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On algebroid solutions of algebraic differential equations in the complex plane, II

Dedicated to Professor Kikuji Matsumoto on the occasion of his sixtieth birthday

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1. Introduction.

The main purpose of this paper is to investigate algebroid solutions of some algebraic differential equations in the complex plane with the aid of the Nevanlinna theory of meromorphic or algebroid functions. Throughout the paper the term "algebroid" or "meromorphic" will mean algebroid or meromorphic in the complex plane.

Let a_{jk} $(j=0, 1, \cdots, n; k=0, 1, \cdots, q_j)$ be entire functions without common zeros such that $a_{0q_0}\neq 0$ and $a_{nq_n}\neq 0$. We put

$$Q_j(w)=\sum_{k=0}^{q_j} a_{jk}w^k, \qquad q_j=\deg_w Q_j$$

$(j=0, 1, \cdots, n)$ and consider the differential equation

$$\sum_{j=0}^{n} Q_j(w)(w')^j=0 \tag{1}$$

under the condition

$$q_n+n>q_j+j \qquad (j=1, 2, \cdots, n-1). \tag{2}$$

We suppose that (1) is irreducible over the field of meromorphic functions and that it admits at least one nonconstant algebroid solution.

We say that a transcendental algebroid solution $w=w(z)$ of the differential equation (1) is admissible if it satisfies

$$T(r, a_{jk}/a_{nq_n})=S(r, w)$$

for all a_{jk}. For example, any transcendental algebroid solution of the differ-

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ential equation (1) is admissible when a_{jk} are polynomials.

Our differential equation (1) under the condition (2) is a generalization of the binomial differential equation

$$Q(w)(w')^n = P(w), \tag{3}$$

where $Q(w)$ and $P(w)$ are polynomials in w with entire coefficients. There are many interesting results concerning the differential equation (3) with meromorphic or algebroid solutions (see [**1**], [**3**], [**6**], [**7**], [**8**], [**9**], [**12**], [**17**], [**18**] etc.).

We would like to generalize those results to the case of our differential equation (1) under the condition (2). We shall often add the following condition to (2):

$$q_n + n > q_0 . \tag{4}$$

We put

$$\max\{q_j + j : j = 0, 1, \cdots, n-1\} = p ,$$

then the conditions (2) and (4) imply $q_n + n > p$.

A few years ago, we proved the following

THEOREM A. *Suppose that all a_{jk} are polynomials and that $q_n + n > p$. Then, any algebroid solution of the differential equation* (1) *is algebraic* ([**13**], *Theorem* 3).

As a generalization of this theorem, we would like to prove the problem:

PROBLEM A. *Is any algebroid solution of the differential equation* (1) *inadmissible when $q_n + n > p$?*

This is a generalization of the conjecture of Gackstatter and Laine ([**3**], p. 266):

"The differential equation with meromorphic coefficients

$$(w')^n = \sum_{j=0}^{m} a_j w^j \qquad (1 \leqq m \leqq n-1)$$

does not possess any admissible meromorphic solution".

This was positively proved by He Yuzan and Laine ([**6**], Corollary 2).

The purpose of this paper is to give a generalization of Theorem A, which is a partial positive answer to our Problem A, and to generalize some results in [**14**] to the differential equation (1). We shall also give a result on the growth of algebroid solutions to the differential equation (1) with constant coefficients under the conditions (2) and (4).

We denote by E, E_1, E_2, $\cdots$ subsets of $[0, \infty)$ for which $m(E) < \infty$, $m(E_j) < \infty$ $(j = 1, 2, \cdots)$, E may be different at different occurrences and K, K_1, K_2, $\cdots$ positive constants in this paper. We use the standard notation of the Nevanlinna theory of meromorphic functions ([**4**]) or of algebroid functions ([**10**], [**15**], [**16**]).

2. Lemmas.

We shall give some lemmas in this section for later use. Let $w=w(z)$ be a nonconstant algebroid solution of the differential equation (1) under the condition (2).

