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Singularities of solutions of equations with noninvolutive characteristics-I; the case of second order Fuchsian equations

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§1. Introduction.

In the present paper we consider singular solutions of second order linear partial differential equations of Fuchsian type. Let $(t, z)=(t, z_1, z_2, \cdots, z_n)$ be the coordinate of $\boldsymbol{C}^1\times\boldsymbol{C}^n=\boldsymbol{C}^{n+1}$ and $(\tau, \xi)=(\tau, \xi_1, \xi_2, \cdots, \xi_n)$ be the variable dual to (t, z). $|z|$ means $\max_i|z_i|$, $\partial_t=\partial/\partial t$, $\partial_{z_i}=\partial/\partial z_i$ and $\partial_z=(\partial_{z_1}, \partial_{z_2}, \cdots, \partial_{z_n})$. For an open set W in $\boldsymbol{C}^N$, $T^*(W)$ is the cotangent space of W and $\widetilde{W}$ means the universal covering space of W. $\mathcal{O}(W)(\mathcal{O}(\widetilde{W}))$ is the set of all holomorphic functions on W (resp. $\widetilde{W}$). $\mathcal{O}(\widetilde{W})$ contains multi-valued functions on W. $\boldsymbol{N}$ is the set of all positive integers and $\boldsymbol{Z}_+$ is the set of all nonnegative integers, that is, $\boldsymbol{N}=\{1, 2, 3, \cdots\}$ and $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$. Let $L_i(t, z, \partial_t, \partial_z)$ $(i=1, 2)$ be linear partial differential operators with order i in the form

$$
(1.1)\qquad \begin{cases} L_2(t, z, \partial_t, \partial_z) = \partial_t^2+A(t, z, \partial_z)\partial_t+B(t, z, \partial_z), \\ L_1(t, z, \partial_t, \partial_z) = a_0(t, z)\partial_t+\sum_{i=1}^n a_i(t, z)\partial_{z_i}+d(t, z). \end{cases}
$$

The coefficients of $L_i(t, z, \partial_t, \partial_z)$ are holomorphic in a polydisk Ω whose center is the origin in $\boldsymbol{C}^{n+1}$.

Now let K_1 be a characteristic surface of $L_2(t, z, \partial_t, \partial_z)$. We study the equations

$$
(1.2)\qquad \begin{cases} L(t, z, \partial_t, \partial_z)u(t, z) = f(t, z), \\ L(t, z, \partial_t, \partial_z) = t^2L_2(t, z, \partial_t, \partial_z)+tL_1(t, z, \partial_t, \partial_z)+c(t, z), \end{cases}
$$

where $c(t, z)\in\mathcal{O}(\Omega)$ and $f(t, z)$ is holomorphic except on K_1. We also study

$$
(1.3)\qquad \begin{cases} L(t, z, \partial_t, \partial_z)u(t, z) = f(t, z), \\ u(0, z) = u_0(z), \\ L(t, z, \partial_t, \partial_z) = tL_2(t, z, \partial_t, \partial_z)+L_1(t, z, \partial_t, \partial_z), \end{cases}
$$

where $f(t, z)$ is holomorphic except on K_1, and $u_0(z)$ is holomorphic except on

$K_1\cap\{t=0\}$. Each $L(t, z, \partial_t, \partial_z)$ in (1.2) or (1.3) is an operator of Fuchsian type with respect to $\{t=0\}$ in the sense of Baouendi-Goulaouic [1]. The aim of this paper is to obtain an integral representation of $u(t, z)$ and the analysis of the singularities of it.

It is well known that the singularities of solutions of linear partial differential equations have relations to the characteristic set of operators. So we give the assumption on the characteristic set of $L_2(t, z, \partial_t, \partial_z)$.

ASSUMPTION 1.1. The principal symbol $(\text{P.S.}L_2)(t, z, \tau, \xi)$ of $L_2(t, z, \partial_t, \partial_z)$ is decomposed as follows:

$$(1.4) \qquad (\text{P.S.}L_2)(t, z, \tau, \xi) = (\tau - H_1(t, z, \xi))(\tau - H_2(t, z, \xi)),$$

where $H_i(t, z, \xi)$ $(i=1, 2)$ are holomorphic in a neighbourhood of $(t, z, \xi)=(0, 0, \hat{\xi})$, $\hat{\xi}=(1, 0, \cdots, 0)$, and $H_1(0, 0, \hat{\xi})\neq H_2(0, 0, \hat{\xi})$.

Let $\varphi_i(t, z)$ be a solution of

$$(1.5) \qquad \partial_t\varphi_i(t, z) = H_i(t, z, \partial_z\varphi_i), \qquad \varphi_i(0, z) = z_1 \qquad (i=1, 2).$$

Let $\Phi(t, z, \zeta)$ $((t, z, \zeta)\in \boldsymbol{C}^1\times\boldsymbol{C}^n\times\boldsymbol{C}^1)$ be the multi-phase function defined by

$$(1.6) \qquad \partial_t\Phi(t, z, \zeta) = H_1(t, z, \partial_z\Phi), \qquad \Phi(t, z, \zeta)|_{t=\zeta} = \varphi_2(\zeta, z).$$

$\Phi(t, z, \zeta)$ exists uniquely in a neighbourhood of $(t, z, \zeta)=(0, 0, 0)$. Put $K_i=\{\varphi_i(t, z)=0\}$ $(i=1, 2)$. We have $K_i\cap\{t=0\}=\{z_1=0\}$ for $i=1, 2$.

The characteristic set of $L(t, z, \partial_t, \partial_z)$ in (1.2) or (1.3) is

$$(1.7) \qquad \{t=0\}\cup\{\tau=H_1(t, z, \xi)\}\cup\{\tau=H_2(t, z, \xi)\}$$

in a neighbourhood of $(0, 0, \hat{\xi})$ in $T^*(\Omega)$ and by Assumption 1.1 $\{\tau=H_1(t, z, \xi)\}\cap\{\tau=H_2(t, z, \xi)\}=\emptyset$. The relation between $\{t=0\}$ and $\{\tau=H_i(t, z, \xi)\}$ is noninvolutive, that is, the Poisson blacket

$$(1.8) \qquad \{t, \tau-H_i(t, z, \xi)\}\neq 0 \quad \text{on} \quad \{t=0\}\cap\{\pi=H_i(t, z, \xi)\} \qquad (i=1, 2),$$

where $\{F, G\}=\partial_\tau F\partial_t G-\partial_t F\partial_\tau G+\sum_{i=1}^n(\partial_{\xi_i}F\partial_{z_i}G-\partial_{z_i}F\partial_{\xi_i}G)$. The characteristic surfaces of $L_2(t, z, \partial_t, \partial_z)$ through $z_1=0$ at $t=0$ are $K_i=\{\varphi_i(t, z)=0\}$ $(i=1, 2)$. $K_0=\{t=0\}$ is also a characteristic surface of the Fuchsian operator $L(t, z, \partial_t, \partial_z)$ in (1.2) (or (1.3)). So it is expected that the singularities of solutions of (1.2) (resp. (1.3)) lie on $K_0\cup K_1\cup K_2$, because K_1 and K_2 are simple characteristic surfaces of $L_2(t, z, \partial_t, \partial_z)$ and K_0 is noninvolutive with respect to K_1 and K_2.

Now we give a class of functions to which solutions belong. Put $\omega=\Omega\cap\{t=0\}$ and $\omega_0=(\Omega-K_1)\cap\{t=0\}=\{z\in\omega;\ z_1\neq 0\}$.

DEFINITION 1.1. We say that $u(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$, if there is a neighbourhood Ω_0 of ω_0 such that $u(t, z)\in\mathcal{O}(\tilde{\Omega}_0)$, where Ω_0 depends on $u(t, z)$.

We note that $u(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$, if and only if it has the expansion

$$u(t, z)=\sum_{n=0}^{+\infty}u_n(z)t^n, \tag{1.9}$$

where $u_n(z)\in\mathcal{O}(\tilde{\omega}_0)$, and (1.9) converges if $|t|<r(|z|)$.

Now let us return to (1.2). Put

$$\ell(L;\ \mu, z)=\mu(\mu-1)+a_0(0, z)\mu+c(0, z), \tag{1.10}$$

which is called the indicial polynomial of $L(t, z, \partial_t, \partial_z)$. Suppose $f(t, z)\in\mathcal{O}(\widetilde{\Omega-K_1})$. According to Baouendi-Goulaouic [1], if

$$\ell(L;\ \mu, 0)=\mu(\mu-1)+a_0(0, 0)\mu+c(0, 0)\neq 0 \quad \text{for} \quad \mu\in\boldsymbol{Z}_+, \tag{1.11}$$

then there is a unique solution $u(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$ of the equation (1.2) for an ω_0 whose Taylor expansion (1.9) converges on $\tilde{\Omega}_0^*$, $\Omega_0^*=\{|t|<r,\ |z|<r,\ k|t|<|z_1|\}$ for some positive constants r and k. The expansion (1.9) is not valid in a full neighbourhood of the origin, because the singularities of $u(t, z)$ lie in $\{k|t|\geqq|z_1|\}$.

In order to give an integral representation of $u(t, z)$ we introduce several spaces of functions. Let $(t, z, \zeta, s)\in\boldsymbol{C}^1\times\boldsymbol{C}^n\times\boldsymbol{C}^1\times\boldsymbol{C}^1=\boldsymbol{C}^{n+3}$. Put

$$\begin{cases} U=U(r)=\{(t, z, \zeta)\in\boldsymbol{C}^{n+2};\ |t|<r,\ |z|<r,\ |\zeta|<r\}, \\ X_0=U\cap\{|\zeta|>c|t|\} \qquad (c\geqq 1), \\ X=U-\{t=0\}\cup\{\zeta=0\}\cup\{\zeta=t\}, \\ Y_0=X_0\times\widetilde{\{s\in\boldsymbol{C}^1;\ 0<|s|<r_1\}}, \\ Y=X\times\widetilde{\{s\in\boldsymbol{C}^1;\ 0<|s|<r_1\}}. \end{cases} \tag{1.12}$$

We define spaces $\mathcal{O}_0(\tilde{X})\subset\mathcal{O}(\tilde{X})$ and $\mathcal{O}_0(\tilde{Y})\subset\mathcal{O}(\tilde{Y})$.

DEFINITION 1.3. (1) $\mathcal{O}_0(\tilde{X})$ is the class of all holomorphic functions $W(t, z, \zeta)$ on X_0 such that it can be holomorphically extensible onto $\tilde{X}$.

(2) $\mathcal{O}_0(\tilde{Y})$ is the class of all holomorphic functions $U(t, z, \zeta, s)$ on Y_0 such that it can be holomorphically extensible onto $\tilde{Y}$.

It follows from the definition that $\mathcal{O}_0(\tilde{X})$ is the class of all holomorphic functions $W(t, z, \zeta)$ on $\tilde{X}$ such that there is a branch of $W(t, z, \zeta)$ which is holomorphic on $X_1=U\cap\{|\zeta|>|t|\}$, that is, we can take $c=1$ in Definition 1.3. But when we show in the following sections that a function $W(t, z, \zeta)$ belongs to $\mathcal{O}_0(\tilde{X})$, we show firstly $W(t, z, \zeta)\in\mathcal{O}(X_0)$ for some c and next that it is extensible to $\tilde{X}$. From the definition $W(t, z, \zeta)$ has the Taylor's expansion on X_0,

$$W(t, z, \zeta)=\sum_{k=0}^{\infty}w_k(z, \zeta)t^k, \tag{1.13}$$

where $w_k(z, \zeta)$ $(k\geqq 0)$ are holomorphic on $\{(z, \zeta)\in\boldsymbol{C}^{n+1};\ |z|<r,\ 0<|\zeta|<r\}$ and the series (1.13) converges if $|t|<|\zeta|/c$. The similar properties hold for functions in $\mathcal{O}_0(\tilde{Y})$.

We have a representation of solutions of (1.2).

THEOREM 1.4. *Suppose that* (1.11) *is satisfied and* $f(t, z)\in\mathcal{O}(\widetilde{\Omega-K_1})$ *in* (1.2). *Let* $u(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$ *be a solution of* (1.2). *Then* $u(t, z)$ *is represented in the form*

$$(1.14)\qquad u(t, z)=\int_\Gamma U^*(t, z, \zeta, \Phi(t, z, \zeta))d\zeta+\text{ a holomorphic function at the origin},$$

in $\tilde{\Omega}_0'$, *where* $\Omega_0'=\{(t, z);\ |t|<r,\ |z|<r,\ k|t|<|z_1|\}$ *for a small* $r>0$ *and some* $k>0$, $U^*(t, z, \zeta, s)\in\mathcal{O}_0(\tilde{Y})$ *and the integration path* Γ *in* ζ*-space is closed and surrounding once* $\zeta=0$ *in* $X_1=\{|\zeta|>|t|\}$ *and* $\{\zeta;\ \Phi(t, z, \zeta)=0\}$ *is outside of* Γ (*see Fig.* 1.1).

Since $U^*(t, z, \zeta, s)\in\mathcal{O}_0(\tilde{Y})$, we can deform Γ homotopically in X and remove the condition that $\{\zeta;\ \Phi(t, z, \zeta)=0\}$ is in $\{|\zeta|>|t|\}$ (see Fig. 1.2).

