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Limit shape of the cross-section of shrinking doughnuts

By Naoyuki ISHIMURA

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§ 1. Introduction.

In this article we are concerned with the asymptotic behaviour of symmetric 2-tori which are shrinking to a circle by the mean curvature flow.

The mean curvature flow problem, in a normal parametrization, is to find the family of hypersurfaces $F(M_0, t)=M_t \hookrightarrow \boldsymbol{R}^{n+1}$ ($n \geqq 2$) satisfying

$$
(1) \qquad \begin{cases} \dfrac{\partial F}{\partial t}(x, t) = -H(x, t)\cdot N(x, t) \\ F(x, 0) = F_0(x) : M_0 \hookrightarrow \boldsymbol{R}^{n+1}, \end{cases}
$$

where N denotes the outward unit normal and H is the mean curvature with respect to N. Notice that in terms of the induced metric on M_t the right hand side of (1) is the Laplace-Beltrami operator Δ_{M_t} on M_t.

First we briefly recall some known facts on this problem. When the initial surface M_0 is strictly convex, Huisken [**18**], inspired by the work of Hamilton [**17**], showed that (1) shrinks M_0 to a round point within finite time, and also proved that for the area preserving rescaled flow M_0 really converges to a sphere in the C^∞-topology. Later Grayson [**15**] gave the counterexample which shows the convexity assumption in Huisken's theorem cannot be omitted; not all compact hypersurfaces with genus zero shrink to a point without singularity. Our previous joint work [**1**] with Ahara, on the other hand, dealt with the symmetric 2-torus and proved that under some technical hypothesis the torus might be shrunk to a circle by the mean curvature flow (see Theorem 2.1, below). Symmetry enables (1) to reduce essentialy a one-dimensional parabolic equation and hence our idea of the proof is based on applying the method of Gage and Hamilton [**11**], which discuss the plane curve shortening problem, to the equation for the generating curve. Although in our case there appears a lower order term in addition to the plane curve equation (see (6) below), our hypothesis in [**1**] makes it possible to apply the method of [**11**]. Indeed this

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hypothesis is imposed so that the generating curve stays convex.

The aim of the present article is to discuss the shape of the generating curve of symmetric tori which are shrinking to a circle by the mean curvature flow. The main result is Theorem 2.2 in section 2, which says that if the isoperimetric ratio of the generating curve stays bounded as shrinking then its asymptotic shape is a circle. It is also shown that the boundedness of the ratio follows from the convexity of curves. Therefore in a situation of our previous work [**1**], which is the starting point of our research, the limit shape of the generating curve is a circle (see Corollary 2.3, below).

The method of our proof is to utilize the backward heat kernel, which is introduced by Struwe [**21**] in the study of heat flow for harmonic mappings and later used cleverly by Huisken [**19**] for the mean curvature flow. Note that this kind of analysis was perhaps first given by Giga and Kohn in their study of semilinear heat equations [**12**][**13**]. We follow the idea of Huisken but some modifications are made. The difficulty of the appearance of a lower order term can be overcome by the assumption of the bounded isoperimetric ratio. The fact that the limit shape is a circle is derived from purely geometric consideration stated in Proposition 5.1, which is not contained in Huisken [**19**]. Remark that Gage and Hamilton [**11**] used Bonnesen type inequality to prove the above fact. Unfortunately this inequality does not hold in our case.

We make several comments on our problem: Recent remarkable work of Altschuler, Angenent and Giga [**2**] investigate the mean curvature flow for surfaces of rotation. They deal with the initial surface homeomorphic to a sphere, not a torus. They discuss the pinching near the axis of rotation and astonishingly they succeed in treating the behaviour *through* pinching off; after the singurarity the solution instantaneously becomes smooth if it did not extinct. Almost the same time Soner and Souganidis [**20**] discuss a similar problem. One of their results states that symmetric tori with its generating curve being a circle can be classified into two open sets and one point. One set is a family of tori shrinking to a circle, and the other set is pinching at the axis. For the latter case they also treat the through singularity behaviour and obtain a similar result as in [**2**]. For the former case, however, they do not concern its asymptotic behaviour. Remark that both above works employ viscosity solution approach, which is formulated in Chen, Giga and Goto [**7**] and Evans and Spruck [**8**], to deal with the behaviour through singularities. It is to be noted that Angenent [**6**] found a self-similarly shrinking torus, i.e., the solution M_t which shrinks according to $M_t=\sqrt{2(T-t)}M_0$ for some $T>0$.