LEMMA 1. *Let* d_i $(i=0, 1, \cdots, s)$ *be meromorphic functions such that* $d_s\neq 0$. *Then, we have*

$$m\left(r, \sum_{i=0}^{s} d_i w^i\right) \leqq sm(r, w)+\sum_{i=0}^{s} m(r, d_i)+O(1).$$

This lemma follows by a simple inductive argument.

LEMMA 2. *If* $q_n+n\geqq q_0$, *the poles of* w *are contained in the set of zeros of* a_{nq_n} ([**13**], *Theorem* 1).

LEMMA 3. *Suppose that* a_{nq_n} *is a polynomial and* $q_n+n>p$. *Then,*

$$\min\{n, q_n+n-p\}\log^+M(r, w)\leqq K\sum_{j,k}\log^+M(r, a_{jk})+O(\log r)$$

for $r\notin E$ ([**13**], *Theorem* 2).

Let $f(z)$ be a nonconstant entire function and $T_0(r, f)$ be the Ahlfors-Shimizu characteristic function of f ([**4**]):

$$T_0(r, f)=\int_0^r \frac{A(t, f)}{t}\,dt.$$

LEMMA 4. *For* $0\leqq r<R$

$$\log M(r, f)\leqq \frac{R+r}{R-r}\left\{T_0(R, f)+\frac{1}{2}\log(1+|f(0)|^2)\right\} \qquad (\textit{see } [\mathbf{5}]).$$

This is a revised inequality of (9.3) in [**5**]. We can easily prove this inequality by the method given in [**5**], p. 257-p. 258, but not the original one, so we use this lemma in the followings.

Let G be a measurable set contained in $[1, \infty)$ and we put

$$G(r)=G\cap[1, r] \qquad (r>1).$$

The lower logarithmic density of G is defined by

$$\lambda(G)=\liminf_{r\to\infty}\left(\int_{G(r)}\frac{1}{r}\,dr\right)/\log r.$$

It is clear that $\lambda(G)=0$ if $m(G)<\infty$.

LEMMA 5. *Suppose that $f_1, \cdots, f_m$ are nonconstant entire functions. Then, we have the inequality*

$$\sum_{i=1}^{m} \log M(r, f_i) \leqq Ke\left[\sum_{i=1}^{m}\left\{T_o(r, f_i)+2A(r, f_i)+\frac{1}{2}\log(1+|f_i(0)|^2)\right\}\right]$$

on a set G of r having positive lower logarithmic density.

PROOF. Substitute $f=f_i$ in Lemma 4 and add the inequalities for i from 1 to m. We then have for $0\leqq r<R$

$$\sum_{i=1}^{m} \log M(r, f_i) \leqq \frac{R+r}{R-r}\left[\sum_{i=1}^{m}\left\{T_o(R, f_i)+\frac{1}{2}\log(1+|f_i(0)|^2)\right\}\right].$$

As in [5], we put $r=e^x$, $R=e^{x+h}$ and write

$$\sum_{i=1}^{m}\left\{T_o(r, f_i)+\frac{1}{2}\log(1+|f_i(0)|^2)\right\} = g(x).$$

Then, $g(x)$ is nonnegative, increasing and convex for $x\geqq 0$ and

$$g'(x) = r\frac{d}{dr}\sum_{i=1}^{m} T_o(r, f_i) = \sum_{i=1}^{m} A(r, f_i). \tag{5}$$

Further, applying the method used in the proof of Theorem 6 ([5]) to our case, we can easily obtain our lemma.