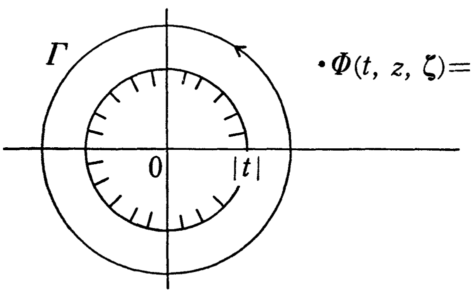

Fig. 1.1.

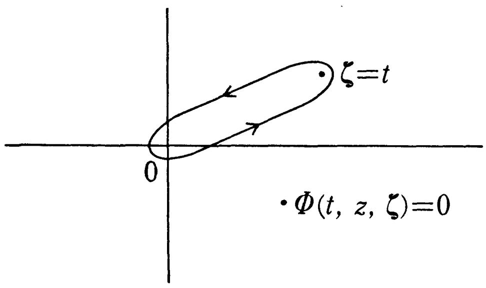

Fig. 1.2.

The singularities of $U^*(t, z, \zeta, s)$ are on $\{t=0\}\cup\{\zeta=0\}\cup\{t=\zeta\}\cup\{s=0\}$. Hence we have

THEOREM 1.5. *Suppose that* (1.11) *is satisfied and* $f(t, z)\in\mathcal{O}(\widetilde{\Omega-K_1})$ *in* (1.2). *Let* $u(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$ *be a solution of* (1.2). *Then* $u(t, z)$ *has the holomorphic prolongation to* $(\widetilde{\Omega'-K_0\cup K_1}\cup K_2$ *in a small neighbourhood* Ω' *of the origin.*

Next we consider (1.2) when (1.11) does not hold: there is an $n_0\in\boldsymbol{Z}_+$ such that

$$(1.15)\qquad n_0(n_0-1)+a_0(0, 0)n_0+c(0, 0)=0.$$

Put $S=\{z\in\omega;\ n_0(n_0-1)+a_0(0, z)n_0+c(0, z)=0\}$, $\omega=\Omega\cap\{t=0\}$. We have

THEOREM 1.6. *Suppose that* (1.15) *holds for an* $n_0\in\boldsymbol{Z}_+$ *and*

$$(1.16)\qquad \mu(\mu-1)+a_0(0, 0)\mu+c(0, 0)\neq 0\quad \textit{for } \mu\in\boldsymbol{Z}_+-\{n_0\}.$$

Further assume $S=\{z\in\omega;\ z_1=0\}$ *and* $f(t, z)\in\mathcal{O}(\Omega)$. *Let* $u(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$ *be a solu-*

tion of (1.2). *Then* $u(t, z)$ *has the holomorphic prolongation to* $\widetilde{\Omega' - K_0 \cup K_1 \cup K_2}$ *in a small neighbourhood* Ω' *of the origin.*

Now we proceed to study the equation (1.3). We have

THEOREM 1.7. *Suppose that*

$$\mu - 1 + a_0(0, 0) \neq 0 \qquad \text{for} \quad \mu \in \boldsymbol{N}, \tag{1.17}$$

$f(t, z) \in \mathcal{O}(\widetilde{\Omega - K_1})$ *and* $u_0(z) \in \mathcal{O}(\tilde{\omega}_0)$ *in* (1.3). *Let* $u(t, z) \in \mathcal{O}_{\tilde{\omega}_0}$ *be a solution of* (1.3). *Then* $u(t, z)$ *has the holomorphic prolongation to* $\widetilde{\Omega' - K_0 \cup K_1 \cup K_2}$ *in a small neighbourhood* Ω' *of the origin.*

We also consider, when (1.17) does not hold: there is an $n_0 \in \boldsymbol{N}$ such that

$$n_0 - 1 + a_0(0, 0) = 0. \tag{1.18}$$

Put $S = \{z \in \omega;\ n_0 - 1 + a_0(0, z) = 0\}$.

THEOREM 1.8. *Suppose* (1.18) *holds for an* $n_0 \in \boldsymbol{N}$ *and* $S = \{z \in \omega;\ z_1 = 0\}$. *Let* $f(t, z) \in \mathcal{O}(\Omega)$ *and* $u_0(z) \in \mathcal{O}(\omega)$. *Let* $u(t, z) \in \mathcal{O}_{\tilde{\omega}_0}$ *be a solution of* (1.3). *Then* $u(t, z)$ *has the holomorphic prolongation to* $\widetilde{\Omega' - K_0 \cup K_1 \cup K_2}$ *in a small neighbourhood* Ω' *of the origin.*

REMARK 1.9. In all the cases (Theorems 1.4–1.8) the existence of a solution $u(t, z) \in \mathcal{O}_{\tilde{\omega}_0}$ and the uniqueness in $\mathcal{O}_{\tilde{\omega}_0}$ follow from Baouendi-Goulaouic [**1**]. In Theorems 1.6 and 1.8 the assumption on S means that S is an $(n-1)$-dimensional complex manifold.

Theorem 1.4 is fundamental in this paper from which Theorem 1.5 follows. Other theorems follow from Theorem 1.5.

Finally we state results about Fuchsian equations in complex domains and integral representations of singular solutions of partial differential equations in complex domains. The existence and uniqueness of holomorphic solutions at the origin for Fuchsian equations were firstly treated in Hasegawa [**5**] and generally in Baouendi-Goulaouic [**1**]. Solutions which admit singularities at $\{t=0\}$ were studied in Tahara [8]. As for the investigations which are much relevant to this paper, Urabe [**9**] and Fujiie [**2**] are cited. In Urabe [**9**] the Cauchy problem with a singular initial data $u_0(z)$ is considered for a special class of Fuchsian operators in $\boldsymbol{C}^2$ with the principal part $t\partial_t^2 - \partial_x^2$, which is reduced to (1.3) by changing $t = s^2$. He used the hypergeometric functions to study singularities of solutions. In Fujiie [**2**] the statements of Theorem 1.8 was obtained for a special class of Fuchsian equations in $\boldsymbol{C}^2$. As for integral representations of solutions with singularities on a characteristic surface, we refer to Ōuchi [6] and [7], where behaviours of solutions near the characteristic surface were also considered, but Fuchsian operators were not treated.

In the following sections we often write only the noteworthy variables in the notations of functions, operators or sets, omitting other variables for simplicity. Many constants will appear. So we denote by $A, B, \cdots$ various constants. Constants $R, R', r, \cdots$, which define neighbourhoods of the origin, are chosen so small, if necessary. The author thanks the referee who read the manuscript carefully and gave him useful advices.

§2. New operator $\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)$.

In order to show Theorem 1.4, we define in §2 a new operator $\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)$ which is derived from the operator $L(t, z, \partial_t, \partial_z)=t^2L_2(t, z, \partial_t, \partial_z)+tL_1(t, z, \partial_t, \partial_z)+c(t, z)$ in (1.2), and give equations (2.12) and (2.30) below. In the following we construct $U^*(t, z, \zeta, s)$ in the integral representation (1.14) by using a solution $W(t, z, \zeta, \lambda)$ of the equation (2.12).

Now let us define an operator $\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)$ after some simple calculations. In the following $d(t, z, \zeta)$ means various holomorphic functions which are not important and z_0 means t.

Put

(2.1) $$H(t, z, \zeta) = H_1(t, z, \partial_z\Phi(t, z, \zeta))-H_2(t, z, \partial_z\Phi(t, z, \zeta)).$$

For the multi-phase function $\Phi(t, z, \zeta)$ we have

LEMMA 2.1. (1) *$K_1=\{\Phi(t, z, 0)=0\}$ and $K_2=\{\Phi(t, z, t)=0\}$ in a neighbourhood of $(t, z)=(0, 0)$.*

(2) *There are holomorphic functions $\chi(t, z, \zeta)$ and $\chi_0(t, z, \zeta)$ in a neighbourhood of $(t, z, \zeta)=(0, 0, 0)$ such that*

(2.2) $$\begin{cases} H(t, z, \zeta) = -\partial_\zeta\Phi(t, z, \zeta)\chi(t, z, \zeta), \\ \chi(t, z, \zeta) = 1+(\zeta-t)\chi_0(t, z, \zeta). \end{cases}$$

PROOF. The assertion (1) is obvious. We show (2). Since $\Phi(t, z, \zeta)=\varphi_2(\zeta, z)+\int_\zeta^t H_1(s, z, \partial_z\Phi(s, z, \zeta))ds$, we have

$$\begin{aligned}\partial_\zeta\Phi(t, z, \zeta) &= \partial_\zeta\varphi_2(\zeta, z)-H_1(\zeta, z, \partial_z\Phi(\zeta, z, \zeta))+O(|t-\zeta|)\\ &= H_2(\zeta, z, \partial_z\varphi_2(\zeta, z))-H_1(\zeta, z, \partial_z\Phi(\zeta, z, \zeta))+O(|t-\zeta|)\\ &= -H(\zeta, z, \zeta)+O(|t-\zeta|) = -H(t, z, \zeta)+O(|t-\zeta|).\end{aligned}$$

By Assumption 1.1, $H(0, 0, 0)\neq 0$. Hence there are $\chi(t, z, \zeta)$ and $\chi_0(t, z, \zeta)$ with $\chi(t, z, \zeta)=1+(\zeta-t)\chi_0(t, z, \zeta)$ such that $H(t, z, \zeta)=-\partial_\zeta\Phi(t, z, \zeta)\chi(t, z, \zeta)$.

In the following ((2.3)-(2.11)) $W(t, z, \zeta, \lambda)$ is a holomorphic function on X_0 with a parameter λ.

LEMMA 2.2. *The following identity holds:*

$$L(t, z, \partial_t, \partial_z)(\exp(\lambda\Phi(t, z, \zeta))W(t, z, \zeta, \lambda)) \\ = \exp(\lambda\Phi(t, z, \zeta))\{(\lambda\mathcal{L}_1(t, z, \zeta, \partial_t, \partial_z)+L(t, z, \partial_t, \partial_z))W(t, z, \zeta, \lambda)\}, \tag{2.3}$$

where

$$\mathcal{L}_1(t, z, \zeta, \partial_t, \partial_z) = H(t, z, \zeta)t^2(\partial_t-\textstyle\sum_{i=1}^n \partial_{\xi_i}H_1(t, z, \partial_z\Phi)\partial_{z_i}) \\ +t(\textstyle\sum_{i=0}^n a_i(t, z)\partial_{z_i}\Phi+td(t, z, \zeta)). \tag{2.4}$$

PROOF. We have

$$\exp(-\lambda\Phi(t, z, \zeta))L(t, z, \partial_t, \partial_z)\exp(\lambda\Phi(t, z, \zeta))W(t, z, \zeta, \lambda) \\ = \lambda\{t^2(\partial_t\Phi-H_2(t, z, \partial_z\Phi))(\partial_t-\textstyle\sum_{i=1}^n \partial_{\xi_i}H_1(t, z, \partial_z\Phi)\partial_{z_i}) \\ +t(\textstyle\sum_{i=0}^n a_i(t, z)\partial_{z_i}\Phi+td(t, z, \zeta))\}W(t, z, \zeta, \lambda)+L(t, z, \partial_t, \partial_z)W(t, z, \zeta, \lambda).$$

From $\partial_t\Phi(t, z, \zeta)=H_1(t, z, \partial_z\Phi(t, z, \zeta))$, we have (2.3) and (2.4).

From Lemmas 2.1 and 2.2

$$\mathcal{L}_1(t, z, \zeta, \partial_t, \partial_z) = -(\partial_\zeta\Phi(t, z, \zeta))\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z), \tag{2.5}$$

where

$$\mathcal{M}(t, z, \zeta, \partial_t, \partial_z) = t^2(\partial_t-\textstyle\sum_{i=1}^n \partial_{\xi_i}H_1(t, z, \partial_z\Phi)\partial_{z_i})+tb^*(t, z, \zeta), \\ b^*(t, z, \zeta) = (\textstyle\sum_{i=0}^n a_i(t, z)\partial_{z_i}\Phi+td(t, z, \zeta))/H(t, z, \zeta). \tag{2.6}$$

Now we introduce an operator

$$\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)W \\ = (1-t\chi_0(t, z, \zeta))^{-1}(L(t, z, \partial_t, \partial_z)W+\partial_\zeta(\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W)), \tag{2.7}$$

using $\chi_0(t, z, \zeta)$ given in (2.2). $\mathcal{L}=\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)$ can be also written as follows:

$$\mathcal{L} = \mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta) \\ = t^2\{(1-t\chi_0(t, z, \zeta))^{-1}(\partial_t^2+\chi(t, z, \zeta)\partial_\zeta\partial_t) \\ +A_1(t, z, \zeta, \partial_z)\partial_t+B_1(t, z, \zeta, \partial_z)\partial_\zeta+C_2(t, z, \zeta, \partial_z)\} \\ +t\{a(z)\partial_t+b(z, \zeta)\partial_\zeta+\textstyle\sum_{i=1}^n a_i(z)\partial_{z_i}+d(t, z, \zeta)\}+c(z), \tag{2.8}$$

where $\mathrm{ord.}A_1(t, z, \zeta, \partial_z)\leqq 1$, $\mathrm{ord.}B_1(t, z, \zeta, \partial_z)\leqq 1$, $\mathrm{ord.}C_2(t, z, \zeta, \partial_z)\leqq 2$, and

$$\begin{cases} a(z) = a_0(0, z), \qquad a_i(z) = a_i(0, z) \qquad (i \geqq 1), \\ b(z, \zeta) = b^*(0, z, \zeta), \qquad c(z) = c(0, z). \end{cases} \tag{2.9}$$

We remark that $\mathcal{L}=\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)$ is also an operator of Fuchsian type with respect to $\{t=0\}$, as for the indicial polynomial $\ell(\mathcal{L}; \mu, z, \zeta)=\ell(L; \mu, z)$, and $\{\zeta=0\}$ and $\{t=\zeta\}$ are its characteristic surfaces.

