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Some conformal properties of p-harmonic maps and a regularity for sphere-valued p-harmonic maps

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1. Introduction.

Let $u: M\rightarrow N$ be a smooth map between Riemannian manifolds and p a real number $1<p<\infty$. We call u a p-harmonic map if it is a critical point of the p-energy functional $\int_{\Omega}|du(x)|^{p}dv_{g}$ for every compact domain $\Omega\subset M$. Since the p-energy functional is a natural generalization of the energy functional ($p=2$) for a harmonic map, it is an important problem to study the difference between p-harmonic maps ($p\neq 2$) and harmonic maps. In this paper we shall focus our study on conformal properties of p-harmonic maps and the regularity for sphere-valued p-harmonic maps.

Our main results are as follows. In Section 3, we show that for $p'\neq p$, $p\neq\dim M$, any p'-harmonic map becomes a p-harmonic one by some conformal change of a given metric on M. We also discuss their stability under this conformal change. In Section 4, we investigate p-harmonic conformal maps and, in particular, show relations between the mean curvature vectors and p-tension fields of these maps. Based on this observation we prove that if $\dim M=p$ and $\dim M<\dim N$, then a conformal map u is p-harmonic if and only if $u(M)$ is a minimal submanifold in N (Corollary 4). If $\dim M=p$ and $\dim M>\dim N$, then the fibres of p-harmonic horizontal conformal maps are minimal submanifolds in M (Proposition 7).

In Section 5, we discuss the regularity for sphere-valued weakly p-harmonic maps which are not necessarily minimum. Helein [**11**] has shown that any weakly harmonic map from a two-dimensional surface into a sphere is smooth. Evans [**7**] generalized this to higher dimensions. We prove a regularity theorem similar to the one of Evans for weakly p-harmonic maps ($p\geqq 2$) into a sphere. Namely, if U is a smooth open subset in $\boldsymbol{R}^{m}$ and S^{n-1} is the unit sphere in $\boldsymbol{R}^{n}$, then a weakly p-harmonic map from U into S^{n-1} is locally Hölder continuous on $\Omega\setminus\mathcal{S}_{u}$ for some compact set $\mathcal{S}_{u}$ whose $(m-p)$-dimensional Hausdorff measure is 0. In particular, in the case $m=p$, p-harmonic maps are

Hölder continuous on Ω everywhere (Corollary 12).

There have been several papers about p-harmonic maps. Hardt and Lin [**10**], Luckhaus [**12**], and Fusco and Hutchinson [**8**] discussed regularity of minimizing p-harmonic maps. Roughly speaking, they proved that a minimizer u of M to N is locally Hölder continuous on $M\setminus\mathcal{S}$ for some compact set $\mathcal{S}$. Duzaar and Fuchs [**4**] proved an existence theorem of p-harmonic maps, which extends a theorem of Eells and Sampson [**6**] for harmonic maps. Coron and Gulliver [**3**] have investigated minimizing p-harmonic maps from a Euclidian ball to a sphere. The stability and Liouville type properties of p-harmonic maps have been discussed in [**16**].

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2. Preliminaries.

Let (M, g) be a Riemannian manifold of dimension m and (N, h) a complete Riemannian manifold of dimension n. Let Ω be a bounded domain in M. For a number $1<p<\infty$ and a smooth map $u: M\to N$, we define a p-energy functional $E_p(u)$ of u on Ω by

$$E_p(u)=\int_\Omega |du(x)|^p dv_g\,,$$

where $|du(x)|$ is the norm of the differential $du(x)$ of u at $x\in\Omega$ and dv_g stands for the volume element of M. We denote by ∇ and ${}^N\nabla$ the Levi-Civita connections of M and N respectively. Let $\overline{\nabla}$ be the induced connection on the induced bundle $u^{-1}TN$. For an orthonormal frame field $\{e_i\}_{i=1}^m$ with respect to g on M, the p-energy density $|du|_g^p$ is given by

$$|du|^p=|du|_g^p=\Big(\sum_{i=1}^m \langle due_i, due_i\rangle\Big)^{p/2},$$

where $\langle\cdot,\cdot\rangle=h(\cdot,\cdot)$. When there is no confusion, we shall often drop the subscript g. We call u a *p-harmonic map* if it is a critical point of the p-energy functional for every compact domain $\Omega\subset M$. We denote the p-tension field $\tau_p(u)$ of u by

$$\tau_p(u)=\sum_{i=1}^m \{\overline{\nabla}_{e_i}(|du|^{p-2}due_i)-|du|^{p-2}du(\nabla_{e_i}e_i)\} \tag{1}$$

$$=\sum_{i=1}^m (\overline{\nabla}_{e_i}(|du|^{p-2}du))(e_i).$$

(The first variational formula.) Let u_t be a one parameter family of maps $u_t: \Omega \to N$ with $u_0 = u$ and $du_t/dt|_{t=0} = V$, V being a given vector field along u. Then we have

$$\frac{d}{dt}E_p(u_t)|_{t=0} = -p\int_\Omega \langle V, \tau_p(u)\rangle dv_g .$$

Therefore a smooth map $u: M \to N$ is a p-harmonic map if and only if the p-tension field $\tau_p(u) = 0$. It should be noted that if the p-energy density $|du(x)|^p$ is constant, then the notion of p-harmonic maps coincides with that of harmonic maps. For example, the p-energy density for an isometric immersion $u: M \to N$ is $|du|^p = m^{p/2}$.

(The second variational formula.) Let $u: M \to N$ be a p-harmonic map. We consider a one parameter family of maps u_t as above. For a compact domain $\Omega \subset M$, we have

$$
\begin{aligned}
(2) \qquad I_p(V, V) &= \frac{d^2}{dt^2}E_p(u_t)|_{t=0} \\
&= p(p-2)\int_\Omega |du|^{p-4} \sum_{i=1}^{m} \langle \nabla_{e_i} V, due_i\rangle^2 dv_g \\
&\quad + p\int_\Omega |du|^{p-2} \sum_{i=1}^{m} \{|\nabla_{e_i} V|^2 - \langle {}^N R(V, due_i)due_i, V\rangle\} dv_g ,
\end{aligned}
$$

where ${}^N R$ is the curvature tensor of manifold N. A p-harmonic map is called *stable* if $I_p(V, V) \geqq 0$ for any vector field V along u and every compact domain $\Omega \subset M$. We call *unstable* otherwise.

