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A remark on the Kawamata rationality theorem

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Introduction.

Let X be a projective variety with Gorenstein, rational singularities. Let $\varphi: X \to Y$ be a surjective morphism with connected fibers from X to a normal projective variety Y. Let L be a φ-ample line bundle and assume that K_X is not φ-nef. Then the Kawamata rationality theorem states that there is a positive fraction $\tau = u/v$, where u, v are positive coprime integers, and such that

a) $K_X + \tau L$ is φ-nef but not φ-ample;

b) $u \leqq \max_{y \in Y}\{\dim \varphi^{-1}(y)\} + 1$.

If u takes on the maximal value, $\max_{y \in Y}\{\dim \varphi^{-1}(y)\} + 1$, allowed by the Kawamata rationality theorem, then X is a $\boldsymbol{P}^{u-1}$ bundle over Y (see (2.2)). Moreover there is an ample line bundle $\mathcal{L}$ on X such that $K_X \otimes \mathcal{L}^u \approx \varphi^* H$ for an ample line bundle H on Y, and thus $X = \boldsymbol{P}(\mathcal{E})$ for the ample vector bundle $\mathcal{E} = \varphi_* \mathcal{L}$.

If L is ample and K_X is not nef, the Kawamata rationality theorem and the Kawamata-Shokurov base point free theorem imply that there is a fraction, $\tau = u/v$, with u, v positive coprime integers (called the *nef value* of the pair (X, L)) and a morphism $\phi: X \to Y$ with connected fibers (called the *nef value morphism* of the pair (X, L)) onto a normal projective variety Y such that

i) $vK_X + uL \approx \phi^* H$ for an ample line bundle H on Y,

ii) $u \leqq \max_{y \in Y}\{\dim \phi^{-1}(y)\} + 1$.

We saw that $u = \max_{y \in Y}\{\dim \phi^{-1}(y)\} + 1$ implies that $\phi: X \to Y$ is very special. In our main result, (1.4.2), we study the structure of the nef value morphism, ϕ, in the case when $u = \max_{y \in Y}\{\dim \phi^{-1}(y)\}$. If the nef value morphism is birational we need a smoothness assumption on X.

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We would like to thank the referee for suggesting improvements of Theorem (2.2). The original proof was longer and only worked in the smooth case.

§ 0. Background material.

(0.1) Notation. We work over the complex field $\boldsymbol{C}$. By *variety* we mean an irreducible and reduced projective scheme, V. We denote its structure sheaf by $\boldsymbol{O}_V$.

Basically we use the standard notation from algebraic geometry. We almost always follow the notation of [**BS1**]. We refer to it and to [**KMM**] for definitions of the following: $\boldsymbol{Q}$-divisor, $\boldsymbol{Q}$-Cartier divisor, $\boldsymbol{Q}$-factorial, $\boldsymbol{Q}$-Gorenstein, numerically effective (nef, for short), numerical equivalence (denoted by $\sim$), linear equivalence (denoted by $\approx$) of $\boldsymbol{Q}$-divisors, intersection of cycles (denoted by "$\cdot$"), canonical divisor, terminal, log-terminal, and canonical singularities. Note that for Gorenstein varieties, rational, canonical, and log-terminal singularities are all equivalent (see [**KMM**], (0.2)).

The smallest positive integer, r, such that rK_V is a line bundle, where K_V is the canonical divisor of a normal variety V, is called the *index* of V.

Linear equivalence classes of Weil divisors on a normal variety and isomorphism classes of reflexive sheaves of rank 1 are used with little (or no) distinction. Hence we shall freely switch back and forth between the multiplicative and additive notation for divisors.

We fix some more notation (here $\mathcal{L}$ denotes a rank 1 reflexive sheaf on a variety V).

$|\mathcal{L}|$, the complete linear system associated to $\mathcal{L}$;

$\Gamma(\mathcal{L})$, the space of the global sections of $\mathcal{L}$; we say that $\mathcal{L}$ is *spanned* if $\mathcal{L}$ is spanned at all points of V by $\Gamma(\mathcal{L})$;

$f(\varphi)$, the dimension of the general fiber of a surjective morphism with connected fibers, $\varphi: V\to Y$, of two varieties V, Y.

Let $\varphi: V\to Y$ be a surjective morphism of varieties. Let $\mathcal{L}$ be a rank 1 reflexive sheaf such that $r\mathcal{L}$ is a line bundle for some positive integer r. We say that $\mathcal{L}$ is *φ-ample* if $r\mathcal{L}$ is φ-ample in the ordinary sense (see e. g. [**I**], Chapter 7).

