 2 & \cdots & m\\ \tau(1) & \tau(2) & \cdots & \tau(m)\end{pmatrix}$$

be permutations and denote the pair of σ and τ by $\lambda=(\sigma,\tau)$. Let Λ be the totality of λ. Thus Λ consists of $(l\,!)(m\,!)$ elements. For each λ let E_λ be the set of all $u>0$ satisfying

$$f_{\sigma(1)}(u)\leqq f_{\sigma(2)}(u)\leqq\cdots\leqq f_{\sigma(l)}(u)$$

and

$$g_{\tau(1)}(u)\leqq g_{\tau(2)}(u)\leqq\cdots\leqq g_{\tau(m)}(u)\,.$$

Since each of the sets $\{u>0: f_{\sigma(j)}(u)\leqq f_{\sigma(j+1)}(u)\}$ and $\{u>0: g_{\tau(k)}(u)\leqq g_{\tau(k+1)}(u)\}$ is an interval (bounded or unbounded), their intersection E_λ is empty set, one point set, or interval. We have $\bigcup_{\lambda\in\Lambda}E_\lambda=(0,\infty)$. Let

$$A_\lambda=\{u>0: 2^{-1}f_{\sigma(l)}(u)\geqq mg_{\tau(m)}(u)\}\,,$$

$$B_\lambda=\{u>0: lf_{\sigma(l)}(u)\leqq 2^{-1}g_{\tau(m)}(u)\}\,.$$

Note that A_λ and B_λ are intervals. Use α_* in the above. Then, for every $u\in E_\lambda\cap A_\lambda$,

$$|J(u)|\leqq\alpha_* nf_{\sigma(l)}(u)\,, \tag{3.9}$$

and

$$|I(u)|\geqq f_{\sigma(l)}(u)-mg_{\tau(m)}(u)\geqq 2^{-1}f_{\sigma(l)}(u)\,. \tag{3.10}$$

Hence we have

$$\frac{|J(u)|}{I(u)^2}\leqq\frac{4n\alpha_*}{f_{\sigma(l)}(u)} \tag{3.11}$$

for $u\in E_\lambda\cap A_\lambda$. Let $[c_1, c_2]=[M, N]\cap E_\lambda\cap A_\lambda$ if it is not empty. Then we obtain from (3.7), (3.8), (3.10), and (3.11) that

(3.12) $$\left|\int_{c_1}^{c_2}\sin F(u)\frac{du}{u}\right| \leqq \frac{4}{f_{\sigma(l)}(C)}+\int_C^\infty \frac{4n\alpha_*}{f_{\sigma(l)}(u)}\frac{du}{u}$$

$$\leqq 4\alpha_{\sigma(l)}^{-1}+4n\alpha_*\alpha_{\sigma(l)}^{-2} = K_6\,,$$

noting that $f_{\sigma(l)}(C)\geqq\alpha_{\sigma(l)}$. Similarly let $[c_3, c_4]=[M, N]\cap E_\lambda\cap B_\lambda$ if it is not empty. We get

(3.13) $$\left|\int_{c_3}^{c_4}\sin F(u)\frac{du}{u}\right| \leqq K_7\,.$$

Let $\theta=|\beta_{\tau(m)}-\alpha_{\sigma(l)}|^{-1}$ and $K_8=(4lm)^\theta$. Let the superscript C denote the complement of a set. We have $A_\lambda^C\cap B_\lambda^C=(c_5, K_8c_5)$, where

$$c_5 = \left(\frac{a_{\sigma(l)}\alpha_{\sigma(l)}}{2mb_{\tau(m)}\beta_{\tau(m)}}\right)^\theta \quad \text{if } \alpha_{\sigma(l)}<\beta_{\tau(m)}\,,$$

$$c_5 = \left(\frac{b_{\tau(m)}\beta_{\tau(m)}}{2la_{\sigma(l)}\alpha_{\sigma(l)}}\right)^\theta \quad \text{if } \alpha_{\sigma(l)}>\beta_{\tau(m)}\,.$$

Let $F_\lambda=[M, N]\cap E_\lambda\cap A_\lambda^C\cap B_\lambda^C$. Since $F_\lambda\subset(c_5, K_8c_5)$,