(The Weitzenböck type formula.) We have the following formula by a direct calculation. Let u be a p-harmonic map from M to N. Then

$$
\begin{aligned}
(3) \qquad \frac{1}{p}\Delta |du|^p &= \operatorname{div}(\omega^\#) + (p-2)|du|^{p-4} \sum_{k=1}^{m} \Big(\sum_i \langle \nabla_{e_k} due_i, due_i\rangle\Big)^2 \\
&\quad + |du|^{p-2}\Big\{|\nabla du|^2 + \sum_k \langle du({}^M Ric(e_k)), due_k\rangle \\
&\quad - \sum_{i,j} \langle {}^N R(due_i, due_j)due_j, due_i\rangle\Big\} ,
\end{aligned}
$$

where $\omega^\#$ is a vector field on M defined by

$$g(\omega^\#, X) = \omega(X) = |du|^{p-2} \sum_{k=1}^{m} \langle (\nabla_{e_k} du)(e_k), duX\rangle$$

for any vector field X on M. We denote by ${}^M Ric$ the Ricci curvature of M, and by div the divergence.

If M is a compact Riemannian manifold without boundary and has nonnegative Ricci curvature, and if a complete Riemannian manifold N has non-positive sectional curvature, then any p-harmonic map u of M to N is totally geodesic. Thus $|du|^p$ is constant. As a result, all p-harmonic maps become harmonic maps.

(Examples of p-harmonic maps.)

(1) p-harmonic functions ($p\neq 1$).

Let $M=\boldsymbol{R}^m\setminus\{0\}$, $N=\boldsymbol{R}$. We suppose that u depends only on $r=|x|$, $x\in\boldsymbol{R}^m$. Then u is a p-harmonic function if $u(r)=Cr^{(p-m)/(p-1)}(m\neq p)$, $u(r)=C\log r(m=p)$, where C is a constant independent of r.

(2) Equator maps.

Let $M=B^m\setminus\{0\}\subset\boldsymbol{R}^m$ be the punctured unit ball and $N=S^n\subset\boldsymbol{R}^{n+1}$ the standard unit sphere. Define $u: M\to N$ as follows. When $m-1>n$, $u(y, z)=y/|y|$, for $y\in\boldsymbol{R}^{n+1}$, $z\in\boldsymbol{R}^{m-n-1}$. When $m-1=n$, $u(y)=y/|y|$, for $y\in\boldsymbol{R}^m$. When $m-1<n$, $u(y)=(y/|y|, 0)$, for $y\in\boldsymbol{R}^m$, $0\in\boldsymbol{R}^{n-m+1}$. Then the map u is p-harmonic.

3. Deformation of p-harmonic maps.

Throughout this section we assume that u is a smooth map between Riemannian manifolds M and N.

PROPOSITION 1. *Let (M, g) and (N, h) be two Riemannian manifolds and* $\dim M=m$. *For $1<p'<\infty$, let $u:(M, g)\to(N, h)$ be a p'-harmonic map and suppose $p\neq m$. Define a new Riemannian metric $\tilde{g}$ on M by $\tilde{g}=|du|_g^{2(p'-p)/(m-p)}g$. Then the map $u:(M_+, \tilde{g})\to(N, h)$ is a p-harmonic map, where $M_+=\{x\in M;\ |du(x)|>0\}$.*

REMARK. Proposition 1 holds for all p if $p'=m$.

PROOF. Deform g conformally to $\tilde{g}=fg$, where f is a positive function on M_+. Then the p-energy functional of u with respect to $\tilde{g}$ is given by

$$
\begin{aligned}
E_p(u) &= \int_{M_+}|du|_{\tilde{g}}^p dv_{\tilde{g}} \\
&= \int_{M_+}|du|_g^p f^{(m-p)/2} dv_g \\
&= \int_{M_+}|du|_g^{p'} dv_g \qquad (\text{since } p\neq m) \\
&= E_{p'}(u)\,.
\end{aligned}
\tag{4}
$$

Thus if u is p'-harmonic with respect to g, then u is p-harmonic with respect to $\tilde{g}$. ∎

Furthermore we have the following.

PROPOSITION 2. *Under the same hypotheses as in Proposition* 1, *the following hold*:

1. *If* $p \geqq p'$ *and* u *is stable as* p'*-harmonic map, then* u *is also stable as* p*-harmonic map.*

2. *If* $p < p'$ *and* u *is unstable as* p'*-harmonic map, then* u *is also unstable as* p*-harmonic map.*

PROOF. Set $\tilde{g} = fg$ in Proposition 1. Then it follows from the second variational formula (2) with respect to $\tilde{g}$ that

$$(5)\qquad I_p(V, V) \geqq p \cdot \inf_{\Omega} f^{(m-p)/2}\Big[(p-p')\int_{\Omega} |du|_g^{p-4}\Big(\sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i} V,\ du e_i\rangle^2\Big) dv_g + (p'-2)\int_{\Omega} |du|_g^{p-4}\Big(\sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i} V,\ du e_i\rangle^2\Big) dv_g + \int_{\Omega} |du|_g^{p-2} \sum_{i=1}^{m} \{|\tilde{\nabla}_{e_i} V|^2 - \langle {}^N R(V, du e_i) du e_i,\ V\rangle\}\, dv_g\Big],$$

where $\{e_i\}$ is an orthonormal frame field with respect to g. The last two terms in the right hand side in (5) coincide with $I_{p'}(V, V)$. Therefore the hypothesis in 1, i.e. $p - p' \geqq 0$ and the $I_{p'}(V, V) \geqq 0$, implies $I_p(V, V) \geqq 0$. Replacing inf into sup, we get the upper estimate of $I_p(V, V)$ contrary to (5). This completes the proof.

4. Properties of p-harmonic conformal maps.

We first study p-harmonic conformal maps.

