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The adjoint action of a Lie group on the space of loops

By Akira KONO and Kazumoto KOZIMA

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1. Introduction.

Let G be a compact, connected, simply connected Lie group and e its unit. Denote by ΛG the space of free loops on G and by ΩG the space of based loops on G the base point e. By the multiplication of G and compact open topology ΛG is a topological group and ΩG is a closed normal subgroup. There is an extension of groups

$$1 \longrightarrow \Omega G \xrightarrow{\ i\ } \Lambda G \xrightarrow{\ \pi\ } G \longrightarrow 1$$

with a canonical section $s: G\to\Lambda G$ defined by $s(g)(t)=g$ for any $g\in G$ and $t\in[0, 1]$. We denote the multiplications of G and ΛG by μ and λ respectively and the multiplication of ΩG by the same symbol λ. We also define maps $\mathrm{Ad}: G\times\Omega G\to\Omega G$ by $\mathrm{Ad}(g, l)(t)=gl(t)g^{-1}$ for $g\in G$, $l\in\Omega G$ and $t\in[0, 1]$ and $\Phi: \Omega G\times G\to\Lambda G$ by $\Phi(l, g)=\lambda(l, s(g))$. Then Φ is a homeomorphism and the diagram

$$\begin{array}{ccccc} \Omega G\times G\times\Omega G\times G & \xrightarrow{\ \omega\ } & \Omega G\times\Omega G\times G\times G & \xrightarrow{\ \lambda\times\mu\ } & \Omega G\times G \\ {\scriptstyle\Phi\times\Phi}\downarrow & & & & {\scriptstyle\Phi}\downarrow \\ \Lambda G\times\Lambda G & & \xrightarrow{\ \lambda\ } & & \Lambda G \end{array} \tag{1.1}$$

is commutative where ω is the composition

$$(1_{\Omega G}\times T\times 1_G)\circ(1_{\Omega G\times G}\times\mathrm{Ad}\times 1_G)\circ(1_{\Omega G}\times\Delta_G\times 1_{\Omega G\times G}).$$

The purpose of this paper is to show the following:

THEOREM 1. *Let G be a compact, connected, simply connected Lie group and p a prime. Then the following three conditions are equivalent:*

(1) *$H^*(G; \boldsymbol{Z})$ is p-torsion free,*

(2) *$H^*(\mathrm{Ad}; \boldsymbol{Z}/p)=H^*(p_2; \boldsymbol{Z}/p)$, where p_2 is the second projection,*

(3) *$H^*(B\Lambda G; \boldsymbol{Z}/p)$ is isomorphic to $H^*(BG; \boldsymbol{Z}/p)\otimes H^*(G; \boldsymbol{Z}/p)$ as an algebra.*

If p is an odd prime, then $H^*(G; \boldsymbol{Z}/p)$ is primitively generated if and only if $H^*(G; \boldsymbol{Z})$ is p-torsion free (cf. [4]). On the other hand, if $p=2$, then $H^6(G_2; \boldsymbol{Z}) \cong \boldsymbol{Z}/2$ but $H^*(G_2; \boldsymbol{Z}/2)$ is primitively generated. Therefore, Theorem 1 is a good characterization of the triviality of the p-torsion part of $H^*(G; \boldsymbol{Z})$.

As is well known, G is isomorphic to $G_1 \times G_2 \times \cdots \times G_k$ as a Lie group where $G_1, G_2, \cdots, G_k$ are compact, connected, simply connected and simple Lie groups. If G is isomorphic to $G' \times G''$ as a Lie group, then ΛG is isomorphic to $\Lambda G' \times \Lambda G''$ and ΩG is isomorphic to $\Omega G' \times \Omega G''$ as topological groups. Moreover the diagram

$$\begin{array}{ccccc} G\times\Omega G & & \xrightarrow{\quad\quad\quad\quad \mathrm{Ad} \quad\quad\quad\quad} & & \Omega G \\ \cong\downarrow & & & & \cong\downarrow \\ G'\times G''\times\Omega G'\times\Omega G'' & \xrightarrow{1_{G'}\times T\times 1_{\Omega G''}} & G'\times\Omega G'\times G''\times\Omega G'' & \xrightarrow{\mathrm{Ad}\times\mathrm{Ad}} & \Omega G'\times\Omega G'' \end{array}$$

is commutative. Therefore, for the proof of Theorem 1, we may assume that G is simple.

This paper is organized as follows: In section 2, certain relations between Ad^*, λ^* and μ^* (where $\mathrm{Ad}^* = H^*(\mathrm{Ad}; \boldsymbol{Z}/p)$ etc.) are proved. Using the classification of simple Lie algebras, we prove the equivalence of (1) and (2) of Theorem 1 in section 3. The equivalence of (1) and (3) is proved in section 4 using the following lemma:

Lemma 4.1. *If x_3 is a generator of $H^3(B\Lambda G_2; \boldsymbol{Z}/2)$, then $x^n \neq 0$ for any n.*

Remark. $H^*(B\Lambda G_2; \boldsymbol{Z}/2)$ is isomorphic to $H^*(BG_2; \boldsymbol{Z}/2) \otimes H^*(G_2; \boldsymbol{Z}/2)$ as an $H^*(BG_2; \boldsymbol{Z}/2)$-module.

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2. Basic properties of Ad and Γ.

Let G be a compact, 1-connected Lie group and ΩG the space of loops on G. We denote the group multiplication of G by $\mu: G \times G \to G$. The letter λ denotes the multiplication of ΩG defined by the formula

$$\lambda(l, l')(t) = \mu(l(t), l'(t))$$

where $l, l' \in \Omega G$. We write gg' for $\mu(g, g')$ as usual.