LEMMA 6. *Suppose that $f_1, \cdots, f_m$ are nonconstant entire functions such that the lower order of*

$$\sum_{i=1}^{m} T_o(r, f_i)$$

is finite. Let G be any subset of $[1, \infty)$ having positive lower logarithmic density. Then, there is a sequence $\{r_\nu\}$ in G such that

(i) *$r_\nu\to\infty$ as $\nu\to\infty$;*

(ii) *$\sum_{i=1}^{m} A(r_\nu, f_i) = O\left(\sum_{i=1}^{m} T_o(r_\nu, f_i)\right)$ for $\nu\to\infty$.*

PROOF. Suppose that there is a $G_0\subset[1, \infty)$ having positive lower logarithmic density such that

$$\lim_{G_0\ni r\to\infty}\left(\sum_{i=1}^{m} A(r, f_i)\right)/\sum_{i=1}^{m} T_o(r, f_i) = \infty.$$

Then, for any arbitrarily large M, there is an r_0 in G_0 such that

$$\left(\sum_{i=1}^{m} A(r, f_i)\right)/\sum_{i=1}^{m} T_o(r, f_i) \geqq M$$

for $r \geqq r_0$ and $r \in G_0$. This inequality reduces to the inequality

$$\liminf_{r\to\infty}\left(\log \sum_{i=1}^{m} T_0(r, f_i)\right)/\log r \geqq M\lambda(G_0)$$

by using (5). Since M is arbitrarily large and $\lambda(G_0)$ is positive, the lower order of $\sum_{i=1}^{m} T_0(r, f_i)$ must be infinity. This contradicts with our hypothesis.

LEMMA 7. *The absolute values of roots of the algebraic equation*

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

are bounded by

$$\max\{n|a_1|, (n|a_2|)^{1/2}, \cdots, (n|a_n|)^{1/n}\} \qquad ([\mathbf{11}]).$$

3. Theorems.

Let $w=w(z)$ be a nonconstant algebroid solution of the differential equation (1) under the condition (2).

THEOREM 1. *Suppose that* a_{nq_n} *is a polynomial and that*

$$q_n + n > q_0 . \tag{4}$$

If the lower order of

$$\sum_{j,k} T(r, a_{jk})$$

is finite, then $w=w(z)$ *is not admissible.*

PROOF. Suppose that w is admissible. Then, since w is transcendental and a_{nq_n} is a polynomial, we have

$$T(r, a_{jk})/T(r, w) \longrightarrow 0 \qquad (r\to\infty,\ r \notin E) \tag{6}$$

for all a_{jk} by the definition of admissibility of the solution.

Let $f_1, \cdots, f_m$ be the nonconstant functions in $\{a_{jk}\}$. Applying Lemma 5 to $f_1, \cdots, f_m$ and using Lemmas 2, 3 and 6, there is a sequence $\{r_\nu\} \subset E^c \cap G$ such that

(i) $r_\nu \to \infty$ $(\nu\to\infty)$;

(ii) $T(r_\nu, w) \leqq \min(n, q_n+n-p)\log M(r_\nu, w) + O(\log r_\nu)$
$= O(\sum_{j,k} T(r_\nu, a_{jk}))$ $(\nu\to\infty)$

since $\lim_{r\to\infty} T_0(r, f)/T(r, f) = 1$ (see [4], p. 13) and $\lambda(E)=0$. This is a contradiction to (6). $w=w(z)$ can not be admissible.

COROLLARY 1. *Under the same hypotheses as in Theorem* 1, *if the orders of all* a_{jk} *are finite,* $w=w(z)$ *is not admissible.*

In fact, it is trivial that the lower order of

$$\sum_{j,k} T(r, a_{jk})$$

is finite in this case.

With respect to meromorphic solutions of (1), we recall the following result, see Eremenko [2]:

THEOREM B. *If the differential equation* (1) *admits an admissible meromorphic solution, then*

$$q_j \leqq 2(n-j) \qquad (j=0, 1, \cdots, n).$$

Taking this theorem into consideration, we are able to give the following problem which is a special case of Problem A but contains the conjecture of Gackstatter and Laine given in § 1.

PROBLEM B. *Is any meromorphic solution of the differential equation* (1) *inadmissible when*

$$q_n = 0 \quad \text{and } q_j \leqq n-j-1 \quad (j=0, 1, \cdots, n-1)?$$

This question was settled when all a_{jk} are polynomials ([13]). As a generalization of this case we have the following from Corollary 1.

