

Isomorphism of meromorphic function fields on Riemann surfaces

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1. Introduction.

It is known that two Riemann surfaces are conformally equivalent if and only if meromorphic function fields on them are \mathcal{C} -isomorphic. In this paper, we consider not \mathcal{C} -isomorphisms but field isomorphisms. A \mathcal{C} -isomorphism means that the restriction of the isomorphism to \mathcal{C} is the identity mapping. We always assume that a field isomorphism maps the imaginary unit $\sqrt{-1}$ to itself. If two Riemann surfaces are conformally equivalent, then their meromorphic function fields are obviously field isomorphic. The converse was studied by Nakai-Sario [7]. Their main theorem is that if meromorphic function fields on two Riemann surfaces are field isomorphic, then the two Riemann surfaces are homeomorphic. As a consequence, if meromorphic function fields on two Riemann surfaces are field isomorphic, then the Riemann surfaces are simultaneously compact or noncompact. If the Riemann surfaces are noncompact, then their meromorphic function fields are \mathcal{C} -isomorphic, see Iss'sa [2] and Nakai-Sario [7]. Thus they are conformally equivalent. In contrast with noncompact case, if the Riemann surfaces are compact, then they do not need to be conformally equivalent. A counter example was shown by Heins [1]. Accordingly we shall be concerned with compact case.

For a Riemann surface R , let $M(R)$ denote the meromorphic function fields on R . We introduce an equivalence relation in the set of all Riemann surfaces. Let R and S be Riemann surfaces. We say that R and S are equivalent if and only if $M(R)$ is field isomorphic to $M(S)$. It is clear that this is an equivalence relation. Each equivalence class consists of all field isomorphic Riemann surfaces. For noncompact Riemann surfaces, each equivalence class consists of only one Riemann surface. We note that two conformally equivalent Riemann surfaces are regarded as the same surface. Let I be the set of compact Riemann surfaces R such that the equivalence class containing R consists of only R itself. Then $R \in I$ if $M(S)$ is field isomorphic to $M(R)$ implies that S is conformally equivalent to R . Let I_g be the set of all elements of I of genus g .

Next we explain our results in this paper. We shall first construct a compact Riemann surface R^σ by a compact Riemann surface R and a field isomorphism σ of \mathbf{C} onto itself and show an alternative proof of compact case of Nakai-Sario's theorem in Section 2. In Section 3, we define an equivalence relation in algebraic function fields. We say that two algebraic function fields are equivalent if and only if the two fields are field isomorphic. We show that the set of all equivalence classes is countably infinite (Theorem 4). As a corollary, we see that the set I is countably infinite. This is an affirmative answer to the problem posed by Kato [5]. In the final section, we deal with a compact Riemann surface which is defined by $y^n = P(x)$, where P denotes a polynomial in x . We define branch divisors for such compact Riemann surfaces and discuss whether two meromorphic function fields are field isomorphic or not. More exactly, for Riemann surface defined by $y^n = x(x-1)(x-\alpha)$, we can count how many equivalent Riemann surfaces are there, where $\alpha \in \mathbf{C} \setminus \{0, 1\}$ and n denotes a natural number such that $n \geq 2$ and $n \not\equiv 0 \pmod{3}$. For compact Riemann surfaces of this type, we show that if $(\alpha^2 - \alpha + 1)^3 / \alpha^2 (\alpha - 1)^2$ is algebraic over $\mathbf{Q}(\sqrt{-1})$, then the number of equivalent Riemann surfaces is equal to the degree of $(\alpha^2 - \alpha + 1)^3 / \alpha^2 (\alpha - 1)^2$ over $\mathbf{Q}(\sqrt{-1})$ and if $(\alpha^2 - \alpha + 1)^3 / \alpha^2 (\alpha - 1)^2$ is transcendental over $\mathbf{Q}(\sqrt{-1})$, then the number of equivalent Riemann surfaces is uncountable.

