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Nonradial solutions of semilinear elliptic equations on annuli

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1. Introduction.

Let $\Omega_a=\{x\in\boldsymbol{R}^N : a<|x|<a+1\}$ with $a>0$ and $g:\boldsymbol{R}\rightarrow\boldsymbol{R}$ is continuous. We are concerned with the problem

$$\left\{\begin{array}{ll} -\Delta u=\lambda u+g(u) & \text{in } \Omega_a \\ u=0 & \text{on } \partial\Omega_a \\ u>0 & \text{in } \Omega_a . \end{array}\right. \tag{1}$$

It was known that any solution of (1) is radially symmetric in the case that the domain Ω_a is a disk instead of an annulus in Gidas, Ni and Nirenberg [4].

On the other hand, the existence of nonradial solutions of (1) in an annulus Ω_a was first obtained when $\lambda=0$, $g(t)=t^p$ with p close to $(N+2)/(N-2)$ and $N\geqq 3$ by Brezis and Nirenberg [2]. Later Coffman [3] showed the generation of essentially infinitely many nonradial solutions as $a\rightarrow+\infty$ for $\lambda=-1$ and $g(t)=t^p$, where $N=2$ and $1<p<\infty$. This result was generalized by Kawohl [7] and Suzuki [11] in the case of $N=2$ and then by Li [9] when $N\geqq 4$. These arguments can be applied only to the case of homogeneous nonlinearities or nonlinear eigenvalue problems because the Lagrangean multiplier principle played a crucial role there.

In order to prove the existence of nonradial solutions of the problem (1) with a general nonlinearity, Lin [10] used a spectral analysis for solutions produced by the Nehari variation and Suzuki [12] was based on estimates the critical values obtained by the mountain pass lemma.

Our purpose of the present paper is to give a simple proof of the above results. We make use of estimates of the Morse indices of the critical points given by the mountain pass lemma, which was first employed to get a sequence of subharmonic solutions of an elliptic equation on a strip-like domain in [5]. Our method enables us to weaken the growth condition of the nonlinearity g of (1) because we do not need information about the order of critical values as $a\rightarrow+\infty$.

2. Case of $N=2$.

Throughout this paper, the following conditions are assumed on $g\in C^1(\boldsymbol{R})$:

i) $g(0)=g'(0)=0$

ii) there are $p>2$ and C_1, $C_2>0$ such that

$$0\leqq g'(t)\leqq C_1t^{p-1}+C_2 \qquad \text{for } t>0$$

iii) there exists $\mu>1$ satisfying

$$\mu g(t)\leqq g'(t)t \qquad \text{for } t>0\,.$$

For $\lambda\in\boldsymbol{R}$ and $a>0$, we denote by $N_\lambda(a)$ the number of rotationally non-equivalent solutions of the problem (1). Let $\|\cdot\|_a$, $|\cdot|_a$ and $\langle\cdot,\cdot\rangle_a$ be the norm of $H_0^1(\Omega_a)$ and $L^2(\Omega_a)$ and the pairing between $H_0^1(\Omega_a)$ and $H^{-1}(\Omega_a)$, respectively. For each nonnegative integer m, H_m^a means the closed subspace of $H_0^1(\Omega_a)$ spanned by

$$\{\varphi^a(r)\sin(km\theta),\ \varphi^a(r)\cos(km\theta):\ k\geqq 0,\ \varphi^a\in H_0^1(a,\ a+1)\}\,.$$

In particular, H_0^a is the subspace of all radially symmetric functions in $H_0^1(\Omega_a)$. Define $J_a: H_0^1(\Omega_a)\to\boldsymbol{R}$ by

$$J_a(u)=\frac{1}{2}\int_{\Omega_a}|\nabla u|^2dx-\frac{1}{2}\int_{\Omega_a}\lambda u^2dx-\int_{\Omega_a}\int_0^{u(x)}g(t)dtdx\,,$$

where $g(t)=0$ for $t<0$. Then J_a satisfies the Palais-Smale condition on each H_m^a. It is said for J_a to have the Morse index k at $u\in H_0^1(\Omega_a)$ provided that the dimension of the maximal subspace on which $J_a''(u)$ is negative definite is equal to k.

We seek for a solution of the problem (1) as a critical point of J_a.

THEOREM 1. *Under the hypotheses* i)-iii), *if* $\lambda<1$, *the number* $N_\lambda(a)$ *diverges to* $+\infty$ *as* $a\to+\infty$.