COROLLARY 2. *Suppose that a_{nq_n} is a polynomial and that*

$$q_n = 0, \quad q_j \leqq n-j-1 \quad (j=0, 1, \cdots, n-1).$$

If the orders of all a_{jk} are finite, the differential equation (1) *does not possess any admissible meromorphic solution.*

As a special case of Theorem A, we can give a sharp estimate of the growth of algebroid solutions of (1) with constant coefficients under the conditions (2) and (4).

THEOREM 2. *Suppose that the coefficients of the differential equation* (1) *are constants and that $q_n+n>p$.*

Let $w=w(z)$ be a nonconstant algebroid solution of the differential equation (1). *Then, there exists a positive constant r_0 such that*

$$\{\min(n, q_n+n-p)+\max(0, q_n-p)\}\log M(r, w) \leqq n \log r+O(1) \quad (r\geqq r_0). \tag{7}$$

PROOF. We first note that w has no poles by Lemma 2 since a_{nq_n} is constant and $q_n+n>q_0$. It is clear that there is an r_1 such that

$$M(r, w) \geqq 1 \qquad (r\geqq r_1)$$

and w has no branch points in $|z|\geqq r_1$ since w is a nonconstant algebraic solution of (1) by Theorem A. We put

$$U(z)=\sum_{k=0}^{q_n} a_{nk}w^{k+1}/(k+1)$$

where $w=w(z)$. Then,

$$U'(z)=Q_n(w)w' \tag{8}$$

and

$$\sum_{j=0}^{n} Q_j(w)Q_n(w)^{n-j-1}(U'(z))^j=0 \tag{9}$$

since $w=w(z)$ satisfies the equation

$$\sum_{j=0}^{n} Q_j(w)Q_n(w)^{n-j-1}(Q_n(w)w')^j=0\,.$$

We here note that $Q_n(w(z))\not\equiv 0$ since w is not constant.

Applying the method used in the proof of Hilfssatz 7.2 ([**9**]) to obtain the inequality (7.10) ([**9**], p. 82) to our U, we have

$$M(r,U)\leqq K_1+K_2rM(r,U') \qquad (r\geqq r_1)\,. \tag{10}$$

Let z_r be a point such that

$$M(r,U')=|U'(z_r)|,\quad |z_r|=r \quad (r\geqq r_1)\,.$$

Then, applying Lemma 7 to (9) at $z=z_r$ we obtain

$$M(r,U')\leqq K_3M(r,w)^{\max\{h_j:j=0,1,\cdots,n-1\}}, \tag{11}$$

where $h_j=(q_j+q_n(n-j-1))/(n-j)$, since

$$|Q_j(w)Q_n(w)^{n-j-1}|\leqq K_4M(r,w)^{q_j+q_n(n-j-1)}\,.$$

As

$$M(r,U)\geqq\frac{|a_{nq_n}|}{(q_n+1)}M(r,w)^{q_n+1}-K_5M(r,w)^{q_n} \qquad (r\geqq r_1),$$

we have from (10) and (11)

$$\begin{aligned} M(r,w)^{q_n+1} &\leqq K_6\{M(r,w)^{q_n}+rM(r,w)^{\max\{h_j:j=0,1,\cdots,n-1\}}\} \\ &\leqq K_6\{M(r,w)^{q_n}+rM(r,w)^{q_n+(p-q_n)/n}\} \qquad (r\geqq r_1) \end{aligned} \tag{12}$$

since $h_j\leqq q_n+(p-q_n)/n$ ($j=0,1,\cdots,n-1$). Dividing the inequality (12) by

$$M(r,w)^{\max(q_n,q_n+(p-q_n)/n)},$$

we have for $r\geqq r_1$

(13) $$M(r,\ w)^{\min(1,\ (q_n+n-p)/n)} \leqq K_6\{1+r/M(r,\ w)^{\max(0,\ (q_n-p)/n)}\}$$