2. Construction of R^σ .

It is well-known that the meromorphic function field on a compact Riemann surface is an algebraic function field of one variable and vice versa. Let R be a compact Riemann surface and let $M(R)$ be the meromorphic function field on R . There are $x, y \in M(R)$ such that $M(R) = \mathbf{C}(x, y)$, where x is transcendental over \mathbf{C} and y is algebraic over $\mathbf{C}(x)$. Let X, Y be indeterminate elements. There is an irreducible polynomial $F(X, Y)$ of Y in $\mathbf{C}(X)[Y]$ such that $F(x(p), y(p)) = 0$ for all $p \in R$ and $\mathbf{C}(x, y)$ is \mathbf{C} -isomorphic to $\mathbf{C}(X)[Y] / \{F(X, Y)\}$, where $\{F(X, Y)\}$ denotes a principal ideal $F(X, Y)\mathbf{C}(X)[Y] = \{F(X, Y)H(X, Y) \mid H(X, Y) \in \mathbf{C}(X)[Y]\}$. The functions x and y correspond to $X + \{F(X, Y)\}$ and $Y + \{F(X, Y)\}$, respectively. Conversely, an arbitrary irreducible polynomial defines an algebraic function field. To be concrete, for an irreducible polynomial $F(X, Y)$, $\mathbf{C}(X)[Y] / \{F(X, Y)\}$ is an algebraic function field. There exists a compact Riemann surface R such that $\mathbf{C}(X)[Y] / \{F(X, Y)\}$ is \mathbf{C} -isomorphic to $M(R)$. Let x and y be functions in $M(R)$ which correspond to $X + \{F(X, Y)\}$ and $Y + \{F(X, Y)\}$, respectively. Then $F(x(p), y(p)) = 0$ for all $p \in R$ and $M(R) = \mathbf{C}(x, y)$. In what follows, we identify $M(R)$ and $\mathbf{C}(X)[Y] / \{F(X, Y)\}$ and say that R is a Riemann surface defined by $F(x, y) = 0$ or $F(X, Y)$. We

always assume $F(X, Y)$ is an irreducible polynomial in $C[X, Y]$.

Let R be a compact Riemann surface and let σ be an automorphism of C , that is, σ is a field isomorphism of C onto itself and maps $\sqrt{-1}$ onto itself. Suppose R is defined by $F(X, Y)$. First of all we construct a new compact Riemann surface R^σ . Let $F(X, Y) = \sum_{i,j} c_{i,j} X^i Y^j$ and set $F^\sigma(Z, W) = \sum_{i,j} \sigma(c_{i,j}) Z^i W^j$, where Z and W are indeterminate elements. Then $F^\sigma(Z, W)$ is also an irreducible polynomial in $C[Z, W]$. Let R^σ be a compact Riemann surface defined by $F^\sigma(Z, W)$. We define a mapping τ from $C(X)[Y]$ onto $C(Z)[W]$ by putting $\tau(X)=Z, \tau(Y)=W$ and $\tau|C=\sigma$, where $\tau|C$ means the restriction of τ to C . The mapping τ induces a field isomorphism of $C(X)[Y]/\{F(X, Y)\}$ onto $C(Z)[W]/\{F^\sigma(Z, W)\}$, we denote it again by τ . The meromorphic function field $M(R)$ (resp. $M(R^\sigma)$) is identified with the algebraic function field $C(X)[Y]/\{F(X, Y)\}$ (resp. $C(Z)[W]/\{F^\sigma(Z, W)\}$). Thus we have a field isomorphism $\tau: M(R) \rightarrow M(R^\sigma)$. The Riemann surface R^σ does not depend on the choice of the defining polynomial F . Suppose R is defined by $G(X, Y)$. Define $R^{\sigma'}$ and a field isomorphism τ' by replacing $F(X, Y)$ with $G(X, Y)$. The composite mapping $\tau' \circ \tau^{-1}$ is a field isomorphism of $M(R^\sigma)$ onto $M(R^{\sigma'})$ and $(\tau' \circ \tau^{-1})|C$ is the identity mapping. Thus $\tau' \circ \tau^{-1}$ is a C -isomorphism. Hence R^σ and $R^{\sigma'}$ are conformally equivalent.

LEMMA 1. *Let R be a compact Riemann surface defined by $F(X, Y) = \sum_{i,j} c_{i,j} X^i Y^j$ and let S be a Riemann surface which is equivalent to R . If $\sigma: M(R) \rightarrow M(S)$ is a field isomorphism, then $R^{\sigma|C}$ and S are conformally equivalent.*

PROOF. We note that $\sigma(C)=C$. For a proof of this fact, see [7]. Since $\sigma|C$ is an automorphism of C , we have a compact Riemann surface $R^{\sigma|C}$ and a field isomorphism τ of $M(R)$ onto $M(R^{\sigma|C})$ mentioned before Lemma 1. The composite mapping $\sigma \circ \tau^{-1}$ is a field isomorphism of $M(R^{\sigma|C})$ onto $M(S)$ and the restriction of $\sigma \circ \tau^{-1}$ to C is the identity mapping. Thus $\sigma \circ \tau^{-1}$ is a C -isomorphism of $M(R^{\sigma|C})$ onto $M(S)$. Hence $R^{\sigma|C}$ and S are conformally equivalent.