PROPOSITION 3. *Let* u *be a conformal immersion from an* m *dimensional Riemannian manifold* (M, g) *to an* n *dimensional Riemannian manifold* (N, h), *i.e.* $u^*h = \sigma^2 g$, *where* σ *is a positive function on* M. *Then the* p*-tension field of* u *is*

$$\tau_p(u) = |du|_g^{p-2}\{m\sigma^2 H + (p-m) du(\mathrm{grad}(\log \sigma))\},$$

where H *is the mean curvature vector with respect to the metric induced by* u *and* grad *denotes the gradient.*

From this proposition, we easily obtain the following corollary.

COROLLARY 4. *Let* M, N *and* u *be as in Proposition* 3. *When* $m = p$, *a conformal immersion* u *from* M *to* N *is* p*-harmonic if and only if* $u(M)$ *is a minimal submanifold in* N. *When* $m \neq p$, *a conformal immersion* u *is* p*-harmonic if and only if* $u(M)$ *is a minimal submanifold in* N *and* u *is homothetic. When* $m = n = p$, *a conformal immersion is always p-harmonic.*

PROOF OF PROPOSITION 3. Let u be a conformal immersion from (M, g) to (N, h). We denote by $\tilde{\nabla}$ the Levi-Civita connection of (M, u^*h) and by $B(\cdot, \cdot)$ the second fundamental form of (M, u^*h) in (N, h). Then the 2-tension field of u is given by

$$\tau_2(u) = \sum_{i=1}^{m} \{{}^N\nabla_{due_i}(due_i) - du(\nabla_{e_i}e_i)\} \tag{6}$$

$$= \sum_{i=1}^{m} \{B(e_i, e_i) + du(\tilde{\nabla}_{e_i}e_i - \nabla_{e_i}e_i)\}$$

$$= m\sigma^2 H + (2-m)du(\mathrm{grad}(\log\sigma)),$$

where $\{e_i\}$ is an orthonormal frame field with respect to g. From the conformality of u we get $|du|^2 = m\sigma^2$. Therefore the p-tension field $\tau_p(u)$ of u is given by

$$\tau_p(u) = |du|_g^{p-2}\{m\sigma^2 H + (2-m)du(\mathrm{grad}(\log\sigma)) \tag{7}$$

$$+(p-2)/2(du(\mathrm{grad}(\log|du|^2)))\}$$

$$= |du|_g^{p-2}\{m\sigma^2 H + (p-m)du(\mathrm{grad}(\log\sigma))\}. \quad \square$$

Next we define *the stress p-energy tensor* $S_p(u)$ of u by

$$S_p(u) = \frac{1}{2}|du|^p g - |du|^{p-2}u^*h,$$

which is a symmetric 2-tensor on M. We then define the divergence of $S_p(u)$ by

$$(\mathrm{div}\, S_p(u))(\cdot) = \sum_{i=1}^{m}((\nabla_{e_i}S_p(u))(e_i, \cdot)).$$

The relation between the stress p-energy tensor and the p-harmonic map is given by following

PROPOSITION 5. *Let $u:(M, g)\to(N, h)$ be a smooth map. For any vector field X on M, we have*

$$\int_M \langle\tau_p(u), duX\rangle dv_g = \int_M \langle X, \mathrm{div}\, S_p(u)\rangle dv_g.$$

PROOF. We prove this by modifying the method of Baird and Eells [**2**], [**5**]. Computing the Lie derivative along any vector field X on M, we get

$$L_X(|du|^p dv_g) = di(X)(|du|^p dv_g), \tag{8}$$

where $i(X)$ denotes the interior product by X.

On the other hand we compute

$$
\begin{aligned}
(9)\qquad & L_X((1/p)|du|^p dv_g) \\
&= L_X((1/p)|du|^p)dv_g+(1/p)|du|^p L_X(dv_g) \\
&= \{\langle\nabla(duX),\ du\rangle-1/2\langle L_Xg,\ u^*h\rangle\}dv_g+(1/2p)|du|^p\langle L_Xg,\ g\rangle dv_g \\
&= |du|^{p-2}\langle du,\ \nabla(duX)\rangle dv_g+1/2\langle L_Xg,\ S_p(u)\rangle dv_g\,.
\end{aligned}
$$

From (8) and (9), we obtain

$$
\begin{aligned}
& 0=\int_M L_X((1/p)|du|^p dv_g) \\
(10)\qquad & =\int_M |du|^{p-2}\langle du,\ \nabla(duX)\rangle dv_g+\int_M\langle\nabla X,\ S_p(u)\rangle dv_g\,. \quad \square
\end{aligned}
$$

We have the following corollary of Proposition 5.

COROLLARY 6. *Let* $u:(M, g)\to(N, h)$ *be a smooth map. If* u *is p-harmonic, then* $\operatorname{div} S_p(u)=0$. *Conversely, if* u *is a submersion, i.e.* $\operatorname{rank}(du)=n$, *and* $\operatorname{div} S_p(u)=0$, *then* u *is p-harmonic.*

Next we study the case where $\dim M=m$ is greater than $\dim N=n$. For each $x\in M$ satisfying $du(x)\neq 0$, we decompose M into $V_x=\ker du(x)$ and $H_x=$ the orthogonal complement with respect to g. We call V_x the vertical space at x, and H_x the horizontal space at x. The map u is said to be horizontal conformal if for $x\in M$, $du(x)\neq 0$, $du(x): H_x\to T_{u(x)}N$ is conformal and surjective. That is, $u^*h|_{H_x\times H_x}=\lambda^2\cdot g|_{H_x\times H_x}$ for some positive function λ, which is called the dilation of u. We have the following properties of p-harmonic and horizontal conformal maps.

PROPOSITION 7. *Let* (M, g), (N, h) *be Riemannian manifolds of dimension* m, n *respectively. Suppose* $m>n$. *Let* $u: M\to N$ *be a p-harmonic and horizontal conformal map. Then:*

(a) *If* $n=p$, *all the fibers are minimal submanifolds.*

(b) *If* $n\neq p$, *the following properties are equivalent.*

  (1) *All the fibers are minimal submanifolds.*

  (2) $\operatorname{grad}(\lambda^p)$ *is vertical.*

  (3) *The horizontal integral manifold has the mean curvature* $\{\operatorname{grad}(\lambda^p)/(p\lambda^p)\}$ *as a submanifold of* M.