The adjoint map $\mathrm{Ad}: G \times \Omega G \to \Omega G$ is defined by

$$\mathrm{Ad}(g, l)(t) = gl(t)g^{-1}$$

where $g\in G$ and $l\in \Omega G$. We denote also by Γ the commutator map

$$\Gamma(g, l)(t)=gl(t)g^{-1}l(t)^{-1}.$$

First we observe some basic properties of these maps.

PROPOSITION 2.1. *The following diagrams commute:*

(1)
$$\begin{array}{ccc} G\times\Omega G & \xrightarrow{\ \mathrm{Ad}\ } & \Omega G \\ {\scriptstyle 1\times\Delta}\downarrow & & \uparrow{\scriptstyle\lambda} \\ G\times\Omega G\times\Omega G & \xrightarrow{\ \Gamma\times 1\ } & \Omega G\times\Omega G \end{array}$$

(2)
$$\begin{array}{ccc} G\times\Omega G & \xrightarrow{\ \Gamma\ } & \Omega G \\ {\scriptstyle 1\times\Delta}\downarrow & & \uparrow{\scriptstyle\lambda} \\ G\times\Omega G\times\Omega G & \xrightarrow{\ \mathrm{Ad}\times\iota\ } & \Omega G\times\Omega G \end{array}$$

where Δ is a diagonal map and ι the inverse map defined by $\iota(l)(t)=l(t)^{-1}$.

We denote the composition

$$G\times\Omega G\times\Omega G\xrightarrow{\ \Delta\times1\times1\ }G\times G\times\Omega G\times\Omega G\xrightarrow{\ 1\times T\times1\ }G\times\Omega G\times G\times\Omega G$$

by D for simplicity, where T is a switching map.

PROPOSITION 2.2. *The following diagrams commute:*

(1)
$$\begin{array}{ccc} G\times G\times\Omega G & \xrightarrow{\ 1\times\mathrm{Ad}\ } & G\times\Omega G \\ {\scriptstyle \mu\times1}\downarrow & & \downarrow{\scriptstyle\mathrm{Ad}} \\ G\times\Omega G & \xrightarrow{\ \mathrm{Ad}\ } & \Omega G \end{array}$$

(2)
$$\begin{array}{ccccc} G\times\Omega G\times\Omega G & \xrightarrow{\ D\ } & G\times\Omega G\times G\times\Omega G & \xrightarrow{\ \mathrm{Ad}\times\mathrm{Ad}\ } & \Omega G\times\Omega G \\ {\scriptstyle 1\times\lambda}\downarrow & & & & \downarrow{\scriptstyle\lambda} \\ G\times\Omega G & & \xrightarrow{\ \mathrm{Ad}\ } & & \Omega G \end{array}$$

(3)
$$\begin{array}{ccccc} G\times\Omega G & \xrightarrow{\ \Delta\times\Delta\ } & G\times G\times\Omega G\times\Omega G & \xrightarrow{\ 1\times T\times1\ } & G\times\Omega G\times G\times\Omega G \\ {\scriptstyle \mathrm{Ad}}\downarrow & & & & \downarrow{\scriptstyle\mathrm{Ad}\times\mathrm{Ad}} \\ \Omega G & & \xrightarrow{\ \Delta\ } & & \Omega G\times\Omega G \end{array}$$

The above propositions are deduced directly from the definitions.

PROPOSITION 2.3. *The diagram*

$$\begin{array}{ccccc} G\times\Omega G\times\Omega G & \xrightarrow{D} & G\times\Omega G\times G\times\Omega G & \xrightarrow{\Gamma\times\Gamma} & \Omega G\times\Omega G \\ {\scriptstyle 1\times\lambda}\downarrow & & & & \downarrow{\scriptstyle\lambda} \\ G\times\Omega G & & \xrightarrow{\qquad\Gamma\qquad} & & \Omega G \end{array}$$

commutes up to homotopy.

PROOF. We write $l*l'$ for $\lambda(l, l')$. Since

$$(\lambda\circ(\Gamma\times\Gamma)\circ D(g, l, l'))(t) = gl(t)g^{-1}l(t)^{-1}(gl'(t)g^{-1})l'(t)^{-1}$$

and λ is a homotopy commutative product, we have

$$\begin{aligned}(\lambda\circ(\Gamma\times\Gamma)\circ D)(g, l, l') &= \mathrm{Ad}(g, l)*l^{-1}*\mathrm{Ad}(g, l')*l'^{-1} \\ &\simeq \mathrm{Ad}(g, l)*\mathrm{Ad}(g, l')*l'^{-1}*l^{-1} \\ &= \mathrm{Ad}(g, l*l')*l'^{-1}*l^{-1} \\ &= \Gamma(g, l*l') \\ &= (\Gamma\circ(1\times\lambda))(g, l, l')\,.\end{aligned}$$

We will deduce the algebraic formula on cohomology theory from the above propositions. Let h^* be the cohomology theory $H^*(\ ; R)$, where R represents mainly $\boldsymbol{Z}/p$, but the some arguments can be applied to the case of the more general R. Notice that $H^*(\Omega G; \boldsymbol{Z})$ is free and has no odd dimensional elements. We will use this fact implicitly in the following arguments.

We denote by α the induced map

$$\mathrm{Ad}^*: h^*(\Omega G) \longrightarrow h^*(G)\otimes h^*(\Omega G)$$

and Γ^* by γ. Put $\phi=\mu^*$ and $\psi=\lambda^*$ as usual. They are the coproducts of the Hopf algebras $h^*(G)$ and $h^*(\Omega G)$. We put also

$$\bar{\phi}(x) = \phi(x)-(x\otimes 1+1\otimes x) \qquad \text{and}$$

$$\bar{\psi}(x) = \psi(x)-(x\otimes 1+1\otimes x)$$

as the reduced coproducts.

