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On strong C^0-equivalence of real analytic functions

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Let $\mathcal{E}_{[\omega]}(n, 1)$ be the set of analytic function germs: $(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}, 0)$. In [7]-[12], T.C. Kuo has introduced some notions of blow-analycity as desirable (natural) equivalence relations for elements of $\mathcal{E}_{[\omega]}(n, 1)$, and has given important results to construct blow-analytic theory. Stimulated by his work, several singularists started on studying blow-analycity and introduced notions similar to but different from those of Kuo's ([2], [3], [5], [6], [16], [18]). These blow-analytic equivalences are slightly weaker than bianalycity, and much stronger than homeomorphism. In this note, we introduce the notion of strong C^0-equivalence as one of blow-analytic equivalences. Roughly speaking, it is a C^0-equivalence which preserves the tangency of analytic arcs at $0\in\boldsymbol{R}^n$. It seems that this equivalence is not so strong. In fact, this is weaker than some other blow-analytic equivalences. Our purposes in this paper are to formulate two conditions which imply strong C^0-equivalence and to show the Briançon-Speder family ([1]) and the Oka family ([15]) are not strongly C^0-trivial.

In the complex case, the Briançon-Speder family is well-known as an example that topological triviality does not imply the Whitney regularity, in other words, the Milnor number constancy does not imply μ^*-constancy. The Oka family also is μ-constant but not μ^*-constant. Both families have a *weak simultaneous resolution*, but have no *strong simultaneous resolution* (see [13], [14], [15], [17]). In the real case, however, the families are topologically trivial, but not even strongly C^0-trivial.

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§1. Results.

At first we define the notion of strong C^0-equivalence.

NOTATION. (1) By an analytic arc at $0\in\boldsymbol{R}^n$, we mean the germ of an analytic map $\lambda: [0, \varepsilon)\rightarrow\boldsymbol{R}^n$ with $\lambda(0)=0$, $\lambda(s)\neq0$, $s>0$. The set of all such arcs

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is defined by $\mathcal{A}(\boldsymbol{R}^n, 0)$.

(2) For λ, $\mu \in \mathcal{A}(\boldsymbol{R}^n, 0)$, $O(\lambda, \mu)>1$ (resp. $O(\lambda, \mu)=1$) means that arcs λ, μ are *tangent* (resp. *crossing without touching*) at $0 \in \boldsymbol{R}^n$.

DEFINITION 1. Given f, $g \in \mathcal{E}_{[\omega]}(n, 1)$, we say they are *strongly C^0-equivalent*, if there exists a local homeomorphism $\sigma: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that

(i) $f=g\circ\sigma$,

(ii) if $\lambda \in \mathcal{A}(\boldsymbol{R}^n, 0)$ with $\lambda \subset f^{-1}(0)$ (resp. $g^{-1}(0)$), then $\sigma(\lambda)$ (resp. $\sigma^{-1}(\lambda)) \in \mathcal{A}(\boldsymbol{R}^n, 0)$, and

(iii) for any λ, $\mu \in \mathcal{A}(\boldsymbol{R}^n, 0)$ with λ, $\mu \subset f^{-1}(0)$, $O(\lambda, \mu)=1$ if and only if $O(\sigma(\lambda), \sigma(\mu))=1$.

Let S^{n-1} denote the $(n-1)$-dimensional unit sphere. For $a=(a_1, \cdots, a_n) \in S^{n-1}$, let $L(a): [0, \delta) \to \boldsymbol{R}^n$ $(\delta>0)$ be a mapping defined by

$$L(a)(t) = (a_1 t, \cdots, a_n t)\,.$$

Then $L(a) \in \mathcal{A}(\boldsymbol{R}^n, 0)$. For any $\lambda \in \mathcal{A}(\boldsymbol{R}^n, 0)$, there exists unique $a \in S^{n-1}$ such that $O(\lambda, L(a))>1$. Then we write $L(a)=T(\lambda)$.

REMARK 1. For λ, $\mu \in \mathcal{A}(\boldsymbol{R}^n, 0)$, $O(\lambda, \mu)>1$ if and only if $T(\lambda)=T(\mu)$.