Put

$$U(t, z, \lambda)=\int_\Gamma \exp(\lambda\Phi(t, z, \zeta))W(t, z, \zeta, \lambda)d\zeta, \tag{2.10}$$

where the path Γ is chosen so that the following calculations are valid. We have, from Lemma 2.2 and by integration by parts,

$$\begin{aligned}
&L(t, z, \partial_t, \partial_z)U(t, z, \lambda) \\
&=\int_\Gamma \exp(\lambda\Phi(t, z, \zeta))\{\lambda\mathcal{L}_1(t, z, \zeta, \partial_t, \partial_z)+L(t, z, \partial_t, \partial_z)\}W(t, z, \zeta, \lambda)d\zeta \\
&=\int_\Gamma \exp(\lambda\Phi(t, z, \zeta))\{-\lambda\partial_\zeta\Phi(t, z, \zeta)\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W(t, z, \zeta, \lambda) \\
&\quad +L(t, z, \partial_t, \partial_z)W(t, z, \zeta, \lambda)\}\,d\zeta \\
&=\int_\Gamma \{(-\partial_\zeta \exp(\lambda\Phi(t, z, \zeta)))\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W(t, z, \zeta, \lambda) \\
&\quad +\exp(\lambda\Phi(t, z, \zeta))L(t, z, \partial_t, \partial_z)W(t, z, \zeta, \lambda)\}\,d\zeta \\
&=\int_\Gamma \exp(\lambda\Phi(t, z, \zeta))\{\partial_\zeta(\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W(t, z, \zeta, \lambda)) \\
&\quad +L(t, z, \partial_t, \partial_z)W(t, z, \zeta, \lambda)\}\,d\zeta \\
&=\int_\Gamma \exp(\lambda\Phi(t, z, \zeta))(1-t\chi_0(t, z, \zeta))(\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)W(t, z, \zeta, \lambda))d\zeta,
\end{aligned} \tag{2.11}$$

which will be justified later.

Now we give an equation to determine $W(t, z, \zeta, \lambda)$ in (2.10) as follows:

$$\begin{cases} \mathcal{L}W(t, z, \zeta, \lambda)\equiv \dfrac{g(t, z, \zeta, \lambda)}{2\pi i\zeta} \quad (\text{mod. holomorphic functions at } \zeta=0), \\ g(t, z, \zeta, \lambda)=\dfrac{\hat{f}(t, z, \lambda)}{(1-t\chi_0(t, z, \zeta))}, \end{cases} \tag{2.12}$$

where $\hat{f}(t, z, \lambda)$ is determined by $f(t, z)$ given in (1.2), which is holomorphic in $\{(t, z, \lambda);\ |t|\leqq R,\ |z|\leqq R,\ |\lambda|<+\infty\}$ (see §6). Note that the right hand side of the first formula of (2.12) has a pole at $\zeta=0$. Our next aim is to construct a singular solution $W(t, z, \zeta, \lambda)$ of the equation (2.12) in a neighbourhood of $(t, z, \zeta)=(0, 0, 0)$.

In order to construct $W(t, z, \zeta, \lambda)$, we introduce auxilliary functions $\{f_n(\zeta)\}_{n\in\mathbf{Z}}$ used in Hamada [3]:

$$
(2.13)\qquad f_n(\zeta)=\begin{cases}\dfrac{\zeta^n}{(2\pi i)n!}\Big(\log\zeta-\Big(1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}\Big)\Big) & n\geqq 1,\\[2mm] \dfrac{1}{2\pi i}\log\zeta & n=0,\\[2mm] (-1)^{n-1}\dfrac{(-n-1)!}{2\pi i}\zeta^n & n<0.\end{cases}
$$

It holds that

$$
(2.14)\qquad \frac{d}{d\zeta}f_n(\zeta)=f_{n-1}(\zeta)
$$

and

$$
(2.15)\qquad \begin{cases}\zeta f_n(\zeta)=(n+1)f_{n+1}(\zeta) & n<0,\\ \zeta f_n(\zeta)=(n+1)f_{n+1}(\zeta)+C_n\zeta^{n+1} & n\geqq 0.\end{cases}
$$

We find $W(t, z, \zeta, \lambda)$ in the form

$$
(2.16)\qquad W(t, z, \zeta, \lambda)=\sum_{n=-1}^{+\infty}\int_{|\rho|=d}f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)d\rho\,.
$$

We have a formula, putting $\chi_1(t, z, \zeta)=(1-t\chi_0(t, z, \zeta))^{-1}$,

$$
\begin{aligned}
(2.17)\qquad & \mathcal{L}f(\zeta-\rho t)v(t, z, \zeta, \rho, \lambda)\\
&=f''(\zeta-\rho t)\{t^2\chi_1(t, z, \zeta)(\rho^2-\rho\chi(t, z, \zeta))v(t, z, \zeta, \rho, \lambda)\}\\
&\quad+f'(\zeta-\rho t)\{t^2(\chi_1(t, z, \zeta)((\chi(t, z, \zeta)-2\rho)\partial_t-\rho\chi(t, z, \zeta)\partial_\zeta)\\
&\quad+B_1(t, z, \zeta, \partial_z)-\rho A_1(t, z, \zeta, \partial_z))+t(b(z, \zeta)-\rho a(z))\}v(t, z, \zeta, \rho, \lambda)\\
&\quad+f(\zeta-\rho t)\{t^2(\chi_1(t, z, \zeta)(\partial_t^2+\chi(t, z, \zeta)\partial_t\partial_\zeta)\\
&\quad+A_1(t, z, \zeta, \partial_z)\partial_t+B_1(t, z, \zeta, \partial_z)\partial_\zeta+C_2(t, z, \zeta, \partial_z))\\
&\quad+t(\textstyle\sum_{i=0}^n a_i(z)\partial_{z_i}+b(z, \zeta)\partial_\zeta+d(t, z, \zeta))+c(z)\}v(t, z, \zeta, \rho, \lambda)\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(2.18)\quad & \mathcal{L}W(t, z, \zeta, \lambda)\\
&=\sum_{n=-1}^{\infty}\int_{|\rho|=d}[f_{n-2}(\zeta-\rho t)\{t^2\chi_1(t, z, \zeta)(\rho^2-\rho\chi(t, z, \zeta))w_n(t, z, \zeta, \rho, \lambda)\}\\
&\quad+f_{n-1}(\zeta-\rho t)\{t^2(\chi_1(t, z, \zeta)((\chi(t, z, \zeta)-2\rho)\partial_t-\rho\chi(t, z, \zeta)\partial_\zeta)+B_1(t, z, \zeta, \partial_z)\\
&\quad-\rho A_1(t, z, \zeta, \partial_z))+t(b(z, \zeta)-\rho a(z))\}w_n(t, z, \zeta, \rho, \lambda)\\
&\quad+f_n(\zeta-\rho t)\{t^2(\chi_1(t, z, \zeta)(\partial_t^2+\chi(t, z, \zeta)\partial_t\partial_\zeta)\\
&\quad+A_1(t, z, \zeta, \partial_z)\partial_t+B_1(t, z, \zeta, \partial_z)\partial_\zeta+C_2(t, z, \zeta, \partial_z))\\
&\quad+t(\textstyle\sum_{i=0}^n a_i(z)\partial_{z_i}+b(z, \zeta)\partial_\zeta+d(t, z, \zeta))+c(z)\}w_n(t, z, \zeta, \rho, \lambda)]d\rho\,.
\end{aligned}
$$

We note that

$$tf_{n-1}(\zeta-\rho t)=-\partial_\rho f_n(\zeta-\rho t) \tag{2.19}$$

and from (2.15)

$$\begin{aligned}(\zeta-\rho t)f_{n-1}(\zeta-\rho t)&=nf_n(\zeta-\rho t)+\text{ a polynomial of }(\zeta-\rho t)\\ &\equiv nf_n(\zeta-\rho t)\text{ mod. holomorphic functions at }\zeta=0.\end{aligned} \tag{2.20}$$

In the following $\equiv$ means modulo holomorphic functions at $\zeta=0$ as in (2.12). The following calculations are formal but they are justified later. Since $\chi(t, z, \zeta)=1+(\zeta-t)\chi_0(t, z, \zeta)$, we have

$$\rho^2-\rho\chi(t, z, \zeta)=(\rho^2-\rho)(1-t\chi_0(t, z, \zeta))-\chi_0(t, z, \zeta)\rho(\zeta-\rho t) \tag{2.21}$$

and

$$\begin{cases}(1-t\chi_0(t, z, \zeta))^{-1}(\rho^2-\rho\chi(t, z, \zeta))=(\rho^2-\rho)-\chi_0^*(t, z, \zeta)\rho(\zeta-\rho t),\\ \chi_0^*(t, z, \zeta)=(1-t\chi_0(t, z, \zeta))^{-1}\chi_0(t, z, \zeta).\end{cases} \tag{2.22}$$

Hence, by (2.19), (2.20), (2.22) and integrations by parts, we have

$$\begin{aligned}&(1-t\chi_0(t, z, \zeta))^{-1}\int_{|\rho|=d}f_{n-2}(\zeta-\rho t)t^2(\rho^2-\rho\chi(t, z, \zeta))w_n(t, z, \zeta, \rho, \lambda)d\rho\\ &=\int_{|\rho|=d}f_{n-2}(\zeta-\rho t)t^2((\rho^2-\rho)-\chi_0^*(t, z, \zeta)\rho(\zeta-\rho t))w_n(t, z, \zeta, \rho, \lambda)d\rho\\ &\equiv\int_{|\rho|=d}\{f_n(\zeta-\rho t)((\rho^2-\rho)w_n(t, z, \zeta, \rho, \lambda))_{\rho\rho}\\ &\quad-f_{n+1}(\zeta-\rho t)(n-1)(\chi_0^*(t, z, \zeta)\rho w_n(t, z, \zeta, \rho, \lambda))_{\rho\rho}\}\,d\rho.\end{aligned} \tag{2.23}$$

Then we have

$$\begin{aligned}\mathcal{L}W(t, z, \zeta, \lambda)\equiv{}&\textstyle\sum_{n=-1}^{\infty}\int_{|\rho|=d}[f_n(\zeta-\rho t)\{((\rho^2-\rho)w_n(t, z, \zeta, \rho, \lambda))_{\rho\rho}\\ &+((b(z, \zeta)-\rho a(z))w_n(t, z, \zeta, \rho, \lambda))_\rho+c(z)w_n(t, z, \zeta, \rho, \lambda)\}\\ &+f_{n+1}(\zeta-\rho t)\{((\chi_1(t, z, \zeta)((\chi(t, z, \zeta)-2\rho)\partial_t-\rho\chi(t, z, \zeta)\partial_\zeta)\\ &+B_1(t, z, \zeta, \partial_z)-\rho A_1(t, z, \zeta, \partial_z)-(n-1)\chi_0^*(t, z, \zeta)\rho)w_n(t, z, \zeta, \rho, \lambda))_{\rho\rho}\\ &+((\textstyle\sum_{i=0}^{n}a_i(z)\partial_{z_i}+b(z, \zeta)\partial_\zeta+d(t, z, \zeta))w_n(t, z, \zeta, \rho, \lambda))_\rho\}\\ &+f_{n+2}(\zeta-\rho t)\{(\chi_1(t, z, \zeta)(\partial_t^2+\chi(t, z, \zeta)\partial_t\partial_\zeta)\\ &+A_1(t, z, \zeta, \partial_z)\partial_t+B_1(t, z, \zeta, \partial_z)\partial_\zeta+C_2(t, z, \zeta, \partial_z))w_n(t, z, \zeta, \rho, \lambda)\}_{\rho\rho}]d\rho.\end{aligned} \tag{2.24}$$

Define

$$Mw=M(t, z, \zeta, \partial_\rho)w=\{(\rho^2-\rho)w\}_{\rho\rho}+\{(b(z, \zeta)-\rho a(z))w\}_\rho+c(z)w \tag{2.25}$$

and

$$N_n(u,v)=\partial_\rho^2\{(\rho A_1^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)+B_1^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta))u(t,z,\zeta,\rho)\}$$
$$+C_1^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)\partial_\rho u(t,z,\zeta,\rho)+(n-2)C_0^*(t,z,\zeta)\partial_\rho^2(\rho u(t,z,\zeta,\rho))$$
$$+D_2^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)\partial_\rho^2 v(t,z,\zeta,\rho), \tag{2.26}$$

where

$$\begin{cases} A_1^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)=\chi_1(t,z,\zeta)(2\partial_t+\chi(t,z,\zeta)\partial_\zeta)+A_1(t,z,\zeta,\partial_z) \\ B_1^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)=-B_1(t,z,\zeta,\partial_z)-\chi_1(t,z,\zeta)\chi(t,z,\zeta)\partial_t \\ C_1^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)=-\{\sum_{i=0}^n a_i(z)\partial_{z_i}+b(z,\zeta)\partial_\zeta+d(t,z,\zeta)\} \\ C_0^*(t,z,\zeta)=\chi_0^*(t,z,\zeta) \\ D_2^*(t,z,\zeta,\partial_t,\partial_z,\partial_\zeta)=-\{\chi_1(t,z,\zeta)(\partial_t^2+\chi(t,z,\zeta)\partial_t\partial_\zeta) \\ \qquad +A_1(t,z,\zeta,\partial_z)\partial_t+B_1(t,z,\zeta,\partial_z)\partial_\zeta+C_2(t,z,\zeta,\partial_z)\}. \end{cases} \tag{2.27}$$