Now we can state some equations which put strong restrictions on the algebraic structure of $h^*(G)$ and $h^*(\Omega G)$.

PROPOSITION 2.4.

(1) $$(\phi\otimes 1)\circ\alpha = (1\otimes\alpha)\circ\alpha\,,$$

(2) $$(1\otimes\phi)\circ\alpha = d\circ(\alpha\otimes\alpha)\circ\phi\,,$$

(3) $$\alpha(xy) = \alpha(x)\alpha(y) \qquad \textit{for } x,\ y\in h^*(\Omega G),$$

where $d=D^*$.

PROOF. We deduce these formulas simply by applying the functor h^* to the diagrams in Proposition 2.2.

Similarly, from Proposition 2.3, we obtain

PROPOSITION 2.5.
$$d\circ(\gamma\otimes\gamma)\circ\phi = (1\otimes\phi)\circ\gamma\,.$$

We define $\bar{\alpha}: h^*(\Omega G)\to h^*(G)\otimes h^*(\Omega G)$ by the equation
$$\bar{\alpha}(a) = \alpha(a)-1\otimes a\,,$$
where $a\in h^*(\Omega G)$. Let $i: \Omega G\to G\times\Omega G$ be the map defined by $i(l)=(e, l)$. Since $\mathrm{Ad}\circ i=1$, one can easily show that
$$\alpha(a) = 1\otimes a+\sum_{\substack{0<|a'|<|a|\\ |x'|+|a'|=|a|}} x'\otimes a', \quad \text{where } |x| \text{ indicates the degree of } x.$$

PROPOSITION 2.6.

(1) $$(\bar{\phi}\otimes 1)\circ\bar{\alpha} = (1\otimes\bar{\alpha})\circ\bar{\alpha}$$

(2) $$(1\otimes\bar{\phi})\circ\bar{\alpha} = d\circ(\bar{\alpha}\otimes\bar{\alpha}+\bar{\alpha}\otimes 1+1\otimes\bar{\alpha})\circ\bar{\phi}\,.$$

PROOF. As mensioned above, $\alpha(a)=1\otimes a+\bar{\alpha}(a)=1\otimes a+\sum x'\otimes a'$. Then
$$\begin{aligned}((\phi\otimes 1)\circ\alpha)(a) &= 1\otimes 1\otimes a+\sum\phi(x')\otimes a'\\ &= 1\otimes 1\otimes a+\sum\{(x'\otimes 1+1\otimes x')\otimes a'+\bar{\phi}(x')\otimes a'\}\,.\end{aligned}$$
By (1) of Proposition 2.4, this equals to
$$\begin{aligned}((1\otimes\alpha)\circ\alpha)(a) &= 1\otimes\alpha(a)+\sum x'\otimes\alpha(a')\\ &= 1\otimes 1\otimes a+\sum(x'\otimes 1\otimes a'+1\otimes x'\otimes a'+x'\otimes\bar{\alpha}(a'))\,.\end{aligned}$$
So we obtain $\sum\bar{\phi}(x')\otimes a'=\sum x'\otimes\bar{\alpha}(a')$. This implies (1). The second equation can be deduced from (2) of Proposition 2.4 in the same way.

By using Proposition 2.5, we can get the following proposition for γ in a similar fashion.

PROPOSITION 2.7.
$$d\circ(\gamma\otimes\gamma)\circ\bar{\phi} = (1\otimes\bar{\phi})\circ\gamma\,.$$

By (1) of Proposition 2.6, we can compute $\bar{\alpha}$ inductively from the lower degree up to (primitive elements) $\otimes h^*(\Omega G)$. α can be also determined inductively up to $h^*(G)\otimes$(primitive elements) by using (2) of Proposition 2.6. Similarly, Proposition 2.7 says that γ can be computed inductively up to $h^*(G)\otimes$ (primitive elements).

Now let T be a maximal torus of G. We denote by T_G^* the set of all transgressive elements with respect to the principal fibration

$$G \xrightarrow{\pi} G/T \longrightarrow BT .$$

Let $P(A)$ be the primitive module for a Hopf algebra A.

The following two propositions are due to Ishitoya, Kono and Toda [**10**].

THEOREM 2.8. *For any element $x \in h^+(G)$, the following are equivalent:*

(1) $$x \in T_G^* ;$$

(2) $$\phi(x)-1\otimes x \in h^*(G)\otimes \operatorname{Im}\pi^* ;$$

(3) $$\phi(x)-1\otimes x \in \operatorname{Im} T_G^* \otimes \operatorname{Im}\pi^* .$$

PROPOSITION 2.9.

(1) $$x \in T_G^{2*} = \operatorname{Im}\pi^+ ,$$

(2) $$P(h^*(G)) \subset T_G^* ,$$

Let $\bar{\alpha}(a)=\sum x'\otimes a'$. If $\bar{\alpha}(a')=0$, then $((\bar{\phi}\otimes 1)\circ\bar{\alpha})(a)=0$ and by using (1) of Proposition 2.6, we deduce that x' is primitive and $x' \in T_G^{2*}$. If we assume that $x' \in T_G^{2*}$ for a whose degree is less than a certain degree n, we conclude that it holds in the degree n by using (1) of Proposition 2.6 and Theorem 2.8. It follows inductively:

PROPOSITION 2.10. *For any element $a \in h^*(\Omega G)$,*

$$\bar{\alpha}(a) \in T_G^{2*}\otimes h^*(\Omega G) .$$

We remark the naturality of α or γ and their behavior under the cohomology operations.