□

Applying Lemma 1, we obtain a direct proof of the theorem of Nakai-Sario for compact case.

THEOREM 2 (Nakai-Sario [7]). *Let R and S be equivalent Riemann surfaces. Then R and S are simultaneously compact or noncompact. If they are noncompact, then they are conformally equivalent. If they are compact, then they are homeomorphic.*

PROOF. Let σ be a field isomorphism of $M(R)$ onto $M(S)$. If R is compact, by Lemma 1, $R^{\sigma|C}$ is conformally equivalent to S . Since $R^{\sigma|C}$ is compact, S is compact. Conversely, if S is compact, since $S^{\sigma^{-1}|C}$ is conformally equivalent

to R , then R is compact. Thus R and S are simultaneously compact or non-compact.

If two Riemann surfaces are noncompact, by Nakai-Sario [7] (section 6), σ is \mathbf{C} -isomorphic, and so they are conformally equivalent.

Now suppose R and S are compact. We shall show that R and S are homeomorphic. To prove this, it is enough to show that R and $R^{\sigma^1\mathbf{C}}$ are homeomorphic, because $R^{\sigma^1\mathbf{C}}$ and S are conformally equivalent by Lemma 1. We shall show that the Euler characteristic of R is equal to that of $R^{\sigma^1\mathbf{C}}$. The following paragraph is well-known facts in the algebraic function theory. (see [3] Chapter 4.)

Let R be a Riemann surface defined by $F(x, y)=0$ and let $R^{\sigma^1\mathbf{C}}$ be a Riemann surface defined by $F^{\sigma^1\mathbf{C}}(z, w)=0$. We regard R and $R^{\sigma^1\mathbf{C}}$ as covering surfaces of \mathbf{P}^1 and x and z as covering mappings. The degree of ramification at each branch point of R (resp. $R^{\sigma^1\mathbf{C}}$) is determined by the algebraic structure of $M(R)$ (resp. $M(R^{\sigma^1\mathbf{C}})$). That is, the order of $f \in M(R)$ at $p \in R$ determines a normalized valuation v of $M(R)$ and every normalized valuation of $M(R)$ can be expressed as the order of $f \in M(R)$ at some $p \in R$. The value group of v is equal to \mathbf{Z} and the value group of the restriction of v to $\mathbf{C}(x)$ is an additive subgroup of \mathbf{Z} . Hence there is a natural number e such that the additive subgroup can be expressed as $e\mathbf{Z}$. This number e is called the ramification index at p and $e-1$ is called the degree of ramification at p . Each normalized valuation v of $M(R)$ determines a valuation ring $\{f \in M(R) | v(f) \geq 0\}$. Conversely, for every valuation ring V of $M(R)$, we say V is a valuation ring of $M(R)$ if $f \in V$ or $1/f \in V$ holds for every $f \in M(R)$, V contains \mathbf{C} and V is contained in $M(R)$ properly, there is a normalized valuation v such that $V = \{f \in M(R) | v(f) \geq 0\}$. Thus there is a one-to-one correspondence between the set of valuation rings of $M(R)$ and the set of points of R . We denote by V_p the valuation ring of $M(R)$ which corresponds to $p \in R$.