Thus we have

$$\mathcal{L}W(t,z,\zeta,\lambda)$$
$$\equiv \sum_{n=-1}^{\infty}\int_{|\rho|=d} f_n(\zeta-\rho t)\{Mw_n(t,z,\zeta,\rho,\lambda)-N_n(w_{n-1},w_{n-2})\}\,d\rho. \tag{2.28}$$

Now let us return to (2.12). We note that, if $|\zeta|>d|t|$,

$$\frac{g(t,z,\zeta,\lambda)}{2\pi i\zeta}=\frac{g(t,z,\zeta,\lambda)}{(2\pi i)^2}\int_{|\rho|=d}\frac{d\rho}{(\zeta-\rho t)\rho}=\frac{g(t,z,\zeta,\lambda)}{2\pi i}\int_{|\rho|=d}\frac{f_{-1}(\zeta-\rho t)}{\rho}d\rho. \tag{2.29}$$

Hence, considering (2.28) and (2.29), we determine $w_n(t,z,\zeta,\rho,\lambda)$ in the following way:

$$\begin{cases} M(t,z,\rho,\partial_\rho)w_{-1}(t,z,\zeta,\rho,\lambda)=\dfrac{g(t,z,\zeta,\lambda)}{2\pi i\rho}, \\ M(t,z,\rho,\partial_\rho)w_n(t,z,\zeta,\rho,\lambda)=N_n(w_{n-1},w_{n-2}) \qquad (n\geqq 0). \end{cases} \tag{2.30}$$

M is a second order linear ordinary differential operator of ρ and it is Fuchsian on $\bar{C}$ which has regular singularities at $\rho=0,1,\infty$. In the following sections we solve (2.30) under the condition

$$\ell(L;\mu,0)=\ell(\mathcal{L};\mu,0,0)=\mu(\mu-1)+a_0(0,0)\mu+c(0,0)\neq 0 \quad \text{for } \mu\in \boldsymbol{Z}_+. \tag{2.31}$$

In §3 we construct $W(t,z,\zeta,\lambda)=\sum_{n=-1}^{+\infty}\int_{|\rho|=d} f_n(\zeta-\rho t)w_n(t,z,\zeta,\rho,\lambda)d\rho$ and show that it is holomorphic in $X_0=U(r)\cap\{|\zeta|>c|t|\}$, $U(r)=\{(t,z,\zeta);\ |t|<r, |z|<r, |\zeta|<r\}$, for some $c\geqq 1$ and $r>0$ as a function of (t,z,ζ). As we remarked above, the operator $\mathcal{L}$ has the characteristic surfaces $\{t=0\}$, $\{\zeta=0\}$ and $\{t=\zeta\}$. So it will be shown in §5 that $W(t,z,\zeta,\lambda)$ is holomorphic except these surfaces.

§ 3. Construction of $W(t, z, \zeta, \lambda)$.

In § 3 we solve (2.30) and construct a solution $W(t, z, \zeta, \lambda)$ of (2.12). Before solving it, we prepare majorant functions which are used to show convergence of $W(t, z, \zeta, \lambda)$ and other functions. Then we obtain majorant estimates as functions of (t, z, ζ), regarding other variables as parameters.

Let $z=(z_0, z_1, \cdots, z_N)$ be the coordinate of $\boldsymbol{C}^{N+1}$, $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_N)\in \boldsymbol{Z}_+^N$ be the multi-index and $z^\alpha=z_0^{\alpha_0}z_1^{\alpha_1}\cdots z_N^{\alpha_N}$. Let $A(z)=\sum_\alpha A_\alpha z^\alpha$ and $B(z)=\sum_\alpha B_\alpha z^\alpha$ be formal power series of z. $A(z)\gg B(z)$ means $A_\alpha \geqq |B_\alpha|$ for all α and $A(z)\gg 0$ means $A_\alpha\geqq 0$. We have

LEMMA 3.1 (Wagschal). *Let $\Theta(t)$ be a formal power series of one variable t, such that $\Theta(t)\gg 0$ and $(R-t)\Theta(t)\gg 0$. Let $L(z, \partial_z)$ be a linear partial differential operator of order m with holomorphic coefficients in $\{|z|\leqq R'\}(R'>R)$. Then*

$$0 \ll (R-t)\Theta'(t), \tag{3.1}$$

$$(R'-t)^{-1}\Theta(t) \ll (R'-R)^{-1}\Theta(t), \tag{3.2}$$

and

$$L(z, \partial_z)\Theta(x) \ll A\Theta^{(m)}(x), \tag{3.3}$$

where $x=\sum_{i=0}^N z_i$ and A is independent of $\Theta(t)$.

This lemma is fundamental and for the proof we refer to Wagschal [**10**]. In this paper we only use

$$\theta(r\,;\, t)=(r-t)^{-1}, \tag{3.4}$$

which satisfies $(r-t)\theta(r\,;\, t)\gg 0$.

Now let us return to solve the equation (2.30) under the assumption (2.31). Under the notation (2.9) put

$$m_\infty(z, \mu) = (2-\mu)(1-\mu)-(1-\mu)a(z)+c(z). \tag{3.5}$$

The condition (2.31) is equivalent to

$$m_\infty(z, s) \neq 0 \quad \text{for} \quad s = 1, 2, \cdots \text{ in } \{z\in \boldsymbol{C}^n\,;\, |z|\leqq R'\} \quad \text{for an } R'>0. \tag{3.6}$$

We try to find $w_n(\rho)=w_n(t, z, \zeta, \rho, \lambda)$ in the form

$$w_n(\rho) = \sum_{s=n+2}^{\infty} w_{n,s}(t, z, \zeta, \lambda)\rho^{-s}. \tag{3.7}$$

This means that $w_n(\rho)$ $(n\geqq -1)$ are holomorphic at $\rho=\infty$. Recall $U(r)=\{(t, z, \zeta)\,;\, |t|<r,\ |z|<r,\ |\zeta|<r\}$ and put

$$Z(\infty, b^{-1}) = U(r)\times\{|\rho|>b\}. \tag{3.8}$$

r is chosen so small, if necessary, and

$$x=t+\textstyle\sum_{i=1}^{n}z_i+\zeta. \tag{3.9}$$

Now we have

LEMMA 3.2. *There exists a constant $A>0$ such that*

$$\begin{cases} m_\infty(z,\ s)^{-1}\ll A(1+s^2)^{-1}\theta(R';\ x), \\ \{(2-s)(1-s)-(1-s)b(z,\ \zeta)\}/m_\infty(z,\ s)\ll A\theta(R';\ x) \end{cases} \tag{3.10}$$

where $b(z, \zeta)$ is given in (2.9) *and $s\in \boldsymbol{N}$.*

PROOF. We have from (3.6) $|m_\infty(z, s)^{-1}|\leqq A(1+s^2)^{-1}$ in $U(R')$. Hence the first estimate in (3.10) holds. Since $(2-s)(1-s)-(1-s)b(z, \zeta)\ll A(1+s^2)\theta(R'';\ x)$ for $R''>R'$, we have the second estimate from Lemma 3.1.

In the following of this section we assume in (2.30)

$$g(t,\ z,\ \zeta,\ \lambda)\ll M(\lambda)\theta(R';\ x). \tag{3.11}$$

We have

PROPOSITION 3.3. *There exist $w_n(\rho)\in\mathcal{O}(Z(\infty, b^{-1}))$ $(n\geqq -1)$ satisfying* (2.30) *for some constant $b>1$, each of which has the form* (3.7) *and converges on $Z(\infty, b^{-1})$. The estimates*

$$w_{n,s}(t,\ z,\ \zeta,\ \lambda)\ll AM(\lambda)a^{n+1}b^{s-n-2}\theta^{(n+1)}(R;\ x) \tag{3.12}$$

and

$$w_n(\rho)\ll AM(\lambda)(b+1)a^{n+1}|\rho|^{-n-2}\theta^{(n+1)}(R;\ x)\quad \textit{for}\ \rho\in Z(\infty,\ (b+1)^{-1}) \tag{3.13}$$

hold for some constants A and a.

PROOF. We have

$$\begin{aligned} Mw_n(\rho)=\textstyle\sum_{s=n+2}^{\infty}\{&m_\infty(z,\ s)w_{n,s}(t,\ z,\ \zeta,\ \lambda) \\ &+((1-s)b(z,\ \zeta)-(2-s)(1-s))w_{n,s-1}(t,\ z,\ \zeta,\ \lambda)\}\rho^{-s} \end{aligned} \tag{3.14}$$

and

$$N_n(w_{n-1},\ w_{n-2})=\textstyle\sum_{s=n+2}^{\infty}\eta_{n,s}(t,\ z,\ \zeta,\ \lambda)\rho^{-s}, \tag{3.15}$$

where

$$\begin{aligned} \eta_{n,s}(t,\ z,\ \zeta,\ \lambda)=\ &(2-s)(1-s)A_1^*(t,\ z,\ \zeta,\ \partial_t,\ \partial_z,\ \partial_\zeta)w_{n-1,s-1}(t,\ z,\ \zeta,\ \lambda) \\ &+(1-s)C_1^*(t,\ z,\ \zeta,\ \partial_t,\ \partial_z,\ \partial_\zeta)w_{n-1,s-1}(t,\ z,\ \zeta,\ \lambda) \\ &+(n-2)(2-s)(1-s)C_0^*(t,\ z,\ \zeta)w_{n-1,s-1}(t,\ z,\ \zeta,\ \lambda) \\ &+(2-s)(1-s)B_1^*(t,\ z,\ \zeta,\ \partial_t,\ \partial_z,\ \partial_\zeta)w_{n-1,s-2}(t,\ z,\ \zeta,\ \lambda) \\ &+(2-s)(1-s)D_2^*(t,\ z,\ \zeta,\ \partial_t,\ \partial_z,\ \partial_\zeta)w_{n-2,s-2}(t,\ z,\ \zeta,\ \lambda). \end{aligned} \tag{3.16}$$

Therefore we determine $w_n(\rho)=w_n(t, z, \zeta, \rho, \lambda)=\sum_{s=n+2}^{\infty} w_{n,s}(t, z, \zeta, \lambda)\rho^{-s}$ in the following way:

$$(3.17)_{-1} \qquad m_\infty(z, s)w_{-1,s} = ((2-s)(1-s)-(1-s)b(z, \zeta))w_{-1,s-1}+\delta_{s,1}\frac{g}{2\pi i},$$

$$(3.17)_n \qquad m_\infty(z, s)w_{n,s} = ((2-s)(1-s)-(1-s)b(z, \zeta))w_{n,s-1}+\eta_{n,s}, \qquad n\geqq 0.$$

It follows from (3.6) that $w_{n,s}$ are successively determined. We show the estimate (3.12) of $w_{n,s}=w_{n,s}(t, z, \zeta, \lambda)$ by induction. We have $w_{-1,1} \ll AM(\lambda)\theta(R; x)$. Assume (3.12) is valid for $n\leqq N-1$ and $n=N$ with $s\leqq S-1$. Then, by Lemma 3.1,

$$(3.18) \qquad \begin{aligned}&(2-S)(1-S)A_1^*(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)w_{N-1,S-1}\\ &\ll AM(\lambda)C(1+S^2)a^N b^{S-N-2}\theta^{(N+1)}(R; x)\end{aligned}$$

and since $(n-1)\theta^{(n)}(R; x)\ll\theta^{(n+1)}(R; x)$,

$$(3.19) \qquad \begin{aligned}&(N-1)(2-S)(1-S)C_0^*(t, z, \zeta)w_{N-1,S-1}\\ &\ll AM(\lambda)C(1+S^2)a^N b^{S-N-2}\theta^{(N+1)}(R; x).\end{aligned}$$

Other terms in $\eta_{N,S}(t, z, \zeta, \lambda)$ are estimated in the same way. Hence we have

$$(3.20) \qquad \eta_{N,S} \ll AM(\lambda)C(1+S^2)a^N b^{S-N-2}\theta^{(N+1)}(R; x).$$

So it follows from Lemma 3.2 that for large a and b

$$(3.21) \qquad \begin{aligned}w_{N,S} &\ll C_1(1+S^2)^{-1}\theta(R'; x)\{AM(\lambda)C(1+S^2)(a^N b^{S-N-2}+a^{N+1}b^{S-N-3})\\ &\quad\times\theta^{(N+1)}(R; x)\} \ll AM(\lambda)a^{N+1}b^{S-N-2}\theta^{(N+1)}(R; x).\end{aligned}$$

Thus $w_n(\rho)=w_n(t, z, \zeta, \rho, \lambda)=\sum_{s=n+2}^{\infty} w_{n,s}(t, z, \zeta, \lambda)\rho^{-s}$ converges in $Z(\infty, b^{-1})$ and (3.13) holds as a function of (t, z, ζ) for $\rho\in Z(\infty, (b+1)^{-1})$.