Since the following diagram commutes

$$\begin{array}{ccc} G'\times\Omega G' & \xrightarrow{\text{Ad(resp. }\Gamma)} & \Omega G' \\ {\scriptstyle f\times\Omega f}\downarrow & & \downarrow{\scriptstyle \Omega f} \\ G'\times\Omega G & \xrightarrow{\text{Ad(resp. }\Gamma)} & \Omega G \end{array}$$

where $f: G'\to G$ is a homomorphism of Lie groups, α (resp. γ) is natural in the following sense.

PROPOSITION 2.11.

(1) $$(f^*\otimes\Omega f^*)\circ\alpha=\alpha\circ\Omega f^*,$$

(2) $$(f^*\otimes\Omega f^*)\circ\gamma=\gamma\circ\Omega f^*.$$

Since α (resp. γ) is the composition of the induced homomorphisms from continuous maps and the Künneth isomorphism, the Steenrod power operations commute with α (resp. γ) like as the following:

PROPOSITION 2.12. *Put* $\boldsymbol{P}=\sum\mathfrak{p}^i$ *where* $\mathfrak{p}^i$ *is the i-th power operation. Then*

(1) $$(\boldsymbol{P}\otimes\boldsymbol{P})\circ\alpha=\alpha\circ\boldsymbol{P},$$

(2) $$(\boldsymbol{P}\otimes\boldsymbol{P})\circ\gamma=\gamma\circ\boldsymbol{P}.$$

In some cases, α can be computed from the Hopf algebra structure of $h^*(G)$. This method is a key computational tool in this paper. We define $\mathrm{ad}\colon G\times G\to G$ by $\mathrm{ad}(g, g')=gg'g^{-1}$ as usual. Ad can be extended naturally on

$$LG=\{l\colon I\to G,\ l(0)=e\}$$

and the diagram

$$\begin{array}{ccccccc}
G\times\Omega G & \longrightarrow & G\times LG & \longrightarrow & G\times(LG,\Omega G) & \xrightarrow{1\times p} & G\times(G,e)\\
\mathrm{Ad}\downarrow & & \mathrm{Ad}\downarrow & & \mathrm{Ad}\downarrow & & \mathrm{ad}\downarrow\\
\Omega G & \longrightarrow & LG & \longrightarrow & (LG,\Omega G) & \xrightarrow{p} & (G,e)
\end{array}$$

commutes. Let σ be the composition

$$h^*(G,e)\xrightarrow{p^*}h^*(LG,\Omega G)\xrightarrow{\delta}\tilde{h}^{*-1}(\Omega G).$$

PROPOSITION 2.13. *The following diagram*

$$\begin{array}{ccc}
\tilde{h}^*(G) & \xrightarrow{\sigma} & \tilde{h}^{*-1}(\Omega G)\\
\mathrm{ad}^*\downarrow & & \alpha\downarrow\\
h^*(G)\otimes\tilde{h}^*(G) & \xrightarrow{1\otimes\sigma} & h^*(G)\otimes\tilde{h}^{*-1}(\Omega G)
\end{array}$$

commutes.

3. Some calculations of α.

First, we handle the case that $\bar{\alpha}$ vanishes. This is also a part of the proof of our main theorem. Assume that $a\in h^*(G)$ is primitive. Then $\mathrm{ad}^*(a)=1\otimes a$ and we can deduce

$$\alpha(\sigma(a))=1\otimes\sigma(a)=p_2^*(\sigma(a))$$

by Proposition 2.13. If $H^*(G;\boldsymbol{Z}_{(p)})$ is torsion free, the horizontal localizations are injective in the following commutative diagram

$$\begin{array}{ccc} H^*(\Omega G;\boldsymbol{Z}_{(p)}) & \xrightarrow{\otimes \boldsymbol{Q}} & H^*(\Omega G;\boldsymbol{Q}) \\ \alpha\downarrow & & \alpha\downarrow \\ H^*(G;\boldsymbol{Z}_{(p)})\otimes H^*(\Omega G;\boldsymbol{Z}_{(p)}) & \xrightarrow{\otimes \boldsymbol{Q}} & H^*(G;\boldsymbol{Q})\otimes H^*(\Omega G;\boldsymbol{Q}). \end{array}$$

Since $H^*(\Omega G;\boldsymbol{Q})$ is generated by σ(primitive elements of $H^*(G;\boldsymbol{Q})$), $\alpha=p_2^*$ in this case. By the mod p reduction, we obtain

PROPOSITION 3.1. *If $H^*(G;\boldsymbol{Z})$ is p-torsion free, $\bar{\alpha}$ vanishes on $H^*(\Omega G;\boldsymbol{Z}/p)$.*

In the case that $H^*(G;\boldsymbol{Z})$ has p-torsion, $\bar{\alpha}$ does not vanish. The most typical case is $G=E_6$, $p=2$. By the result of Kono-Mimura [15] and Toda [21], one has

$$h^*(E_6)=\boldsymbol{Z}/2[x_3]/(x_3^4)\otimes E(Sq^2x_3,\ Sq^4Sq^2x_3,\ x_{15},\ x_{17},\ x_{23})$$

where x_i is a generator of degree i and $E(\ )$ represents an exterior algebra and we can choose x_{15} so as to satisfy

$$\bar{\phi}(x_{15})=x_3^2\otimes Sq^4Sq^2x_3\,.$$

We will show a lemma to determine $\mathrm{ad}^*(x_{15})$.

