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On the hyperbolicity of projective plane with lacunary curves

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Introduction.

1°. Let X be a complex manifold of dim $=n$, and, M, a dense subdomain of X. Denote by $d_M(p, q)$ be the intrinsic pseudodistance of two points p and q of M introduced by Kobayashi [5]. In [2], we extended d_M onto X as follows. For p, q of X, we define

$$d_M(p, q)=\varliminf_{p'\to p, q'\to q} d_M(p', q'), \qquad p', q' \in M.$$

It is clear that $0\leqq d_M(p, q)\leqq\infty$ and $d_M(p, r)\leqq d_M(p, q)+d_M(q, r)$ for p, q, r of X.

A point $p\in X$ is called a degeneracy point of d_M on X, if there exists a point $q\in X\setminus\{p\}$ such that $d_M(p, q)=0$. We denote by $S_M(X)$ the set of all degeneracy points of d_M on X and call $S_M(X)$ the degeneracy locus of d_M in X.

Let S be an analytic subset of X. According to Kiernan-Kobayashi [4], M is hyperbolically imbedded modulo S in X, if every distinct points p, q of X such that $d_M(p, q)=0$ are contained in S. In this case, $S_M(X)\subset S$. M is hyperbolically imbedded in X if $S_M(X)=\emptyset$.

We showed in [2] that $S_M(X)$ is a pseudoconcave subset of order 1 in X and that, if $S_M(X)$ is not empty and is contained in an analytic subset of dimension 1 of X, then $S_M(X)$ is also an analytic subset of dimension 1 of X composed of irreducible components of genus $\leqq 1$.

2°. Let X be a compact complex manifold of dim $=2$, and let A be a curve in X. An irreducible curve C in X will be called a nonhyperbolic curve with respect to A, if the following condition is satisfied: In case $C\not\subset A$, the normalization of $C\setminus A$ is isomorphic to either a smooth elliptic curve, $\boldsymbol{P}$, $\boldsymbol{C}$ or $\boldsymbol{C}^*=\boldsymbol{C}\setminus\{0\}$. In case $C\subset A$, the normalization of $C\setminus A'$ is isomorphic to either a smooth elliptic curve, $\boldsymbol{P}$, $\boldsymbol{C}$ or $\boldsymbol{C}^*$, where A' is the union of the components of A except C. So if we set $M=X\setminus A$, then $C\subset S_M(X)$ in case $C\not\subset A$.

The main result of this paper is

THEOREM. *Let A be a curve in $\boldsymbol{P}^2$. Set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\setminus A$. If $S_M(X)$ is a curve in X, then $S_M(X)$ is composed of nonhyperbolic curves with respect to A.*

We obtain

COROLLARY. *Let A be a curve with $l(l\geqq 4)$ irreducible components in $\boldsymbol{P}^2$. Set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\setminus A$.*

(1) *If the number of the nonhyperbolic curves in $\boldsymbol{P}^2$ with respect to A is finite (respectively zero), $S_M(X)$ consists of at most finite number of nonhyperbolic curves with respect to A (respectively, $S_M(X)$ is empty).*

(2) *If the number of the nonhyperbolic curves in $\boldsymbol{P}^2$ with respect to A is infinite, then $S_M(X)=X$.*

1. Regular exhaustion.

Let $\bar{S}$ be a compact bordered Riemann surface with k real analytic simple closed curves $\alpha_1, \cdots, \alpha_k (k\geqq 1)$. We set $\partial S=\alpha_1\cup\cdots\cup\alpha_k$ and $S=\bar{S}\setminus\partial S$. Let ds^2 be a conformal metric on $\bar{S}$. Consider a sequence of discs $\Delta(R_j)$ $(j=1, 2, \cdots)$ and an open subset D_j in each $\Delta(R_j)$ bounded by a finite number of real analytic arcs and curves. We set

$$\Gamma_j=\partial D_j\cap\Delta(R_j), \qquad L_j=\partial D_j\cap\partial\Delta(R_j).$$

Suppose that for each j there exists a nonconstant holomorphic mapping $\varphi_j: \bar{D}_j\to\bar{S}$ such that $\varphi_j(\Gamma_j)\subset\partial S$. We denote by $\varphi_j^* ds^2=h_j(z)|dz|^2$ the pull back of ds^2 by φ_j on D_j. We set

$$|D_j|=\int_{D_j} h_j(z)\frac{i}{2}dz\wedge d\bar{z}, \qquad |L_j|=\int_{L_j}\sqrt{h_j(z)}\,|dz|.$$