(3.14) $$\left|\int_{F_\lambda}\sin F(u)\frac{du}{u}\right| \leqq \log K_8\,.$$

It follows from (3.12), (3.13), and (3.14) that

$$\left|\int_{[M,N]\cap E_\lambda}\sin F(u)\frac{du}{u}\right| \leqq K_\lambda\,,$$

where K_λ is a positive constant independent of M, N, $a_1, \cdots, a_n$. Now we get

(3.15) $$\left|\int_M^N\sin F(u)\frac{du}{u}\right| \leqq \sum_{\lambda\in\Lambda}K_\lambda\,,$$

completing the proof of (3.1).

PROOF OF (3.2) IN LEMMA 3.1. Proof is again by induction in n. If $n=1$, then (3.2) is evident by (2.8). Note that

$$\left|\int_M^N(\cos a_1u^{\alpha_1}-e^{-|a_1u^{\alpha_1}|})\frac{du}{u}\right| = \alpha_1^{-1}\left|\int_{M'}^{N'}(\cos u-e^{-u})\frac{du}{u}\right|$$

with some $M'>0$ and $N'>0$. Now let $n\geqq2$ and assume that (3.2) is valid with n replaced by $n-1$. Proof for n is given in three steps as before. Use C defined in the proof of (3.1) again.

First step. We show that, if $M<C$, then

(3.16) $$\left|\int_M^C[\cos F(u)-e^{-|F(u)|}]\frac{du}{u}\right| \leqq K_9\,.$$

As in the proof of (3.6), we can assume that $l\geqq1$ and $C=a_1^{-1/\alpha_1}$. Define $F_*(u)$ in the same way. We find from the induction hypothesis that

$$\left|\int_M^N [\cos F_*(u) - e^{-|F_*(u)|}] \frac{du}{u}\right| \leqq K_{10}\,.$$

On the other hand, we get

$$\left|\int_M^C [\cos F(u) - \cos F_*(u)] \frac{du}{u}\right| \leqq K_{11}\,,$$

$$\left|\int_M^C [e^{-|F(u)|} - e^{-|F_*(u)|}] \frac{du}{u}\right| \leqq K_{12}$$

as before, since $|\cos x - \cos y| \leqq |x-y|$ and $|e^{-|x|} - e^{-|y|}| \leqq |x-y|$. This proves (3.16).

Second step. We prove (3.2) when $l=0$ or $m=0$. By the first step we may assume (3.7). Further, we can assume that $m=0$. As in the proof of (3.1) we see that

$$\left|\int_M^N \cos F(u) \frac{du}{u}\right| \leqq K_{13} \tag{3.17}$$

by using integration by parts. On the other hand, we find from (3.7) that

$$\int_M^N e^{-|F(u)|} \frac{du}{u} \leqq K_{14}\,, \tag{3.18}$$

since the integral is

$$\leqq \int_C^\infty \exp(-a_1 u^{\alpha_1}) \frac{du}{u} \leqq \alpha_1^{-1} \int_1^\infty e^{-v} \frac{dv}{v}\,.$$

Third step. We prove (3.2) when $l \geqq 1$ and $m \geqq 1$. We can assume (3.7). Then we claim that (3.17) and (3.18) hold. The proof of (3.17) is completely analogous to the third step of the proof of (3.1). For the proof of (3.18), we change the definitions of f_j and g_k to $f_j(u) = a_j u^{\alpha_j}$ for $1 \leqq j \leqq l$ and $g_k(u) = b_k u^{\beta_k}$ for $1 \leqq k \leqq m$. Using these new f_j and g_k, we define E_λ, F_λ, A_λ, B_λ and c_1, c_2, c_3, c_4 in the same manner as in the proof of (3.1). We find from (3.7) that

$$\int_{c_1}^{c_2} e^{-|F(u)|} \frac{du}{u} \leqq \int_C^\infty \exp(-2^{-1} f_{\sigma(l)}(u)) \frac{du}{u}$$

$$\leqq \alpha_{\sigma(l)}^{-1} \int_1^\infty e^{-v/2} \frac{dv}{v} = K_{15}$$

noting that $f_{\sigma(l)}(C) \geqq 1$. Similarly we get

$$\int_{c_3}^{c_4} e^{-|F(u)|} \frac{du}{u} \leqq K_{16}\,.$$

Write as $E_\lambda \cap A_\lambda^c \cap B_\lambda^c = (c_6, K_{17} c_6)$ with some c_6 and K_{17} as before. Then, since F_λ is contained in this interval,

$$\int_{F_\lambda} e^{-|F(u)|}\frac{du}{u} \leqq \log K_{17}\,.$$

Combining these estimates, we obtain (3.18) as in (3.15). The proof is complete.