PROOF. Since u is horizontal conformal, $|du|^2=n\lambda^2$. Thus the stress p-energy tensor is given by

$$
S_p(u)=(1/p)n^{p/2}\lambda^p g-n^{(p-2)/2}\lambda^{p-2}u^*h\,.
$$

Take a point $x_0 \in M$ and an orthonormal frame field $\{e_a\}_{1\leq a\leq m}$ such that $\{e_i\}_{1\leq i\leq n}$ are horizontal and $\{e_r\}_{n+1\leq r\leq m}$ are vertical. Since u is p-harmonic, $\operatorname{div} S_p(u)=0$. We get, for any $1\leqq b\leqq m$,

$$0=\sum_a(\nabla_{e_a}S_p(u))(e_a, e_b) \tag{11}$$

$$=\sum_a\Big[\frac{1}{p}(\nabla_{e_a})g(e_b, e_a)-\{e_a(|du|^{p-2}u^*h(e_b, e_a))$$

$$-|du|^{p-2}u^*h(\nabla_{e_a}e_b, e_a)-|du|^{p-2}u^*h(e_b, \nabla_{e_a}e_a)\}\Big].$$

We choose $e_b=e_j$ $(1\leqq j\leqq n)$. From $u^*h(\cdot, e_r)=0$, we have

$$\frac{1}{p}\nabla_{e_j}(n^{p/2}\lambda^p)-n^{(p/2-1)}\nabla_{e_j}\lambda^p \tag{12}$$

$$+\sum_i\{u^*h(\nabla_{e_i}e_j, e_i)+u^*h(e_j, \nabla_{e_i}e_i)\}+\sum_r u^*h(e_j, \nabla_{e_r}e_r).$$

$$=n^{p/2-1}\Big\{\frac{(n-p)}{p}\nabla_{e_j}\lambda^p+\lambda^p(m-n)H(e_j)\Big\}=0,$$

where $H(e_j)$ denotes the mean curvature of the fibre in the e_j direction. From this formula, we get (a). Because, if $n=p$, we have $H(e_j)=0$ for $1\leqq j\leqq n$ (i.e. the fibers through x_0 are minimal). For (b), if $n\neq p$, then the fibers through x_0 are minimal if and only if $\nabla_{e_j}\lambda^p=0$, that is, if and only if grad λ^p is vertical. This proves (1)↔(2).

Choose $e_b=e_r$ $(n+1\leqq r\leqq m)$ in (11). Then the equation (11) becomes

$$0=n^{p/2}\{(1/p)e_r(\lambda^p)-\lambda^p H(e_r)\}.$$

If grad λ^p is vertical, i.e., grad $\lambda^p=\sum_{r=n+1}^m(\nabla_{e_r}\lambda^p)e_r$, we conclude

$$H(e_r)=\frac{e_r(\lambda^p)}{p\lambda^p}=\frac{\operatorname{grad}\lambda^p}{p\lambda^p}\cdot e_r,$$

which proves (2)→(3). We recall the definition of the mean curvature vector of the horizontal distribution, namely, $H(e_r)=(1/n)(\sum_{i=1}^n\langle\nabla_{e_i}e_i, e_r\rangle)$. By the assumption of (3) of (b) in Proposition 7, $\nabla_{e_r}\lambda^p=(p\lambda^p/n)\sum_{i=1}^m\langle\nabla_{e_i}e_i, e_r\rangle$. Since e_r is a vertical vector, $\nabla_{e_r}\lambda^p$ has vertical component. This implies (3)→(2). □

PROPOSITION 8. *Let u be a p-harmonic and horizontal conformal map from M to N. Then for an open set $V\subset N$, if f is a p-harmonic function on V, then the function $f\circ u$ is a p-harmonic function on $u^{-1}(V)$.*

To show this, we need the following composition law.

LEMMA 9. *Let (M, g), (N, h) and (Q, k) be three Riemannian manifolds. Let $\phi: M\to N$ be a smooth map satisfying $|d\phi|^2\neq 0$, and $\psi: N\to Q$ a smooth map.*

Then

(13) $$\tau_p(\psi\circ\phi) = \theta^{p-2}d\psi(\tau_p(\phi)) + |d\phi|^{p-2}\{\theta^{p-2}\,\mathrm{trace}\,(\nabla d\psi)(d\phi,\, d\phi)+d(\psi\circ\phi)(\mathrm{grad}\,\theta^{p-2})\},$$

where $\theta=|d(\psi\circ\phi)|/|d\phi|$.

PROOF OF PROPOSITION 8. Since u is a conformal map with dilation λ, we have $|du|^2=n\lambda^2$, and $|d(f\circ u)|^2=\lambda^2|df|^2$. By the p-harmonicity of u and the composition law above, we get

$$\tau_p(f\circ u) = \lambda^p\tau_p(f).$$

Thus, if f is p-harmonic function, then $\tau_p(f\circ u)=0$. □

5. A regularity for sphere-valued p-harmonic maps.

In this section, we prove a regularity result for a p-harmonic map into sphere by modifying the method of Evans [7]. Suppose m, n and $p\geqq 2$. Let Ω be a smooth open subset of the Euclidean space $\boldsymbol{R}^m$ and S^{n-1} the unit sphere in $\boldsymbol{R}^n$. Let $L^{1,p}(\Omega, \boldsymbol{R}^n)$ be the Sobolev space of all functions u such that u and their first derivative ∇u belong to $L^p(\Omega, \boldsymbol{R}^n)$. We define

$$L^{1,p}(\Omega, S^{n-1}) = \{u\in L^{1,p}(\Omega, \boldsymbol{R}^n)\,;\, u(x)\in S^{n-1} \text{ a.e. on } \Omega\}.$$