For each $0<r<R_j$, set

$$D_j(r)=D_j\cap\Delta(r), \qquad L_j(r)=D_j\cap\partial\Delta(r),$$

$$|D_j(r)|=\int_{D_j(r)} h_j(z)\frac{i}{2}dz\wedge d\bar{z}, \qquad |L_j(r)|=\int_{L_j(r)}\sqrt{h_j(z)}\,|dz|.$$

DEFINITION 1. We call the sequence of the pairs (D_j, φ_j) a regular exhaustion of (S, ds), if $\underline{\lim}_{j\to\infty}(|L_j|/|D_j|)=0$.

We shall say that φ_j converges uniformly to a holomorphic mapping φ: $\Delta(r)\to S$, if there exists a positive integer j_0 such that $\Delta(r)\subset D_j$ for all $j\geqq j_0$ and $\{\varphi_j\}_{j\geqq j_0}$ converges uniformly to φ on $\Delta(r)$.

LEMMA 1. *Assume that each D_j contains the origin 0 and that the following three conditions are satisfied:*

(i) $\lim_{j\to\infty}R_j=\infty$, (ii) $\{\varphi_j(0)\}$ *converges to a point p of S,* (iii) $\{\varphi_j\}$ *has no subsequence which converges uniformly to the constant $\varphi(z)\equiv p$ on $\Delta(=\Delta(1))$.*

Then, there exist a subsequence $\{j_\lambda\}_{\lambda=1,2,\cdots}$ of $\{j\}$ and a sequence of positive *numbers $r_{j_\lambda}<R_{j_\lambda}$ such that the sequence of the pairs $(D_{j_\lambda}(r_{j_\lambda}), \varphi_{j_\lambda})$ is a regular exhaustion of (S, ds).*

PROOF. 1°. Let $R>1$. We first prove that $|D_j(R)|$ is bounded from below by a positive constant. Take j_0 such that $R_j>R$ for all $j\geqq j_0$, and consider the graph G_j in $\Delta(R)\times S$:

$$G_j=\{(z, w)\in \Delta(R)\times S\,;\ w=\varphi_j(z),\ z\in D_j(R)\}.$$

Then each G_j $(j\geqq j_0)$ is a closed analytic subset of dimension 1 in $\Delta(R)\times S$ and the area $|G_j|$ of G_j measured by $d\sigma^2=|dz|^2+ds^2$ is given by

$$|G_j|=\int_{D_j(R)}(1+h_j(z))\frac{i}{2}dz\wedge d\bar{z}\leqq \pi R^2+|D_j(R)|.$$

It follows by the Oka [8]-Nishino [6]-Bishop [3] theorem that, if the sequence $\{|D_j(R)|\}$ has a bounded subsequence, then there exists a subsequence $\{G_{j_\lambda}\}$ which converges uniformly to a closed analytic subset of dimension 1 on each compact subset of $\Delta(R)\times S$. Hence, if we assume that $\lim_{j_\lambda\to\infty}|D_{j_\lambda}(1)|=0$, then we can choose a subsequence of φ_{j_λ} which converges uniformly to the constant $\varphi(z)\equiv p$ on $\Delta(1)$. This contradicts (iii). Thus 1° is proved, namely there exists a positive constant A such that $|D_j(R)|>A$ for all $j\geqq j_0$.

2°. For $\varepsilon>0$, set $E_j(\varepsilon)=\{r\in[R, R_j)\,;\ |L_j(r)|\geqq\varepsilon\cdot|D_j(r)|\}$. Define $h_j(z)$ to be zero on $\Delta(R_j)\setminus D_j$. Then by the Schwarz inequality,

$$|L_j(r)|^2=\left|\int_0^{2\pi}\sqrt{h_j(re^{i\theta})}rd\theta\right|^2$$

$$\leqq 2\pi r\int_0^{2\pi}h_j(re^{i\theta})rd\theta=2\pi r\frac{d|D_j(r)|}{dr}.$$

Therefore,

$$\int_{E_j(\varepsilon)}\frac{dr}{2\pi r}\leqq\int_{E_j(\varepsilon)}\frac{d|D_j(r)|}{|L_j(r)|^2}<\frac{1}{\varepsilon^2}\int_{E_j(\varepsilon)}\frac{d|D_j(r)|}{|D_j(r)|^2}$$

$$\leqq\frac{1}{\varepsilon^2}\left(\frac{1}{|D_j(R)|}-\frac{1}{|D_j(R_j)|}\right)\leqq\frac{1}{A\varepsilon^2}.$$