Put $Z=\{\rho\in\bar{C}-\{0, 1, \infty\}\}$ and

$$(3.22) \qquad w(t, z, \zeta, \rho, \lambda) = \sum_{n=-1}^{\infty} f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda),$$

whose convergence is given in Proposition 3.4. In the following the constant b is that in Proposition 3.3. Then we have

PROPOSITION 3.4. (1) *$w_n(\rho)=w_n(t, z, \zeta, \rho, \lambda)\in\mathcal{O}(\tilde{Z})$ as a function of ρ.*

(2) *There exist h and $r>0$ such that $w(t, z, \zeta, \rho, \lambda)$ converges in $\{(t, z, \zeta, \rho);$ $(t, z, \zeta)\in U(r), 0<|\zeta-\rho t|<h|\rho|, |\rho|>b+1\}$.*

PROOF. $w_n(\rho)$ $(n\geqq -1)$ satisfy ordinary differential equations (2.30). Hence $w_n(\rho)\in\mathcal{O}(\tilde{Z})$ as a function of ρ. The convergence of $w(t, z, \zeta, \rho, \lambda)$ follows from (2.13), (3.13) and $|\theta^{(n)}(R; x)|\leqq AB^n n!$ for $|x|\leqq r(r<R)$.

Let the constant h be that in Proposition 3.4. Choose positive constants d and r satisfying $d>b+1$ and $r(d^{-1}+1)<h$. Then $0<|\zeta-\rho t|<h|\rho|$ for $|\rho|=d$ and $(t, z, \zeta)\in U(r)$ with $|\zeta|>d|t|$. Now we can define for $\{(t, z, \zeta);\ (t, z, \zeta)\in U(r),\ |\zeta|>d|t|\}$

$$W_n(t, z, \zeta, \lambda)=\int_{|\rho|=d} w_n(t, z, \zeta, \rho, \lambda) f_n(\zeta-\rho t)d\rho \tag{3.23}$$

and

$$W(t, z, \zeta, \lambda)=\int_{|\rho|=d} w(t, z, \zeta, \rho, \lambda)d\rho\,. \tag{3.24}$$

We have $W(t, z, \zeta, \lambda)=\sum_{n=-1}^{\infty} W_n(t, z, \zeta, \lambda)$. Put $X_0=U(r)\cap\{|\zeta|>c|t|\}$, $c=b+1$, and $X=U(r)-\{t=0\}\cup\{\zeta=0\}\cup\{t=\zeta\}$. Recall $\mathcal{O}_0(\tilde{X})$ (see § 1). As for the holomorphy of $W_n(t, z, \zeta, \lambda)$, we have

PROPOSITION 3.5. *$W_n(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$ as a function of (t, z, ζ) and it is an entire function of λ.*

PROOF. Let us show that $W_n(t, z, \zeta, \lambda)$ is holomorphic on X_0, namely, $W_n(t, z, \zeta, \lambda)\in\mathcal{O}(X_0)$. The integrand $w_n(t, z, \zeta, \rho, \lambda)f_n(\zeta-\rho t)$ in (3.23) is multivalued with respect to ζ for $n\geqq 0$. It is due to $\log(\zeta-\rho t)$. Hence the difference of two branches of the integrand is $2\pi mi$ for some $m\in\boldsymbol{Z}$. It follows from (3.7) that $\int_{|\rho|=d}(\zeta-\rho t)^n w_n(t, z, \zeta, \rho, \lambda)d\rho=0$. Hence $W_n(t, z, \zeta, \lambda)\in\mathcal{O}(X_0)$. Let us prolong $W_n(t, z, \zeta, \lambda)$ from X_0 to $\tilde{X}$. Since $w_n(t, z, \zeta, \rho, \lambda)\in\mathcal{O}(\tilde{Z})$ by Proposition 3.4, the singularities of the integrand in (3.23) are $\rho=0$, 1, ∞ and ζ/t. Let $\tilde{x}\in\tilde{X}$ and $\sigma=\{\sigma(s);\ 0\leqq s\leqq 1\}$ be a continuous path in $\tilde{X}$ such that $\sigma(0)\in X_0$ and $\sigma(1)=\tilde{x}$. Hence $\zeta/t\neq 0$, 1, ∞ on σ. So we can deform the integration path $|\rho|=d$ in (3.23) homotopically in $\tilde{Z}$ to $\gamma=\gamma_\sigma\subset\tilde{Z}$ so that the holomorphic prolongation of $W_n(t, z, \zeta, \lambda)$ along σ is given by $W_{n,\gamma}(t, z, \zeta, \lambda)$

$$W_{n,\gamma}(t, z, \zeta, \lambda)=\int_\gamma f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)d\rho \tag{3.25}$$

(see Fig. 3.1). $W_n(t, z, \zeta, \lambda)$ can be holomorphically extensible from X_0 to $\tilde{X}$ by this method. Thus $W_n(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$ as a function of (t, z, ζ).

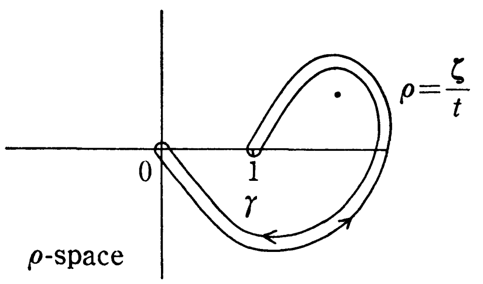

Fig. 3.1.

REMARK 3.6. It follows from (3.7) and (2.20) that

$$\int_{|\rho|=d} f_{n-2}(\zeta-\rho t)\rho(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)d\rho$$
$$= (n-1)\int_{|\rho|=d} f_{n-1}(\zeta-\rho t)\rho w_n(t, z, \zeta, \rho, \lambda)d\rho .$$

Hence we can replace $\equiv$ by $=$ in (2.23) and (2.24). The method in Proposition 3.5 to prolong $W_n(t, z, \zeta, \lambda)$ from X_0 to $\tilde{X}$ will be used again to show $W(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$ in Theorem 5.7.

We have the existence of a solution of (2.12).

THEOREM 3.7. *$W(t, z, \zeta, \lambda)$ defined by (3.24) is holomorphic in X_0, that is, $W(t, z, \zeta, \lambda)\in\mathcal{O}(X_0)$, and*

$$\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)W(t, z, \zeta, \lambda) = \frac{g(t, z, \zeta, \lambda)}{2\pi i\zeta} \tag{3.26}$$

holds.

PROOF. It follows from Proposition 3.4-(2) and proposition 3.5 that $W(t, z, \zeta, \lambda)$ is holomorphic on X_0. The calculations in §2 ((2.16)-(2.28)) are justified. We have (2.12) without modulo parts by Remark 3.6.

Consequently it is shown that $w_n(t, z, \zeta, \rho, \lambda)$ is holomorphic on $\tilde{Z}$ for $(t, z, \zeta)\in U(r)$ and $\lambda\in \boldsymbol{C}$ and $W_n(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$. $W(t, z, \zeta, \lambda)=\sum_{n=-1}^{+\infty} W_n(t, z, \zeta, \lambda)$ converges in X_0 and belongs to $\mathcal{O}(X_0)$. Since $W_n(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$, it is expected that $W(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$. We show it in §4 and §5.

§4. Holomorphic extension of $W(t, z, \zeta, \lambda)$ - I.

We investigate the singularities of $W(t, z, \zeta, \lambda)$ more precisely. As we have shown in §3, $W(t, z, \zeta, \lambda)$ is represented in the form (3.24) on X_0. So in order to show $W(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$, $X = \{(t, z, \zeta);\ 0<|t|<r,\ |z|<r,\ 0<|\zeta|<r,\ t\neq\zeta\}$, we study $w_n(t, z, \zeta, \rho, \lambda)$ $(n\geqq-1)$ as functions of ρ which are determined by the relation (2.30). Our aim in §4 is to make preparations for showing $W(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$. In this section we denote $(t, z, \zeta)=(t, z_1, \cdots, z_n, \zeta)$ by $z=(z_0, z_1, \cdots, z_n, z_{n+1})$, $z_0=t$, $z_{n+1}=\zeta$, for simplicity. For $\rho\in\tilde{Z}$, $\pi(\rho)$ is the projection from $\tilde{Z}$ on $Z=\{\rho\in\bar{\boldsymbol{C}}-\{0, 1, \infty\}\}$.

Now let us study the Cauchy problem at $\rho=\kappa\in\tilde{Z}$.

$$\begin{cases} M(z, \rho, \partial_\rho)V(z, \rho) = G(z, \rho), \\ V(z, \rho)|_{\rho=\kappa} = A(z, \kappa), \qquad (\rho\partial_\rho)V(z, \rho)|_{\rho=\kappa} = B(z, \kappa), \end{cases} \tag{4.1}$$

where $M(z, \rho, \partial_\rho) = \{(\rho^2-\rho)\cdot\}_{\rho\rho}+\{(b(z)-\rho a(z))\cdot\}_\rho+c(z)\cdot$. Firstly we consider (4.1) when the initial point κ does not equal to 0 but near 0. Put $m_0(z, \mu)=$

$\mu(b(z)-\mu-1)$, $\mu_0^1(z)=0$ and $\mu_0^2(z)=b(z)-1$. The path C_0 in μ-space is defined as follows. If $\mu_0^2(0)-\mu_0^1(0)=b(0)-1\notin \boldsymbol{Z}$, C_0 is a closed path surrounding $\mu_0^i(z)(i=1,2)$ once and points $\mu_0^i(z)+s(i=1, 2, s\in \boldsymbol{Z}-\{0\})$ are outside of C_0. If $\mu_0^2(0)-\mu_0^1(0)=b(0)-1=n_0\in \boldsymbol{Z}$, C_0 is a closed path surrounding $\{\mu_0^2(z)-n_0, \mu_0^2(z), \mu_0^1(z), \mu_0^1(z)+n_0\}=\{\mu_0^2(z)-n_0, \mu_0^2(z), 0, n_0\}$ once and points $\mu_0^i(z)+s(i=1, 2, s\in \boldsymbol{Z}-\{0\}, (i, s)\neq(1, n_0), (2, -n_0))$ are outside of C_0. Δ_0 is a compact neighbourhood of C_0 on which $m_0(z, \mu+s)\neq 0$ for $s\in \boldsymbol{Z}$ and $\{|z|\leqq R'\}$.

By the change $\rho=\tau\kappa$, (4.1) becomes

$$(4.2)\qquad \begin{cases} M(z, \tau, \kappa, \partial_\tau)V(z, \tau, \kappa) = G(z, \tau, \kappa), \\ V(z, 1, \kappa) = A(z, \kappa), \qquad \partial_\tau V(z, 1, \kappa) = B(z, \kappa), \end{cases}$$

where $V(z, \tau, \kappa)=V(z, \tau, \kappa)$, $G(z, \tau, \kappa)=\kappa G(z, \tau\kappa)$ and

$$(4.3)\qquad M(z, \tau, \kappa, \partial_\tau) = \{\tau(\kappa\tau-1)\cdot\}_{\tau\tau}+\{(b(z)-\kappa a(z)\tau)\cdot\}_\tau+\kappa c(z)\cdot\,.$$

We treat (4.2) instead of (4.1).

Assume $A(z, \kappa), B(z, \kappa)\in\mathcal{O}(U(R)\times\{0<|\kappa|<r_0\})(r<R, 0<r_0<1)$ and

$$(4.4)\qquad G(z, \tau, \kappa) = \sum_{s=s^*-1}^{+\infty}\tau^s\int_{C_0}\tau^\mu g_s(z, \mu, \kappa)d\mu,$$

where $s^*\leqq 0$, $g_s(z, \mu, \kappa)\in\mathcal{O}(U(R)\times\Delta_0\times\widetilde{\{0<|\kappa|<r_0\}})$. In the following we consider κ as a parameter and we often omit the domain of κ in the notations, for example $g_s(z, \mu, \kappa)\in\mathcal{O}(U(R)\times\Delta_0)$. We also assume

$$(4.5)\qquad \begin{cases} A(z, \kappa), \quad B(z, \kappa) \ll K(\kappa)\Theta(R\,;\, x), \\ g_s(z, \mu, \kappa) \ll K(\kappa)(1+s^2)\left(\dfrac{1}{2}\right)^{s-s^*+1}\Theta(R\,;\, x) \qquad \text{for} \quad \mu\in\Delta_0, \end{cases}$$

where $\Theta(R\,;\, x)=n\,!/(R-x)^{n+1}$ for some $n\in \boldsymbol{Z}_+$, and $x=\sum_{i=0}^{n+1}z_i$. Hence $G(z, \tau, \kappa)\in\mathcal{O}(U(R)\times\widetilde{\{0<|\tau|<2\}})$. We have

Proposition 4.1. *Suppose κ is small, namely, $0<|\kappa|<\beta$ for some $\beta<r_0$. Let $V(z, \tau, \kappa)$ be a unique solution of* (4.2). *Then*