For $f \in \mathcal{E}_{[\omega]}(n, 1)$, let $C_0(f)$ denote the set of connected components of $f^{-1}(0)-\{0\}$ as germs at $0 \in \boldsymbol{R}^n$. We put

$$C_0(f) = \{C_1, \cdots, C_m\} \quad (m \in \{0\} \cup \boldsymbol{N})\,.$$

Here we consider the following problem:

PROBLEM. *Let $\{f_t\}$ be a family where $f_t \in \mathcal{E}_{[\omega]}(n, 1)$ (with an isolated singularity). Find the condition so that $\{f_t\}$ is topologically trivial, but is not strongly C^0-trivial or there exist f_s, f_r $(s \neq r)$ such that f_s is not strongly C^0-equivalent to f_r.*

In the case $m=0$ i.e. $f_t^{-1}(0)=\{0\}$, C^0-equivalence and strong C^0-equivalence are same notions. Therefore we consider the case $m \geqq 1$. Assume λ, $\mu \in \mathcal{A}(\boldsymbol{R}^n, 0)$ do not satisfy the condition (iii) of Definition 1. Then we can consider the following two situations:

(I) There exists $C_k \in C_0(f)$ such that λ, $\mu \subset \overline{C}_k$.

(II) There exist C_i, C_j $(i \neq j)$ such that $\lambda \subset \overline{C}_i$ and $\mu \subset \overline{C}_j$.

REMARK 2. In the case $n=3$, the situation (I) has a deep relation with a problem of position of arcs in $\overline{C}_k$.

Set

$$D(f) = \{a \in S^{n-1} \mid \exists \lambda \in \mathcal{A}(\boldsymbol{R}^n, 0),\ \lambda \subset f^{-1}(0) \text{ and } T(\lambda)=L(a)\}\,.$$

For $1 \leqq i \leqq m$, set

$$D_i(f) = \{a \in S^{n-1} \mid \exists \lambda \in \mathcal{A}(\boldsymbol{R}^n, 0),\ \lambda \subset \bar{C}_i \text{ and } T(\lambda) = L(a)\}.$$

Then $D(f) = D_1(f) \cup \cdots \cup D_m(f)$.

Now we introduce certain quantities $e(f)$, $D_{ij}(f)$ corresponding to the above situations (I) and (II), respectively.

(I) Let $f \in \mathcal{E}_{[\omega]}(3, 1)$. For each i, we denote by $E_i(f)$ the cardinal number of the set consisting of $a \in S^2$ which satisfies the following conditions:

(i) There exist $\lambda_1, \lambda_2 \in \mathcal{A}(\boldsymbol{R}^3, 0)$ such that $\lambda_1, \lambda_2 \subset \bar{C}_i$, and $T(\lambda_1) = T(\lambda_2) = L(a)$.

(ii) There exist $\mu_1, \mu_2 \in \mathcal{A}(\boldsymbol{R}^3, 0)$ such that $\mu_1, \mu_2 \subset \bar{C}_i$, $T(\mu_1) \neq L(a)$, $T(\mu_2) \neq L(a)$, and $\mu_j - \{0\}$ $(j=1, 2)$ are contained in the different components of $C_i - \lambda_1 \cup \lambda_2$.

We put $e(f) = \#\{i \mid E_i(f) = 0\}$. In the case $m=0$, put $e(f) = -1$ for convenience.

(II) For $1 \leqq i,\ j \leqq m$ $(i \neq j)$, define

$$D_{ij}(f) = \#(D_i(f) \cap D_j(f)).$$

We call $D_{ij}(f)$ *the cardinal number of common directions* of C_i and C_j.

PROPOSITION. (1) *If f, $g \in \mathcal{E}_{[\omega]}(3, 1)$ are strongly C^0-equivalent, then $e(f) = e(g)$.*

(2) *If f, $g \in \mathcal{E}_{[\omega]}(n, 1)$ are strongly C^0-equivalent, then the cardinal number of common directions of elements of $C_0(f)$ is equal to the corresponding one of $C_0(g)$.*

Let $D = \{x \in \boldsymbol{R} \mid |x| < 1 + \varepsilon\}$ where ε is a sufficiently small positive number. Applying Proposition, we have the following results.