Let $\{\varepsilon_\lambda\}$ be a sequence of positive numbers tending to zero. By (i), for each λ we can take j_λ such that

$$\frac{1}{A\varepsilon_\lambda^2}<\frac{1}{2\pi}(\log R_{j_\lambda}-\log R)=\int_R^{R_{j_\lambda}}\frac{dr}{2\pi r},$$

so that $[R, R_{j_\lambda})\setminus E_{j_\lambda}(\varepsilon_\lambda)\neq\emptyset$. If we choose, for $\lambda=1, 2, \cdots$, an $r_{j_\lambda}\in[R, R_{j_\lambda})$ $E_{j_\lambda}(\varepsilon_\lambda)$, then

$$\lim_{\lambda\to\infty}\frac{|L_{j_\lambda}(r_{j_\lambda})|}{|D_{j_\lambda}(r_{j_\lambda})|}\leqq\lim_{\lambda\to\infty}\varepsilon_\lambda=0. \qquad \text{Q.E.D.}$$

Now, for each boundary component α_i $(1\leqq i\leqq k)$ of S, denote by $N_i(j)$ the number of the closed curves on $\varphi_j^{-1}(\alpha_i)\cap\Delta(R_j)$ and by $m_i(j)$ the minimum of the degree of φ_j on these closed curves. If $\varphi_j^{-1}(\alpha_i)\cap\Delta(R_j)$ contains no closed curve, we set $m_i(j)=\infty$. Thus $1\leqq m_i(j)\leqq\infty$.

DEFINITION 2. Setting

$$m_i=\varliminf_{j\to\infty} m_i(j),$$

we say that the sequence $\{(D_j, \varphi_j)\}$ ramifies at least m_i-ply along α_i.

LEMMA 2. *Assume that the sequence $\{(D_j, \varphi_j)\}$ is a regular exhaustion of (S, ds) and ramifies at least m_i-ply along α_i $(i=1, \cdots, k)$. Then we have*

$$(1)\quad \sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)\leqq 2-2g(S),$$

where $g(S)$ is the genus of S. In particular, $g(S)\leqq 1$.

PROOF. (Cf. [9] Chap. VI, [7] n°6.) The area and the length with respect to ds or φ_j^*ds are denoted by $|\cdot|$. Let D_j have $l(j)$ connected components $D_j^1, \cdots, D_j^{l(j)}$ and let the border of D_j^ν consist of q_j^ν contours $(1\leqq\nu\leqq l(j))$. First, we note that

$$\sum_{\nu=1}^{l(j)}(q_j^\nu-2)\leqq\sum_{i=1}^{k}N_i(j)-l(j)\quad\text{and}\quad q_j^\nu-2\geqq -1.$$

Hence, we have

$$(2)\quad \sum_{\nu=1}^{l(j)}\max\{q_j^\nu-2, 0\}\leqq\sum_{i=1}^{k}N_i(j).$$

Next, by Ahlfors' second covering theorem ([9], p. 141), there exists a positive constant h_1 depending only on (S, ds) such that

$$\left|\frac{|D_j|}{|S|}-\frac{|\varphi_j^{-1}(\alpha_i)|}{|\alpha_i|}\right|\leqq h_1|L_j|.$$

This yields

$$(3)\quad N_i(j)\leqq\frac{|\varphi_j^{-1}(\alpha_i)|}{|\alpha_i|\cdot m_i(j)}\leqq\frac{|D_j|}{|S|\cdot m_i(j)}+h_1|L_j|.$$

On the other hand, by Ahlfors' main theorem ([9], p. 148), there exists a positive constant h_2 depending only on (S, ds) such that

$$(4)\quad \max\{q_j^\nu-2, 0\}\geqq\frac{|D_j^\nu|}{|S|}(2g(S)+k-2)-h_2|L_j^\nu|.$$

From (2), (3) and (4), it follows that

$$\sum_{i=1}^{k}\frac{|D_j|}{|S|\cdot m_i(j)}\geqq\frac{|D_j|}{|S|}(2g(S)+k-2)-(kh_1+h_2)|L_j|,$$

namely,

$$\sum_{i=1}^{k}\left(1-\frac{1}{m_i(j)}\right) \leqq 2-2g(S)+\frac{(kh_1+h_2)|S|\cdot|L_j|}{|D_j|}.$$

On letting $j\to\infty$ in this inequality, we obtain (1). Q.E.D.