(1) *$V(z, \tau, \kappa)$ is represented in the form*

$$(4.6)\qquad V(z, \tau, \kappa) = \sum_{s=s^*}^{+\infty}\tau^s\int_{C_0}\tau^\mu v_s(z, \mu, \kappa)d\mu,$$

where $v_s(z, \mu, \kappa)\in\mathcal{O}(U(R)\times\Delta_0)$ and

$$(4.7)\qquad v_s(z, \mu, \kappa) \ll \gamma K(\kappa)\left(\frac{1}{2}\right)^{s-s^*}\Theta(R\,;\, x)$$

holds for some constant γ independent of κ and s, and

(2) *for any $\omega>0$, the series* (4.6) *converges on $U(R)\times\{0<|\tau|<2\,;\, |\arg\tau|<\omega\}$, and there exist $A(\omega)$ and c such that*

(4.8) $$V(z, \tau, \kappa) \ll A(\omega)K(\kappa)|\tau|^{c+s^*}(2-|\tau|)^{-1}\Theta(R\,;\,x).$$

Before the proof we give a formula used often. Let $V(z, \tau, \mu, \kappa)=\sum_{s=\hat{s}}^{+\infty} v_s(z, \mu, \kappa)\tau^{\mu+s}$ be a formal series. Then

(4.9) $$M(z, \tau, \kappa, \partial_\tau)V(z, \tau, \mu, \kappa)=\sum_{s=\hat{s}-1}^{+\infty}\{m_0(z, \mu+s+1)v_{s+1}(z, \mu, \kappa) +\kappa((\mu+s+2)(\mu+s+1)-a(z)(\mu+s+1)+c(z))v_s(z, \mu, \kappa)\}\tau^{\mu+s}.$$

In order to show Proposition 4.1 we give two lemmas. First we consider the equation without the initial conditions:

(4.10) $$M(z, \tau, \kappa, \partial_\tau)E(z, \tau, \kappa)=G(z, \tau, \kappa),$$

where $G(z, \tau, \kappa)$ has the form (4.4) with $g_s(z, \mu, \kappa)$ $(s\geqq s^*-1)$ satisfying the assumption (4.5). We have

LEMMA 4.2. *Suppose κ is small, namely, $0<|\kappa|\leqq\beta$ for some $\beta<r_0$. Then*
(1) *there exists a solution $E(z, \tau, \kappa)$ of (4.10) of the form*

(4.11) $$E(z, \tau, \kappa)=\sum_{s=s^*}^{+\infty}\tau^s\int_{C_0}\tau^\mu e_s(z, \mu, \kappa)d\mu,$$

such that $e_s(z, \mu, \kappa)\in\mathcal{O}(U(R)\times\Delta_0)$ as a function of (z, μ), and for $\mu\in\Delta_0$

(4.12) $$e_s(z, \mu, \kappa)\ll\gamma_1 K(\kappa)\left(\frac{1}{2}\right)^{s-s^*}\Theta(R\,;\,x)$$

holds for some constant γ_1 independent of κ and s, and
(2) *for any ω, the series (4.11) converges in $\tau\in\{0<|\tau|<2,\ |\arg\tau|<\omega\}$ and there exist $A(\omega)$ and c such that*

(4.13) $$E(z, \tau, \kappa)\ll\gamma_1 C(\omega)K(\kappa)|\tau|^{c+s^*}(2-|\tau|)^{-1}\Theta(R\,;\,x).$$

PROOF. Assume $E(z, \tau, \kappa)$ has the form (4.11). We substitute it into (4.10) and use (4.9). Then we obtain

(4.14) $$m_0(z, \mu+s+1)e_{s+1}+\kappa((\mu+s+2)(\mu+s+1)-a(z)(\mu+s+1)+c(z))e_s = g_s(z, \mu, \kappa) \qquad (s\geqq s^*-1).$$

Since $m_0(z, \mu+s)\neq 0$ on Δ_0, $e_s(z, \mu)$ $(s\geqq s^*)$ are successively determined. We show (4.12). The assumption $m_0(z, \mu+s)\neq 0$ on $\overline{U(R')}\times\Delta_0$ means

(4.15) $$m_0(z, \mu+s)^{-1}\ll A(1+s^2)^{-1}(R'-x)^{-1}, \qquad x=\sum_{i=0}^{n+1}z_i.$$

So from Lemma 3.1, $e_{s^*}(z, \mu, \kappa)=m_0(z, \mu+s)^{-1}g_{s^*-1}(z, \mu, \kappa)\ll AK(\kappa)\,\Theta(R\,;\,x)$. Assume that (4.12) holds for $s\leqq S$. Then we have

(4.16) $$e_{S+1}=m_0(z, \mu+S+1)^{-1}\{g_S-\kappa((\mu+S+2)(\mu+S+1)-a(z)(\mu+S+1)+c(z))e_S\}$$

$$\ll K(\kappa)\left(\frac{1}{2}\right)^{S+1-s^*}(A+|\kappa|B\gamma_1)\Theta(R\,;\,x).$$

If $|\kappa|B<1/2$ and γ_1 is large, then $A+|\kappa|B\gamma_1<\gamma_1$. This means (4.12) for $s=S+1$. Let us show (2). We have, if $|\arg\tau|<\omega$,

$$\tau^{\mu+s}e_s(z,\,\mu,\,\kappa)\ll C(\omega)|\tau|^{\mathrm{Re}\,\mu+s}\gamma_1 K(\kappa)\left(\frac{1}{2}\right)^{s-s^*}\Theta(R\,;\,x) \tag{4.17}$$

for $\mu\in\Delta_0$. Hence, if $|\arg\tau|<\omega$, there are $C'(\omega)$ and c such that

$$\int_{C_0}\tau^{\mu+s}e_s(z,\,\mu,\,\kappa)d\mu\ll C'(\omega)|\tau|^{s+c}\gamma_1 K(\kappa)\left(\frac{1}{2}\right)^{s-s^*}\Theta(R\,;\,x). \tag{4.18}$$

Therefore, for τ with $0<|\tau|<2$ and $|\arg\tau|<\omega$

$$E(z,\,\tau,\,\kappa)\ll\gamma_1 C(\omega)K(\kappa)|\tau|^{s^*+c}\Theta(R\,;\,x)/(2-|\tau|). \tag{4.19}$$

Next let us consider a homogeneous equation with initial conditions:

$$\begin{cases} M(z,\,\tau,\,\kappa,\,\partial_\tau)H(z,\,\tau,\,\kappa)=0\,, \\ H(z,\,1,\,\kappa)=A(z,\,\kappa), \qquad \partial_\tau H(z,\,1,\,\kappa)=B(z,\,\kappa)\,, \end{cases} \tag{4.20}$$

where $A(z,\kappa)$ and $B(z,\kappa)$ satisfy (4.5). The existence of a unique solution $H(z,\tau,\kappa)$ is obvious. Moreover we have also a representation of $H(z,\tau,\kappa)$ such as (4.11) by the following lemma.

LEMMA 4.3. *Assume* $0<|\kappa|\leqq\beta$ *for some small* β. *Then the unique solution* $H(z,\tau,\kappa)$ *of* (4.20) *has the form*

$$H(z,\,\tau,\,\kappa)=\textstyle\sum_{s=0}^{+\infty}\tau^s\displaystyle\int_{C_0}\tau^{\mu}h_s(z,\,\mu,\,\kappa)d\mu\,. \tag{4.21}$$

Here $h_s(z,\mu,\kappa)\in\mathcal{O}(U(R)\times\Delta_0)$ *as a function of* (z,μ), *for* $\mu\in\Delta_0$

$$h_s(z,\,\mu,\,\kappa)\ll\gamma_2 K(\kappa)\left(\frac{1}{2}\right)^{s}\Theta(R\,;\,x) \tag{4.22}$$

holds for some constant γ_2 *independent of* κ *and* s, *for any* ω *the series* (4.21) *converges in* $\{0<|\tau|<2,\ |\arg\tau|<\omega\}$ *and*

$$H(z,\,\tau,\,\kappa)\ll\gamma_2 A(\omega)K(\kappa)|\tau|^{c}(2-|\tau|)^{-1}\Theta(R\,;\,x) \tag{4.23}$$

holds, where the constant c *is that in the inequality* (4.13).

PROOF. Put $g_{-1}(z,\mu,\kappa)=1$ and $g_s(z,\mu,\kappa)=0$ for $s\neq-1$. Obviously

$$G(z,\,\tau,\,\kappa)=\textstyle\sum_{s=-1}^{+\infty}\tau^s\displaystyle\int_{C_0}\tau^{\mu}g_s(z,\,\mu,\,\kappa)d\mu=0\,. \tag{4.24}$$

Determine $e_s(z,\mu,\kappa)$ $(s\geqq0)$ by (4.14). Each $e_s(z,\mu,\kappa)$ is a rational function of μ whose poles are in $\{\mu\,;\,\prod_{j=0}^{s}m_0(z,\,\mu+j)=0\}$ and we have $e_0(z,\mu,\kappa)=m_0(z,\mu)^{-1}$ and $e_s(z,\mu,\kappa)=\kappa^s e_s^*(z,\mu)$ for $s\geqq1$. Put

$$(4.25)\quad \begin{cases} H^0(z,\tau,\kappa)=\sum_{s=0}^{\infty}\tau^s\int_{C_0}\tau^\mu e_s(z,\mu,\kappa)d\mu=\int_{C_0}\tau^\mu/m_0(z,\mu)d\mu \\ \qquad +\sum_{s=1}^{+\infty}(\kappa\tau)^s\int_{C_0}\tau^\mu e_s^*(z,\mu)d\mu, \\ H^1(z,\tau,\kappa)=\sum_{s=0}^{+\infty}\tau^s\int_{C_0}\tau^\mu \mu e_s(z,\mu,\kappa)d\mu=\int_{C_0}\mu\tau^\mu/m_0(z,\mu)d\mu \\ \qquad +\sum_{s=1}^{+\infty}(\kappa\tau)^s\int_{C_0}\tau^\mu \mu e_s^*(z,\mu)d\mu. \end{cases}$$

It follows from (4.24) and the definition of the integration path C_0 that $H^i(z,\tau,\kappa)$ $(i=0, 1)$ are linearly independent solutions of $M(z,\tau,\kappa,\partial_\tau)H(z,\tau,\kappa)=0$. Hence we have

$$(4.26)\qquad H(z,\tau,\kappa)=\phi^0(z,\kappa)H^0(z,\tau,\kappa)+\phi^1(z,\kappa)H^1(z,\tau,\kappa),$$

where

$$(4.27)\quad \begin{cases} \phi^0(z,\kappa)=\dfrac{A(z,\kappa)\partial_\tau H^1(z,1,\kappa)-B(z,\kappa)H^1(z,1,\kappa)}{W(z,1,\kappa)}, \\ \phi^1(z,\kappa)=\dfrac{-A(z,\kappa)\partial_\tau H^0(z,1,\kappa)+B(z,\kappa)H^0(z,1,\kappa)}{W(z,1,\kappa)}, \\ W(z,\tau,\kappa)=\begin{vmatrix} H^0(z,\tau,\kappa) & H^1(z,\tau,\kappa) \\ \partial_\tau H^0(z,\tau,\kappa) & \partial_\tau H^1(z,\tau,\kappa)\end{vmatrix}. \end{cases}$$

It follows from (4.25) that $W(z,\tau,\kappa)=W(z,\tau,0)+\kappa W^*(z,\tau,\kappa)$. If $0<|\kappa|\leqq\beta$ for a small β, $|W(z,1,\kappa)|\geqq C>0$ for some constant C independent of κ and $W(z,1,\kappa)^{-1}\ll A(R'-x)^{-1}$. Therefore $\phi^i(z,\kappa)\ll AK(\kappa)\Theta(R;x)$ $(i=0, 1)$. By putting $h_s(z,\mu,\kappa)=(\phi^0(z,\kappa)+\mu\phi^1(z,\kappa))e_s(z,\mu,\kappa)$, we have (4.21)-(4.23).

Now we give the proof of Proposition 4.1.

Proof of Proposition 4.1. By Lemma 4.2 there is an $E(z,\tau,\kappa)$ such that $M(z,\tau,\kappa,\partial_\tau)E(z,\tau,\kappa)=G(z,\tau,\kappa)$. Let $H(z,\tau,\kappa)$ be a solution of

$$(4.28)\quad \begin{cases} M(z,\tau,\kappa,\partial_\tau)H(z,\tau,\kappa)=0, \\ H(z,1,\kappa)=A(z,\kappa)-E(z,1,\kappa),\ \partial_\tau H(z,1,\kappa)=B(z,\kappa)-\partial_\tau E(z,1,\kappa). \end{cases}$$

It holds by Lemma 4.2 that $A(z,\kappa)-E(z,1,\kappa)\ll CK(\kappa)\Theta(R;x)$ and $B(z,\kappa)-\partial_\tau E(z,1,\kappa)\ll CK(\kappa)\Theta(R;x)$. We have, by Lemma 4.3, a unique solution $H(z,\tau,\kappa)$ of (4.28) in the form of (4.21). By putting $v_s(z,\mu,\kappa)=e_s(z,\mu,\kappa)+h_s(z,\mu,\kappa)$ and $V(z,\tau,\kappa)=E(z,\tau,\kappa)+H(z,\tau,\kappa)$, we have (4.6). The estimates (4.7) and (4.8) follow from Lemmas 4.2 and 4.3.