LEMMA 3.2. *If $\bar{\phi}(x)=\sum_i a_i\otimes b_i$ for $x\in h^*(G)$ and a_i, b_i are primitive, then*

$$\mathrm{ad}^*(x)=\sum_i (a_i\otimes b_i-(-1)^{\deg a_i \deg b_i}b_i\otimes a_i)$$

PROOF. Since $\Delta^*\circ(1\times\iota)^*\circ\phi=0$, one has

$$0=x+\iota^*x+\sum_i a_i\iota^*b_i$$

and $\iota^*b_i=-b_i$, we obtain $\iota^*x=-x+\sum_i a_ib_i$.

Since

$$\mathrm{ad}=\mu\circ(1\times\mu)\circ(1\times1\times\iota)\circ(1\times T)\circ(\Delta\times1)\,,$$

we can calculate ad^*x as follows:

$$\begin{aligned} x &\xrightarrow{\mu^*} x\otimes1+1\otimes x+\sum a_i\otimes b_i \\ &\xrightarrow{1\otimes\mu^*} x\otimes1\otimes1+1\otimes(x\otimes1+1\otimes x+\sum a_i\otimes b_i)+\sum a_i\otimes(b_i\otimes1+1\otimes b_i) \\ &\xrightarrow{1\otimes1\otimes\iota^*} x\otimes1\otimes1+1\otimes(x\otimes1+1\otimes(-x+\sum a_ib_i)+\sum a_i\otimes(-b_i)) \\ &\qquad +\sum a_i\otimes(b_i\otimes1-1\otimes b_i) \end{aligned}$$

$$\xrightarrow{1\otimes T} x\otimes 1\otimes 1+1\otimes(1\otimes x+(-x+\sum a_ib_i)\otimes 1-\sum(-1)^{\deg a_i \deg b_i}b_i\otimes a_i)$$
$$+\sum a_i\otimes(1\otimes b_i-b_i\otimes 1)$$
$$\xrightarrow{\Delta^*\otimes 1} x\otimes 1+1\otimes x+(-x+\sum a_ib_i)\otimes 1-\sum(-1)^{\deg a_i \deg b_i}b_i\otimes a_i$$
$$+\sum(a_i\otimes b_i-a_ib_i\otimes 1)$$
$$=1\otimes x+\sum_i(a_i\otimes b_i-(-1)^{\deg a_i \deg b_i}b_i\otimes a_i)\,.$$

On the other hand, we can show easily (for example, by the result of [**12**] and [**13**])

$$h^*(\Omega E_6)=\boldsymbol{Z}/2[a_2]\otimes E(a_8,\ a_{10},\ a_{14}) \qquad \text{for } *<15\,,$$

where a_i is a generator of degree i and the following equations hold:

$$a_2=\sigma(x_3)\,, \qquad a_{14}=\sigma(x_{15})\,, \qquad a_{14}=Sq^4Sq^2a_8\,.$$

Then we obtain

PROPOSITION 3.3. *In $H^*(\Omega E_6;\boldsymbol{Z}/2)$, we have*

$$\alpha(a_8)=1\otimes a_8+x_3^2\otimes a_2\,,$$
$$\alpha(a_{14})=1\otimes a_{14}+x_3^2\otimes a_2^4\,.$$

PROOF. For simplicity, we put $x_9=Sq^4Sq^2x_3$. Since $\sigma(x_9)=Sq^4Sq^2a_2=a_2^4$, by applying (2.13) and (3.2) to x_{15}, we obtain

$$\alpha(a_{14})=1\otimes a_{14}+x_3^2\otimes a_2^4\,.$$

Then, by the relation $Sq^4Sq^2a_8=a_{14}$, we can conclude that $\alpha(a_8)$ must be $1\otimes a_8+x_3^2\otimes a_2$.

In the case of $p=2$, by the naturality of α, one can show easily that there are elements

$$a_8\in h^*(\Omega G) \text{ such that } \bar{\alpha}\neq 0 \quad \text{for } G=G_2,\ F_4,\ Spin(7), \text{ and}$$
$$a_{14}\in h^*(\Omega G) \text{ such that } \bar{\alpha}\neq 0 \quad \text{for } G=E_7,\ E_8\,.$$

The most complicated case is $(G,\ p)=(Spin(2^r+1),\ 2)$. Let $\tilde{G}$ be its 3-connected cover and X the two stage Postnikov space obtained from $SO(2^r+1)$ by killing its homotopy group of $\dim\geqq 4$. Then, there is a fibration

$$K(\boldsymbol{Z},\ 3)\longrightarrow X\longrightarrow K(\boldsymbol{Z}/2,\ 1)$$

and one can show easily

$$h^*(X)\cong h^*(K(\boldsymbol{Z},\ 3))\otimes h^*(K(\boldsymbol{Z}/2,\ 1))\,.$$

Let $j: SO(2^r+1) \to X$ be the inclusion and $Bj: BSO(2^r+1) \to BX$ the map induced from j which is an H-map between associative H-spaces $SO(2^r+1)$ and X. Since $Sq^1 w_4 = w_5$, we have

$$\begin{aligned} Sq^1\sigma(Bj^* w_4) &= \sigma(Sq^1 Bj^* w_4) \\ &= \sigma(Bj^* w_5) \\ &= j^*\sigma(w_5) = z_1^4 \end{aligned}$$

where $z_1 \in h^1(X)$ is a generator. We put $z_3 = \sigma(Bj^* w_4) \in Ph^3(X)$. Then

$$h^*(X) = \boldsymbol{Z}/2[z_1, z_3, Sq^2 z_3, Sq^4 Sq^2 z_3, \cdots, Sq^{2^k} Sq^{2^{k-1}} \cdots Sq^2 z_3, \cdots].$$