PROOF. First it holds

$$|u|_a^2\leqq\left(1+\frac{1}{a}\right)\|u\|_a^2 \qquad \text{for } u\in H_0^1(\Omega_a)\,. \tag{2}$$

Indeed, from Hölder's inequality

$$|u|_a^2=\int_0^{2\pi}\int_a^{a+1}u(r,\ \theta)^2rdrd\theta$$

$$=\int_0^{2\pi}\int_a^{a+1}\left(\int_a^r\frac{\partial u}{\partial\rho}(\rho,\ \theta)d\rho\right)^2rdrd\theta$$

$$\leqq \int_0^{2\pi}\int_a^{a+1}\frac{r}{a}\Big(\int_a^{a+1}\Big(\frac{\partial u}{\partial\rho}(\rho,\,\theta)\Big)^2\rho d\rho\Big)drd\theta$$

$$\leqq\Big(1+\frac{1}{a}\Big)\|u\|_a^2\,.$$

This implies that the first eigenvalue of $-\Delta$ in $H_0^1(\Omega_a)$ is greater than $\lambda(<1)$ for $a>0$ sufficiently large. According to the mountain pass lemma, for each $a>0$ and $m\in\boldsymbol{N}$ there exists a nonzero critical point $u_m^a\in H_m^a$ of $J_a|_{H_m^a}$. Then the Morse index of $J_a|_{H_m^a}$ at u_m^a is less than or equal to 1 (see [**6**], [8]). It was known that $H_0^1(\Omega_a)$ is generated by

$$\{\varphi_{mk}^a(r)\sin(m\theta),\ \varphi_{mk}^a(r)\cos(m\theta):m\in\boldsymbol{Z}^+,\ k\in\boldsymbol{N},\ \varphi_{mk}^a\in H_0^1(a,\,a+1)\}\,.$$

Since u_m^a is a critical point of $J_a|_{H_m^a}$, we have

$$\langle-\Delta u_m^a-\lambda u_m^a-g(u_m^a),\ v\rangle_a=0\qquad\text{for } v\in H_m^a\,.$$

From $g:H_m^a\to H_m^a$, it follows for any $n\neq m$ and $k\in\boldsymbol{N}$

$$\begin{aligned}&\langle-\Delta u_m^a-\lambda u_m^a-g(u_m^a),\ \varphi_{nk}^a(r)\sin(n\theta)\rangle_a\\&=\int_a^{a+1}\int_0^{2\pi}\Big\{\frac{\partial u_m^a}{\partial r}\frac{\partial}{\partial r}(\varphi_{nk}^a(r)\sin(n\theta))+\frac{1}{r^2}\frac{\partial u_m^a}{\partial\theta}\frac{\partial}{\partial\theta}(\varphi_{nk}^a(r)\sin(n\theta))\\&\quad-\lambda u_m^a\varphi_{nk}^a(r)\sin(n\theta)-g(u_m^a)\varphi_{nk}^a(r)\sin(n\theta)\Big\}rd\theta dr\\&=0\,.\end{aligned}$$

Similarly, we have

$$\langle-\Delta u_m^a-\lambda u_m^a-g(u_m^a),\ \varphi_{nk}^a(r)\cos(n\theta)\rangle_a=0\,.$$

Therefore u_m^a is a critical point of J_a in $H_0^1(\Omega_a)$. Now, from (2), we get

$$\limsup_{a\to\infty}\frac{\langle J_a''(u_m^a)u_m^a,\ u_m^a\rangle_a}{|u_m^a|_a^2}<0\tag{3}$$

uniformly for $m\in\boldsymbol{N}$. In fact,

$$\begin{aligned}&\limsup_{a\to\infty}\frac{\langle J_a''(u_m^a)u_m^a,\ u_m^a\rangle_a}{|u_m^a|_a^2}\\&=\limsup_{a\to\infty}\frac{\langle-\Delta u_m^a-\lambda u_m^a-g'(u_m^a)u_m^a,\ u_m^a\rangle_a}{|u_m^a|^2}\\&\leqq\limsup_{a\to\infty}\frac{\langle-\Delta u_m^a-\lambda u_m^a-\mu g(u_m^a),\ u_m^a\rangle_a}{|\mu_m^a|_a^2}\\&\leqq(1-\mu)\Big(\liminf_{a\to\infty}\frac{\|u_m^a\|_a^2}{|u_m^a|_a^2}-\lambda\Big)\\&<0\,.\end{aligned}$$

uniformly for m. Let $m, n \in \boldsymbol{N}$, satisfying $m=kn$ with $k \geqq 3$. It is obvious that $H_m^a \subset H_n^a$. Suppose that $u_n^a \in H_m^a$. Let $v_m^a \in H_m^a$ be an eigenfunction corresponding to the first eigenvalue of $J_a''(u_n^a)$ in H_m^a. By (3), it holds that

$$\limsup_{a \to \infty} \frac{\langle J_a''(u_n^a)v_m^a,\ v_m^a \rangle_a}{|v_m^a|_a^2} < 0 . \tag{4}$$