(0.2) Nef values. The following Theorem, due to Kawamata, inspired this paper. We state here the results in the case of terminal singularities which occur in the adjunction theory (see e. g. [**BS1**]), even though they hold true in the more general case of log-terminal singularities (see [**KMM**], 4.1).

(0.2.1) Kawamata Rationality Theorem. *Let V be a normal variety of dimension n with terminal singularities and let r be the index of V. Let $\pi: V\to S$ be a projective morphism onto a variety S. Let L be a π-ample line bundle on V. If K_V is not π-nef, then*

$$\tau := \min\{t\in\boldsymbol{R},\ K_V+tL \text{ is } \pi\text{-nef}\}$$

is a positive rational number. Furthermore expressing $r\tau=u/v$ *with* u, v *positive coprime integers, we have* $u\leqq r(b+1)$ *where* $b=\max_{s\in S}\{\dim\pi^{-1}(s)\}$.

With the notation as in (0.2.1) we say that the rational number τ is the π-*nef value* of (V, L). If S is a point, τ is called the *nef value* of (V, L). Note also that, if S is a point, then $K_V+\tau L$ is nef and hence by the Kawamata-Shokurov Base Point Free Theorem ([**KMM**], §3) we know that $|m(K_V+\tau L)|$ is base point free for $m\gg 0$ such that $m\tau$ and m/r are integral, and defines a morphism, ϕ, which we call the *nef value morphism* of (V, L).

(0.2.2) REMARK. Let (V, L) be as in (0.2.1). Let τ be the nef value of (V, L) and let ϕ be the nef value morphism of (V, L). Then

$$\tau=\min\{t\in\boldsymbol{R},\ K_V+tL \text{ is nef}\}=\min\{t\in\boldsymbol{R},\ K_V+tL \text{ is } \phi\text{-nef}\}.$$

That is, τ coincides with the ϕ-nef value of (V, L).

(0.2.3) LEMMA ([**BS1**], (0.8.3)). *Let* V *be a normal variety with terminal singularities. Let* L *be an ample line bundle on* V. *A rational number* τ *is the nef value of* (V, L) *if and only if* $K_V+\tau L$ *is nef but not ample.*

(0.3) Let V be an n-dimensional normal variety with canonical singularities. Define

$Z_1(V)=$ the free abelian group generated by reduced irreducible curves;
$N_1(V)=\{Z_1(V)/\sim\}\otimes\boldsymbol{R}$;
$NE(V)=$ the convex cone in $N_1(V)$ generated by the effective 1-cycles;
$\overline{NE}(V)=$the closure of $NE(V)$ with respect to the Euclidean topology.

A part of Mori's theory of extremal rays is to be used throughout the paper. We will use freely the notion of extremal ray, extremal rational curve, as well as the basic theorems such as Cone Theorem and Contraction Theorem. We refer the reader to [**M**] and [**KMM**].

In particular we will denote by $\rho=\mathrm{cont}_R: V\to Y$ the morphism given by the contraction of an extremal ray R. We also simply refer to ρ as *Mori contraction*. We say that ρ is of fiber type if $n>\dim Y$ and R is nef in this case. If R is not nef then ρ is a birational morphism. If γ is a 1-dimensional cycle in V we will denote by $\boldsymbol{R}_+[\gamma]$, where $\boldsymbol{R}_+=\{x\in\boldsymbol{R},\ x\geqq 0\}$, or $[\gamma]$ its class in $\overline{NE}(V)$.

If V is smooth, the *length* $l(R)$ of an extremal ray R is defined as

$$l(R):=\min\{-K_V\cdot C,\ C \text{ rational curve and } [C]\in R\}.$$

We will denote by $E(R)$ the *locus* of R, that is the locus of curves whose numerical classes are in R. We say that R is of *divisorial type* if $\dim E(R)=$

$n-1$. Let $\mathcal{E}$ be any irreducible component of the locus, $E(R)$, of R and let Δ be an irreducible component of any fiber of the restriction $\rho_{\mathcal{E}}: \mathcal{E} \to \rho(\mathcal{E})$. If V is smooth, we have the following result of Wisniewski [**W**], (1.1) (see also [**Io**], (0.4))

$$\dim \mathcal{E} + \dim \Delta \geqq n + l(R) - 1 . \tag{0.3.1}$$

(0.3.2) LEMMA ([**BS1**], (0.4.3)). *Let V be a normal projective variety with at most canonical singularities. Let L be an ample line bundle on V and let t be some positive rational number such that K_V+tL is nef. Let C be an effective curve in $NE(V)$ such that $(K_V+tL)\cdot C=0$. Then C can be written in $NE(V)$ as a finite sum $C=\sum_i \lambda_i C_i$ where $\lambda_i \in \boldsymbol{R}_+$ and $\boldsymbol{R}_+[C_i]$ are extremal rays such that $(K_V+tL)\cdot C_i=0$ for all i. In particular if V is nonsingular the curves C_i can be taken to be extremal rational curves.*