2. Application of the regular exhaustion.

Let X be a complex manifold of dim$=n$ and, M, a dense subdomain of X. In [2], we proved

LEMMA 3. *For any point p of $S_M(X)$ and any compact subset K of $X\setminus S_M(X)$, there exists a sequence of holomorphic mappings f_j: $\overline{\Delta(R_j)}\to M$ such that*

(i) $\lim_{j\to\infty}R_j=\infty$, (ii) $\lim_{j\to\infty}f_j(0)=p$, (iii) $\|f_j'(0)\|=1$ *and* (iv) $f_j(\overline{\Delta(R_j)})\cap K=\emptyset$ *for all j, where $f_j'(0)=df((d/dz)|_{z=0})$ and $\|*\|$ is the norm of the vector $*$ with respect to a fixed hermitian metric on X.*

Let A be a curve in $\boldsymbol{P}^2$ and set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\setminus A$. Assume that $S_M(X)$ is a curve in X. We denote by Sing$(S_M(X))$ the singular points of $S_M(X)$. Let Σ be any irreducible component of $S_M(X)$. We take a closed subdomain $\bar{S}$ of Σ such that

1) $\bar{S}\cap\mathrm{Sing}(S_M(X))=\emptyset$,

2) S is bordered by k real analytic simple closed curves $\alpha_1,\cdots,\alpha_t,\alpha_{t+1},\cdots,\alpha_k$ where t is determined as follows:

Case I. $\Sigma\not\subset A$. We set

$$\Sigma\cap A=\{p_1,\cdots,p_m\};$$

$$\Sigma\cap(\mathrm{Sing}(S_M(X))\setminus A)=\{p_{m+1},\cdots,p_n\}.$$

For each $p_l\,(1\leqq l\leqq n)$, we take a small neighborhood U_l of p_l such that

1) $U_i\cap U_j=\emptyset$ for $i\neq j$; $1\leqq i,j\leqq n$.

2) If we denote by $\{\Sigma_{l_1},\cdots,\Sigma_{l_{\nu_l}}\}$ the set of irreducible components of $\Sigma\cap U_l$, then each $\Sigma_{l_i}\,(1\leqq i\leqq\nu_l)$ contains p_l and is irreducible at p_l. On each $\Sigma_{l_k}\,(1\leqq l\leqq n;\ 1\leqq k\leqq\nu_l)$ we draw a real analytic simple closed curve α_{l_k} around p_l. We remember

$$\alpha_{11},\alpha_{12},\cdots,\alpha_{m\nu_m}=\alpha_1,\cdots,\alpha_t,$$

$$\alpha_{m+11},\alpha_{m+12},\cdots,\alpha_{n\nu_m}=\alpha_{t+1},\cdots,\alpha_k.$$

Case II. $\Sigma\subset A$. Let A' be the union of the components of A except Σ. We set

$$\Sigma\cap A'=\{p_1,\cdots,p_m\};$$

$$\Sigma\cap(\mathrm{Sing}(S_M(X))\setminus A')=\{p_{m+1},\cdots,p_n\}.$$

For each $p_l\,(1\leqq l\leqq n)$, we take a small neighborhood U_l of p_l and draw $\alpha_1, \cdots, \alpha_t, \alpha_{t+1}, \cdots, \alpha_k$ by the same manner as above Case I.

By Lemma 1 of Nishino-Suzuki [7], there exists a relatively compact subdomain V of X and a holomorphic mapping $\pi: \bar{V}\rightarrow\bar{S}$ such that

1) $\bar{V}\cap S_M(X)=\bar{S}$,

2) $\pi|_{\bar{S}}=\mathrm{id.}$,

3) $\bar{V}\xrightarrow{\pi}\bar{S}$ is topologically a locally trivial fiber bundle with fibers homeomorphic to the real 2-dimensional closed disk.

By reading the proof of the lemma carefully, we conclude the following: for any sufficiently small $\varepsilon>0$ we can take $\bar{V}$ such that $d(p, q)\leqq\varepsilon$ for any $p\in\bar{S}$ and any $q\in\pi^{-1}(p)$, where d is the distance defined from a hermitian metric of X. So

4) for any $p\in S$ there exists a neighborhood $U(p)\subset S$ such that $\pi^{-1}(U)$ is contained in a local coordinate neighborhood of X,

5) $\pi^{-1}(\alpha_i)\subset U_l$, $\pi^{-1}(\alpha_i)\cap A=\emptyset$ in case $\Sigma\not\subset A$ and $\pi^{-1}(\alpha_i)\cap A'=\emptyset$ in case $\Sigma\subset A$.