Concerning the shrinking torus there still exists a problem on the behaviour before singularity; if the initial generating curve merely embedded one and it shrinks to a point off the axis, then does it become convex before extinction?

For the plane curve Grayson [14] already gave a positive answer. Our result may be interpreted as a partial answer to this: if the isoperimetric ratio stays bounded then it becomes circular. However it seems to be a difficult task to give a general answer. Finally about the formation of singularity of curves on surfaces with various normal velocity, we refer to papers of Angenent [3], [4], [5].

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§2. Notation and results.

Let M_t be a family of an embedding of a 2-torus $F_t: T^2 \hookrightarrow \boldsymbol{R}^3$ such that they are rotationally symmetric about the z-axis. We represent them by

$$F_t(u, \varphi) = (f(t, u)\cos\varphi,\ f(t, u)\sin\varphi,\ g(t, u)),$$

where $u \in S^1$ is a parameter independent of t and $0 \leqq \varphi < 2\pi$. We call M_t doughnuts hereafter. Let C_t be their generating curves, i.e., C_t are the intersection of M_t with the half xz-plane $\{(x, 0, z) \mid x > 0\}$. C_t are represented by

$$C_t(u) = (f(t, u), 0, g(t, u)).$$

We define the speed $v(t, u)$ of C_t by

$$v(t, u)^2 \equiv f'(t, u)^2 + g'(t, u)^2,$$

where $' = \partial/\partial u$. The mean curvature $H(t, u)$ of M_t is then given by

$$H(t, u) = \frac{f'g'' - f''g'}{v^3} + \frac{g'}{fv} \equiv k_m + k_l.$$

Here

$$k_m = \frac{f'g'' - f''g'}{v^3} : \text{the meridional sectional curvature.}$$

$$k_l = \frac{g'}{fv} \qquad : \text{the latitudinal sectional curvature.}$$

Notice that k_m is a planar curvature of the generating curve C_t and k_l is a curvature of rotation. Since the outer unit normal N on M_t is given by

$$N = \left(\frac{g'}{v}\cos\varphi,\ \frac{g'}{v}\sin\varphi,\ -\frac{f'}{v}\right),$$

the equation (1) is described as the one for the generating curve:

(2) $$\frac{\partial}{\partial t}\binom{f}{g} = -(k_m+k_l)\cdot\frac{1}{v}\binom{g'}{-f'},$$

or explicitly

(3) $$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{v}\left(\dfrac{f'}{v}\right)' - \dfrac{g'^2}{fv^2} \\ \dfrac{\partial g}{\partial t} = \dfrac{1}{v}\left(\dfrac{g'}{v}\right)' + \dfrac{f'g'}{fv^2}, \end{cases}$$

with the periodic condition

$$\begin{cases} f(t,\ u+2\pi) = f(t,\ u) \\ g(t,\ u+2\pi) = g(t,\ u) \end{cases}$$

and the initial condition.

For plane curve shortening, second terms of the right hand side of (3) does not appear. Therefore we regard (2) as the perturbed equation and so, dropping the y-coordinate, we take a coordinate (x, z) only in the sequel.

For later use we denote the length of C_t and the area enclosed by C_t by L and A, respectively:

$$L = \int_{C_t} ds, \qquad A = \frac{1}{2}\int_{C_t} \langle F,\ N\rangle ds\,,$$

where $ds = vdu$ is the arc-length parameter.

Our previous result is now stated as follows.