since $q_n-\max(q_n,\ q_n+(p-q_n)/n)\leqq 0,\ q_n+(p-q_n)/n-\max(q_n,\ q_n+(p-q_n)/n)=\min(0,\ (p-q_n)/n)$ and $M(r,\ w)\geqq 1$ for $r\geqq r_1$. As $q_n+n-p>0$, there is an $r_0(\geqq r_1)$ from (13) such that

$$r/M(r,\ w)^{\max(0,\ (q_n-p)/n)} \geqq 1 \qquad (r\geqq r_0).$$

This gives us the following inequality by calculating $\log^+$ of the both sides of (13) for $r\geqq r_0$.

$$\begin{aligned}&\min(1,\ (q_n+n-p)/n)\log M(r,\ w)\\ &\qquad\leqq \log r-\max(0,\ (q_n-p)/n)\log M(r,\ w)+O(1),\end{aligned}$$

which reduces to our inequality to be proved.

EXAMPLE 1. $w=2z^{1/2}$ is a nonconstant algebroid solution of the differential equation

$$ww'-1=0$$

with constant coefficients.

This example shows that Theorem 2 is sharp.

We next generalize some results obtained for binomial differential equations with polynomial coefficients in [**14**]. We suppose that the differential equation (1) under the condition (2) admits at least one admissible algebroid solution $w=w(z)$.

THEOREM 3. *Suppose that a_{nq_n} is a polynomial, the orders of all a_{jk} are finite and that*

$$q_0 > \max_{1\leq j\leq n-1}(q_j+j)$$

in (1). *Then, the following three statements are equivalent.*

1) $\delta(\infty,\ w)>0$

2) $q_0=q_n+n$

3) ∞ *is a Picard exceptional value of* w.

PROOF. (i) Suppose that $\delta(\infty,\ w)>0$. If $q_0>q_n+n$, we obtain from (1)

$$w^{q_0}=-\frac{1}{a_{0q_0}}\left\{\sum_{j=1}^{n}Q_j(w)w^j(w'/w)^j-\sum_{k=0}^{q_0-1}a_{0k}w^k\right\}$$

and by Lemma 1

$$q_0 m(r,\ w)\leqq (q_0-1)m(r,\ w)+\sum_{j,\ k}m(r,\ a_{jk})+Km(r,\ w'/w)+m(r,\ 1/a_{0q_0})+O(1)$$

which reduces to

$$m(r, w) = S(r, w)$$

since w is admissible. This means that

$$\delta(\infty, w) = 0,$$

which is a contradiction, It must be $q_0 \leqq q_n + n$. If $q_0 < q_n + n$, then $p < q_n + n$ and w cannot be admissible by Theorem 1. We have

$$q_0 = q_n + n.$$

(ii) Suppose that $q_0 = q_n + n$. Then, by Lemma 2, ∞ is a Picard exceptional value of w since a_{nq_n} is a polynomial.

(iii) Suppose that ∞ is a Picard exceptional value of w. Then, it is clear that $\delta(\infty, w) = 1$ since w is admissible and so it is transcedental.

To obtain a similar result to this theorem for a finite value τ, we define the nonnegative integers $q_j(\tau)$ by the following way:

(i) When $Q_j \neq 0$,

$$Q_j(w) = (w-\tau)^{q_j(\tau)} \tilde{Q}_j(w),$$

where $\tilde{Q}_j(w)$ is polynomial in w with coefficients which are linear combinations of $a_{j0}, \cdots, a_{jq_j}$ with constant coefficients and $\tilde{Q}_j(\tau) \neq 0$. It is trivial that

$$0 \leqq q_j(\tau) \leqq q_j.$$

(ii) When $Q_j = 0$, we put for convenience

$$q_j(\tau) = \max(q_n + 2n, q_0) - 2j \quad \text{and} \quad \tilde{Q}_j = 0.$$

We have the relation

$$q_0(\tau) q_1(\tau) \cdots q_n(\tau) = 0 \tag{14}$$

since (1) is irreducible over meromorphic functions.