By the result of [**10**], the Hopf algebra structure of the $\boldsymbol{Z}/2$-cohomology of $Spin(n)$ are known:

$$h^*(Spin(n)) \cong \Delta(x_j \mid 3 \leqq j < n,\ j \neq 2^l) \otimes E(x_{2^s-1}),$$

$$\bar{\varphi}(x_j) = 0 \quad \text{for } j \neq 2^s-1, \qquad \bar{\varphi}(x_{2^s-1}) = \sum_{i+j=2^s-1} x_{2i} \otimes x_{2j-1},$$

where $\Delta(\ )$ indicates the module generated by the simple system of generators and $2^{s-1} < n \leqq 2^s$. So, if $n = 2^r+1$ or 2^r+2, then $s = r+1$. One can determine $h^*(\tilde{G})$, by using the result of Kono [**12**]:

$$\begin{aligned} h^*(\tilde{G}) \cong\ & \Delta(x_j \mid 3 \leqq j < 2^r+1,\ j \neq 2^l,\ 2^l+1) \otimes E(x_{2^{r+1}-1}) \\ & \otimes \boldsymbol{Z}/2[u_{2^r}] \otimes \Delta(u_{2^r+1}, u_{2^r+2+1}, \cdots, u_{2^r+2^{r-2}+\cdots+1}), \end{aligned}$$

$$Sq^2 u_{2^r+1} = u_{2^r+2+1}, \quad Sq^4 u_{2^r+2+1} = u_{2^r+4+2+1}, \cdots,$$

$$Sq^{2^{r-2}} u_{2^r+2^{r-3}+\cdots+1} = u_{2^r+2^{r-2}+\cdots+1}.$$

Let us show the following proposition:

PROPOSITION 3.4. $Sq^{2^{r-1}} u_{2^r+2^{r-1}-1} = x_{2^{r+1}-1}$.

PROOF. Since $\tau(x_{2^{r+1}-1}) \neq 0$ in the Serre spectral sequence of the fibration $G \to SO(2^r+1) \to K(\boldsymbol{Z}/2, 1)$, we have $\tau(x_{2^{r+1}-1}) = z_1^{2^r+1}$ in the Serre spectral sequence of the fibration $\tilde{G} \to SO(2^r+1) \to X$. Similarly, by comparing the Serre spectral sequences of the fibrations which are the horizontal raws of the following diagram

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{\pi} & G & \longrightarrow & K(\boldsymbol{Z}, 3) \\ {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow \\ \tilde{G} & \longrightarrow & SO(2^r+1) & \longrightarrow & X, \end{array}$$

we obtain $\tau(u_{2^r}) = Sq^{2^{r-1}} Sq^{2^{r-2}} \cdots Sq^2 z_3$.

So,

$$\begin{aligned} Sq^{2^{r-1}}\tau(u_{2^r+2^{r-1}-1}) &= Sq^{2^{r-1}}Sq^{2^{r-2}}\cdots Sq^2Sq^1\tau(u_{2^r}) \\ &= Sq^{2^{r-1}}Sq^{2^{r-2}}\cdots Sq^2Sq^1Sq^{2^{r-1}}Sq^{2^{r-2}}\cdots Sq^2z_3 \\ &= Sq^{2^{r-1}}Sq^{2^{r-2}}\cdots Sq^2Sq^{2^{r-1}+1}(Sq^{2^{r-2}}\cdots Sq^2z_3) \\ &= Sq^{2^{r-1}}Sq^{2^{r-2}}\cdots Sq^2(Sq^{2^{r-2}}\cdots Sq^2z_3)^2 \\ &= Sq^{2^{r-1}}Sq^{2^{r-2}}\cdots (Sq^1Sq^{2^{r-2}}\cdots Sq^2z_3)^2 \\ &\quad\cdots \\ &= Sq^{2^{r-1}}z_3^{2^{r-1}} \\ &= z_1^{2^{r+1}}. \end{aligned}$$

Thus $Sq^{2^{r-1}}u_{2^r+2^{r-1}-1}$ maps transgressively to $z_1^{2^{r+1}}$. Since $x_{2^{r+1}-1}$ is a unique element whose image of the transgression is $z_1^{2^{r+1}}$, the result follows.

Since $\Omega\pi^*: h^{2*}(\Omega G)\to h^{2*}(\Omega\tilde{G})$ is epic, there is an element $u\in h^{2^r+2^{r-1}-2}(\Omega G)$ satisfying

$$\Omega\pi^*u = \sigma(u_{2^r+2^{r-1}-1}) \in h^{2^r+2^{r-1}-2}(\Omega\tilde{G}).$$

If we put $a=Sq^{2^{r-1}}u$, then $\Omega\pi^*a=\sigma(x_{2^{r+1}-1})$.

Let $i: Spin(2^r+1)\to Spin(2^r+2)$ be the natural inclusion. Then, the generator x_j corresponds by i^*. In $h^*(Spin(2^r+2))\otimes h^*(Spin(2^r+2))$, we have the equation

$$\mathrm{ad}^*(x_{2^{r+1}-1}) = 1\otimes x_{2^{r+1}-1}+x_{2^r-2}\otimes x_{2^r+1}+x_{2^r+1}\otimes x_{2^r-2}.$$

By Proposition 2.13, we have

$$\bar{\alpha}(\sigma(x_{2^{r+1}-1})) = x_{2^r-2}\otimes t^{2^{r-1}},$$

because $\sigma(x_{2^r+1})=t^{2^{r-1}}$ where t is a generator of $h^2(\Omega\, Spin(2^r+2))$. So, $\bar{\alpha}(u)$ can not vanish.

Now, we turn to the case of odd primes.