PROOF OF LEMMA 3.2. The lemma is concerning the functions $G(u)$ and $H(u)$. Again we may assume that $a_j \neq 0$ for $1 \leqq j \leqq n$. Moreover we may assume that $0 < p_j \leqq 1$ for $1 \leqq j \leqq n$ by changing the sign of a_j if necessary. As before we assume (3.4) with l being an integer satisfying $0 \leqq l \leqq n$ and use the notations (3.5). Our argument is similar to the proof of Lemma 3.1. We have two steps.

First step. Let us prove (3.3) when $l=0$ or $m=0$. We may assume $m=0$. In this case,

$$\begin{aligned}
&\int_M^N [e^{-|G(u)|}-e^{-H(u)}]\frac{du}{u}\\
&\leqq \int_0^\infty \Big[\exp\Big(-\sum_{j=1}^n p_j a_j u^{\alpha_j}\Big)-\exp\Big(-\sum_{j=1}^n a_j u^{\alpha_j}\Big)\Big]\frac{du}{u}\\
&\leqq \sum_{j=1}^n \int_0^\infty [\exp(-p_j a_j u^{\alpha_j})-\exp(-a_j u^{\alpha_j})]\frac{du}{u} = \sum_{j=1}^n \alpha_j^{-1}\log\frac{1}{p_j},
\end{aligned}$$

which proves (3.3). Here we employed

$$\int_0^\infty (e^{-au^\alpha}-e^{-bu^\alpha})\frac{du}{u} = \frac{1}{\alpha}\log\frac{b}{a} \tag{3.19}$$

for $\alpha > 0$ and $0 < a < b$, shown as in (2.12).

Second step. We prove (3.3) when $l \geqq 1$ and $m \geqq 1$. We change the definitions of f_j and g_k to $f_j(u) = p_j a_j u^{\alpha_j}$ $(1 \leqq j \leqq l)$ and $g_k(u) = p_{l+k} b_k u^{\beta_k}$ $(1 \leqq k \leqq m)$. Using these new f_j and g_k, we define E_λ, F_λ, A_λ, B_λ, and c_1, c_2, c_3, c_4 in the same way as in the proof of (3.1). Notice that, for $u \in E_\lambda \cap A_\lambda$,

$$\begin{aligned}
a_j u^{\alpha_j} &= p_j^{-1} f_j(u) \leqq p_j^{-1} f_{\sigma(l)}(u)\\
b_k u^{\beta_k} &= p_{l+k}^{-1} g_k(u) \leqq p_{l+k}^{-1} f_{\sigma(l)}(u)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{c_1}^{c_2} [e^{-|G(u)|}-e^{-H(u)}]\frac{du}{u}\\
&\leqq \int_0^\infty \Big[\exp(2^{-1}f_{\sigma(l)}(u))-\exp\Big(-\sum_{j=1}^n p_j^{-1} f_{\sigma(l)}(u)\Big)\Big]\frac{du}{u}\\
&= \alpha_{\sigma(l)}^{-1}\log\Big(2\sum_{j=1}^n p_j^{-1}\Big) \leqq \alpha_{\sigma(l)}^{-1}\Big(\log 2n + \sum_{j=1}^n \log\frac{1}{p_j}\Big)
\end{aligned}$$

by (3.19) and by $\sum_{j=1}^n p_j^{-1} \leqq n \prod_{j=1}^n p_j^{-1}$. Similarly we get

$$\int_{c_3}^{c_4}[e^{-|G(u)|}-e^{-H(u)}]\frac{du}{u} \leqq \beta_{\tau(m)}^{-1}\Big(\log 2n+\sum_{j=1}^{n}\log\frac{1}{p_j}\Big).$$

We see as in (3.14) that

$$\int_{F_\lambda}[e^{-|G(u)|}-e^{-H(u)}]\frac{du}{u} \leqq K$$

with a constant K independent of $M, N, a_1, \cdots, a_n, p_1, \cdots, p_n$. Therefore, as in (3.15), we obtain (3.3) when $l \geqq 1$ and $m \geqq 1$. Proof of Lemma 3.2 is complete.