THEOREM 2.1 ([**1**]). *Suppose M_0 satisfies the following assumption (A). Then the mean curvature flow shrinks M_0 to a circle within finite time.*

(*A*) *There exists a positive constant* ε *such that*

$$f > \varepsilon \quad \textit{and} \quad k_m > \frac{1}{\varepsilon}\frac{1+\sqrt{5}}{2}.$$

The key observation in [**1**] is that the assumption (A) forces the generating curve to stay convex.

The next question naturally arises; how is the shape of the generating curve becomes? Is it becoming circular as in the case of plane curve shortening [**9**], [**10**]? The answer is positive when the isoperimetric ratio stays bounded. This is the focus of this article.

Now let (f, g) be the solution of (2). We *assume* (f, g) converges to (1, 0) smoothly as $t \to T$. Let $\rho(X, t)$ be the backward heat kernel at ((1, 0), T), namely (see [**19**], [**21**]),

(4) $$\rho(X,\ t) = \frac{1}{\sqrt{4\pi(T-t)}}\exp\left\{-\frac{|X|^2}{4(T-t)}\right\} \qquad t < T\,.$$

Here we put $X=(f-1, g)$.

We next define the rescaled immersions $\tilde{X}\equiv(\tilde{f}-1, \tilde{g})$ by

$$\tilde{X}(\cdot, \tilde{t})=\frac{1}{\sqrt{2(T-t)}}X(\cdot, t), \qquad \tilde{t}(t)=-\frac{1}{2}\log(T-t). \tag{5}$$

Similarly we denote the rescaled quantities by $\tilde{\ }$ (for example, $\tilde{A}$, $\tilde{L}$, $\cdots$). Notice that we take the point (1, 0) to be the center of rescaling.

Our main result is then stated as follows:

THEOREM 2.2. *Suppose the solution* (f, g) *of* (2) *converges smoothly to* (1, 0) *as* $t\rightarrow T$. *Suppose also that the isoperimetric ratio* L^2/A *of* C_t *is bounded as it converges. Then the generating curve* $\tilde{C}_{\tilde{t}}$ *of* $\tilde{M}_{\tilde{t}}\equiv\tilde{F}(\cdot, \tilde{t})=(\tilde{f}(\tilde{t}), \tilde{g}(\tilde{t}))$ *converges smoothly to a unit circle centered at* (1, 0).

In particular when C_t *stays convex as it converges then the corresponding isoperimetric ratio remains bounded and hence* $\tilde{C}_{\tilde{t}}$ *converges smoothly to a circle centered at* (1, 0).

COROLLARY 2.3. *If the initial torus satisfies the assumption* (*A*) *in Theorem* 2.1, *then the limit shape of its generating curve is a circle.*

We remark here that the boundedness of the isoperimetric ratio seems to be an unpleasant assumption. However it is relevant to the curve shortening problem. Indeed in [**16**] Grayson showed that in a figure-eight curve shortening the unboundedness of the isoperimetric ratio is equivalent to that the loops bound regions of equal area. We also notice that in a convex plane curve shortening Gage [**9**] proved that the isoperimetric ratio is monotone decreasing and so it is bounded.

§ 3. Preliminaries.

Before proving the theorem we make some preliminaries. Many of them are variants of [**1**]. But we present them for completeness.

We first show the curvature k_m evolves according to

PROPOSITION 3.1.

$$\frac{\partial k_m}{\partial t}=\frac{\partial^2}{\partial s^2}H+k_m{}^2H.$$

For the proof we need some lemmas.

LEMMA 3.2.

(i) $\dfrac{\partial v}{\partial t}=-k_m H v$

(ii) $\frac{\partial}{\partial t}\frac{\partial}{\partial s}=\frac{\partial}{\partial s}\frac{\partial}{\partial t}+k_m H\frac{\partial}{\partial s}.$

PROOF.

(i) Using Frenet's equality $\partial N/\partial s=k_m T$ we have

$$\frac{\partial(v^2)}{\partial t}=\frac{\partial}{\partial t}\left\langle\frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\right\rangle=2\left\langle\frac{\partial}{\partial u}(-HN), \frac{\partial F}{\partial u}\right\rangle$$

$$=-2H\left\langle\frac{\partial N}{\partial u}, \frac{\partial F}{\partial u}\right\rangle=-2k_m Hv^2 .$$

Here $\langle\cdot, \cdot\rangle$ denotes the inner product.