(0.3.3) LEMMA. *Let V be an n-dimensional smooth projective variety. Let R_1, R_2 be two distinct extremal rays of length $l(R_i)$, $i=1, 2$. Let $E(R_i)$ be the loci of R_i, $i=1, 2$. If $l(R_1)+l(R_2) \geqq n+3$ then $E(R_1) \cap E(R_2) = \emptyset$.*

Furthermore if R_1, R_2 are not nef and $l(R_1)+l(R_2) \geqq n+1$ then $E(R_1) \cap E(R_2) = \emptyset$.

PROOF. Assume $E(R_1) \cap E(R_2) \neq \emptyset$ and take a point $v \in E(R_1) \cap E(R_2)$. Let ρ_i be the contraction of R_i and let Δ_i be an irreducible component of a fiber of ρ_i with $v \in \Delta_i$, $i=1, 2$. If $\dim(\Delta_1 \cap \Delta_2) \geqq 1$, there exists a curve, C, contained in $\Delta_1 \cap \Delta_2$ which contracts to a point under ρ_1, ρ_2. Therefore $[C] \in R_1$, $[C] \in R_2$. This leads to the contradiction $R_1 = R_2$. Thus we have

$$0 = \dim(\Delta_1 \cap \Delta_2) \geqq \dim \Delta_1 + \dim \Delta_2 - n$$

and hence

$$\dim \Delta_1 + \dim \Delta_2 \leqq n . \tag{0.3.3.1}$$

By (0.3.1) we have, for $i=1, 2$,

$$n + \dim \Delta_i \geqq \dim E(R_i) + \dim \Delta_i \geqq n + l(R_i) - 1 .$$

This gives $\dim \Delta_1 + \dim \Delta_2 \geqq l(R_1) + l(R_2) - 2 \geqq n+1$, which contradicts (0.3.3.1).

If ρ_1, ρ_2 are birational, (0.3.1) yields, for $i=1, 2$,

$$n-1+\dim \Delta_1 \geqq \dim E(R_i) + \dim \Delta_i \geqq n + l(R_i) - 1 .$$

This gives $\dim \Delta_1 + \dim \Delta_2 \geqq l(R_1) + l(R_2) \geqq n+1$, the same contradiction as above. Q. E. D.

REMARK (0.3.3.2). Notation as in (0.3.3). The proof above shows that $E(R_1) \cap E(R_2) = \emptyset$ if $l(R_1)+l(R_2) \geqq n+3-\mathrm{cod}_V E(R_1) - \mathrm{cod}_V E(R_2)$. Note that in this case the assumption that R_1, R_2 are not nef is not needed.

(0.4) Some special varieties. Let V be a normal variety of dimension n and index r, L an ample line bundle on V. We say that (V, L) is a *scroll* (respectively a *quadric fibration*) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibers $p: V \to Y$, such that $r(K_V+(n-m+1)L) \approx p^*\mathcal{L}$ (respectively $r(K_V+(n-m)L) \approx p^*\mathcal{L}$) for some *ample* line bundle $\mathcal{L}$ over Y.

We say that (V, L) is a $\boldsymbol{P}^d$ *bundle* over Y if there exists a surjective morphism $p: V \to Y$ such that all fibers F of p are $\boldsymbol{P}^d$ and $L_F \cong \boldsymbol{O}_{\boldsymbol{P}^d}(1)$.

For any further background material we refer to [**BS1**] and [**KMM**].

§1. On the structure of the nef value morphism.

Let X be a projective, irreducible, variety of dimension n with Gorenstein, rational singularities. Hence in particular, X has log-terminal singularities (see [**KMM**], (0.2)). Let L be an ample line bundle on X. Let $\tau = u/v$ be the nef value of (X, L), u, v coprime integers. Let $\phi: X \to W$ be the nef value morphism of (X, L). From the Kawamata Rationality Theorem we know that $u \leqq \max_{w \in W}\{\dim \phi^{-1}(w)\}+1$. In this section we study the structure of ϕ in the cases when either $u = \max_{w \in W}\{\dim \phi^{-1}(w)\}+1$ or $u = \max_{w \in W}\{\dim \phi^{-1}(w)\}$.

First we need the following preparatory lemmas.