Let $C_0^\infty(\Omega, \boldsymbol{R}^n)$ be the set of all $\boldsymbol{R}^n$-valued smooth functions with compact supports in Ω. When $u\in L^{1,p}(\Omega, S^{n-1})$ is a weak solution of the Euler-Lagrange equation associated to $E_p(u)$, we call *u a weakly p-harmonic map* of Ω to S^{n-1}. That is, u satisfies the following equation

(14) $$\int_\Omega |Du|^{p-2}Du\cdot Dw\,dx = \int_\Omega |Du|^p u\cdot w\,dx$$

for each test function $w\in C_0^\infty(\Omega, \boldsymbol{R}^n)$. Here $Du=du$ and $Du\cdot Dw=du\cdot dw$ are

$$Du = \left(\left(\frac{\partial u^\alpha}{\partial x_i}\right)\right)_{1\leq i\leq m;1\leq\alpha\leq n}, \qquad Du\cdot Dw = \sum_{i,\alpha}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i}.$$

In addition, if u is a critical point of $E_p(u)$ with respect to compact variations of a parameter domain Ω, we call *u a weakly stationary p-harmonic map.* The Hölder continuity of the weak solution is trivial in the case $p>m$ because of the Sobolev imbedding theorem. Set

$$\tilde{E}_p(u, B(x, r)) = r^{p-m}\int_{B(x,r)}|Du|^p dx,$$

where $B(x, r)=\{y\in\Omega\,;\,|y-x|\leqq r\}$ is the ball in $\boldsymbol{R}^m$, centered at x with radius r.

PROPOSITION 10. *Let N be a Riemannian submanifold in $\boldsymbol{R}^n$. Suppose u is a weakly stationary p-harmonic map from $B(x, r)\subset \boldsymbol{R}^m$ into $N\subset \boldsymbol{R}^n$ and $p\leqq m$. For $x\in\Omega$ and $0<\sigma<\rho<\operatorname{dist}(x, \partial\Omega)$, we have*

$$(15)\qquad e^{\Lambda\rho}\rho^{p-m}E_p(u, B(x, \rho))-e^{\Lambda\sigma}\sigma^{p-m}E_p(u, B(x, \sigma))$$
$$\geqq p\int_{B(x,\rho)\backslash B(x,\sigma)} e^{\Lambda r}|Du|^{p-2}|\partial u/\partial r|^2 r^{p-m}dx\,,$$

where Λ is a constant, r is the radial coordinate on $B(x, \rho)$, and $\partial u/\partial r$ is the radial derivative. In particular, we have the following monotonicity formula:

$$(16)\qquad \tilde{E}_p(x, B(x, \rho))\geqq \tilde{E}_p(u, B(x, \sigma))\,.$$

PROOF. Let $\varphi_t: \Omega\to\Omega$ a one parameter family of diffeomorphisms which are $\varphi_0=$identity and $d\varphi_t/dt|_{t=0}=V$. Consider a composition map $u_t=u\circ\varphi_t$. Since u is stationary p-harmonic, we have

$$(17)\qquad \left.\frac{dE_p(u_t)}{dt}\right|_{t=0}=-\int_\Omega\Big\{|du|^p\operatorname{div}(V)-p|du|^{p-2}\sum_i\langle du(\nabla_{e_i}V),\ due_i\rangle\Big\}dx$$
$$=0\,.$$

Let $\{\partial/\partial r, e_1, \cdots, e_{m-1}\}$ be an orthonormal basis on Ω. We take the above diffeomorphisms φ_t satisfying

$$(18)\qquad V=\left.\frac{d\varphi_t}{dt}\right|_{t=0}=\xi(r)r\frac{\partial}{\partial r}\,,$$

where $\xi\in C_0^\infty(\boldsymbol{R})$. Substituting (18) into (17), we obtain

$$\int_\Omega|du|^p\{\xi' r+m\xi-(m-1)\tilde{\Lambda}\xi r\}dx$$
$$\leqq p\int_\Omega|du|^{p-2}\{\xi' r|du(\partial/\partial r)|^2+\xi|du|^2+(m-1)\tilde{\Lambda}\xi r|du|^2\}dx\,,$$

where $\tilde{\Lambda}$ is a positive constant (cf. [**13**], [**15**]). We choose, for $\tau\in(\sigma, \rho)$,

$$(19)\qquad \xi(r)=\xi_\tau(r)=\phi(r/\tau)\,,$$

where ϕ is a smooth function and $\phi(r)=1$ for $r\in[0, 1]$, $\phi(r)=0$ for $r\in(1+\varepsilon, \infty)$, $\varepsilon>0$, $\phi'(r)\leqq 0$. Then we have

$$p\tau\frac{\partial}{\partial\tau}\int_\Omega\left|\frac{\partial u}{\partial r}\right|^2\xi_\tau|du|^{p-2}dx\leqq\tau\frac{\partial}{\partial\tau}\int_\Omega|du|^p\xi_\tau dx$$
$$+(p-m)\int_\Omega|du|^p dx+(p+1)(m-1)\tilde{\Lambda}(1+\varepsilon)\tau\int_\Omega|du|^p\xi_\tau dx\,.$$

We set $\Lambda=2(p+1)(m-1)\tilde{\Lambda}$. Multiplying by $e^{\Lambda\tau}\tau^{p-m-1}$ and integrating from σ to ρ, and taking the limit $\varepsilon\to 0$, we have (15). □

REMARK. When $m=p$, we can observe that the monotonicity formula of the weakly p-harmonic map is always valid.

Next we state the regularity theorem for a weakly p-harmonic map.

THEOREM 11. *Let* $u\in L^{1,p}(\Omega, S^{n-1})$ *be a weakly p-harmonic map which satisfies the monotonicity formula* (16). *Then u is locally Hölder continuous on* $\Omega\setminus\mathcal{S}_u$, *and* $H^{m-p}(\mathcal{S}_u)=0$. *Here*

$$\mathcal{S}_u=\{a\in\Omega : \limsup_{r\to 0} r^{p-m}E_p(u, B(a, r))>0\},$$

and H^{m-p} *denotes* $(m-p)$*-dimensional Hausdorff measure.*

REMARK. When $m=p$, we see $\mathcal{S}_u=\emptyset$.