(ii) $\frac{\partial}{\partial t}\frac{\partial}{\partial s}=\frac{\partial}{\partial t}\frac{1}{v}\frac{\partial}{\partial u}=\frac{\partial}{\partial s}\frac{\partial}{\partial t}+k_m H\frac{\partial}{\partial s}.$ ■

LEMMA 3.3.

(i) $\frac{\partial T}{\partial t}=-\frac{\partial H}{\partial s}N$ (ii) $\frac{\partial N}{\partial t}=\frac{\partial H}{\partial s}T.$

PROOF.

(i) From Lemma 3.2 (ii) we compute

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\frac{\partial F}{\partial s}=\frac{\partial}{\partial s}(-HN)+k_m HT=-\frac{\partial H}{\partial s}N .$$

(ii) $\frac{\partial N}{\partial t}=\left\langle\frac{\partial N}{\partial t}, T\right\rangle T=\left\langle\frac{\partial N}{\partial t}, N\right\rangle N=-\left\langle N, \frac{\partial T}{\partial t}\right\rangle T=\frac{\partial H}{\partial s}N,$

since $\partial N/\partial t$ is orthogonal to N and $\langle N, T\rangle=0$. ■

Locally we take parameter θ which is an angle between the x-axis and the tangent line to the curve C. The unit tangent vector T and the unit outer normal vector N is given by

$$T=(\cos\theta, \sin\theta) \qquad N=(\sin\theta, -\cos\theta) .$$

Then there holds the following

LEMMA 3.4.

(i) $\frac{\partial\theta}{\partial s}=k_m$ (ii) $\frac{\partial\theta}{\partial t}=\frac{\partial H}{\partial s}.$

PROOF.

(i) In view of Frenet's equality we get

$$\frac{\partial\theta}{\partial s}N=-\frac{\partial}{\partial s}(\cos\theta, \sin\theta)=-\frac{\partial T}{\partial s}=k_m N .$$

(ii) Lemma 3.3 (i) leads to

$$\frac{\partial\theta}{\partial t}N=-\frac{\partial}{\partial t}(\cos\theta,\ \sin\theta)=\frac{\partial H}{\partial s}N. \qquad \blacksquare$$

PROOF OF THE PROPOSITION 3.1. From Lemmas 3.4 and 3.2(ii) we derive

$$\frac{\partial k_m}{\partial t}=\frac{\partial}{\partial t}\frac{\partial\theta}{\partial s}=\frac{\partial}{\partial s}\frac{\partial\theta}{\partial t}+k_mH\frac{\partial\theta}{\partial s}=\frac{\partial^2 H}{\partial s^2}+k_m{}^2H. \quad \blacksquare$$

We compute the explicit formulation for the evolution of k_m.

COROLLARY 3.5.

$$\frac{\partial k_m}{\partial t}=\frac{\partial^2 k_m}{\partial s^2}+\frac{1}{f}\frac{\partial f}{\partial s}\frac{\partial k_m}{\partial s}+\left(k_l{}^2-\frac{2}{f^2}+\frac{2k_l{}^2}{f^2}\right)k_m+k_m^3+\frac{2}{f^2}(1-k_l{}^2)k_l.$$

PROOF. It is easy to check

$$\frac{\partial k_l}{\partial s}=\frac{\partial}{\partial s}\left(\frac{g'}{fv}\right)=\frac{1}{f}\frac{\partial f}{\partial s}k_m-\frac{1}{f}\frac{\partial f}{\partial s}k_l,$$