(1.1) LEMMA. *Let u, v coprime positive integers. Then there exist positive integers a, b such that $av-bu=1$.*

PROOF. There exist integers a', b' such that $a'v-b'u=1$. If $a'>0$, $b'>0$ we are done. If not, let $a := a'+\lambda u$, $b := b'+\lambda v$ for $\lambda \gg 0$. Therefore

$$av-bu = (a'+\lambda u)v-(b'+\lambda v)u = a'v-b'u = 1$$

and $a>0$, $b>0$ for $\lambda \gg 0$. Q.E.D.

(1.2) LEMMA. *Let X be a normal projective variety with log-terminal singularities and let r be the index of X. Let L be an ample line bundle on X. Let $\varphi: X \to Y$ be a surjective morphism onto a normal variety Y. Assume that φ has at least one positive dimensional fiber and that $rvK_X+uL \approx \varphi^*H$ for some ample line bundle H on Y and coprime positive integers u, v. Let $\mathcal{L} := brK_X+aL$ where a, b are as in* (1.1). *Then $\mathcal{L}$ is ample, $rK_X+u\mathcal{L} \approx \varphi^*(aH)$ and u/r is the nef value of $(X, \mathcal{L})$.*

PROOF. Since $av-bu=1$ by Lemma (1.1) we have

$$rK_X+u\mathcal{L} = r(1+ub)K_X+auL = ravK_X+auL = a(rvK_X+uL) \approx \varphi^*(aH).$$

Hence in particular $rK_X+u\mathcal{L}$ is nef but not ample, so that u/r is the nef value of $(X, \mathcal{L})$ by (0.2.3). Note that $\mathcal{L}=brK_X+aL$ is ample since rvK_X+uL is nef and $a/br=1/brv+u/rv>u/rv$. Q. E. D.

The following consequence of (1.2) improves (3.1.1.2) of [**BSW**].

(1.3) COROLLARY. *Let X be a smooth connected projective variety, L an ample line bundle on X. Let $\tau=u/v$ be the nef value of (X, L) and let $\phi: X\to Y$ be the morphism with connected fibers associated to $|m(K_X+\tau L)|$ for $m\gg 0$. Assume that ϕ is not birational. If $u\geqq n/2+1$, ϕ is a fiber type contraction of an extremal ray R. Let $\mathcal{L}$ be the ample line bundle on X given by* (1.2). *Then* $\mathrm{Pic}(X)\cong\phi^*\mathrm{Pic}(Y)\oplus\boldsymbol{Z}[\mathcal{L}]$ *unless $u=n/2+1$, $(X, \mathcal{L})\cong(\boldsymbol{P}^{n/2}\times\boldsymbol{P}^{n/2}, \boldsymbol{O}_{\boldsymbol{P}^{n/2}\times\boldsymbol{P}^{n/2}}(1))$ and* $\dim Y=0$.

PROOF. By the Rationality Theorem (0.2.1), $\tau=u/v$ where u, v are coprime positive integers. By Lemma (1.1) there exist positive integers a, b such that $av-bu=1$.

Let $\mathcal{L}:=bK_X+aL$. From Lemma (1.2) we know that $\mathcal{L}$ is ample, $K_X+u\mathcal{L}=a(vK_X+uL)\approx\phi^*(aH)$ for an ample line bundle H on Y and u is the nef value of $(X, \mathcal{L})$. From $K_X+u\mathcal{L}\approx\phi^*(aH)$ it thus follows that the nef value morphism of $(X, \mathcal{L})$ coincides with ϕ. Therefore from [**BSW**], (3.1.1.2) we conclude that, if $u\geqq n/2+1$, ϕ is a fiber type contraction of an extremal ray and $\mathrm{Pic}(X)\cong\phi^*\mathrm{Pic}(Y)\oplus\boldsymbol{Z}[\mathcal{L}]$ unless $u=n/2+1$ and $(X, \mathcal{L})\cong(\boldsymbol{P}^{n/2}\times\boldsymbol{P}^{n/2}, \boldsymbol{O}_{\boldsymbol{P}^{n/2}\times\boldsymbol{P}^{n/2}}(1))$ and $\dim Y=0$. Q. E. D.

We can prove now the main result of this paper.

(1.4) THEOREM. *Let X be a projective, irreducible, variety of dimension n with Gorenstein, rational singularities. Assume K_X is not nef. Let L be an ample line bundle on X. Let $\tau=u/v$ be the nef value of (X, L), u, v coprime positive integers. Let $\phi: X\to W$ be the nef value morphism of (X, L). Let $\mathcal{L}:=bK_X+aL$ be an ample line bundle on X given by* (1.2).