Thus we can get the following corollary.

COROLLARY 12. *Any weakly p-harmonic map* $(p=m)$ *from* $\Omega\subset\boldsymbol{R}^m$ *into* S^{n-1} *is everywhere Hölder continuous.*

To prove this theorem, we need some lemmas.

We denote by $\mathcal{H}^1(\boldsymbol{R}^m)$ the Hardy space. Namely, a function $f\in\mathcal{H}^1(\boldsymbol{R}^m)$ if and only if $f\in L^1(\boldsymbol{R}^m)$ and $f^*\in L^1(\boldsymbol{R}^m)$. Here f^* is defined by

$$f^*(x)=\sup_{r>0}\left|\frac{1}{r^m}\int_{\boldsymbol{R}^m} f(y)\phi\left(\frac{x-y}{r}\right)dy\right|.$$

Here ϕ is any smooth function with support in the unit ball, and $\int_{\boldsymbol{R}^m}\phi dx=1$.

Its norm is defined by

$$\|f\|_{\mathcal{H}^1(\boldsymbol{R}^m)}=\|f^*\|_{L^1(\boldsymbol{R}^m)}.$$

LEMMA 13. *Assume* $u\in L^{1,p}(\boldsymbol{R}^m)$, $v\in L^q(\boldsymbol{R}^m, \boldsymbol{R}^m)$, $q=p/(p-1)$, *and* $\operatorname{div}(v)=0$ *in the distribution sense. Then*

$$Du\cdot v\in\mathcal{H}^1(\boldsymbol{R}^m),$$

and there exists a constant C such that

$$\|Du\cdot v\|_{\mathcal{H}^1(\boldsymbol{R}^m)}\leqq C(\|u\|_{L^{1,p}}^p+\|v\|_{L^q}^q).$$

PROOF. Clearly $Du\cdot v\in L^1(\boldsymbol{R}^m)$. Choose $x\in\boldsymbol{R}^m$, $r>0$. Set

$$\phi_r(y)=\phi\left(\frac{x-y}{r}\right).$$

Then, by the assumption $\operatorname{div}(v)=0$, we get

$$\left|\frac{1}{r^m}\int_{R^m} Du\cdot v\phi_r dy\right| \leqq \frac{C}{r^{m+1}}\int_{B(x,r)} |u-(u)_{x,r}|\,|v|\,dy\,,$$

where

$$(u)_{x,r} = \frac{1}{|B(x,r)|}\int_{B(x,r)} u\,dy\,,$$

$|B(x, r)|$ is denoted the Lebesgue measure of $B(x, r)$, and C is a constant independent of r.

Choose $p<\alpha<p^*=mp/(m-p)\leqq\infty$, and let $1<\beta=\alpha/(\alpha-1)<p/(p-1)$. Then

$$(20)\qquad \left|\frac{1}{r^{m+1}}\int_{B(x,r)}(Du\cdot v\phi_r)dy\right| \leqq \frac{1}{r^{m+1}}\Big(\int_{B(x,r)}|u-(u)_{x,r}|^{\alpha}\Big)^{1/\alpha}\Big(\int_{B(x,r)}|v|^{\beta}\Big)^{1/\beta}$$
$$\leqq \frac{C}{r^{m+1}}\Big(\int_{B(x,r)}|Du|^{\gamma}\Big)^{1/\gamma}\Big(\int_{B(x,r)}|v|^{\beta}\Big)^{1/\beta}$$
$$\leqq C\{(M(|Du|^{\gamma}))^{p/\gamma}+(M(|v|^{\beta}))^{q/\beta}\},$$

where $\gamma=n\alpha/(n+\alpha)$, and $M(\cdot)$ denotes the Hardy-Littlewood maximal function. Now noting $|Du|^{\gamma}\in L^{p/\gamma}$, $p/\gamma>1$, and $|v|\in L^{q/\beta}$, $q/\beta>1$. Then we get

$$(21)\qquad \|M(|Du|^{\gamma})\|_{L^{p/\gamma}} \leqq C\||Du|^{\gamma}\|_{L^{p/\gamma}}\,,$$
$$\|M(|v|^{\beta})\|_{L^{q/\beta}} \leqq C\||v|^{\beta}\|_{L^{q/\beta}}\,.$$

Consequently,

$$(22)\qquad (Du\cdot v)^* := \sup_{r>0}\left|\frac{1}{r^m}\int_{R^m} Du\cdot v\phi_r dy\right| \in L^1\,,$$
$$\|(Du\cdot v)^*\|_{L^1} \leqq C(\|u\|^p_{L^{1,p}}+\|v\|^q_{L^q})\,. \quad \square$$

Next we see the p-energy decay and blow up of the weakly p-harmonic map.

LEMMA 14. *Let $u\in L^{1,p}(\Omega, S^{n-1})$ satisfies the hypothesis of Theorem 11. There exist constants $0<\varepsilon_0$, $\tau<1$ such that $\tilde{E}_p(u, B(x, r))\leqq\varepsilon_0$ implies*

$$\tilde{E}_p(u, B(x, \tau r)) \leqq \frac{1}{2}\tilde{E}_p(u, B(x, r))$$

for all $x\in\Omega$, and $0<r<\mathrm{dist}(x, \partial\Omega)$.