Then

$$\begin{aligned}
&\langle J_a''(u_n^a)v_m^a \cos(n\theta),\ v_m^a \cos(n\theta)\rangle_a \\
&= \int_0^{2\pi}\int_a^{a+1}\Big\{\Big(\frac{\partial}{\partial r}(v_m^a \cos(n\theta))\Big)^2+\frac{1}{r^2}\Big(\frac{\partial}{\partial\theta}(v_m^a\cos(n\theta))\Big)^2 \\
&\quad -\lambda(v_m^a\cos(n\theta))^2-g'(u_n^a)(v_m^a\cos(n\theta))^2\Big\} r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_a^{a+1}\Big[\Big(\frac{\partial v_m^a}{\partial r}\Big)^2\cos^2(n\theta)+\frac{1}{r^2}\Big\{\frac{\partial v_m^a}{\partial\theta}\cos(n\theta)+v_m^a(-n\sin(n\theta))\Big\}^2 \\
&\quad -\lambda(v_m^a)^2\cos^2(n\theta)-g'(u_n^a)(v_m^a)^2\cos^2(n\theta)\Big] r\,dr\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_a^{a+1}\Big\{\Big(\frac{\partial v_m^a}{\partial r}\Big)^a+\frac{1}{r^2}\Big(\frac{\partial v_m^a}{\partial\theta}\Big)^2-\lambda(v_m^a)^2-g'(u_n^a)(v_m^a)^2\Big\}(1+\cos(2n\theta))\,r\,dr\,d\theta \\
&\quad -\int_0^{2\pi}\int_a^{a+1}\frac{n}{r^2}v_m^a\frac{\partial v_m^a}{\partial\theta}\sin(2n\theta)\,r\,dr\,d\theta \\
&\quad +\frac{1}{2}\int_0^{2\pi}\int_a^{a+1}\frac{n^2}{r^2}(v_m^a)^2(1-\cos(2n\theta))\,r\,dr\,d\theta \\
&\leqq \frac{1}{2}\langle J_a''(u_n^a)v_m^a,\ v_m^a\rangle_a+\frac{n^2}{a^2}|v_m^a|_a^2 .
\end{aligned}$$

From (4), it follows that

$$\langle J_a''(u_n^a)v_m^a\cos(n\theta),\ v_m^a\cos(n\theta)\rangle_a < 0$$

for $a>0$ sufficiently large. This contradicts that the Morse index of $J_a|_{H_n^a}$ at u_n^a is less than or equal to 1 because v_m^a and $v_m^a \cos(n\theta)$ are orthogonal. Therefore we have $u_n \notin H_m^a$ if $a>0$ is sufficiently large. This completes the proof.

3. Case of $N \geqq 3$.

In this section, we show the existence of nonradial solutions of (1) with $N \geqq 3$ by reducing the original problem to the case of $N=2$.

THEOREM 2. *Under the assumptions* i), ii) *with* $2<p<(N+2)/(N-2)$ *and* iii), *if* $\lambda \leqq 0$, *the problem* (1) *possesses a nonradial solution for* $a>0$ *sufficiently large.*

PROOF. Similarly to the proof of Theorem 1, there exists a nonzero critical point u_a of J_a such that the Morse index of J_a at u_a is less than or equal to 1. The same way as the proof of (3) indicates

$$\limsup_{a\to\infty}\sup\left\{\frac{\langle J_a''(u)u,\ u\rangle_a}{|u|_a^2} : u \text{ is radial, } J_a'(u)=0\right\} < 0\,. \tag{5}$$

Suppose that u_a is radially symmetric. Let v_a be an eigenfunction corresponding to the first eigenvalue of $J_a''(u_a)$ in the subspace of radial functions in $H_0^1(\Omega_a)$. From (5), it follows that

$$\limsup_{a\to\infty}\frac{\langle J_a''(u_a)v_a,\ v_a\rangle_a}{|v_a|_a^2} < 0\,. \tag{6}$$