THEOREM A. (Briançon-Speder family [1]) *Let $f_t : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}, 0)$, $t \in D$, be a family of weighted homogeneous polynomials with an isolated singularity defined by*

$$f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}.$$

Then f_0 is not strongly C^0-equivalent to f_{-1}.

REMARK 3. (1) T. Fukui [4] has proved the Briançon-Speder family admits a modified analytic trivialization via the weighted blowing-up in his sense.

(2) P. Milman pointed out to me that the Briançon-Speder family is not almost analytically trivial in the sense of Kuo [7].

THEOREM B. (Oka family [15]) *Let $f_t : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}, 0)$, $t \in D$, be a family of polynomials with an isolated singularity defined by*

$$f_t(x, y, z) = x^8 + y^{16} + z^{16} + tx^5z^2 + x^3yz^3 .$$

Then f_0 is not strongly C^0-equivalent to f_1.

§2. Proofs of Proposition and Theorems A, B.

Proof of Proposition. Let $\sigma : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ be a homeomorphism which gives a strong C^0-equivalence between f and g, and let $C_0(f) = C_1 \cup \cdots \cup C_m$ and $C_0(g) = C_1' \cup \cdots \cup C_m'$. Assume that $C_i' = \sigma(C_i)$ for $1 \leqq i \leqq m$. For any $a \in D(f)$, there exists $\lambda \in \mathcal{A}(\boldsymbol{R}^n, 0)$ such that $\lambda \subset f^{-1}(0)$ and $T(\lambda) = L(a)$. Then there exists unique $a' \in S^{n-1}$ such that $T(\sigma(\lambda)) = L(a')$, since $\sigma(\lambda) \in \mathcal{A}(\boldsymbol{R}^n, 0)$ with $\sigma(\lambda) \subset g^{-1}(0)$. We define $\sigma^* : D(f) \to D(g)$ by $\sigma^*(a) = a'$ for $a \in D(f)$. It is easy to see σ^* is a one-to-one correspondence. Moreover the restricted mapping $\sigma^*|_{D_i(f)} : D_i(f) \to D_i(g)$, $1 \leqq i \leqq m$, also gives a one-to-one correspondence. Therefore the statements (1), (2) in Proposition immediately follow.

Proof of Theorem A. We start by giving a sufficient condition for $E_i(f) = 0$ for some i.

Lemma. *Let $f \in \mathcal{E}_{[\omega]}(3, 1)$, and let $C_0(f) = \{C_1, \cdots, C_m\}$, $m \geqq 1$. If there exists a continuous function $h : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ which is differentiable at $0 \in \boldsymbol{R}^2$ and C_i such that* graph $h = \overline{C}_i$, *then $E_i(f) = 0$.*

Proof. Let $\Phi : \boldsymbol{R}^2 \to \mathrm{graph}\, h$ be a mapping defined by

$$\Phi(x, y) = (x, y, h(x, y)),$$

and let $\pi : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ be a projection: $\pi(x, y, z) = (x, y)$. For any $\lambda \in \mathcal{A}(\boldsymbol{R}^3, 0)$ with $\lambda \subset \overline{C}_i$, there exists unique $a \in D_i(f)$ such that $T(\lambda) = L(a)$. It follows from the differentiability of h at $0 \in \boldsymbol{R}^2$ that $\|\pi(a)\| \neq 0$. Then $\pi(\lambda) \in \mathcal{A}(\boldsymbol{R}^2, 0)$ and $T(\pi(\lambda)) = L(\pi(a)/\|\pi(a)\|)$. Let $\pi_i : D_i(f) \to S^1$ be a mapping defined by $\pi_i(a) = \pi(a)/\|\pi(a)\|$. By the differentiability, π_i is a one-to-one correspondence.