since $\partial/\partial s(g'/v)=(f'/v)k_m$ and

$$\begin{aligned}\frac{\partial^2 k_l}{\partial s^2}&=\frac{\partial}{\partial s}\left(\frac{1}{f}\frac{\partial f}{\partial s}\right)(k_m-k_l)+\frac{1}{f}\frac{\partial f}{\partial s}\frac{\partial k_m}{\partial s}-\frac{1}{f}\frac{\partial f}{\partial s}\frac{\partial k_l}{\partial s}\\&=\left(-\frac{1}{f^2}\left(\frac{\partial f}{\partial s}\right)^2-\frac{1}{f}\frac{\partial g}{\partial s}k_m\right)(k_m-k_l)+\frac{1}{f}\frac{\partial f}{\partial s}\frac{\partial k_m}{\partial s}\\&\quad-\frac{1}{f^2}\left(\frac{\partial f}{\partial s}\right)^2(k_m-k_l)\\&=\frac{1}{f}\frac{\partial f}{\partial s}\frac{\partial k_m}{\partial s}-k_mk_l(k_m-k_l)-2\left(\frac{1}{f^2}-k_l^2\right)(k_m-k_l). \quad \blacksquare\end{aligned}$$

One can give the evolution of L and A.

PROPOSITION 3.6.

(i) $\displaystyle\frac{\partial L}{\partial t}=-\int_C k_mHds$ (ii) $\displaystyle\frac{\partial A}{\partial t}=-\int_C Hds.$

PROOF.

(i) From Lemma 3.2(i) one has

$$\frac{\partial L}{\partial t}=\frac{d}{dt}\int_0^{2\pi}vdu=-\int_C k_mHds.$$

(ii) Using Lemma 3.3(ii) and the integration by parts one gets

$$\frac{\partial A}{\partial t}=\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}\langle F, N\rangle v du$$
$$=\frac{1}{2}\int\left(-Hv+\langle F, T\rangle\frac{\partial H}{\partial s}v-k_m H\langle F, N\rangle v\right)du$$
$$=\frac{1}{2}\int\left(-H-H\frac{\partial}{\partial s}\langle F, T\rangle-k_m H\langle F, N\rangle\right)ds$$
$$=-\int_C H ds. \quad \blacksquare$$

Following the idea of Huisken [**19**] we next examine the evolution formula for the backward heat kernel. Due to the effect of the latitudinal curvature k_l we do not expect a monotonicity formula in general.

PROPOSITION 3.7. *Let C_t be a plane curve satisfying* (2) *for $t<T$, then we have*

$$\frac{d}{dt}\int_{C_t}\rho(X, t)ds=-\int_{C_t}\left(k_m-\frac{\langle X, N\rangle}{2(T-t)}\right)^2\rho ds$$
$$-\int_{C_t}k_l\left(k_m-\frac{\langle X, N\rangle}{2(T-t)}\right)\rho ds,$$

where $\rho(X, t)$ is given in (4) *and $X=(f-1, g)$.*

PROOF. From (2) we derive

$$\frac{d}{dt}\int_{C_t}\rho(X, t)ds$$
$$=-\int_{C_t}\left\{k_m H-\frac{1}{2(T-t)}+\frac{X\cdot dX/dt}{2(T-t)}+\frac{|X|^2}{4(T-t)^2}\right\}\rho ds$$
$$=-\int_{C_t}\left\{\left|k_m N-\frac{X}{2(T-t)}\right|^2+\left(k_m k_l-\frac{1}{2(T-t)}\right)+\frac{k_m-k_l}{2(T-t)}\langle X, N\rangle\right\}\rho ds .$$

Invoking the identity

$$\int_{C_t}\left\langle T, \frac{\partial}{\partial s}(\rho X)\right\rangle ds=\int_{C_t}k_m\langle N, \rho X\rangle ds,$$

which follows from integration by parts, we finally obtain

$$\frac{d}{dt}\int_{C_t}\rho(X, t)ds$$
$$=-\int_{C_t}\left\{\left|k_m N-\frac{X}{2(T-t)}\right|^2+\left(k_m k_l-\frac{1}{2(T-t)}\right)-\frac{k_l}{2(T-t)}\langle X, N\rangle\right\}\rho ds$$
$$-\int_{C_t}\left\{\frac{1}{2(T-t)}-\frac{\langle X, T\rangle^2}{4(T-t)^2}\right\}\rho ds$$