4. Proof of Theorem B.

We will use the following fact. Let ρ and $\alpha_1, \cdots, \alpha_n$ be those in Theorem B.

LEMMA 4.1. *Let $S_1, \cdots, S_n$ be matrices, and U_1, U_2, R be invertible matrices. Let c_1, c_2 be positive reals. Then*

$$\int_0^1\frac{dv}{v}\exp\Big[\int_v^1\frac{du}{u}\int_{|U_1x|\geqq c_1}\Big(\exp\Big(-\sum_{j=1}^{n}u^{\alpha_j}|S_jx|\Big)-1\Big)\rho(dx)\Big]=\infty \tag{4.1}$$

if and only if

$$\int_0^1\frac{dv}{v}\exp\Big[\int_v^1\frac{du}{u}\int_{|U_2x|\geqq c_2}\Big(\exp\Big(-\sum_{j=1}^{n}u^{\alpha_j}|RS_jx|\Big)-1\Big)\rho(dx)\Big]=\infty. \tag{4.2}$$

PROOF. It is enough to show that (4.1) implies (4.2). Assume (4.1). There exist positive reals a, b, a_j, b_j $(j=1, 2)$ such that $a|x|\leqq|Rx|\leqq b|x|$ and $a_j|x| \leqq |U_jx|\leqq b_j|x|$. Without loss of generality, we can assume that $\alpha_1<\alpha_2<\cdots<\alpha_n$. The equality (4.1) remains true with c_1 replaced by any $c>0$, because, for $c<c_1$,

$$\begin{aligned}&\int_0^1\frac{du}{u}\int_{c\leqq|U_1x|<c_1}\Big[1-\exp\Big(-\sum_{j=1}^{n}u^{\alpha_j}|S_jx|\Big)\Big]\rho(dx)\\ &\leqq\int_0^1\frac{du}{u}\int_{c/b_1\leqq|x|<c_1/a_1}[1-\exp(-u^{\alpha_1}|x|s)]\rho(dx)\\ &=\int_{c/b_1\leqq|x|<c_1/a_1}\rho(dx)\int_0^{|x|s}(1-e^{-u})\frac{du}{\alpha_1 u}<\infty,\end{aligned}$$

where $s=\sum_{j=1}^{n}\|S_j\|$. Define

$$f_k(u, x)=\sum_{j=1}^{k-1}u^{\alpha_j}|S_jx|+\sum_{j=k}^{n}bu^{\alpha_j}|S_jx| \qquad \text{for } 1\leqq k\leqq n+1.$$

By using (2.12) we obtain that

$$\int_0^1 \frac{du}{u} \left| \int_{|U_1 x| \geq c} [\exp(-f_k(u, x)) - \exp(-f_{k+1}(u, x))] \rho(dx) \right|$$

$$\leqq \int_0^1 \frac{du}{u} \int_{|U_1 x| \geq c} |\exp(-u^{\alpha_k} b |S_k x|) - \exp(-u^{\alpha_k} |S_k x|)| \rho(dx)$$

$$\leqq \alpha_k^{-1} \log (b \vee b^{-1}) \int_{|U_1 x| \geq c} \rho(dx) < \infty$$

for every k $(1 \leqq k \leqq n)$. Choose $c = c_2 a_1 / b_2$. Then the left-hand side of (4.2) is

$$\geqq \int_0^1 \frac{dv}{v} \exp \left[\int_v^1 \frac{du}{u} \int_{|U_1 x| \geq c} \left(\exp\left(-\sum_{j=1}^n u^{\alpha_j} b |S_j x| \right) - 1 \right) \rho(dx) \right]$$

$$\geqq \mathrm{const} \int_0^1 \frac{dv}{v} \exp \left[\int_v^1 \frac{du}{u} \int_{|U_1 x| \geq c} \left(\exp\left(-\sum_{j=1}^n u^{\alpha_j} |S_j x| \right) - 1 \right) \rho(dx) \right] = \infty .$$

Thus (4.2) holds. □

PROOF OF THEOREM B. The eigenspaces $V_1, \cdots, V_n$ of the eigenvalues $\alpha_1, \cdots, \alpha_n$ are not orthogonal in general. But they are orthogonal if Q is diagonal. The first step of our proof is under the assumption that Q is diagonal. The second step is reduction to the case of diagonal Q.