(1.4.1) *Assume that $u=\max_{w\in W}\{\dim\phi^{-1}(w)\}+1$. Then $(X, \mathcal{L})$ is a scroll over W under ϕ. If X is smooth, or more generally if* $\operatorname{codim}\mathrm{Sing}(X)>\dim W$, *then $(X, \mathcal{L})$ is in fact a $\boldsymbol{P}^{u-1}$ bundle over W under ϕ. Furthermore ϕ is a fiber type contraction of an extremal ray.*

(1.4.2) *Assume that $u=\max_{w\in W}\{\dim\phi^{-1}(w)\}$. If ϕ is not birational, then either*

i) *$(X, \mathcal{L})$ is a scroll over W under ϕ; or*

ii) *$(X, \mathcal{L})$ is a quadric fibration over W under ϕ, and all fibers are equidimensional.*

If ϕ is birational, X is smooth, and $u\geqq(n+1)/2$, then

iii) *ϕ is the simultaneous contraction of a finite number of extremal rays* *and is an isomorphism outside of $\phi^{-1}(\mathcal{B})$ where $\mathcal{B}\subset W$ is an algebraic subset of W which is the disjoint union of irreducible components of dimension $n-u-1$. Let B be an irreducible component of $\mathcal{B}$ and let $E=\phi^{-1}(B)$. The general fiber, $\varDelta$, of the restriction, ϕ_E, of ϕ to E is a linear $\boldsymbol{P}^u$, $(\varDelta, \mathcal{L}_\varDelta)\cong(\boldsymbol{P}^u, \boldsymbol{O}_{\boldsymbol{P}^u}(1))$, $\mathcal{N}^X_{E|\varDelta}\cong\boldsymbol{O}_{\boldsymbol{P}^u}(-1)$ and W is factorial with terminal singularities.*

PROOF. We only prove (1.4.2). Indeed (1.4.1) is essentially contained in Theorem (2.2). So for a proof of it we simply refer to (2.2) below.

To prove (1.4.2), let us first consider the case when ϕ is not birational. Since $vK_X+uL\approx\phi^*H$ for some ample line bundle H on W we have from Lemma (1.2) that $K_X+u\mathcal{L}\approx\phi^*(aH)$, where a is a positive integer. Let F be a general fiber of ϕ and let $\mathcal{L}_F$ be the restriction of $\mathcal{L}$ to F. Then $K_F+u\mathcal{L}_F\approx\boldsymbol{O}_F$ and hence

$$u\leqq\dim F+1=f(\phi)+1\,,$$

where $f(\phi)$ denotes the dimension of the general fiber of ϕ. Since $u=\max_{w\in W}\{\dim\phi^{-1}(w)\}\geqq f(\phi)$, then either $u=f(\phi)+1$ or $u=f(\phi)$.

Let $u=f(\phi)+1$. Therefore $K_X+u\mathcal{L}\approx\phi^*(aH)$ where $u=n-\dim W+1$. This means that $(X, \mathcal{L})$ is a scroll under ϕ as in (1.4.2), i).

Let $u=f(\phi)$. In this case $K_X+u\mathcal{L}\approx\phi^*(aH)$ with $u=n-\dim W$, so that $(X, \mathcal{L})$ is a quadric fibration under ϕ as in (1.4.2), ii). Since $u=f(\phi)$, clearly ϕ has equidimensional fibers.

Now, let us assume that the nef value morphism $\phi: X\to W$ is birational, X is smooth and $u\geqq(n+1)/2$. By Lemma (0.3.2) we know that there exists an extremal ray R such that $(K_X+\tau L)\cdot R=0$. Let $\rho=\mathrm{cont}_R: X\to Z$ be the contraction of R. Then ϕ factors through ρ, $\phi=\alpha\circ\rho$. Let $\mathcal{E}$ be any irreducible component of the locus, $E(R)$, of R. Let $\varDelta$ be a general fiber of the restriction $\rho_{\mathcal{E}}: \mathcal{E}\to\rho(\mathcal{E})$. Then by (0.3.1) we have

$$\dim\mathcal{E}+\dim\varDelta\geqq n+l(R)-1\,. \tag{1.4.3}$$

Note that we can choose an extremal rational curve, C, such that $(K_X+\tau L)\cdot C=0$ and $-K_X\cdot C=l(R)$ (see e. g. [**BS1**], (0.7)). Therefore $l(R)=\tau L\cdot C=(uL\cdot C)/v$ or

$$vl(R)=uL\cdot C\,. \tag{1.4.4}$$

Since $(u, v)=1$, we see from (1.4.4) that u divides $l(R)$ and hence in particular $l(R)\geqq u$. Thus (1.4.3) yields

$$\dim\mathcal{E}+\dim\varDelta\geqq n+u-1\,. \tag{1.4.5}$$

Since $\phi=\alpha\circ\rho$, clearly $u:=\max_{w\in W}\{\dim\phi^{-1}(w)\}\geqq\dim\varDelta$, so that (1.4.5) gives

$$\dim E(R)=\dim\mathcal{E}=n-1 \tag{1.4.6}$$

and hence

$$u=\dim \Delta . \tag{1.4.7}$$

In view of the Contraction Theorem we conclude from (1.4.6) that $E(R)$ is a prime divisor in X and from (1.4.7) we have that the image $\rho(E(R))$ is of dimension $n-u-1$. Let $\Omega=\rho(E(R))$.