PROOF. Suppose the conclusion would not be hold. Then $\tau>0$ may be selected as follows. There exist balls $B(x_k, r_k)\subset\Omega$ such that $\tilde{E}_p(u, B(x_k, r_k))=\lambda_k^p\to 0$, and $\tilde{E}_p(u, B(x_k, \tau r_k))>(1/2)\lambda_k^p$. Rescale the variable to the unit ball $B(0, 1)\subset \boldsymbol{R}^m$. If $z\in B(0, 1)$, put

$$v_k(z) = \frac{u(x_k+r_k z)-a_k}{\lambda}\,,$$

where $a_k=(u)_{x_k, r_k}$ denotes the average of u over $B(x_k, r_k)$, $(k=1, 2, \cdots)$. Then we verify that

$$\sup_k \int_{B(0,1)} |v_k|^p dz < \infty\,, \qquad \int_{B(0,1)} |Dv_k|^p dz = 1\,,$$

but

$$\frac{1}{\tau^{m-p}} \int_{B(0,\tau)} |Dv_k|^p dz > 1/2 \qquad (k=1, 2, \cdots)\,. \tag{23}$$

Then the sequence $\{v_k\}_{k=1}^{\infty}$ is bounded in $L^{1,p}(B(0, 1), \boldsymbol{R}^n)$. Hence there exists a subsequence such that

$$v_k \longrightarrow v \ \text{ strongly in } L^p(B(0, 1), \boldsymbol{R}^n), \ \text{ and} \tag{24}$$

$$Dv_k \longrightarrow Dv \ \text{ weakly in } L^p(B(0, 1), \boldsymbol{R}^{nm})\,.$$

Next select an arbitrary smooth function $w: B(0, 1) \to \boldsymbol{R}^n$ with compact support. Set

$$w_k(y) = w\Big(\frac{y-x_k}{r_k}\Big) \qquad (y \in B(x_k, r_k)).$$

Since u is a weakly p-harmonic map, we have

$$\int_{B(x_k, r_k)} |Du|^{p-2} Du \cdot Dw_k dy = \int_{B(x_k, r_k)} |Du|^p u \cdot w_k dy\,. \tag{25}$$

Rescaling this equality to the unit ball, we get

$$\int_{B(0,1)} |Dv_k|^{p-2} Dv_k \cdot Dw dz = \lambda_k \int_{B(0,1)} |Dv_k|^p (a_k + r_k v_k) \cdot w dz\,. \tag{26}$$

Send k to infinity in (26). Using the weakly convergence in $L^{p/(p-1)}$ of $|Dv_k|^{p-2} Dv_k$ to $|Dv|^{p-2} Dv$, $\|Dv_k\|_{L^p}=1$, and $a_k + r_k v_k = 1$, we get

$$\int_{B(0,1)} |Dv|^{p-2} Dv \cdot Dw dz = 0\,.$$

That is, v is a weakly p-harmonic map. Using Uhlenbeck's estimate in [**17**, p. 228, Theorem 3.2] which is

$$\sup_{B(x, r/2)} |Dv|^p \leqq C\Big(\frac{1}{r^m} \int_{B(x,r)} |Dv|^p dz\Big),$$

for any $B(x, r) \subset B(0, 1)$, we have

$$\frac{1}{\tau^{m-p}} \int_{B(0,\tau)} |Dv|^p dz \leqq C\tau^p \sup_{B(0,\tau)} |Dv|^p \tag{27}$$

$$\leqq \frac{C}{(2\tau)^{m-p}} \int_{B(0,2\tau)} |Du|^p dz < 1/2\,,$$

for a small $0<\tau<1/2$.

On the other hand, we shall show in Lemma 18 in the next section that

$$Dv_k \longrightarrow Dv \quad \text{strongly in } L^p(B(0, 1/2), \boldsymbol{R}^{mn}).$$

This implies by (23)

$$\frac{1}{\tau^{m-p}}\int_{B(0,\tau)}|Dv|^p dz \geqq 1/2.$$

This contradicts to (27). □

6. Compactness.

In this section, we shall prove in Lemma 18 that the above functions Dv_k converge strongly to Dv in $L^p(B(0, 1/2), \boldsymbol{R}^{mn})$, and shall complete the proof of Theorem 11.

We denote by BMO the space of bounded mean oscillation functions. Namely, the functions $f\in BMO$ if and only if f is locally summable and $\|f\|_{BMO}<\infty$. Here

$$\|f\|_{BMO}=\sup\left\{\frac{1}{|B(x, r)|}\int_{B(x,r)}|f-(f)_{x,r}|dy\,;\ x\in\boldsymbol{R}^m,\ r>0\right\},$$

where

$$(f)_{x,r}=\frac{1}{|B(x, r)|}\int_{B(x,r)}f dy.$$

First select a smooth cutoff function $\zeta:\boldsymbol{R}^m\to\boldsymbol{R}$ satisfying

$$0\leqq\zeta\leqq 1,$$
$$\zeta=1 \quad \text{on } B(0, 1/2),$$
$$\zeta=0 \quad \text{on } \boldsymbol{R}^m\setminus B(0, 5/8).$$

LEMMA 15. *The sequence* $\{\zeta v_k\}_{k=1}^{\infty}$ *is bounded in* $BMO(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

PROOF. Fix any point $z_0\in B(0, 7/8)$ and any radius $0<r\leqq 1/8$.

$$y_k=x_k+r_k z_0\in B(x_k, (7/8)r_k).$$

From the monotonicity formula (16) we have

$$\frac{1}{(rr_k)^{m-p}}\int_{B(y_k, rr_k)}|Du|^p dy\leqq 8^{m-p}\lambda_k^p.$$

Rescaling this estimate, we obtain

$$\frac{1}{r^{m-p}}\int_{B(z_0, r)}|Dv_k|^p dz\leqq 8^{m-p}$$

for all $k=1, 2, \cdots$ and all $0<r\leqq 1/8$, $z_0\in B(0, 7/8)$. Using the Poincaré and Hölder inequalities, we get

$$\frac{1}{r^m}\int_{B(z_0, r)}|v_k-(v_k)_{z_0, r}|dz \leqq C < \infty .$$

This implies $v_k\in BMO$. Using the John-Nirenberg inequality, for any $1\leqq s<\infty$, we have

$$\left(\frac{1}{|B(0, 7/8)|}\int_{B(0, 7/8)}|v_k|^s dz\right)^{1/s} \leqq C_1\|v_k\|_{BMO}+C_2\|v_k\|_{L^p} .$$