First step (Q diagonal). We assume that $\alpha_1 < \cdots < \alpha_n$ without loss of generality. The line of proof is the same as that of Theorem A. Suppose that $\boldsymbol{X}$ is transient. We claim that

$$\int_0^1 \frac{dv}{v} \exp \left[\int_v^1 \frac{du}{u} \int_{|x| \geq c} \left(\exp\left(-\sum_{j=1}^n u^{\alpha_j} |T_j x| \right) - 1 \right) \rho(dx) \right] < \infty \tag{4.3}$$

for some $c > 0$. We have

$$\hat{p}_t^c(0, z) = \exp \left[\int_0^t ds \int (\exp(i \langle e^{-sQ} z, x \rangle) - 1) \rho^c(dx) \right],$$

$$\langle e^{-sQ} z, x \rangle = \sum_{j=1}^n e^{-s\alpha_j} \langle T_j z, T_j x \rangle .$$

Without the orthogonality of $V_1, \cdots, V_n$, this expression is not obtained. Let

$$F_c(t, z) = \int_0^t ds \int \sin \langle e^{-sQ} z, x \rangle \rho^c(dx),$$

$$G_c(t, z) = \int_0^t ds \int (\cos \langle e^{-sQ} z, x \rangle - 1) \rho^c(dx),$$

$$H_c(t, z) = \int_0^t ds \int [\cos \langle e^{-sQ} z, x \rangle - \exp(-|\langle e^{-sQ} z, x \rangle|)] \rho^c(dx).$$

Use Lemma 2.1 and use the function $h(x)$ of (2.7). Then, for any $c > 0$, we have

$$\int_0^\infty dt \int \hat{h}(z)(\cos F_c(t, z))(\exp G_c(t, z))dz < \infty .$$

Denote by K_1, K_2 constants which depend only on $\alpha_1, \cdots, \alpha_n$. We have

$$(4.4) \qquad \left|\int_0^t \sin\langle e^{-sQ}z, x\rangle ds\right| = \left|\int_{e^{-t}}^1 \sin\Big(\sum_{j=1}^n u^{\alpha_j}\langle T_j z, T_j x\rangle\Big)\frac{du}{u}\right| \leqq K_1$$

by virtue of Lemma 3.1. Choosing c large enough, we get $\cos F_c(t, z) \leqq 1/\sqrt{2}$ for all t and z. Now, for some z with $0<|z|\leqq 1$, we get

$$(4.5) \qquad \int_0^\infty \exp G_c(t, z)dt < \infty .$$

By the same change of variables as in (4.4) we get

$$\left|\int_0^t [\cos\langle e^{-sQ}z, x\rangle - \exp(-|\langle e^{-sQ}z, x\rangle|)]ds\right| \leqq K_2$$

from Lemma 3.1. Hence $H_c(t, z)$ is bounded in t (and z, too). Now it follows from (4.5) that

$$\int_0^\infty dt \exp\left[\int_0^t ds \int (\exp(-|\langle e^{-sQ}z, x\rangle|)-1)\rho^c(dx)\right] < \infty .$$

This implies (4.3), since $|\langle e^{-sQ}z, x\rangle| \leqq \sum_{j=1}^n e^{-\alpha_j s}|T_j x|$.