$$= -\int_{C_t}\left(k_m - \frac{\langle X, N\rangle}{2(T-t)}\right)^2 \rho ds - \int_{C_t} k_l\left(k_m - \frac{\langle X, N\rangle}{2(T-t)}\right)\rho ds,$$

which proves the desired formula. ■

In the rescaled setting we define a corresponding function $\tilde{\rho}$ by

$$\tilde{\rho}(\tilde{X}) = \exp\left\{-\frac{|\tilde{X}|^2}{2}\right\}.$$

PROPOSITION 3.8. *For the rescaled immersions* (5) *we have*

$$\frac{d}{d\tilde{t}}\int_{\tilde{C}_{\tilde{t}}} \tilde{\rho}(\tilde{X}, \tilde{t})d\tilde{s} = -\int_{\tilde{C}_{\tilde{t}}} (\tilde{k}_m - \langle \tilde{X}, \tilde{N}\rangle)^2 \tilde{\rho} d\tilde{s} - \int_{\tilde{C}_{\tilde{t}}} \sqrt{2T} e^{-\tilde{t}} k_l(\tilde{k}_m - \langle \tilde{X}, \tilde{N}\rangle)\tilde{\rho} d\tilde{s}. \tag{6}$$

Here $\tilde{X} = (\tilde{f}-1, \tilde{g})$.

PROOF. From the identity

$$\frac{d}{d\tilde{t}}\int_{\tilde{C}_{\tilde{t}}} \tilde{\rho} d\tilde{s} = 2(T-t)\frac{d}{dt}\int_{C_t} \sqrt{2\pi}\rho ds$$

and an analogous calculation to the proof of Proposition 3.7 we obtain the result. ■

One can give the rescaled evolution for the curvature $\tilde{k}_m$. See Corollary 3.5.

PROPOSITION 3.9.

$$\frac{\partial \tilde{k}_m}{\partial \tilde{t}} = \frac{\partial^2 \tilde{k}_m}{\partial \tilde{s}^2} + \sqrt{2\pi}\frac{e^{-\tilde{t}}}{f}\frac{\partial \tilde{f}}{\partial \tilde{s}}\frac{\partial \tilde{k}_m}{\partial \tilde{s}} + 2Te^{-2\tilde{t}}\left(k_l^2 - \frac{2}{f^2} + \frac{2k_l^2}{f^2}\right)\tilde{k}_m - \tilde{k}_m + \tilde{k}_m^3 + (2T)^{3/2}e^{-3\tilde{t}}\frac{2}{f^2}(1-k_l^2)k_l. \tag{7}$$

We end this section with the following formula on the evolution for L, which will be useful in the next section.

LEMMA 3.10. *We have*

$$\frac{dL}{dt} = -\int_C k_m^2 ds + \int_C \frac{(f')^2}{f^2 v^2} ds.$$

PROOF. From Proposition 3.2 (i) one derives

$$\begin{aligned}\frac{dL}{dt} &= -\int k_m^2 ds - \int k_m k_l ds \\ &= -\int k_m^2 ds - \int \frac{f'g'g'' - (g')^2 f''}{f v^4} ds \\ &= -\int k_m^2 ds - \int \left(-\frac{f''}{f v^2} + \frac{f'}{f}\frac{v'}{v^3}\right) ds \\ &= -\int k_m^2 ds + \int \frac{(f')^2}{f^2 v^2} ds .\end{aligned}$$

In the last equality we performed the integration by parts. ■

§4. Proof of the main theorem.

The assumption that (f, g) converges to $(1, 0)$ smoothly as $t \to T$ implies that $L \to 0$, $A \to 0$ and the latitudinal curvature k_l is uniformly bounded. We first show that the boundedness of the isoperimetric ratio yields a bound for the rescaled length $\tilde{L}$.