On the other hand, it is clear that there does not exist $b \in S^1$ satisfying the following conditions:

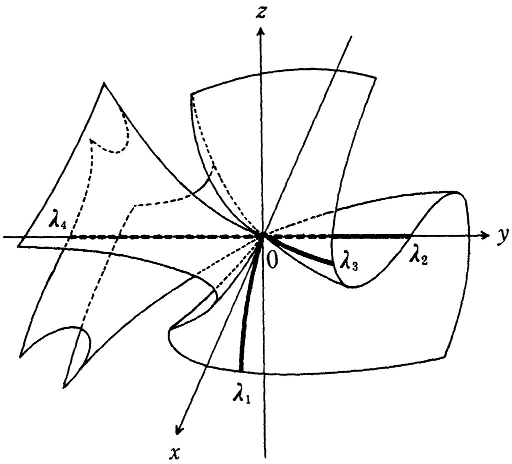
(i) There exist $\lambda_1, \lambda_2 \in \mathcal{A}(\boldsymbol{R}^2, 0)$ such that $T(\lambda_1) = T(\lambda_2) = L(b)$.

(ii) There exist $\mu_1, \mu_2 \in \mathcal{A}(\boldsymbol{R}^2, 0)$ such that $T(\mu_1) \neq L(b)$, $T(\mu_2) \neq L(b)$, and $\mu_j - \{0\}$ $(j=1, 2)$ are contained in the different components of $\boldsymbol{R}^2 - \lambda_1 \cup \lambda_2$.

Since $\pi|_{\mathrm{graph}\, h} : \mathrm{graph}\, h \to \boldsymbol{R}^2$ is a homeomorphism, $E_i(f) = 0$.

We show Theorem A by using this lemma. Put $f = f_0$ and $g = f_{-1}$. Let us define $h : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ by $h(x, y) = -(y^7x + x^{15})^{1/5}$. Then h is continuous and differentiable at $0 \in \boldsymbol{R}^2$. Remark that h is not of class C^1 at $0 \in \boldsymbol{R}^2$. Moreover we have graph $h = f^{-1}(0)$. By Lemma, $E(f) = 0$. Therefore it follows that $e(f) = 1$.

Let us consider the variety $g^{-1}(0)$ around $0\in \boldsymbol{R}^3$:



(Figure 1.)

Pick a point $P_0=(1, y_1, z_1)$ on $g^{-1}(0)$ with $y_1>0$, $z_1>0$. Define analytic arcs $\lambda_j\in\mathcal{A}(\boldsymbol{R}^3, 0)$ $(1\leqq j\leqq 4)$ as follows:

$$\begin{cases} \lambda_1(s)=(s, 0, -s^3), \\ \lambda_2(s)=(0, s, 0), \\ \lambda_3(s)=(s, y_1s^2, z_1s^3), \\ \lambda_4(s)=(0, -s, 0) \qquad (s\geqq 0). \end{cases}$$

Then $\quad T(\lambda_1)=L((1, 0, 0)), \quad T(\lambda_2)=L((0, 1, 0)),$

$$T(\lambda_3)=L((1, 0, 0)), \text{ and } T(\lambda_4)=L((0, -1, 0)).$$

Moreover $\lambda_2-\{0\}$ and $\lambda_4-\{0\}$ are contained in the different components of $g^{-1}(0)-\lambda_1\cup\lambda_3$. Therefore $E(g)\neq 0$. It follows that $e(g)=0$.

By Proposition (1), f is not strongly C^0-equivalent to g.

PROOF OF THEOREM B. Put

$$f(x, y, z)=f_0(x, y, z)=x^8+y^{16}+z^{16}+x^3yz^3\,.$$

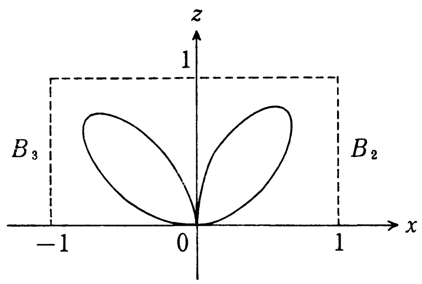
In each coordinate plane, $f^{-1}(0)-\{0\}=\emptyset$. Here we put

$$\begin{cases} B_1=\{x>0, y>0, z<0\}, \\ B_2=\{x>0, y<0, z>0\}, \\ B_3=\{x<0, y>0, z>0\}, \\ B_4=\{x<0, y<0, z<0\}. \end{cases}$$