Let τ be the field isomorphism of $M(R)$ onto $M(R^{\sigma^1\mathbf{C}})$. Then $\tau(V_p)$ is a valuation ring of $M(R^{\sigma^1\mathbf{C}})$ and there is a point $q \in R^{\sigma^1\mathbf{C}}$ such that $\tau(V_p) = V_q$, where V_q is a valuation ring of $M(R^{\sigma^1\mathbf{C}})$ corresponding to q . Since $M(R)$ and $M(R^{\sigma^1\mathbf{C}})$ are field isomorphic and $v \circ \tau^{-1}$ is the normalized valuation of $M(R^{\sigma^1\mathbf{C}})$ associated with the valuation ring V_q , the degree of ramification at each point p of R is equal to that at the corresponding point q of $R^{\sigma^1\mathbf{C}}$. The Riemann-Hurwitz formula asserts that

$$\chi(R) = n\chi(\mathbf{P}^1) - N(R),$$

where $\chi(R)$ denotes the Euler characteristic of R , n denotes the number of sheets of R as a covering surface and $N(R)$ denotes the sum of the degrees of ramification. Since $N(R) = N(R^{\sigma^1\mathbf{C}})$ and the number of sheets of R is equal to

that of $R^{\sigma^1 C}$ from their definitions, $\chi(R)=\chi(R^{\sigma^1 C})$. Thus R and S are homeomorphic. \square

3. Countability of I .

Let H_g be the set of all hyperelliptic Riemann surfaces of genus g . Every hyperelliptic compact Riemann surface of genus g can be expressed as $y^2=x(x-1)(x-\alpha_1)\cdots(x-\alpha_{2g-1})$, namely, a tuple $(\alpha_1, \dots, \alpha_{2g-1})$ of distinct $2g-1$ complex numbers defines a hyperelliptic Riemann surface, where $\alpha_j \in \mathbb{C} \setminus \{0, 1\}$. We denote by $\pi((\alpha_1, \dots, \alpha_{2g-1}))$ the Riemann surface. Then the set $\pi^{-1}(H_g \cap I_g)$ is of Lebesgue measure zero in \mathbb{C}^{2g-1} . Moreover, for the case of $g=1$, $H_1 \cap I_1$ is equal to I_1 and I_1 is a countably infinite set. These facts were shown by Kato [5]. In this section, we shall show that I is a countably infinite set.

We shall construct a subset G of \mathbb{C} . At first, we inductively construct a sequence $\{\varepsilon_j\}_{j=1}^\infty$. Let $\varepsilon_1 \in \mathbb{C}$ be a transcendental element over $\mathbb{Q}(\sqrt{-1})$. For $j \geq 2$, we choose an ε_j so that $\varepsilon_j \in \mathbb{C}$ is transcendental over $\mathbb{Q}(\sqrt{-1}, \varepsilon_1, \dots, \varepsilon_{j-1})$. Let G_j be the set of all algebraic elements of \mathbb{C} over $\mathbb{Q}(\sqrt{-1}, \varepsilon_1, \dots, \varepsilon_{j-1})$. Since the set of all algebraic elements over a countable field are countable, every $G_n (n=1, 2, \dots)$ is a countable set. We note that a finite number of elements of $\{\varepsilon_j\}_{j=1}^\infty$ are algebraically independent over $\mathbb{Q}(\sqrt{-1})$. Put $G = \bigcup_{n=1}^\infty G_n$ and $G^n = \{(a_1, \dots, a_n) \mid a_j \in G, j=1, \dots, n\}$. Then G^n is countable for any natural number n . Next theorem is important to prove Lemma 3.

THEOREM OF KESTELMAN [6]. *We can extend an automorphism of an arbitrary subfield of the complex number field \mathbb{C} to an automorphism of \mathbb{C} , i.e. for any subfield K of \mathbb{C} and its automorphism σ , there exists an automorphism $\hat{\sigma}$ of \mathbb{C} such that $\hat{\sigma}|K = \sigma$.*

LEMMA 3. *For any element $(a_1, \dots, a_n) \in \mathbb{C}^n$, there is an automorphism σ of \mathbb{C} such that $(\sigma(a_1), \dots, \sigma(a_n)) \in G^n$.*