(1.4.8) CLAIM. The general fiber, Δ, of the restriction $\rho_{E(R)}: E(R)\to\Omega$ is a linear $\boldsymbol{P}^u$ with $\mathcal{L}_\Delta\approx\boldsymbol{O}_{\boldsymbol{P}^u}(1)$. Furthermore $\mathcal{N}^X_{E(R)|\Delta}\cong\boldsymbol{O}_{\boldsymbol{P}^u}(-1)$.

PROOF. First let us show that for a suitable very ample line bundle $\mathcal{A}$ on Z the linear system $|K_X+u\mathcal{L}+\rho^*\mathcal{A}|$ defines the contraction ρ. To see this, note that $K_X+u\mathcal{L}\approx\phi^*\mathcal{H}=\rho^*\alpha^*\mathcal{H}$ for some ample line bundle $\mathcal{H}$ on W. Then the assertion follows from the fact that $\alpha^*\mathcal{H}+\mathcal{A}$ is very ample if $\mathcal{A}$ is very ample enough on Z and that

$$K_X+u\mathcal{L}+\rho^*\mathcal{A}\approx\rho^*(\alpha^*\mathcal{H}+\mathcal{A}).$$

Let A_i, $i=1, \dots, n-u-1$, be $n-u-1$ general elements of $|\mathcal{A}|$ and let $A=\mathcal{A}_1\cap\cdots\cap\mathcal{A}_{n-u-1}$ be the $(u+1)$-dimensional subvariety of Z given by the transversal intersection of the A_i's. By Bertini's Theorem we can assume that $A':=\rho^{-1}(A)$ is a smooth $(u+1)$-dimensional subvariety of X. By the above and by noting that

$$K_{A'}\approx(K_X+(n-u-1)\rho^*\mathcal{A})_{A'}$$

we have that the restriction $\rho_{A'}$ of ρ to A' is given by the linear system associated to the divisor

$$(K_X+u\mathcal{L}+(n-u-1)\rho^*\mathcal{A})_{A'}\approx K_{A'}+u\mathcal{L}_{A'}\approx K_{A'}+(\dim A'-1)\mathcal{L}_{A'},$$

where $\mathcal{L}_{A'}$ denotes the restriction of $\mathcal{L}$ to A'. This means that $\rho_{A'}$ is the morphism associated to the adjoint line bundle of $(A', \mathcal{L}_{A'})$. Therefore we can apply Theorem (3.1) of [**BS1**] to the pair $(A', \mathcal{L}_{A'})$ to conclude that $\rho_{A'}$ has disjoint exceptional divisors $D_i\cong\boldsymbol{P}^u$ with the restrictions $\mathcal{L}_{A'|D_i}$ isomorphic to $\boldsymbol{O}_{\boldsymbol{P}^u}(1)$ for each index i. Since Ω and $\mathcal{A}$ intersect in a finite number of points it thus follows that the general fiber, Δ, of $\rho_{E(R)}: E(R)\to\Omega$ is a linear $\boldsymbol{P}^u$ with $\mathcal{L}_\Delta\cong\boldsymbol{O}_{\boldsymbol{P}^u}(1)$, where $\mathcal{L}_\Delta$ is the restriction of $\mathcal{L}$ to Δ.

To show that $\mathcal{N}^X_{E(R)|\Delta}\cong\boldsymbol{O}_{\boldsymbol{P}^u}(-1)$ note that $(K_X+u\mathcal{L})_\Delta\cong\boldsymbol{O}_\Delta$. Then $K_{X|\Delta}\approx -u\mathcal{L}_\Delta\approx\boldsymbol{O}_{\boldsymbol{P}^u}(-u)$ and therefore the adjunction formula yields

$$\boldsymbol{O}_{\boldsymbol{P}^u}(-u-1)\cong K_\Delta\cong K_{E(R)|\Delta}\cong K_{X|\Delta}+\mathcal{N}^X_{E(R)|\Delta},$$

whence $\mathcal{N}^X_{E(R)|\Delta}\cong\boldsymbol{O}_{\boldsymbol{P}^u}(-1)$. ∎