In B_i $(1 \leqq i \leqq 4)$, $f^{-1}(0) \neq \emptyset$. In other octant, $f^{-1}(0) = \emptyset$. Put $C_i = f^{-1}(0) \cap B_i$ $(1 \leqq i \leqq 4)$. Then it is easy to see that C_i is connected, in particular, $\overline{C}_i = C_i \cup \{0\}$ is homeomorphic to S^2. Therefore $D_{ij}(f) \leqq 1$ $(i \neq j)$. We consider the curve defined by

$$f^{-1}(0) \cap \{y = -x\} \qquad \text{i.e.} \quad x^8 + x^{16} + z^{16} - x^4 z^3 = 0$$

(see Figure 2).



(Figure 2.)

Then there exist $\lambda_i \in \mathcal{A}(\boldsymbol{R}^3, 0)$ with $\lambda_i \subset \overline{C}_i$ $(i = 2, 3)$ such that $T(\lambda_2) = T(\lambda_3) = L((0, 0, 1))$. Therefore $D_{23}(f) \geqq 1$. It follows that $D_{23}(f) = 1$. Similarly, $D_{ij}(f) = 1$ $(i \neq j)$.

Next put

$$g(x, y, z) = f_1(x, y, z) = x^8 + y^{16} + z^{16} + x^5 z^2 + x^3 y z^3 .$$

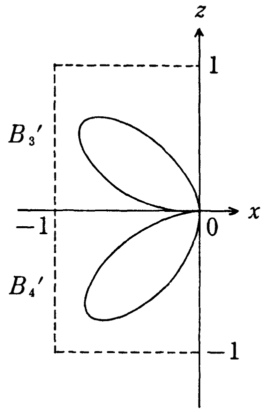
In (x, y)-plane or (y, z)-plane, $g^{-1}(0) - \{0\} = \emptyset$. Here we put

$$\begin{cases} B_1' = \{x > 0,\ y > 0,\ z < 0\}, \\ B_2' = \{x > 0,\ y < 0,\ z > 0\}, \\ B_3' = \{x < 0,\ z > 0\}, \\ B_4' = \{x < 0,\ z < 0\}. \end{cases}$$

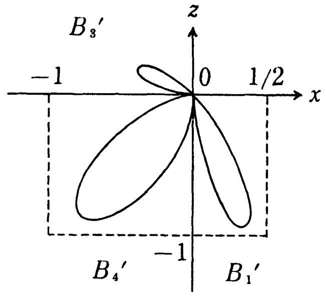
In B_i' $(1 \leqq i \leqq 4)$, $g^{-1}(0) \neq \emptyset$. Put $C_i' = g^{-1}(0) \cap B_i$ $(1 \leqq i \leqq 4)$. Then C_i' is connected and $\overline{C}_i'$ is homeomorphic to S^2. We consider the curve defined by

$$g^{-1}(0) \cap \{y = 0\} \qquad \text{i.e.} \quad x^8 + z^{16} + x^5 z^2 = 0$$

(see Figure 3).



(Figure 3.)



(Figure 4.)

Then there exist $\mu_i \in \mathcal{A}(\boldsymbol{R}^3, 0)$ with $\mu_i \subset \bar{C}_i'$ $(i=3, 4)$ such that $T(\mu_3)=T(\mu_4)=L((-1, 0, 0))$. Next consider the curve defined by

$$g^{-1}(0) \cap \{y=x\} \qquad \text{i.e.} \quad x^8+x^{16}+z^{16}+x^5z^2+x^4z^3=0$$

(see Figure 4). Then there exist $\nu_i \in \mathcal{A}(\boldsymbol{R}^3, 0)$ with $\nu_i \subset \bar{C}_i'$ $(i=3, 4)$ such that $T(\nu_3)=T(\nu_4)=L((-1/\sqrt{2}, -1/\sqrt{2}, 0))$. Therefore $D_{34}(g) \geqq 2$.

By Proposition (2), f is not strongly C^0-equivalent to g.

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