Now let us apply Proposition 4.1 to a series of Cauchy problems:

$$(4.29)_n \quad \begin{cases} M(z, \rho, \partial_\rho)V_n(z, \rho, \kappa) = N_n(V_{n-1}, V_{n-2}) + \delta_{n,-1}\dfrac{g(z)}{\rho}, \\ V_n(z, \rho, \kappa)|_{\rho=\kappa} = a_n(z, \kappa), (\rho\partial_\rho)V_n(z, \rho, \kappa)|_{\rho=\kappa} = b_n(z, \kappa), \quad n \geqq -1, \end{cases}$$

where $N_n(U, V)$ is defined by (2.26). We assume that $a_n(z, \kappa), b_n(z, \kappa) \in \mathcal{O}(U(R) \times \widetilde{\{0<|\kappa|\leqq r_0\}})$ and they satisfy

$$(4.30) \qquad a_n(z, \kappa), b_n(z, \kappa) \ll \frac{K(\kappa)}{|\kappa|^{n+1}}\alpha^{n+1}\theta^{(n+1)}(R; x),$$

$$(4.31) \qquad g(z) \ll A\theta(R; x) \qquad (A \leqq K(\kappa)).$$

By the change $\rho = \kappa\tau$, the equation $(4.29)_n$ becomes

$$(4.32)_n \quad \begin{cases} M(z, \tau, \kappa, \partial_\tau)V_n^*(z, \tau, \kappa) = N_n(\kappa; V_{n-1}^*, V_{n-2}^*) + \delta_{n,-1}\dfrac{g(z)}{\tau}, \\ V_n^*(z, \tau, \kappa)|_{\tau=1} = a_n(z, \kappa), \partial_\tau V_n^*(z, \tau, \kappa)|_{\tau=1} = b_n(z, \kappa), \end{cases}$$

where $V_n^*(z, \tau, \kappa) = V_n(z, \rho, \kappa)|_{\rho=\kappa\tau}$, and

$$(4.33) \qquad N_n(\kappa; U^*, V^*) = \frac{1}{\kappa}\{(\kappa\tau A_1^*(z, \partial_z) + B_1^*(z, \partial_z))U^*\}_{\tau\tau}$$
$$+\{C_1^*(z, \partial_z)U^*\}_\tau + (n-2)\{\tau C_0^*(z)U^*\}_{\tau\tau} + \frac{1}{\kappa}\{D_2^*(z, \partial_z)V^*\}_{\tau\tau},$$

where $A_1^*(z, \partial_z)$ etc. are the same operators as those in (2.26) with the notations of independent variables of this section.

We have

LEMMA 4.4. *There exist $\alpha^*>1$ and $0<\beta<r_0$ such that the following holds. Suppose $\alpha \geqq \alpha^*$ and $0<|\kappa|\leqq\beta$. Then*

(1) *the unique solution $V_n^*(z, \tau, \kappa)$ of $(4.32)_n$ has the form*

$$(4.34) \qquad V_n^*(z, \tau, \kappa) = \sum_{s=-(n+1)}^{+\infty}\tau^s\int_{C_0}\tau^\mu v_{n,s}^*(z, \mu, \kappa)d\mu,$$

where $v_{n,s}^*(z, \mu, \kappa) \in \mathcal{O}(U(R)\times\Delta_0\times\widetilde{\{0<|\kappa|\leqq\beta\}})$ with

$$(4.35) \qquad v_{n,s}^*(z, \mu, \kappa) \ll \gamma K(\kappa)|\kappa|^{-n-1}\alpha^{n+1}\left(\frac{1}{2}\right)^{s+n+1}\theta^{(n+1)}(R; x),$$

γ being a constant independent of κ, n, s and α, and

(2) *for any $\omega>0$ the series (4.34) converges in $\{0<|\tau|<2, |\arg\tau|<\omega\}$ and there exist $\gamma_1(\omega)$ and c such that*

$$(4.36) \qquad V_n^*(z, \tau, \kappa) \ll \gamma_1 K(\kappa)|\kappa|^{-n-1}\alpha^{n+1}\frac{|\tau|^{-n-1+c}}{(2-|\tau|)}\theta^{(n+1)}(R; x).$$

PROOF. By Proposition 4.1 we have (4.34)-(4.35) for $n=-1$. Assume (4.34)-(4.35) are valid for $n \leqq N-1$. Then we have

$$N_N(\kappa;\ V^*_{N-1},\ V^*_{N-2}) = \sum_{s=-(N+2)}^{+\infty} \tau^s \int_{C_0} \tau^\mu \eta^*_{N,s}(z,\ \mu,\ \kappa) d\mu, \tag{4.37}$$

where

$$\begin{aligned}\eta^*_{n,s}(z,\ \mu,\ \kappa) &= (\mu+s+2)(\mu+s+1)A^*_1(z,\ \partial_z)v^*_{n-1,s+1}(z,\ \mu,\ \kappa) \\ &+(\mu+s+1)C^*_1(z,\ \partial_z)v^*_{n-1,s+1}(z,\ \mu,\ \kappa) \\ &+(n-2)(\mu+s+2)(\mu+s+1)C^*_0(z)v^*_{n-1,s+1}(z,\ \mu,\ \kappa) \\ &+\frac{1}{\kappa}(\mu+s+2)(\mu+s+1)B^*_1(z,\ \partial_z)v^*_{n-1,s+2}(z,\ \mu,\ \kappa) \\ &+\frac{1}{\kappa}(\mu+s+2)(\mu+s+1)D^*_2(z,\ \partial_z)v^*_{n-2,s+2}(z,\ \mu,\ \kappa).\end{aligned} \tag{4.38}$$

We can show in the same way as in Proposition 3.3 that there exists a constant A independent of γ, κ, α, N and s such that

$$\eta^*_{N,s}(z,\ \mu,\ \kappa) \ll A(1+s^2)\gamma K(\kappa)|\kappa|^{-(N+1)}\alpha^N\left(\frac{1}{2}\right)^{s+N+2}\theta^{(N+1)}(R;\ x). \tag{4.39}$$

We choose α^* to satisfy $A\gamma/\alpha^*<1$. Hence if $\alpha \geqq \alpha^*$,

$$\eta^*_{N,s}(z,\ \mu,\ \kappa) \ll (1+s^2)K(\kappa)|\kappa|^{-(N+1)}\alpha^{N+1}\left(\frac{1}{2}\right)^{s+N+2}\theta^{(N+1)}(R;\ x). \tag{4.40}$$

Then we have from Proposition 4.1 that $v^*_{N,s}(z,\ \mu,\ \kappa)$ has the form (4.34) for $n=N$ and that

$$v^*_{N,s}(z,\ \mu,\ \kappa) \ll \gamma K(\kappa)|\kappa|^{-(N+1)}\alpha^{N+1}\left(\frac{1}{2}\right)^{s+N+1}\theta^{(N+1)}(R;\ x) \quad s \geqq -(N+1), \tag{4.41}$$

holds. We have also (2) in the same way as in Proposition 4.1.

Now let us return to the equations $(4.29)_n$ $(n \geqq -1)$. We have, by the change $\tau=\rho/\kappa$,

PROPOSITION 4.5. *There exist $\alpha^*>1$ and $\beta>0$ such that the following holds. Suppose $\alpha \geqq \alpha^*$ and $0<|\kappa| \leqq \beta$. Then*

(1) *the unique solution $V_n(z,\ \rho,\ \kappa)$ of $(4.29)_n$ is represented in the form*

$$V_n(z,\ \rho,\ \kappa) = \sum_{s=-(n+1)}^{+\infty}\left(\frac{\rho}{\kappa}\right)^s \int_{C_0}\left(\frac{\rho}{\kappa}\right)^\mu v_{n,s}(z,\ \mu,\ \kappa)d\mu, \tag{4.42}$$

where $v_{n,s}(z,\ \mu,\ \kappa) \in \mathcal{O}(U(R)\times\Delta_0\times\widetilde{\{0<|\kappa|\leqq\beta\}})$ with

$$v_{n,s}(z,\ \mu,\ \kappa) \ll \gamma K(\kappa)|\kappa|^{-n-1}\alpha^{n+1}\left(\frac{1}{2}\right)^{s+n+1}\theta^{(n+1)}(R;\ x), \tag{4.43}$$

γ being a constant independent of κ, n, s and α, and

(2) *for any $\omega>0$ there exist $\gamma_1(\omega)$ and c such that the series (4.42) converges and*

(4.44) $$V_n(z,\rho,\kappa) \ll \gamma_1 K(\kappa)|\kappa|^{-c+1}\alpha^{n+1}\frac{|\rho|^{-n+c-1}}{(2|\kappa|-|\rho|)}\theta^{(n+1)}(R\,;\,x)$$

holds for $\rho\in\{0<|\rho|<2|\kappa|,\ |\arg\rho/\kappa|<\omega\}$.

The results similar to Lemma 4.4 and Proposition 4.5 hold for the Cauchy problems $(4.29)_n$ $(n\geqq -1)$ with initial point κ with $0<|\kappa-1|<r_0$. We explain them shortly. Put $m_1(z,\mu)=\mu(\mu-a(z)+b(z)+1)$, $\mu_1^1(z)=0$ and $\mu_1^2(z)=a(z)-b(z)-1$. The path C_1 in μ-space is defined as follows. If $\mu_1^2(0)-\mu_1^1(0)=a(0)-b(0)-1\notin \boldsymbol{Z}$, C_1 is a closed path surrounding $\mu_1^i(z)$ $(i=1,2)$ once and points $\mu_1^i(z)+s$ $(i=1,2,$ $s\in \boldsymbol{Z}-\{0\})$ are outside of C_1. If $\mu_1^2(0)-\mu_1^1(0)=a(0)-b(0)-1=n_1\in\boldsymbol{Z}$, C_1 is a closed path surrounding $\{\mu_1^2(z)-n_1,\ \mu_1^2(z),\ \mu_1^1(z),\ \mu_1^1(z)+n_1\}=\{\mu_1^2(z)-n_1,\mu_1^2(z),0,n_1\}$ once and points $\mu_1^i(z)+s$ $(i=1,2,\ s\in\boldsymbol{Z}-\{0\},(i,s)\neq(1,-n_1),(2,n_1))$ are outside of C_1. Δ_1 is a compact neighbourhood of C_1 on which $m_1(z,\mu+s)\neq 0$ for $s\in\boldsymbol{Z}$ and $\{|z|\leqq R'\}$. We assume that $a_n(z,\kappa),\ b_n(z,\kappa)\in\mathcal{O}(U(R)\times\widetilde{\{0<|\kappa-1|\leqq r_0\}})$, $g(z)\in\mathcal{O}(U(R))$ and

(4.45) $$\begin{cases} a_n(z,\kappa),\ b_n(z,\kappa)\ll \dfrac{K(\kappa)}{|\kappa-1|^{n+1}}\alpha^{n+1}\theta^{(n+1)}(R\,;\,x), \\ g(z)\ll A\theta(R\,;\,x) \qquad (A\leqq K(\kappa)), \end{cases}$$

in $(4.29)_n$. By expanding $g(z)/\rho=g(z)\sum_{s=0}^{+\infty}(1-\rho)^s$ at $\rho=1$ in the right hand side of $(4.29)_n$ and using the preceding method in this section, we have

PROPOSITION 4.6. *There exist* $\alpha^*>1$ *and* $\beta>0$ *such that the following holds. Suppose* $\alpha\geqq\alpha^*$ *and* $0<|\kappa-1|\leqq\beta$. *Then*

(1) *the unique solution* $V_n(z,\rho,\kappa)$ *of* $(4.29)_n$ *is represented in the form*

(4.46) $$V_n(z,\rho,\kappa)=\sum_{s=-(n+1)}^{+\infty}\left(\frac{\rho-1}{\kappa-1}\right)^s\int_{C_1}\left(\frac{\rho-1}{\kappa-1}\right)^{\mu}v_{n,s}(z,\mu,\kappa)d\mu,$$

where $v_{n,s}(z,\mu,\kappa)\in\mathcal{O}(U(R)\times\Delta_1\times\widetilde{\{0<|\kappa-1|\leqq\beta\}})$ *with*

(4.47) $$v_{n,s}(z,\mu,\kappa)\ll\gamma K(\kappa)|\kappa-1|^{-n-1}\alpha^{n+1}\left(\frac{1}{2}\right)^{s+n+1}\theta^{(n+1)}(R\,;\,x),$$

where γ *is a constant independent of* κ, n, s *and* α, *and*

(2) *for any* $\omega>0$ *there exist* $\gamma_1(\omega)$ *and* c *such that*

(4.48) $$V_n(z,\rho,\kappa)\ll\gamma_1 K(\kappa)|\kappa-1|^{-c+1}\alpha^{n+1}\frac{|\rho-1|^{-n+c-1}}{(2|\kappa-1|-|\rho-1|)}\theta^{(n+1)}(R\,;\,x)$$

for $\rho\in\{0<|\rho-1|<2|\kappa-1|,\ |\arg(\rho-1)/(\kappa-1)|<\omega\}$.

REMARK 4.7. We can obtain the similar result to Propositions 4.5 and 4.6 when the initial point $\rho=\kappa$ is near ∞.