Conversely, suppose that (4.3) holds for some (hence all) $c>0$. We claim that $\boldsymbol{X}$ is transient. To show this, it is enough to exhibit (2.11) for some $c>0$. Let us use $H_c(t, z)$ introduced above and

$$I_c(t, z) = \int_0^t ds \int \Big[\exp(-|\langle e^{-sQ}z, x\rangle|) - \exp\Big(-\sum_{j=1}^n e^{-\alpha_j s}|T_j x|\Big)\Big]\rho^c(dx).$$

We have

$$(4.6) \qquad \int_{|z|<1} |\hat{p}_t^c(0, z)|dz$$

$$= \int_{|z|<1} dz \exp\left[\int_0^t ds \int \Big(\exp\Big(-\sum_{j=1}^n e^{-\alpha_j s}|T_j x|\Big)-1\Big)\rho^c(dx) + H_c(t, z) + I_c(t, z)\right].$$

Now $H_c(t, z)$ is bounded in t, z as explained above. By Lemma 3.2

$$I_c(t, z) = \int \rho^c(dx) \int_{e^{-t}}^1 \Big[\exp\Big(-\Big|\sum_{j=1}^n u^{\alpha_j}\langle T_j z, T_j x\rangle\Big|\Big) - \exp\Big(-\sum_{j=1}^n u^{\alpha_j}|T_j x|\Big)\Big]\frac{du}{u}$$

$$\leqq \int\Big(C_0 + \sum_{j=1}^n C_j \log\frac{|T_j x|}{|\langle T_j z, T_j x\rangle|}\Big)\rho^c(dx).$$

Choose c so large that the total mass A of ρ^c satisfies $AC_j<1$ for $j=1, \cdots, n$. Denote the Lebesgue measure on V_j by $dz^{(j)}$. Then

$$\int_{|z|<1}\exp(\sup_t I_c(t, z))dz \leqq e^{AC_0}\prod_{j=1}^{n}\int_{|z^{(j)}|<1}\exp\Big(C_j\int\log\frac{|T_jx|}{|\langle z^{(j)}, T_jx\rangle|}\rho^c(dx)\Big)dz^{(j)},$$

which is finite by virtue of Lemma 2.2. Now (4.6) yields

$$\int_0^\infty dt\int_{|z|<1}|\hat{p}_t^c(0, z)|dz$$

$$\leqq \text{const}\int_0^\infty dt\exp\Big[\int_0^t ds\int\Big(\exp\Big(-\sum_{j=1}^{n}e^{-\alpha_j s}|T_jx|\Big)-1\Big)\rho^c(dx)\Big],$$

which is finite by the assumption (4.3).

Second step. We reduce the general case to the case of the first step. Since we have assumed that $\boldsymbol{R}^d=V_1\oplus\cdots\oplus V_n$, there exists an invertible matrix R such that $RQR^{-1}=D$ is diagonal. The diagonal entries of D consist of $\alpha_1, \cdots, \alpha_n$ each with multiplicity $\geqq 1$. The equation (1.2) is equivalent to

$$RX_t = Rx+RZ_t-\int_0^t DRX_s ds\,.$$

It follows that the process RX defined by $\{RX_t\}$ is the process of Ornstein-Uhlenbeck type associated with the Lévy process $\{RZ_t\}$ and the matrix D. The Lévy measure of the process $\{RZ_t\}$ is ρR^{-1}, where $(\rho R^{-1})(E)=\rho(R^{-1}E)$. The process RX is recurrent if and only if X is recurrent. For $j=1, \cdots, n$ denote by $\tilde{V}_j$ the eigenspace of α_j for D, and by $\tilde{T}_j$ the projector onto $\tilde{V}_j$ in the orthogonal decomposition $\boldsymbol{R}^d=\tilde{V}_1\oplus\cdots\oplus\tilde{V}_n$. We have $\tilde{V}_j=RV_j$ and $\tilde{T}_j=RT_jR^{-1}$. We know that RX is recurrent if and only if

$$\int_0^1\frac{dv}{v}\exp\Big[\int_v^1\frac{du}{u}\int_{|x|\geq c}\Big(\exp\Big(-\sum_{j=1}^{n}u^{\alpha_j}|\tilde{T}_jx|\Big)-1\Big)\rho R^{-1}(dx)\Big]=\infty\,,$$

that is,

$$\int_0^1\frac{dv}{v}\exp\Big[\int_0^1\frac{du}{u}\int_{|Rx|\geq c}\Big(\exp\Big(-\sum_{j=1}^{n}u^{\alpha_j}|RT_jx|\Big)-1\Big)\rho(dx)\Big]=\infty\,.$$