From Proposition 3.2 (ii) one obtains

$$\begin{aligned}\frac{dA}{dt} &= -\int k_m ds - \int k_l ds \\ &= -2\pi + O(v) .\end{aligned}$$

In view of $v \to 0$ as $t \to T$ we conclude that for all t sufficiently near T we have, for some constant δ,

$$A(t) \leqq (2\pi + \delta)(T - t) ,$$

which implies $\tilde{A} \leqq (2\pi + \delta)$ for all sufficiently large $\tilde{t}$. Since $L^2/A = \tilde{L}^2/\tilde{A}$ is bounded we get a bound for the rescaled length $\tilde{L}$.

We also have a bound for the rescaled coordinate $(\tilde{f}, \tilde{g})$ since we know the relation

$$\tilde{L} \geqq \max\{|\tilde{f} - 1|, |\tilde{g}|\} .$$

We remark that in the rescaled flow analysis the exact determination of blow-up rate is rather a complicated task (see [**5**], [**19**]). In our situation, however, the boundedness of the isoperimetric ratio plays a crucial rôle and we do without determining it.

Let us now turn our attention to the relation (6). The boundedness of $(\tilde{f}, \tilde{g})$ yields

$$\int_{\tilde{t}_0}^{\infty} \left(\frac{d}{d\tilde{t}} \int_{\tilde{C}_{\tilde{t}}} \tilde{\rho}(\tilde{X}, \tilde{t}) d\tilde{s} \right) d\tilde{t} \geqq -\delta(\tilde{t}_0) ,$$

for some function $\delta(\tilde{t}_0)$. Therefore we integrate (6) with respect to $\tilde{t}$ and invoke the Cauchy-Schwarz inequality to obtain

$$-\delta(\tilde{t}_0) \leqq -\int_{\tilde{t}_0}^{\infty}\int_{\tilde{C}_{\tilde{t}}}(\tilde{k}_m-\langle\tilde{X},\ \tilde{N}\rangle)^2\tilde{\rho}\,d\tilde{s}\,d\tilde{t}$$

$$+Ce^{-\tilde{t}_0}\Big(\int_{\tilde{t}_0}^{\infty}\int(\tilde{k}_m-\langle\tilde{X},\ \tilde{N}\rangle)^2\tilde{\rho}\,d\tilde{s}\,d\tilde{t}\Big)^{1/2},$$

for some constant C. Here the constant C is related to the bound

$$\int_{\tilde{C}_{\tilde{t}}} k_l^2 d\tilde{s} < \infty,$$

which follows from the boundedness of k_l and $\tilde{L}$. Elementary calculus now implies

$$\int_{\tilde{t}_0}^{\infty}\int(\tilde{k}_m-\langle\tilde{X},\ \tilde{N}\rangle)^2\tilde{\rho}\,d\tilde{s}\,d\tilde{t} < \infty.$$

Moreover for each sequence $\tilde{t}_j\to\infty$ there exists a subsequence $\tilde{t}_{jk}\to\infty$ such that $\{\tilde{C}_{\tilde{t}_{jk}}\}$ converges smoothly to $\tilde{C}_\infty$ and hence $\int_{\tilde{C}_{\tilde{t}_{jk}}}\tilde{\rho}\,d\tilde{s}$ converges. For this sequence $\tilde{t}_{jk}$ we have

$$\int_{\tilde{t}_{jk}}^{\infty}\int(\tilde{k}_m-\langle\tilde{X},\ \tilde{N}\rangle)^2\tilde{\rho}\,d\tilde{s}\,d\tilde{t} \longrightarrow 0.$$

In view of again the fact that the rescaled coordinate $(\tilde{f},\ \tilde{g})$ are bounded and the parabolic regularity theory applied to the equation (7) we deduce that

$$\tilde{k}_m-\langle\tilde{X},\ \tilde{N}\rangle \longrightarrow 0$$

smoothly as $\tilde{t}_{jk}\to\infty$.

From Proposition 5.1 in the next section we finally conclude that the limiting $\tilde{C}_\infty$ is a unit circle centered at (1, 0). This proves the main theorem.