PROOF. Let (a_1, \dots, a_n) be an arbitrary element of \mathbb{C}^n . Let m be a transcendental degree of $\mathbb{Q}(\sqrt{-1}, a_1, \dots, a_n)$ over $\mathbb{Q}(\sqrt{-1})$ and let b_1, \dots, b_m be elements of \mathbb{C} such that a_1, \dots, a_n are algebraic over $\mathbb{Q}(\sqrt{-1}, b_1, \dots, b_m)$. Choose $\varepsilon_{j_1}, \dots, \varepsilon_{j_m} (1 \leq j_1 < \dots < j_m)$ from $\{\varepsilon_j\}_{j=1}^\infty$ so that $b_1, \dots, b_m, \varepsilon_{j_1}, \dots, \varepsilon_{j_m}$ are algebraically independent over $\mathbb{Q}(\sqrt{-1})$. We define an automorphism σ of $\mathbb{Q}(\sqrt{-1}, b_1, \dots, b_m, \varepsilon_{j_1}, \dots, \varepsilon_{j_m})$ by $\sigma| \mathbb{Q}(\sqrt{-1}) = \text{id.}$, $\sigma(b_k) = \varepsilon_{j_k}$ and $\sigma(\varepsilon_{j_k}) = b_k (k=1, \dots, m)$. By Theorem of Kestelman, we can extend σ to an automorphism of \mathbb{C} , and it maps every algebraic element of \mathbb{C} over $\mathbb{Q}(\sqrt{-1}, b_1, \dots, b_m)$ to algebraic one over $\mathbb{Q}(\sqrt{-1}, \varepsilon_{j_1}, \dots, \varepsilon_{j_m})$. Thus $\sigma(a_j) \in G$ for every j , and so $(\sigma(a_1), \dots, \sigma(a_n)) \in G^n$. \square

We shall define an equivalence relation in the set of all of algebraic function fields. Two algebraic function fields are equivalent if and only if the two fields are field isomorphic. We have the following theorem:

THEOREM 4. *The set of all equivalence classes of algebraic function fields is countably infinite.*

PROOF. It is well-known that every algebraic function field of n -variables is defined by an irreducible polynomial of Y in $\mathbf{C}(X_1, \dots, X_n)[Y]$. We may assume that the polynomial is contained in $\mathbf{C}[X_1, \dots, X_n, Y]$ and is denoted by $F(X_1, \dots, X_n, Y) = \sum c_{j_1, \dots, j_{n+1}} X_1^{j_1} \dots X_n^{j_n} Y^{j_{n+1}}$, where $c_{j_1, \dots, j_{n+1}} \in \mathbf{C}$. Since coefficients $c_{j_1, \dots, j_{n+1}}$ are finite, by Lemma 3, there exists an automorphism σ of \mathbf{C} such that $\sigma(c_{j_1, \dots, j_{n+1}}) \in \mathbf{G}$. Since $F^\sigma(Z_1, \dots, Z_n, W) = \sum \sigma(c_{j_1, \dots, j_{n+1}}) Z_1^{j_1} \dots Z_n^{j_n} W^{j_{n+1}}$ is irreducible in $\mathbf{C}(Z_1, \dots, Z_n)[W]$, it defines an algebraic function field of n -variables. By setting $\sigma(X_j) = Z_j (j=1, \dots, n)$ and $\sigma(Y) = W$, we can extend the automorphism σ to the field isomorphism of the algebraic function field defined by $F(X_1, \dots, X_n, Y)$ onto the algebraic function field defined by $F^\sigma(Z_1, \dots, Z_n, W)$. Thus every algebraic function field of n -variables is isomorphic to the algebraic function field defined by an irreducible polynomial of $\mathbf{G}[Z_1, \dots, Z_n, W]$. Because all polynomials of $\mathbf{G}[Z_1, \dots, Z_n, W]$ is countable, the set of all equivalence classes is at most countable. The restriction of an automorphism of \mathbf{C} to $\mathbf{Q}(\sqrt{-1})$ is identity, and it is known that the set of all equivalence classes defined by an irreducible polynomial of $\mathbf{Q}(\sqrt{-1})[Z_1, \dots, Z_n, W]$ is countably infinite. Thus the set of all equivalence classes is countably infinite. \square

Next corollary is an immediate consequence of Theorem 4.