In the case $(G, p)=(F_4, 3)$, we have

$$h^*(G) = \boldsymbol{Z}/3[u_8]/(u_8^3)\otimes E(u_3, u_7)\otimes E(x_{11}, x_{15})$$

and we can choose the generator x_{11} so as to satisfy

$$\bar{\phi}(x_{11}) = u_3\otimes u_8.$$

Since u_3 and u_8 are primitive for dimensional reasons, we have

$$\mathrm{ad}^*(x_{11}) = 1\otimes x_{11}+u_3\otimes u_8-u_8\otimes u_3.$$

If we put $a_2=\sigma(u_3)$, $a_{10}=\sigma(x_{11})$ and $a_{14}=\sigma(x_{15})=\mathfrak{p}^1a_{10}$, then

$$\bar{\alpha}(a_{10}) = -u_8\otimes a_2 \quad\text{and}\quad \bar{\alpha}(a_{14}) = -u_8\otimes a_2^3,$$

because $\sigma(u_8)\in h^7(\Omega G)=0$. Since a_2 is a generator of $h^2(\Omega G)\cong \boldsymbol{Z}/3$, we deduce that $\alpha\neq p_2^*$. By the naturality, $\alpha\neq p_2^*$ holds for the case $(G, p)=(E_6, 3)$, $(E_7, 3)$, $(E_8, 3)$. The case $(G, p)=(E_8, 5)$ is quite similar and one can prove that there exists an element a_{14} satisfying $\bar{\alpha}(a_{14})\neq 0$. Thus in all cases that the cohomology of G has non-trivial torsion part, we have $\alpha\neq p_2^*$.

4. The proof of the main theorem.

We proved already that (1) and (2) of our main theorem are equivalent.

Since the fibration

$$G_2 \longrightarrow B\Lambda G_2 \longrightarrow BG_2 \tag{4.1}$$

has a section, we obtain

$$H^*(B\Lambda G_2; \boldsymbol{Z}/2) \cong H^*(BG_2; \boldsymbol{Z}/2)\otimes H^*(G_2; \boldsymbol{Z}/2)$$

as the $H^*(BG_2; \boldsymbol{Z}/2)$-module by computing the Serre spectral sequence.

By the result of [**5**], we have

$$H^*(G_2; \boldsymbol{Z}/2) = \boldsymbol{Z}/2[x_3]/(x_3^4)\otimes E(x_5)\,, \qquad Sq^2x_3 = x_5$$

$$H^*(BG_2; \boldsymbol{Z}/2) = \boldsymbol{Z}/2[y_4, y_6, y_7]\,, \quad Sq^2y_4 = y_6, \quad Sq^1y_6 = y_7$$

where x_i and y_i are the generators of degree i.

We will show the following lemma:

LEMMA 4.1. *If x_3 is a generator of $H^3(B\Lambda G_2; \boldsymbol{Z}/2)$, then $x^n\neq 0$ for any n.*

PROOF. We can put $x_5^2=\varepsilon_1x_3^2y_4+\varepsilon_2x_3y_7$ where $\varepsilon_1, \varepsilon_2\in \boldsymbol{Z}/2$ for dimensional reasons.

On the other hand, one can prove easily that

$$\begin{aligned} H^*(\Lambda G_2; \boldsymbol{Z}/2) &\overset{\Phi^*}{\cong} H^*(G_2; \boldsymbol{Z}/2)\otimes H^*(\Omega G_2; \boldsymbol{Z}/2) \\ &= \boldsymbol{Z}/2[x_3]/(x_3^4)\otimes E(x_5)\otimes \boldsymbol{Z}/2[a_2, a_8, a_{10}]/(a_2^4) \end{aligned}$$

for $*\leqq 10$ by calculating the Serre spectral sequence. By using the diagram (1.1),

$$\begin{aligned} (\Phi^*\otimes\Phi^*)\lambda^*a_8 &= \omega^*(\phi\otimes\phi)\Phi^*a_8 \\ &= \omega^*(\phi(a_8)\otimes\phi(1)) \\ &= \omega^*(a_8\otimes 1+1\otimes a_8+a_2^2\otimes a_2^2)\,. \end{aligned}$$

Since

$$\alpha(a_8) = 1\otimes a_8+x_3^2\otimes a_2$$

and

$$\omega^* = (1\otimes\Delta^*\otimes 1)\circ(1\otimes\alpha\otimes 1)\circ(1\otimes T^*\otimes 1),$$

we obtain

$$\lambda^* a_8 = a_8\otimes 1 + 1\otimes a_8 + a_2^2\otimes a_2^2 + x_3^2\otimes a_2 .$$

Now, in the Eilenberg-Moore spectral sequence

$$\mathrm{Cotor}^{H^*(\Lambda G_2;\, \boldsymbol{Z}/2)}(\boldsymbol{Z}/2,\ \boldsymbol{Z}/2) \Longrightarrow H^*(B\Lambda G_2\,;\ \boldsymbol{Z}/2),$$

$s(a_2)$, $s(a_2^2)$, $s(x_3^2)$ correspond to x_3, x_5, y_7 respectively, so the above equation yields a relation

$$x_5^2 + y_7 x_3 + \cdots = 0 .$$

(See **[15]** for the computation of the Cotor and the spectral sequence of this type.)

So ε_2 must be 1. Since $Sq^1 x_5 = x_3^2$ and $Sq^1 x_3 = 0$, we get

$$x_3^4 = Sq^2 x_5^2 = \varepsilon_1 x_3^2 y_6 + \varepsilon_2 x_5 y_7$$

and

$$0 = Sq^1 x_3^4 = \varepsilon_1 x_3^2 y_7 + \varepsilon_2 x_3^2 y_7 .$$

Thus $\varepsilon_1 = \varepsilon_2 = 1$ and we have two relations $x_5^2 = x_3 y_7 + x_3^2 y_4$ and $x_3^4 = x_5 y_7 + x_3^2 y_6$.