We suppose that the differential equation (1) possesses an admissible algebroid solution $w = w(z)$ under the condition (2). If we transform w to v by the relation

$$w - \tau = 1/v, \tag{15}$$

v is a nonconstant algebroid solution of the following differential equation:

$$\sum_{j=0}^{n} v^{\max(q_n+2n, q_0)} R_j(v)(v')^j = 0, \tag{16}$$

where

$$R_j(v) = (-1)^j v^{q_j - q_j(\tau)} \tilde{Q}_j(\tau + 1/v) \qquad (j = 0, 1, \cdots, n).$$

It is clear that (16) is irreducible over meromorphic functions as it is so with (1). We put for $j=0, 1, \cdots, n$

$$v^{\max(q_n+2n, q_0)}R_j(v) = \sum_{k=0}^{p_j} b_{jk}v^k ,$$

where $p_j=\max(q_n+2n, q_0)-2j-q_j(\tau)$. Then, b_{jk} are linear combinations of $a_{j0}, \cdots, a_{jq_j}$ with constant coefficients. We here note that

$$b_{0p_0} = \tilde{Q}_0(\tau) \neq 0 \quad \text{and} \quad b_{np_n} = (-1)^n\tilde{Q}_n(\tau) \neq 0$$

since $Q_0\neq 0$ and $Q_n\neq 0$. Further b_{jk} have no common zeros, because, if they have common zeros, a_{jk} have common zeros since a_{jk} are linear combinations of b_{jk} with constant coefficients by substituting $v=1/(w-\tau)$ into (16) and this contradicts with our hypothesis that a_{jk} have no common zeros.

PROPOSITION 1. *v is an admissible algebroid solution of the differential equation* (16).

PROOF. For any b_{jk}

$$T(r, b_{jk}/b_{np_n}) \leqq \sum_{i=0}^{q_j} T(r, a_{ji}/a_{nq_n}) + \sum_{i=0}^{q_n} T(r, a_{ni}/a_{nq_n}) + O(1)$$
$$= S(r, w) = S(r, v)$$

since $T(r, w)=T(r, v)+O(1)$.

PROPOSITION 2. *Suppose that $a_{n0}, \cdots, a_{nq_n}$ are polynomials, the orders of all other a_{jk} are finite and that*

(17) $$q_j(\tau) > n-j \qquad (j=1, \cdots, n-1).$$

Then,

$$0 \leqq q_0(\tau) \leqq n .$$

PROOF. When $q_0(\tau)=0$, there is nothing to prove. Suppose now that $q_0(\tau) >0$. Then, by (14) and (17), $q_n(\tau)=0$ and (16) satisfies the condition (2) since

$$p_n+n = \max(q_n+2n, q_0)-n > \max(q_n+2n, q_0)-j-q_j(\tau) = p_j+j$$

by (17). Further, b_{np_n} is a polynomial since we have

$$b_{np_n} = (-1)^n\tilde{Q}_n(\tau) = (-1)^n Q_n(\tau) = (-1)^n \sum_{k=0}^{q_n} a_{nk}\tau^k$$

due to $q_n(\tau)=0$.

As v is admissible and the orders of all b_{jk} are finite, it must be

$$p_0 \geqq p_n+n$$

by Corollary 1. This means that

$$\max(q_n+2n,\ q_0)-q_0(\tau) \geqq \max(q_n+2n,\ q_0)-n$$

and we have

$$q_0(\tau) \leqq n\ .$$

THEOREM 4. *Suppose that* $a_{n0}, \cdots, a_{nq_n}$ *are polynomials, the orders of all other* a_{jk} *are finite and that*

$$q_j(\tau) > n-j \qquad (j=1, \cdots, n-1)\ .$$

Further, we suppose that the differential equation (1) *undr the condition* (2) *has an admissible solution* $w=w(z)$. *Then, for a finite value* τ, *the following three statements are equivalent.*

1) $\delta(\tau, w) > 0$
2) $q_0(\tau) = n$
3) τ *is a Picard exceptional value of* w.