COROLLARY. *I is a countably infinite set.*

4. Conditions for field isomorphism.

For some type compact Riemann surfaces considered as covering surfaces of the Riemann sphere \mathbf{P}^1 , we shall determine whether they are contained in I or not. Let R be a compact Riemann surface defined by $y^n = \prod_{j=1}^m (x - \alpha_j)^{k_j}$, where n, m and k_j are natural numbers, $\alpha_j \in \mathbf{C} (j=1, \dots, m)$ and $\alpha_j \neq \alpha_l (j \neq l)$. Define a branch divisor D_R of R by $D_R = k_1(\alpha_1) + \dots + k_m(\alpha_m)$, where (α_j) is a prime divisor on \mathbf{P}^1 . Let S be a Riemann surface defined by $w^n = \prod_{j=1}^m (z - \beta_j)^{l_j}$, where $\{l_j\}$ are natural numbers and $\{\beta_j\}$ are distinct complex numbers. Let D_S be the branch divisor of S . Let us write $D_R \sim D_S$ if and only if there are a linear transformation T and an integral number N (coprime to n) such that $T(\{\alpha_1, \dots, \alpha_m\}) = \{\beta_1, \dots, \beta_m\}$ and $Nk_j \equiv l_i \pmod{n}$, where $T(\alpha_j) = \beta_i$. It is clear that this is an equivalence relation. For any automorphism σ of \mathbf{C} and any branch divisor $D_R = k_1(\alpha_1) + \dots + k_m(\alpha_m)$, we put $D_R^\sigma = k_1(\sigma(\alpha_1)) + \dots + k_m(\sigma(\alpha_m))$.

Under some additional conditions on $n, m, \{l_j\}$ and $\{k_j\}$, we see that R and S are conformally equivalent if and only if $D_R \sim D_S$. For instance, R and S are conformally equivalent if and only if $D_R \sim D_S$, under the following conditions:

- (i) $\sum_{j=1}^m k_j \equiv \sum_{j=1}^m l_j \equiv 0 \pmod{n}$, n is prime and $m \geq 2n+1$.
 - (ii) $k_j = l_j = 1 (j=1, \dots, m)$, m is prime, $n \geq 2m+1$ and $n \not\equiv 0 \pmod{m}$.
 - (iii) $k_j = l_j = 1 (j=1, \dots, m)$, $n=m$.
 - (iv) $\sum_{j=1}^m k_j \equiv \sum_{j=1}^m l_j \equiv 0 \pmod{n}$, n is odd prime, $m \geq 2n$ or $n \geq 11$.
- Conditions (i) to (iii) are due to Namba [8] and (iv) is due to Kato [4].

THEOREM 5. *Let R and S be compact Riemann surfaces satisfying one of (i) to (iv) simultaneously. Then $M(R)$ and $M(S)$ are field isomorphic if and only if there is an automorphism σ of \mathbf{C} such that $D_R^\sigma \sim D_S$. Moreover, R is contained in I if and only if $D_R^\sigma \sim D_R$ for any automorphism σ of \mathbf{C} .*

PROOF. If there is an automorphism σ of \mathbf{C} such that $D_R^\sigma \sim D_S$, then the branch divisor D_{R^σ} of R^σ is equal to D_R^σ from their definitions. Thus $M(R^\sigma)$ and $M(S)$ are \mathbf{C} -isomorphic, because $D_{R^\sigma} \sim D_S$. Since $M(R)$ and $M(R^\sigma)$ are field isomorphic, $M(R)$ and $M(S)$ are field isomorphic.

If $M(R)$ and $M(S)$ are field isomorphic by $\hat{\sigma}$, by Lemma 1, $M(R^{\hat{\sigma}|C})$ is \mathbf{C} -isomorphic to $M(S)$. Thus $D_S \sim D_{R^{\hat{\sigma}|C}}$. On the other hand, $D_{R^{\hat{\sigma}|C}} = D_R^{\hat{\sigma}|C}$. Hence $D_R^{\hat{\sigma}|C} \sim D_S$.

Next, if $R \in I$, then $M(R^\sigma)$ is field isomorphic to $M(R)$ for any automorphism σ of \mathbf{C} . Since $M(R)$ and $M(R^\sigma)$ are \mathbf{C} -isomorphic from assumption, we have $D_R \sim D_R^\sigma (= D_{R^\sigma})$.

If $D_R^\sigma \sim D_R$ for any automorphism σ of \mathbf{C} , then for an arbitrary compact Riemann surface S whose meromorphic function field is isomorphic to $M(R)$ by τ , since S is denoted by $w^n = \prod_{j=1}^m (z - \tau(\alpha_j))^{k_j}$, we have $D_R^{\tau|C} \sim D_S$ by the fact mentioned above. Thus $D_R \sim D_S$. Since R and S are conformally equivalent, we see $R \in I$. \square

Finally we shall show the next theorem. Any compact Riemann surface of genus 1 can be expressed as $y^2 = x(x-1)(x-\alpha)$. Thus the result contains the case of I_1 .