Finally we prove that when $\{C_t\}$ stay convex the isoperimetric ratio is bounded. To do that from Lemmas 3.6 and 3.10

$$(8)\quad \frac{d}{dt}\Big(\frac{L^2}{A}\Big)=2\frac{L}{A}\frac{dL}{dt}-\frac{L^2}{A^2}\frac{dA}{dt}$$

$$=-2\frac{L}{A}\Big(\int k_m^2 ds-\pi\frac{L}{A}\Big)+\frac{L^2}{A^2}\int k_l ds$$

$$+2\frac{L}{A}\int\frac{1}{f^2}\Big(\frac{\partial f}{\partial s}\Big)^2 ds.$$

Gage [9] showed that for convex plane curve, the inequality

$$\int k_m^2 ds \leqq \pi\frac{L}{A}$$

holds. For the second term of (8) we have, since $k_m\geqq 0$ and $f=1+O(v)$

$$\frac{d}{dt}\int k_l ds = \int_0^{2\pi}\left(-\frac{g'}{f^2}\frac{\partial f}{\partial t}+\frac{1}{f}\frac{\partial}{\partial t}\right)g' du$$
$$= \int_C \frac{(g')^2+(f')^2}{f^2 v^2}(k_m+k_l)ds$$
$$\geqq \int_C \frac{1}{f^2}k_l ds = (1+O(v^2))\int_C k_l ds,$$

which implies $\int_C k_l ds \leqq 0$ for all t sufficiently near T. Therefore (8) can be estimated as

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) \leqq C\frac{L^2}{A},$$

for some constant C for all t sufficiently near T, from which we conclude that the isoperimetric ratio is bounded in this convex case.

§ 5. Limit shape.

In this final section we prove the following geometric proposition. Although it is elementary there seems to be no presentation in the literature.

PROPOSITION 5.1. *Suppose that the closed embedded plane curve C satisfies*

$$\langle X, N\rangle = k. \tag{9}$$

Here X, N, k denote the position vector, the outer unit normal, the curvature, respectively. Then C is the unit circle centered at the origin.

PROOF. We introduce the coordinate $X=(x(s), y(s))$ parametrized by arc length. Then the unit tangent vector T and the unit outer normal vector N are given by

$$T=(\dot{x}(s), \dot{y}(s)) \qquad N=(\dot{y}(s), -\dot{x}(s)),$$

where $\dot{}$ denotes the derivation by arc length s. Frenet's equality $dT/ds=-kN$ implies that (9) is equivalent to

$$\left\langle X, -\frac{d^2X}{ds^2}\right\rangle = 1,$$

that is, $1=-x\ddot{x}-y\ddot{y}$. Since at the point where $|k|$ attains its maximum we have $\langle X, T\rangle=0$, we obtain

$$k^2|_{\max} = \langle X, N\rangle\cdot(\dot{x}\ddot{y}-\ddot{x}\dot{y})$$
$$= (x\dot{y}-y\dot{x})(\dot{x}\ddot{y}-\ddot{x}\dot{y})$$
$$= x\dot{x}\dot{y}\ddot{y}-x\ddot{x}\dot{y}^2-\dot{x}^2y\ddot{y}+\dot{x}\ddot{x}y\dot{y}$$

$$= -y\dot{y}^2\ddot{y} - x\ddot{x}\dot{y}^2 - \dot{x}^2 y\ddot{y} - x\dot{x}^2\ddot{x}$$
$$= -x\ddot{x} - y\ddot{y} = 1.$$

On the other hand (9) and Frenet's equality leads

$$\frac{\partial k}{\partial s} = k\langle X, T\rangle$$

(10)
$$\frac{\partial^2 k}{\partial s^2} = \frac{1}{k}\left(\frac{\partial k}{\partial s}\right)^2 + k - k^3,$$

for any interval where $k \neq 0$. Here we note that since $k(s) \leqq 0$ for all s yields an obvious contradiction we do have a point where $k > 0$. Then at the point where $k|_{\max}$ attains there holds $0 \geqq k - k^3$. More precisely $k|_{\max} \geqq 1$. From the uniqueness for (10) we therefore deduce $k \equiv 1$. This immediately implies our statement. ■

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Naoyuki ISHIMURA
Department of Mathematical Sciences
University of Tokyo
Hongo 7-3-1, Tokyo 113
Japan