PROOF. We transform w to v by the relation (15) and we obtain (16) from (1). It is trivial that

$$\delta(\tau,\ w) = \delta(\infty,\ v)\ .$$

(i) Suppose that $\delta(\tau, w)>0$. If $q_0(\tau)<n$, then $p_0>p_j+j$ for all $j\neq 0$ and we have as in (i) of Proof of Theorem 3

$$m(r,\ v) = S(r,\ v),$$

which means that $\delta(\infty, v)=0$ since v is admissible by Proposition 1. This is a contradiction. This shows that $q_0(\tau)=n$ by Proposition 2.

(ii) Suppose that $q_0(\tau)=n$. Then, as in the proof of Proposition 2, (16) satisfies the condition (2), b_{np_n} is a polynomial and $p_0=p_n+n$. By Theorem 3, ∞ is a Picard exceptional value of v and so τ is a Picard exceptional value of w.

(iii) Suppose that τ is a Picard exceptional value of w. Then, it is trivial that $\delta(\tau, w)=1$ since w is transcendental.

COROLLARY 3. *Under the same assumption as in Theorem* 4,

$$q_0(\tau) < n \text{ if and only if } \delta(\tau,\ w) = 0\ .$$

REMARK 1. Proposition 2 and Theorem 4 contain a generalization of Theorem 2 ([**14**]) proved for the differential equation (3) with polynomial coefficients.

At the end of this paper, we give some examples.

EXAMPLE 2. The differential equation

$$4w^2(w')^2+w^4-1 = 0\ .$$

In this case, the coefficients are constants,

$$n=2, \quad q_2=2, \quad q_1=0 \quad \text{and} \quad q_0=4 .$$

By Theorems 3 and 4, for any transcendental algebroid solution $w=w(z)$ of this equation,

1) ∞ is a Picard exceptional value

2) $\delta(\tau, w)=0$ $(\tau \neq \infty)$.

This equation has 2-valued transcendental algebroid solutions

$$w_1=(\sin z)^{1/2} \quad \text{and} \quad w_2=(\cos z)^{1/2} .$$

EXAMPLE 3. The differential equation

$$(w^2-1)^2(w')^2+2w^2(w^2-1)w'+w^2(w^2+zw+1)^2=0 .$$

In this case, the coefficients are polynomials,

$$n=2, \quad q_2=4, \quad q_1=4 \quad \text{and} \quad q_0=6$$

and the condition (2) is satisfied.

For any transcendental algebroid solution $w=w(z)$ of this equation, 0 and ∞ are Picard exceptional values by Theorem 4 and Theorem 3 respectively.

This equation has 2-valued transcendental algebroid solutions

$$w_1=(\sin z-z+((\sin z-z)^2-4)^{1/2})/2$$

and

$$w_2=(\cos z-z+((\cos z-z)^2-4)^{1/2})/2 .$$

EXAMPLE 4. The differential equation

$$p^n w^{n(p-1)}(w')^n=(\cos^n z)(\sin z)(w^p-1)^n ,$$

where n and p are integers such that $n \geqq 1$ and $p \geqq 2$.

In this case, $a_{nq_n}=p^n$ is a constant and

$$q_n=n(p-1), \quad q_j=0 \quad (1 \leqq j \leqq n-1), \quad q_0=np .$$

It is obvious that the condition (2) is satisfied. For any admissible algebroid solution $w=w(z)$ of this equation,

1) ∞ is a Picard exceptional value by Theorem 3.

2) the roots $\zeta_0, \zeta_1, \cdots, \zeta_{p-1}$ of the equation $w^p-1=0$ are Picard exceptional values and

3) $\delta(\tau, w)=0$ $(\tau \neq \infty, \zeta_j\ (j=0, \cdots, p-1))$ by Theorem 4.

This equation has an admissible algebroid solution

$$w=(\exp(\sin z)^{(n+1)/n}+1)^{1/p} .$$

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