THEOREM 6. *Let R be a compact Riemann surface defined by $y^n = x(x-1)(x-\alpha)$, where $\alpha \in \mathbf{C} \setminus \{0, 1\}$ and n denotes a natural number such that $n \geq 2$ and $n \not\equiv 0 \pmod{3}$. Then we can determine the number of compact Riemann surfaces which are equivalent to R : if $(\alpha^2 - \alpha + 1)^3 / \alpha^2(\alpha - 1)^2$ is algebraic over $\mathbf{Q}(\sqrt{-1})$, then the number is equal to the degree of $(\alpha^2 - \alpha + 1)^3 / \alpha^2(\alpha - 1)^2$ over $\mathbf{Q}(\sqrt{-1})$, if $(\alpha^2 - \alpha + 1)^3 / \alpha^2(\alpha - 1)^2$ is transcendental over $\mathbf{Q}(\sqrt{-1})$, then the number is uncountably infinite.*

PROOF. Let S be a compact Riemann surface defined by $w^n = z(z-1)(z-\beta)$

$(\beta \in \mathbb{C} \setminus \{0, 1\})$. At first, we shall show that R and S are conformally equivalent if and only if

$$\frac{(\alpha^2 - \alpha + 1)^3}{\alpha^2(\alpha - 1)^2} = \frac{(\beta^2 - \beta + 1)^3}{\beta^2(\beta - 1)^2}. \tag{1}$$

It is known that R and S are conformally equivalent if and only if

$$\beta \in \left\{ \alpha, \frac{1}{\alpha}, 1 - \alpha, \frac{1}{1 - \alpha}, \frac{\alpha}{\alpha - 1}, \frac{\alpha - 1}{\alpha} \right\}.$$

The case of $n=2$ is well-known. The general case was shown by Namba [8]. Since $\{\alpha, 1/\alpha, 1 - \alpha, 1/(1 - \alpha), \alpha/(\alpha - 1), (\alpha - 1)/\alpha\}$ becomes a group if we regard its elements as mappings and if we take the composition of mappings as the product, R and S are conformally equivalent if and only if

$$\left\{ \alpha, \frac{1}{\alpha}, 1 - \alpha, \frac{1}{1 - \alpha}, \frac{\alpha}{\alpha - 1}, \frac{\alpha - 1}{\alpha} \right\} = \left\{ \beta, \frac{1}{\beta}, 1 - \beta, \frac{1}{1 - \beta}, \frac{\beta}{\beta - 1}, \frac{\beta - 1}{\beta} \right\}. \tag{2}$$

The equation (2) holds if and only if each elementary symmetric polynomial of α is equal to that of β with the same order. Simple calculation shows that the last statement holds if and only if (1) holds. Thus the Riemann surface R is equivalent to the Riemann surface S if and only if there is an automorphism σ of \mathbb{C} such that

$$\sigma\left(\frac{(\alpha^2 - \alpha + 1)^3}{\alpha^2(\alpha - 1)^2}\right) = \frac{(\beta^2 - \beta + 1)^3}{\beta^2(\beta - 1)^2}.$$

We say that two complex numbers a, b are equivalent if there is an automorphism σ of \mathbb{C} such that $\sigma(a) = b$. The number of complex numbers which are equivalent to $(\alpha^2 - \alpha + 1)^3/\alpha^2(\alpha - 1)^2$ is equal to the degree of $(\alpha^2 - \alpha + 1)^3/\alpha^2(\alpha - 1)^2$ over $\mathbb{Q}(\sqrt{-1})$ if $(\alpha^2 - \alpha + 1)^3/\alpha^2(\alpha - 1)^2$ is algebraic over $\mathbb{Q}(\sqrt{-1})$ and uncountably infinite if $(\alpha^2 - \alpha + 1)^3/\alpha^2(\alpha - 1)^2$ is transcendental over $\mathbb{Q}(\sqrt{-1})$. This completes the proof. \square

REMARK. If we substitute λ for α in the function $(\alpha^2 - \alpha + 1)^3/\alpha^2(\alpha - 1)^2$ and multiply $4/27$, then we get the J -function

$$J(z) = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

where λ is the λ -function of z .

COROLLARY. *Let R be a compact Riemann surface as in Theorem 6. Then $R \in I$ if and only if*

$$\frac{(\alpha^2 - \alpha + 1)^3}{\alpha^2(\alpha - 1)^2} \in \mathbb{Q}(\sqrt{-1}).$$

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