So $x_3^8 \equiv x_5^2 y_7^2 \equiv x_3 y_7^3 \bmod (y_4,\ y_6)$. Inductively, we have the equation

$$x_3^{8^m} \equiv x_3 y_7^{3(8^m-1)/7} \bmod (y_4,\ y_6) .$$

Thus $x_3^n \neq 0$ for all n. Since x_3 is a permanent cycle of the Serre spectral sequence of the fibration (4.1), the generator of $H^3(B\Lambda G_2\,;\ \boldsymbol{Z}/2)$ is not nilpotent.

$H^*(G\,;\ \boldsymbol{Z})$ has non trivial 2-torsion part if and only if G is G_2, F_4, E_6, E_7, E_8 or $Spin(n)$ $(n \geqq 7)$. For these cases, there exists an injective homomorphism $i: G_2 \to G$ such that

$$i_*: \pi_3(G_2) \xrightarrow{\ \cong\ } \pi_3(G) \cong \boldsymbol{Z}$$

and, by the naturality, the generator of $H^3(B\Lambda G\,;\ \boldsymbol{Z}/2)$ can not be nilpotent. Thus if $p=2$, (3) of the main theorem implies (1).

In the case of $(G,\ p) = (F_4,\ 3)$, there is an element $a_{10} \in H^{10}(\Omega G\,;\ \boldsymbol{Z}/3)$ satisfying

$$\alpha(a_{10}) = 1\otimes a_{10} - u_8\otimes a_2$$

as shown in section 3. A similar calculation in the proof of Lemma 4.1 gives $\bar{\lambda}^*(a_{10}) = -u_8\otimes a_2$. So, by using the Eilenberg-Moore spectral sequence, one can conclude that there is a relation of the form $y_4 u_8 + \cdots = 0$ in $H^*(B\Lambda G\,;\ \boldsymbol{Z}/3)$. In the Serre spectral sequence of the fibration

$$F_4 \longrightarrow B\Lambda F_4 \longrightarrow BF_4 ,$$

$y_4 \otimes u_8$ must be killed by the differential from $1 \otimes x_{11}$ and this spectral sequence does not collapse. Then, by the naturality, the Serre spectral sequences for $(G, p)=(E_6, 3)$, $(E_7, 3)$ and $(E_8, 3)$ can not collapse.

In the case of $(G, p)=(E_8, 5)$, by using the element $a_{14} \in H^{14}(\Omega E_8; \boldsymbol{Z}/5)$, a quite similar argument shows that the Serre spectral sequence

$$E_8 \longrightarrow B\Lambda E_8 \longrightarrow BE_8$$

does not collapse.

Thus (3) of the main theorem implies (1). To complete the proof of the main theorem, we have only to show that (1) implies (3).

ΛG is considered as the gauge group of the trivial principal G-bundle over S^1 and $\mathrm{Map}(S^1, BG)$ is connected because $\pi_1(BG)=0$. So, by the result of Atiyah-Bott [**4**], one has

$$B\Lambda G \cong \mathrm{Map}(S^1, BG).$$

Denote by $X_{(0)}$ the (0)-localization of a simply connected space X. Then, by the result of Hilton-Mislin-Roitberg [**9**], we have

$$\mathrm{Map}(S^1, BG)_{(0)} \cong \mathrm{Map}(S^1, BG_{(0)}).$$

Since $BG_{(0)}$ is the direct product of the Eilenberg-MacLane spaces, we obtain

$$\mathrm{Map}(S^1, BG_{(0)}) \cong BG_{(0)} \times G_{(0)}.$$

If we assume that (1) of the main theorem, then $H^*(BG; \boldsymbol{Z})$ is also p-torsion free. (See Borel [**5**].) Denote the Poincaré series of $H^*(X; R)$ by $PS(X; R)$. Then $PS(BG; \boldsymbol{Z}/p) \cdot PS(G; \boldsymbol{Z}/p)$ is equal to $PS(BG; \boldsymbol{Q}) \cdot PS(G; \boldsymbol{Q})$. Hence it is equal to $PS(B\Lambda G; \boldsymbol{Q})$. So, if the mod p cohomology Serre spectral sequence of the fibration $G \to B\Lambda G \to BG$ has non-trivial differentials, then there is a coefficient of $PS(B\Lambda G; \boldsymbol{Z}/p)$ which is less than the corresponding coefficient of $PS(B\Lambda G; \boldsymbol{Q})$. But it is impossible by the universal coefficient theorem.

Thus the mod p cohomology Serre spectral sequence of the fibration $G \to B\Lambda G \to BG$ collapses and $H^*(BG; \boldsymbol{Z}/p) \otimes H^*(G; \boldsymbol{Z}/p)$ is the tensor product of a polynomial algebra and an exterior algebra. If $p \neq 2$, this is free and commutative since $x^2=0$ for any element x of odd degree. Thus (3) holds.

Assume $p=2$. Since the mod p Serre spectral sequence is trivial, by comparing the rational case, clearly $H^*(B\Lambda G; \boldsymbol{Z})$ is p-torsion free. So, if x is the mod p reduction of an element $\tilde{x} \in H^{odd}(B\Lambda G; \boldsymbol{Z})$, then $\tilde{x}^2$ is of order 2 and must be zero. Thus x^2 itself is zero and (3) of the main theorem holds.

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Akira KONO
Department of Mathematics
Kyoto University
Sakyo-ku, Kyoto 606
Japan

Kazumoto KOZIMA
Department of Information Sciences
Kanagawa University
Hiratsuka-shi, Kanagawa 259-12
Japan