Let R_i be distinct extremal rays on X such that $(K_X+\tau L)\cdot R_i=0$. Exactly the same argument as above, by using (1.4.4), shows that $l(R_i)\geqq u$ for each index i. Assume that there exist two distinct extremal rays at least R_1, R_2. Then $l(R_1)+l(R_2)\geqq 2u\geqq n+1$. Therefore Lemma (0.3.3) applies to give $E(R_1)\cap E(R_2)=\emptyset$. This shows that the loci, $E(R_i)$, of the extremal rays R_i are disjoint. Thus we can consider the contraction $\sigma: X\to V$ which is a biholomorphism in the complement of all the $E(R_i)$ and which agrees in some complex neighborhood of $E(R_i)$ with the contraction associated to R_i, for all i. Then V is a normal, compact, analytic variety and ϕ factors through σ, $\phi=\beta\circ\sigma$. Note that β has connected fibers since ϕ, σ have connected fibers.

We claim that β is an isomorphism. Assume the converse. Then there exists a curve contained in V, say C, such that $\beta(C)$ is a point. Let C' be an irreducible curve in X such that C' goes onto C under σ. Since C' is contracted to a point under ϕ, we have $(K_X+\tau L)\cdot C'=0$. Hence by Lemma (0.3.2) we can write in $NE(X)$,

$$C' = \textstyle\sum_i \lambda_i C_i$$

where $\lambda_i\in \boldsymbol{R}_+$ and $R_i=\boldsymbol{R}_+[C_i]$ are extremal rays which correspond to the $E(R_i)$ that gave rise to the morphism σ. Recall that $E(R_j)\cdot C_i=0$ since $E(R_i)\cap E(R_j)=\emptyset$ for $i\neq j$ and $E(R_i)\cdot C_i<0$ for all i. Therefore, for each index j,

$$E(R_j)\cdot C' = \textstyle\sum_i \lambda_i E(R_j)\cdot C_i = \lambda_j E(R_j)\cdot C_j<0\,,$$

so that C' is contained in $E(R_j)$ for all j. This leads to a contradiction unless $C'=\lambda_j C_j$ for some index j. In this case $[C']\in R_j$ and hence $\sigma(C')$ is a point. This contradicts the fact that $\sigma(C')=C$. Thus we conclude that β is an isomorphism.

For any fixed index i, let Δ denote a general fiber of the restriction to $E(R_i)$ of the contraction $\rho_i=\mathrm{cont}_{R_i}$. Then the condition $\mathcal{N}^X_{E(R_i)|\Delta}\cong \boldsymbol{O}_{\boldsymbol{P}^u}(-1)$ proved above implies that $(E(R_i)\cdot C_i)=-1$, where $R_i=\boldsymbol{R}_+[C_i]$. By the Contraction Theorem (see e.g. [**BS1**], (0.4.4.2)) it thus follows that W is factorial with terminal singularities. This completes the proof of the Theorem. Q.E.D.

(1.5) REMARK. Note that a converse of (1.4.1) above holds true. Let X be an n-dimensional projective variety with Gorenstein singularities, and let L be an ample line bundle on X. Assume that (X, L) is a scroll, $\phi: X\to W$, over a normal variety W of dimension m. Furthermore assume that ϕ is a $\boldsymbol{P}^{n-m}$ bundle. Since $K_X+(n-m+1)L\approx\phi^*(H)$ for some ample line bundle H on W, the nef value, τ, of (X, L) is $\tau=n-m+1=\max_{w\in W}\{\dim\phi^{-1}(w)\}+1$.

(1.6) REMARK. Notation and assumptions as in (1.4.2). In [**BS2**] we conjectured that if X is a manifold, and (X, L) is a scroll, $p: X\to Y$, over a normal variety Y and $f(p)\geqq \dim Y-1$, then $p: X\to Y$ is a $\boldsymbol{P}^{f(p)}$ bundle. Note that $(X, \mathcal{L})$ in (1.4.2), i) is not a $\boldsymbol{P}^{f(\phi)}$ bundle, since $\max_{w\in W}\{\dim\phi^{-1}(w)\}=f(\phi)+1$

in this case. Then assuming the conjecture above it would follow that $f(\phi)<\dim W-1$ or $2f(\phi)<\dim X-1$, or $u=f(\phi)+1<(\dim X+1)/2$, or $u\leqq\dim X/2$.

§ 2. Final Remarks.