Write $\Omega_a^i=\Omega_a\cap\boldsymbol{R}^i$. Putting $I_k=\int_{-\pi/2}^{\pi/2}\cos^k\theta d\theta$, we easily see $I_k=((k-1)/k)I_{k-2}$ for $k\geqq 2$. In general, for any function f dependent only on ρ and x_N, where $\rho=(\sum_{1\leq i\leq N-1} x_i^2)^{1/2}$ and $(x_1,\ x_2,\ \cdots,\ x_N)\in\boldsymbol{R}^N$, we get

$$\begin{aligned}
&\int_{\Omega_a^N} f(\rho,\ x_N)dx_1\cdots dx_N\\
&=\int_0^{2\pi}\int_{\Omega_a^{N-1}\cap\{r_1>0\}} f(\rho,\ x_N)r_1dr_1d\theta dx_3\cdots dx_N\\
&=2\pi\int_{\Omega_a^{N-1}\cap\{r_1>0\}} f(\rho,\ x_N)r_1dr_1dx_3\cdots dx_N\\
&=2\pi\int_{-\pi/2}^{\pi/2}\int_{\Omega_a^{N-2}\cap\{r_2>0\}} f(\rho,\ x_N)r_2^2\cos\theta dr_2d\theta dx_4\cdots dx_N\\
&=2\pi I_1\int_{\Omega_a^{N-2}\cap\{r_2>0\}} f(\rho,\ x_N)r_2^2dr_2dx_4\cdots dx_N\\
&\ \ \vdots\\
&=2\pi I_1I_2\cdots I_{N-3}\int_{\Omega_a^2\cap\{\rho>0\}} f(\rho,\ x_N)\rho^{N-2}d\rho dx_N\\
&=2\pi I_1I_2\cdots I_{N-3}\int_a^{a+1}\int_{-\pi/2}^{\pi/2} f(r\cos\theta,\ r\sin\theta)r^{N-1}\cos^{N-2}\theta d\theta dr\,.
\end{aligned}$$

Similarly, for $w\in H_0^1(\Omega_a)$ dependent only on ρ and x_N, it holds that

$$\langle -\Delta w,\ w\rangle_a=2\pi I_1I_2\cdots I_{N-3}\int_a^{a+1}\int_{-\pi/2}^{\pi/2}\left\{\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\right\}r^{N-1}\cos^{N-2}\theta d\theta dr.$$

Therefore we have

$$\begin{aligned}
&\langle J_a''(u_a)v_a\sin\theta,\ v_a\sin\theta\rangle_a\\
&=2\pi I_1I_2\cdots I_{N-3}\int_a^{a+1}\int_{-\pi/2}^{\pi/2}\left\{\left(\frac{\partial v_a}{\partial r}\right)^2\sin^2\theta+\frac{1}{r^2}v_a^2\cos^2\theta\right.\\
&\qquad\left.-\lambda v_a^2\sin^2\theta-g'(u_a)v_a^2\sin^2\theta\right\}r^{N-1}\cos^{N-2}\theta d\theta dr
\end{aligned}$$

$$\leqq 2\pi I_1 I_2 \cdots I_{N-3}\int_a^{a+1}\int_{-\pi/2}^{\pi/2}\left\{\left(\frac{\partial v_a}{\partial r}\right)^2-\lambda v_a^2-g'(u_a)v_a^2\right\}r^{N-1}\cos^{N-2}\theta\, d\theta\, dr$$

$$-2\pi I_1 I_2 \cdots I_{N-3}\int_a^{a+1}\int_{-\pi/2}^{\pi/2}\left\{\left(\frac{\partial v_a}{\partial r}\right)^2-\lambda v_a^2-g'(u_a)v_a^2\right\}r^{N-1}\cos^{N}\theta\, d\theta\, dr$$

$$+\frac{1}{a^2}\cdot 2\pi I_1 I_2 \cdots I_{N-3}\int_a^{a+1}\int_{-\pi/2}^{\pi/2} v_a^2 r^{N-1}\cos^{N-2}\theta\, d\theta\, dr$$

$$\leqq \langle J_a''(u_a)v_a,\ v_a\rangle_a-\frac{I_N}{I_{N-2}}\langle J_a''(u_a)v_a,\ v_a\rangle_a+\frac{1}{a^2}|v_a|_a^2$$

$$\leqq \frac{1}{N}\langle J_a''(u_a)v_a,\ v_a\rangle_a+\frac{1}{a^2}|v_a|_a^2 .$$

From the inequality (6), it follows that

$$\langle J_a''(u_a)v_a \sin\theta,\ v_a \sin\theta\rangle_a < 0$$

for $a>0$ sufficiently large. This contradicts that the Morse index of J_a at u_a is less than or equal to 1 since v_a and $v_a \sin\theta$ are orthogonal. Consequently u_a is nonradial if $a>0$ is sufficiently large.

REMARK. As seen from the above proofs, it is sufficient to assume growth conditions of g under which the functional J_a is of class C^2 and satisfies the Palais-Smale condition instead of the condition ii).

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