Recall $\{v_k\}_{k=1}^{\infty}\subset L^p(B(0, 1), \boldsymbol{R}^n)$. This implies $\{v_k\}_{k=1}^{\infty}$ is bounded in $L^s(B(0, 7/8), \boldsymbol{R}^n)$ $(1\leqq s<\infty)$. In a similar fashion to [7, p. 110], we get

$$\frac{1}{r^m}\int_{B(z_0, r)}|\zeta v_k-(\zeta v_k)_{z_0, r}|dz < \infty ,$$

for $z_0\in \boldsymbol{R}^m$, $0<r\leqq 1/8$. Thus we get

$$\sup_k \|\zeta v_k\|_{L^1} < \infty .$$

This completes the proof. □

Next define

$$b_{k,l}^{ij} = |Dv_k|^{p-2}\{v_{k,x_l}^j(a_k^i+\lambda_k v_k^i)-v_{k,x_l}^i(a_k^j+\lambda_k v_k^j)\} ,$$

for $1\leqq i,\ j\leqq n$, $1\leqq l\leqq m$, $k=1, 2, \cdots$, where the subscript x_l means the partial derivative with respect to x_l. Note $b_{k,l}^{ij}\in L^{p/(p-1)}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ for an arbitrary fixed j.

LEMMA 16. *For each functions* $\phi\in C_0^{\infty}(B(0, 1))$,

$$\int_{B(0, 1)}\phi_{x_l} b_{k,l}^{ij} dz = 0 ,$$

for $1\leqq i,\ j\leqq n,\ k=1, 2, \cdots$.

PROOF. A direct calculation.

From Lemmas 13 and 16, we have the following.

LEMMA 17. *For each* $1\leqq i,\ j\leqq n$, *the sequence* $\{\sum_{l=1}^{n}(\zeta v_k^i)_{x_l} b_{k,l}^{ij}\}_{k=1}^{\infty}$ *is bounded in* $\mathcal{H}^1(\boldsymbol{R}^m)$.

Finally we obtain

LEMMA 18. *The functions* $\{Dv_k\}_{k=1}^{\infty}$ *converge strongly to* Dv *in* $L^p(B(0, 1/2), \boldsymbol{R}^{mn})$.

PROOF. First, we can see

$$\int_{B(0,1)}(|Dv_k|^{p-2}Dv_k-|Dv|^{p-2}Dv)\cdot Dwdz = \lambda_k\int_{B(0,1)}|Dv_k|^p(a_k+\lambda v_k)\cdot wdz\,, \tag{28}$$

for a smooth function $w: B(0, 1)\to \boldsymbol{R}^n$ with compact support. We now substitute

$$w=\zeta^2(v_k-v)$$

into (28). Using the weakly convergence in $L^{p/(p-1)}$ of $|Dv_k|^{p-2}Dv_k$ to $|Dv|^{p-2}Dv$ and the strongly convergence in L^p of v_k to v, we have

$$\text{the left hand side of (28)} \geqq C\int_{B(0,1/2)}|Dv_k-Dv|^p dz+o(1)$$

as $k\to\infty$. The right hand side of (28) is

$$\begin{aligned} R_k &\equiv \lambda_k\int_{B(0,1)}\zeta^2|Dv_k|^p(a_k+\lambda_k v_k)\cdot(v_k-v)dz \\ &= \lambda_k\int_{B(0,1)}\zeta^2 v^j_{k,x_l}b^{ij}_{k,l}(v^i_k-v^i)dz \\ &= \lambda_k\int_{R^m}(\zeta v^j_k)_{x_l}b^{ij}_{k,l}(\zeta(v^i_k))dz-\lambda_k\int_{R^m}v^j_k\zeta_{x_l}b^{ij}_{k,l}\zeta(v^i_k-v^i)dz \\ &\equiv \lambda_k(R^1_k+R^2_k)\,. \end{aligned}$$

Since $\{v_k\}_{k=1}^{\infty}$ is bounded in $L^{2p}(B(0, 7/8), \boldsymbol{R}^n)$ and $\{b^{ij}_{k,l}\}_{k=1}^{\infty}$ is bounded in $L^{p/(p-1)}(B(0, 7/8))$, we obtain

$$\sup_k|R^2_k|<\infty\,.$$

Lemmas 15 and 17 imply

$$\sup_k|R^1_k| \leqq \sum_{i,j=1}^n C\cdot\sup\|\zeta(v^i_k-v^i)\|_{BMO}\|(\zeta v^j_k)_{x_l}b^{ij}_{k,l}\|_{\mathcal{H}^1(R^m)}<\infty\,.$$

Thus we get $R_k=o(1)$ as $k\to\infty$. Hence we have

$$\int_{B(0,1/2)}|Dv_k-Dv|^p dz \leqq o(1) \qquad \text{as } k\to\infty\,,$$

which is our desired conclusion. □

PROOF OF THEOREM 11. Let τ be fixed as in Lemma 14. For any $\rho<r$, $B(x, r)\subset\Omega$ there exists some integer $k\in\boldsymbol{N}$ such that $\tau^{k+1}r<\rho<\tau^k r$. Using Lemma 14 inductively, we have

(29)
$$\begin{aligned}\tilde{E}_p(u, B(x, \rho)) &= \frac{1}{\rho^{m-p}}\int_{B(\rho, x)} |Du|^p dx \\ &\leqq \frac{1}{\tau^{m-p}}\tilde{E}_p(u, B(x, \tau^k r)) \\ &\leqq \tau^{p-m-p\beta}(\tau^{k+1})^{p\beta}\tilde{E}_p(u, B(x, r)) \\ &\leqq \tau^{p-m-p\beta}\left(\frac{\rho}{r}\right)^{p\beta}\tilde{E}_p(u, B(x, r)),\end{aligned}$$

where we take β such that $\tau^{p\beta}=1/2$. Thus we have

$$\int_{B(x,\rho)} |Du|^p dx \leqq \tau^{p-m-p\beta}\left(\frac{\rho}{r}\right)^{p-m+p\beta}\int_{B(x,r)} |Du|^p dx .$$

The Hölder continuity of u with Hölder exponent β follows from the Dirichlet growth theorem (cf. [**9**, p. 64, Theorem 1.1]). $H^{m-p}(\mathcal{S}_p(u))=0$ follows from [**9**, p. 101, Theorem 2.2].

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