Let X be a projective irreducible variety with Gorenstein rational (hence log-terminal) singularities, and let $\varphi: X\to Y$ be a surjective morphism. Let L be a φ-ample line bundle on X. Assume that K_X is not φ-nef. Let $\tau:=\min\{t\in\boldsymbol{R},\ K_X+tL$ is φ-nef$\}$. Then, by the Kawamata Rationality Theorem, $\tau=u/v$ where u, v are coprime, positive integers with $u\leqq\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1$. In Theorem (2.2) we describe the structure of φ in the boundary case when $u=\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1$. This result extends (1.4.1). We thank the referee for the proof given here, which is simpler than our original proof, and applies in the more general case where X has Gorenstein, rational singularities.

First we need the following Lemma.

(2.1) LEMMA. *Let X be an irreducible, projective variety with Gorenstein, rational singularities and let $\varphi: X\to Y$ be a surjective morphism with connected fibers onto a normal projective variety Y. Let L be a φ-ample line bundle on X. Let t be a positive rational number such that K_X+tL is φ-nef but not φ-ample. Then there exists an ample line bundle $\mathcal{H}$ on Y such that*

(2.1.1) *$\mathcal{M}:=L+\varphi^*\mathcal{H}$ is ample;*

(2.1.2) *$K_X+t\mathcal{M}$ is nef and not φ-ample.*

PROOF. (2.1.1) is well known, and (2.1.2) follows by standard reasoning by using the relative version of the Kawamata-Shokurov Base Point Free Theorem ([**KMM**], 3.1.1). Q. E. D.

(2.2) THEOREM. *Let X be an irreducible, projective variety with Gorenstein, rational singularities, and let $\varphi: X\to Y$ be a surjective morphism with connected fibers onto a normal projective variety Y. Let L be a φ-ample line bundle on X. Assume that* $\operatorname{codim}\operatorname{Sing}(X)>\dim Y$ *and that K_X is not φ-nef. Let*

$$\tau:=\min\{t\in\boldsymbol{R},\ K_X+tL \text{ is } \varphi\text{-nef}\}.$$

By (0.2.1), $\tau=u/v$ for some coprime positive integers u, v with $u\leqq\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1$. Assume $u=\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1$. Then φ is a $\boldsymbol{P}^{u-1}$ bundle, which is a Mori contraction, and $(X, \mathcal{H})$ is a scroll over Y under φ for some ample line bundle $\mathcal{H}$ on X.

PROOF. Take a, b as in (1.1) and let $\mathcal{L}:=bK_X+aL$. Then $\mathcal{L}$ is φ-ample, $K_X+u\mathcal{L}=a(vK_X+uL)$ is φ-nef, but not φ-ample. Replacing L by $\mathcal{L}$ if necessary we may assume that $v=1$. By Lemma (2.1) we may further assume that

$\mathcal{L}$ is ample.

Let F be a general fiber of $\varphi: X \to Y$. By Lemma (0.3.2) we know that there exists an extremal ray R such that $(K_X+u\mathcal{L})\cdot R=0$. Let $\rho=\mathrm{cont}_R: X\to Z$ be the contraction of R. Then φ factors through ρ, $\varphi=\alpha\circ\rho$. We claim that ρ cannot be birational. Indeed, if it was, let $t:=\dim\rho^{-1}(z)$ for some point $z\in Z$. Then by [**F**], (2.5) we know that $(K_X+t\mathcal{L})\cdot R\geqq 0$ and hence $t\geqq u$. Since

$$u=\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1\geqq\max_{z\in Z}\{\dim\rho^{-1}(z)\}+1>t$$

we find the contradiction $u>t$. Thus ρ is not birational and hence the general fiber, F, of φ is of positive dimension and F contains all curves C such that $[C]\in R$. Therefore $(K_X+u\mathcal{L})_F\approx K_F+u\mathcal{L}_F$ is nef, but not ample. Note that F has log-terminal singularities since X has log-terminal singularities. Therefore [**Ma**], (2.1) applies to say that $u\leqq\dim F+1$.

If $u<\dim F+1$ we get the contradiction $\dim F>\max_{y\in Y}\{\dim\varphi^{-1}(y)\}$.

If $u=\dim F+1$, then $(F,\mathcal{L}_F)\cong(\boldsymbol{P}^{u-1},\boldsymbol{O}_{\boldsymbol{P}^{u-1}}(1))$ by well known results of Kobayashi-Ochiai type. Furthermore $\dim F=u-1=\max_{y\in Y}\{\dim\varphi^{-1}(y)\}$ so that φ is equidimensional. Since $\mathrm{codim}\,\mathrm{Sing}(X)>\dim Y$, the arguments of [**F**], (2.12) (see also [**BS1**], (1.4)) let us conclude that φ is a $\boldsymbol{P}^{u-1}$ bundle. The remaining assertions are now straightforward. Q. E. D.

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