By Lemma 4.1 this condition is equivalent to (1.6). Proof of Theorem B is complete. □

5. Example.

Let us give an example of Theorem A. Let X be a process of Ornstein-Uhlenbeck type on $\boldsymbol{R}^d$ generated by (1.1) with $Q=\alpha I$, $\alpha>0$, and Lévy measure ρ. Suppose that there exist $\gamma>0$, $c>0$, and $b>1$ such that, for every Borel set E in $[b, \infty)$,

$$\int 1_E(|x|)\rho(dx) = c\int_E\frac{dr}{r(\log r)^{\gamma+1}}\,.$$

Then,

(i) *X has a limit distribution if and only if $\gamma>1$.*

(ii) *If $\gamma<1$, then X is transient.*

(iii) *If $\gamma=1$ and $c\leqq\alpha$, then X is recurrent.*

(iv) *If $\gamma=1$ and $c>\alpha$, then X is transient.*

Symmetric one-dimensional case of this example was first treated in Sato and Yamazato [**10**]. Shiga [**12**] handled this example in multidimensional case under the condition that ρ is symmetric and concentrated to the coordinate axes. At the end of his paper [**12**] he made an interesting observation concerning recurrence of the projected processes in case $\gamma=1$.

Proof of (i)-(iv) is as follows. Assertion (i) is evident from Fact 4. To see (ii)-(iv), note that

$$\int_0^1 \frac{dv}{v} \exp\left[\int_v^1 \frac{du}{u}\int_{|x|\geqq b}(e^{-u^{\alpha}|x|}-1)\rho(dx)\right] \tag{5.1}$$

$$= \int_0^1 \frac{dv}{\alpha v}\exp\left[-\frac{c}{\alpha}\int_v^1\frac{du}{u}\int_b^\infty \frac{1-e^{-ur}}{r(\log r)^{\gamma+1}}dr\right].$$

We have

$$\int_v^1 \frac{du}{u}\int_b^\infty \frac{1-e^{-ur}}{r(\log r)^{\gamma+1}}dr = \int_v^1 du\int_b^\infty \frac{dr}{r(\log r)^{\gamma+1}}\int_0^r e^{-us}ds$$

$$=\int_v^1 du\int_0^\infty \frac{e^{-us}ds}{\gamma(\log(s\vee b))^\gamma} = \frac{1}{\gamma}\int_v^1 du\int_b^\infty \frac{e^{-us}ds}{(\log s)^\gamma}+O(1)$$

as $v\downarrow 0$ and, for each $A>0$,

$$\int_v^1 du\int_b^\infty \frac{e^{-us}ds}{(\log s)^\gamma} = \int_b^\infty \frac{e^{-vs}ds}{s(\log s)^\gamma}+\text{const} = \int_b^{A/v}\frac{e^{-vs}}{s(\log s)^\gamma}ds+O(1)\,.$$

If $\gamma<1$, then, letting $A=1$, we have

$$\int_v^1 du\int_b^\infty \frac{e^{-us}ds}{(\log s)^\gamma} \geqq \frac{e^{-1}}{1-\gamma}\left(\log\frac{1}{v}\right)^{1-\gamma}+O(1)$$

and see that the right-hand side of (5.1) is finite, using $(\log(1/v))^{1-\gamma}/\log\log(1/v)\to\infty$. This establishes Assertion (ii). For $\gamma=1$ we get

$$e^{-A}\log\log\frac{A}{v}+O(1)\leqq \int_v^1 du\int_b^\infty \frac{e^{-us}ds}{\log s}\leqq \log\log\frac{A}{v}+O(1)$$

and hence

$$(1+o(1))\log\log\frac{1}{v}\leqq\int_v^1\frac{du}{u}\int_b^\infty\frac{1-e^{-ur}}{r(\log r)^2}dr\leqq\log\log\frac{1}{v}+O(1)\quad\text{as } v\downarrow 0\,. \tag{5.2}$$

Now it follows from (5.2) that, if $\gamma=1$, then the right-hand side of (5.1) is infinite for $c/\alpha\leqq 1$ and finite for $c/\alpha>1$. Thus Assertions (iii) and (iv) are true.

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