From Lemma 3, for any point $p\in S$, there exists a sequence of holomorphic mappings $f_j: \overline{\Delta(R_j)}\rightarrow M$ such that

(i) $\lim_{j\to\infty}R_j=\infty$, (ii) $\lim_{j\to\infty}f_j(0)=p$, (iii) $\|f_j'(0)\|=1$ and (iv) $f_j(\overline{\Delta(R_j)})\cap\bigcup_{q\in\bar{S}}\partial\pi^{-1}(q)=\emptyset$.

Set $\varphi_j=\pi\circ f_j$, $D_j=f_j^{-1}(V)$ and $\Gamma_j=\partial D_j\cap\Delta(R_j)$. Then $\varphi_j(\Gamma_j)\subset\partial S$ and we may assume that each D_j contains the origin 0.

LEMMA 4. *$\{\varphi_j\}$ has no subsequence which converges uniformly to the constant $\varphi(z)\equiv p$ on Δ.*

PROOF. Assume φ_{j_λ} converges uniformly to φ on Δ. Then $\{f_{j_\lambda}\}$ is a normal family since the image $f_{j_\lambda}(\Delta)$ is contained in a local coordinate neighborhood of X for every sufficiently large j_λ. By renumbering $\{j_\lambda\}$, we may assume $f_{j_\lambda}\rightarrow f$ on Δ, where f is a holomorphic mapping of Δ to X. Obviously $f(0)=p$. Let $f(a)=q$ for any $a\in\Delta^*$ where $\Delta^*=\Delta\setminus\{0\}$. Then $q\in\pi^{-1}(p)$.

By distance decreasing property,

$$\begin{aligned} d_M(p, q) &= d_M(f(0), f(a)) \\ &\leqq d_M(f(0), f_{j_\lambda}(0))+d_M(f_{j_\lambda}(0), f_{j_\lambda}(a))+d_M(f_{j_\lambda}(a), f(a)) \\ &\leqq d_M(f(0), f_{j_\lambda}(0))+d_{\Delta(R_{j_\lambda})}(0, a)+d_M(f_{j_\lambda}(a), f(a)) \\ &\longrightarrow 0\ (j_\lambda\longrightarrow\infty). \end{aligned}$$

Since $q\in S_M(X)$, then $q\in\bar{S}$. As $\pi|_{\bar{S}}=\mathrm{id.}$, so $p=q$, and thus $f\equiv p$. It is a contradiction to $\|f'(0)\|=1$. Q.E.D.

From Lemma 1 and Lemma 4, we can replace the sequence $\{(D_j, \varphi_j)\}$ by its subsequence and shift the values of R_j's so that $\{(D_j, \varphi_j)\}$ turns into a regular exhaustion of (S, ds). Then, by Lemma 2, if the sequence $\{(D_j, \varphi_j)\}$ ramifies at least m_i-ply along α_i, we have

$$\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right) \leqq 2-2g(S),$$

where $g(S)$ is the genus of S. In particular, $g(S) \leqq 1$.

3. Lemma A and B.

Let A be a curve in $\boldsymbol{P}^2$ and set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2 \setminus A$. We assume that $S_M(X)$ is a curve in X. Let Σ be any irreducible component of $S_M(X)$ and S, α_i, p_l and U_l are same notations as in the section 2. Let V be a relatively compact subdomain of X and π be a holomorphic mapping from $\bar{V}$ to $\bar{S}$ which satisfy conditions 1) $\sim$ 5) in the section 2.

LEMMA A. *If δ be a simply connected domain in $\boldsymbol{C}$, there is no holomorphic mapping $f: \bar{\delta} \to M$ such that $f(\partial\delta) \subset \pi^{-1}(\alpha_i) \subset U_l$ and $\pi \circ f(\partial\delta)=\alpha_i$ for some $i (1 \leqq i \leqq t)$.*

PROOF. Suppose that there is a holomorphic mapping which satisfies above condition. Let A_0 be an irreducible component of A except Σ which passes through p_l. There exists a rational function F of X where the set of zeros is exactly A_0 and the set of poles in U_l is empty. From Rouché's theorem

$$0 \geqq \frac{1}{2\pi i}\int_{\partial\delta} d\log(F \circ f) = \frac{1}{2\pi i}\int_{\pi \circ f(\partial\delta)} d\log(F),$$

since $\pi^{-1}(\alpha_i) \cap A_0 = \emptyset$. And

$$\frac{1}{2\pi i}\int_{\pi \circ f(\partial\delta)} d\log F = \frac{1}{2\pi i}\int_{\pi \circ f(\partial\delta)} d\log(F \mid \Sigma) > 0.$$

It is absurd. Q.E.D.

LEMMA B. *Let R be a domain of $\boldsymbol{P}$ bounded with q real analytic simple closed curves $(q \geqq 1)$, and S be a compact bordered Riemann surface with k real analytic simple closed curves. If $f: \bar{R} \to \bar{S}$ be a nonconstant holomorphic mapping such that $f(\partial R) \subset \partial S$, then $g(S)=0$.*

PROOF. Let n be a degree of f and χ be the Euler characteristic. By the Hurwitz formula,

$$2-q=\chi(R) \leqq n \cdot \chi(S) = n(2-k-2g(S)).$$

Since $q \leqq n \cdot k$, then $g(S) \leqq (n-1)/n < 1$. Q.E.D.

4. Proof of Theorem.

Let Σ be any irreducible component of $S_M(X)$ and S, α_i, p_l, U_l, V, f_j and φ_j are same notations as in the section 2.

If $g(S)=0$ we want to show that $t\leqq 2$. This will follow from the estimate

$$\sum_{i=1}^{t}\left(1-\frac{1}{m_i}\right)\leqq\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)\leqq 2$$

if we can show that $m_i=\infty$ for $1\leqq i\leqq t$ if $t\geqq 2$. So, we assume $m_i\neq\infty$ for some i $(1\leqq i\leqq t,\ t\geqq 2)$. Then there exists a closed curve on $\varphi_{j_\lambda}^{-1}(\alpha_i)\cap\Delta(R_{j_\lambda})$ for infinite $\{j_\lambda\}$. If for almost all $\{j_\lambda\}$, such curve surrounds a connected component of D_{j_λ} as the outside boundary, there exists a closed curve on $\varphi_{j_\lambda}^{-1}(\alpha_{i'})\cap\Delta(R_{j_\lambda})$ $(i'\neq i,\ 1\leqq i'\leqq t)$ which is not the outside boundary of such connected component of D_{j_λ} for an infinite subsequence of $\{j_\lambda\}$ since $t\geqq 2$. By renumbering $\{j_\lambda\}$, we may assume that for each j_λ there exists a closed curve on $\varphi_{j_\lambda}^{-1}(\alpha_i)\cap\Delta(R_{j_\lambda})$ which is not the outside boundary of a connected component of D_{j_λ} for some i $(1\leqq i\leqq t)$. It is absurd from Lemma A. So $m_i=\infty$ for every i $(1\leqq i\leqq t)$ if $t\geqq 2$.

In case $g(S)=1$, we assume that $m_i\neq\infty$ for some i $(1\leqq i\leqq t)$. Then, there exists a closed curve on $\varphi_{j_\lambda}^{-1}(\alpha_i)\cap\Delta(R_{j_\lambda})$ for infinite $\{j_\lambda\}$. By Lemma B such curve does not surround a connected component of D_{j_λ} as the outside boundary. It is absurd from Lemma A. Since $\sum_{i=1}^{k}(1-m_i^{-1})\leqq 0$, then $t=0$. Q.E.D.

5. Proof of Corollary.

If the number of the nonhyperbolic curves in $\boldsymbol{P}^2$ with respect to A is at most finite, then $S_M(X)$ is contained in some curve from Theorem 3 in [**1**]. So, $S_M(X)$ is a curve or empty by Theorem 2 in [**2**]. If $S_M(X)$ is a curve, then $S_M(X)$ is composed of nonhyperbolic curves with respect to A from Theorem in this paper.

If the number of the nonhyperbolic curves in $\boldsymbol{P}^2$ with respect to A is infinite, then there exists a regular rational function f on $\boldsymbol{P}^2\setminus A$ such that all the irreducible components of the level curves $f^{-1}(a)$ in $\boldsymbol{P}^2\setminus A$ $(a\in\boldsymbol{P}^1)$ are isomorphic to either $\boldsymbol{C}$ or $\boldsymbol{C}^*$ from Theorem 3 in [**1**]. So, it is easy to see that $S_M(X)=X$. Q.E.D.

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