Finally we consider $(4.29)_n$ in a neighbourhood of regular points. Let $\delta>0$ be a fixed positive constant. Put $Z_\delta=\{\rho\in \boldsymbol{C}^1;\ |\rho-1|>\delta \text{ and } |\rho|>\delta\}$. Consider

$$\begin{cases} M(z,\rho,\partial_\rho)V(z,\rho)=G(z,\rho), \\ V(z,\rho)|_{\rho=\kappa}=A(z,\kappa), \qquad (\rho\partial_\rho)V(z,\rho)|_{\rho=\kappa}=B(z,\kappa), \end{cases} \tag{4.49}$$

where $\kappa\in\tilde{Z}_\delta$, $G(z,\rho)\ll A_\delta\Theta(R;x)$ in $\tilde{Z}_\delta$ and $A(z,\kappa)$, $B(z,\kappa)\ll K(\kappa)\Theta(R;x)$. We have

LEMMA 4.8. *Let $V(z,\rho)$ be a solution of* (4.49). *Then there are $\gamma(\delta)$ and $r_1(\delta)$ such that*

$$V(z,\rho),\quad \rho\partial_\rho V(z,\rho)\ll\gamma(\delta)(K(\kappa)+A_\delta)\Theta(R;x) \tag{4.50}$$

holds for ρ with $|\rho-\kappa|<r_1(\delta)$.

Lemma 4.8 follows from the existence and uniqueness theorem for Cauchy problems. Now we assume in $(4.29)_n$ that $a_n(z,\kappa)$, $b_n(z,\kappa)\in\mathcal{O}(U(R)\times\tilde{Z}_\delta)$, $g(z)\in\mathcal{O}(U(R))$ and

$$a_n(z,\kappa),\quad b_n(z,\kappa)\ll K(\kappa)\alpha^{n+1}\theta^{(n+1)}(R;x), \tag{4.51}$$

$$g(z)\ll A\theta(R;x)\qquad (A\leqq K(\kappa)). \tag{4.52}$$

We have, by the similar method to the preceding one,

PROPOSITION 4.9. *Suppose that $\kappa\in\tilde{Z}_\delta$ and the estimates* (4.51) *and* (4.52) *hold in* $(4.29)_n$. *Then there are $\alpha^*>1$, $\gamma(\delta)$, and $r_1(\delta)$ such that the solution $V_n(z,\rho,\kappa)$ of* $(4.29)_n$ *satisfies the following: if $\alpha\geqq\alpha^*$,*

$$V_n(z,\rho,\kappa),\ \rho\partial_\rho V_n(z,\rho,\kappa)\ll\gamma(\delta)K(\kappa)\alpha^{n+1}\theta^{(n+1)}(R;x) \tag{4.53}$$

in $|\rho-\kappa|<r_1(\delta)$.

§ 5. Holomorphic extension of $W(t,z,\zeta,\lambda)$ - II.

In § 5 we show $W(t,z,\zeta,\lambda)\in\mathcal{O}_0(\tilde{X})$, $X=U(r)-\{t=0\}\cup\{t=\zeta\}\cup\{\zeta=0\}$, as a function of (t,z,ζ). Recall $Z=\bar{\boldsymbol{C}}-\{0,1,\infty\}$ and π is the projection from $\tilde{Z}$ to Z. Put $x=t+\sum_{i=1}^N z_i+\zeta$ and for $\delta>0$

$$\begin{cases} S_0=\{\tilde{p}\in\tilde{Z};\ 0<|\pi(\tilde{p})|<3\delta\}, \\ S_1=\{\tilde{p}\in\tilde{Z};\ 0<|\pi(\tilde{p})-1|<3\delta\}, \\ S_\infty=\{\tilde{p}\in\tilde{Z};\ |\pi(\tilde{p})|>\delta,\ |\pi(\tilde{p})-1|>\delta\}. \end{cases} \tag{5.1}$$

We study the prolongation of $w_n(\rho)=w_n(t,z,\zeta,\alpha,\lambda)$ and its estimate. Let $\delta=\beta/4$, where $\beta<1$ is that in Propositions 4.5 and 4.6 in § 4. Choose $\tilde{p}_0\in S_\infty$ so that $w_n(\rho)$ $(n\geqq-1)$ are represented by the convergent power series of ρ^{-1} in a neighbourhood of $\tilde{p}_0$ (see (3.7)). We note that $w_n(\rho)$ $(n\geqq-1)$ are determined by the equations (2.30), which suggests that Propositions in §4 are available for them.

Now let us consider the prolongation of $w_n(\rho)$. Let $V\subset\tilde{Z}$ be any open connected set such that $\tilde{p}_0\in V$ and $\bar{V}$ is compact. Take α so large that $w_n(\tilde{p}_0)$, $\tilde{p}_0\partial_\rho w_n(\tilde{p}_0)\ll AM(\lambda)\alpha^{n+1}\theta^{(n+1)}(R\,;\,x)$ (see Proposition 3.3). We can extend $w_n(\rho)$ from $\tilde{p}_0$ to any $\tilde{p}_1\in V$ along a continuous path $\tilde{p}(s)$ $(0\leqq s\leqq 1)$ with $\tilde{p}(0)=\tilde{p}_0$ and $\tilde{p}(1)=\tilde{p}_1$. Suppose $\tilde{p}_1\in S_\infty$. Then we can continue $w_n(\rho)$ along a path $\tilde{p}(s)$ $(0\leqq s\leqq 1)$ not always contained in V such that $\tilde{p}(s)\subset S_\infty$ for $0\leqq s\leqq 1$. We decompose the path $\tilde{p}(s)$: $\tilde{p}(s)=\sum_{i=1}^m\tilde{p}_i(s)$, where $\tilde{p}_i(s)=\{\tilde{p}(s)\,;\,s_{i-1}\leqq s\leqq s_i\}$ and $0=s_0<s_1<\cdots<s_i<\cdots<s_m=1$. Put $\kappa_i=\tilde{p}(s_i)$ and choose $\{s_i\,;\,0\leqq i\leqq m\}$ to satisfy $|\kappa_i-\kappa_{i-1}|<r_1(\delta)/2$ for all i, where $r_1(\delta)$ is that in Proposition 4.9. By the assumption on V, we may assume that $m\leqq N'$ for some $N'\in\boldsymbol{N}$, which does not depend on $\tilde{p}_1\in V\cap S_\infty$. It follows from Proposition 4.9 that

$$w_n(\rho),\ \rho\partial_\rho w_n(\rho)\ll\gamma_1 AM(\lambda)\alpha^{n+1}\theta^{(n+1)}(R\,;\,x)\qquad\text{at}\quad\rho=\kappa_1.$$

Using Proposition 4.9 repeatedly, we have

$$w_n(\rho),\ \rho\partial_\rho w_n(\rho)\ll\gamma_1^i AM(\lambda)\alpha^{n+1}\theta^{(n+1)}(R\,;\,x)\qquad\text{at}\quad\rho=\kappa_i.$$

Hence there exists a constant $K(V)$ such that for $\rho\in V\cap S_\infty$

$$w_n(\rho),\ \rho\partial_\rho w_n(\rho)\ll K(V)M(\lambda)\alpha^{n+1}\theta^{(n+1)}(R\,;\,x).\tag{5.2}$$

Suppose $\tilde{p}_1\in S_0$. We can prolong $w_n(\rho)$ along a path $\tilde{p}(s)$ $(0\leqq s\leqq 1)$ such that $\{\tilde{p}(s)\,;\,0\leqq s\leqq s'\}\subset S_\infty$ and $\{\tilde{p}(s)\,;\,s'\leqq s\leqq 1\}\subset S_0$. We may put $\tilde{p}(s')=\kappa$ and assume $2\delta<|\kappa|<3\delta=3\beta/4<1$. We have from (5.2)

$$w_n(\kappa),\kappa\partial_\rho w_n(\kappa)\ll K(V)M(\lambda)\alpha^{n+1}\theta^{(n+1)}(R\,;\,x)\ll K(V)M(\lambda)\Big(\frac{\alpha}{|\kappa|}\Big)^{n+1}\theta^{(n+1)}(R\,;\,x).\tag{5.3}$$

So it follows from Proposition 4.5 that in a neighbourhood of $\tilde{p}_1$

$$w_n(\rho)\ll K'(V)\alpha^{n+1}M(\lambda)|\rho|^{c-n-1}(2|\kappa|-\rho)^{-1}\theta^{(n+1)}(R\,;\,x),\tag{5.4}$$

for some positive constant $K'(V)$. When $\tilde{p}_1\in S_1$, an estimate similar to (5.4) holds. Thus we have

PROPOSITION 5.1. *Let $V\subset\tilde{Z}$ be an open connected set such that $\bar{V}$ is compact. Then the following estimate for $w_n(t, z, \zeta, \rho, \lambda)$ holds:*

$$\begin{cases} w_n(t, z, \zeta, \rho, \lambda)\ll A(V)M(\lambda)\alpha^{n+1}\theta^{(n+1)}(R\,;\,x) & \textit{for}\ \rho\in V\cap S_\infty,\\ w_n(t, z, \zeta, \rho, \lambda)\ll A(V)M(\lambda)\alpha^{n+1}|\rho|^{c-n-1}\theta^{(n+1)}(R\,;\,x) & \textit{for}\ \rho\in V\cap S_0,\\ w_n(t, z, \zeta, \rho, \lambda)\ll A(V)M(\lambda)\alpha^{n+1}|\rho-1|^{c-n-1}\theta^{(n+1)}(R\,;\,x) & \textit{for}\ \rho\in V\cap S_1.\end{cases}\tag{5.5}$$

Recall the definition formula (3.23) of $W_n(t, z, \zeta, \lambda)$ on X_0. Then we have for $n \geqq 0$

$$W_n(t, z, \zeta, \lambda) = \frac{1}{2\pi i n!}\Big\{\int_{|\rho|=d}(\zeta-\rho t)^n \log\Big(\frac{\zeta}{t}-\rho\Big) w_n(t, z, \zeta, \rho, \lambda)d\rho \tag{5.6}$$

$$+\log t\int_{|\rho|=d}(\zeta-\rho t)^n w_n(t, z, \zeta, \rho, \lambda)d\rho$$

$$-\int_{|\rho|=d}(\zeta-\rho t)^n\Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)w_n(t, z, \zeta, \rho, \lambda)d\rho\Big\} \quad \text{on } X_0.$$

It follows from Proposition 3.5 that its holomorphic prolongation in $\tilde{X}$ is given by

$$W_{n,\gamma}(t, z, \zeta, \lambda) = \int_\gamma f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)d\rho, \tag{5.7}$$

where $\gamma=\{\gamma(s);\ (0\leqq s\leqq 1)\}$ is a homotopic deformation of the path $|p|=d$ in (5.6). We decompose $\gamma(s)$. Let $0=s_0<s_1<\cdots<s_l=1$ and put $\gamma_i=\{\gamma(s);\ s_{i-1}\leqq s\leqq s_i\}$. Assume that each γ_i is contained entirely in S_0, S_1 or S_∞ and that $\gamma(s_i)\in S_\infty$ $(0\leqq i\leqq l)$. Put

$$W_{n,\gamma_i}(t, z, \zeta, \lambda) = \int_{\gamma_i} f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)d\rho. \tag{5.8}$$

We have $W_{n,\gamma_i}(t, z, \zeta)\in\mathcal{O}(\widetilde{U(R)-\cup_{\rho\in\gamma_i}\{\zeta-\rho t=0\}})$. Let us investigate (5.8). Fix $\kappa\in S_0$ with $2\delta<|\kappa|<3\delta$. Then, by Proposition 4.5, we have in a neighbourhood of $\rho=0$ except $\rho=0$

$$w_n(t, z, \zeta, \rho, \lambda) = \sum_{s=-(n+1)}^{+\infty}\Big(\frac{\rho}{\kappa}\Big)^s\int_{C_0}\Big(\frac{\rho}{\kappa}\Big)^\mu w_{n,s}(t, z, \zeta, \mu, \kappa, \lambda)d\mu. \tag{5.9}$$

Put for $k\in \boldsymbol{Z}_+$

$$w_n^{-k}(t, z, \zeta, \rho, \lambda) \tag{5.10}$$

$$= \sum_{s=-(n+1)}^{+\infty}\Big(\frac{\rho}{\kappa}\Big)^s\int_{C_0}\Big(\frac{\rho}{\kappa}\Big)^\mu \frac{\rho^k}{(\mu+s+1)\cdots(\mu+s+k)} w_{n,s}(t, z, \zeta, \mu, \kappa, \lambda)d\mu.$$

Then

LEMMA 5.2. (1) $w_n^{-k}(t, z, \zeta, \rho, \lambda)\in\mathcal{O}(U(R)\times\widetilde{(0<|\rho|<2|\kappa|)})$.

(2) *For any $\omega>0$, there exist constants $C(\omega)$ and A such that*

$$w_n^{-k}(t, z, \zeta, \rho, \lambda) \tag{5.11}$$

$$\ll C(\omega)M(\lambda)(A^k\alpha^{n+1}|\rho|^{c+k-n-1}/(2|\kappa|-|\rho|)k!)\theta^{(n+1)}(R;\ x)$$

for $\rho\in\{0<|\rho|<2|\kappa|;\ |\arg\rho/\kappa|<\omega\}$.

(3) $$\partial_\rho^s(w_n^{-k}(t, z, \zeta, \rho, \lambda)) = w_n^{s-k}(t, z, \zeta, \rho, \lambda) \quad \textit{for} \quad 0\leqq s\leqq k.$$

Lemma 5.2 follows from Proposition 4.5 and it is not difficult to show Lemma

5.2. So we omit the details of the proof.

Let $\gamma'=\{\tilde{p}(s)\,;\ a\leqq s\leqq b\}$ be a path such that $\tilde{p}(s)\subset S_0$. Then, by integrations by parts, we have

$$\begin{aligned}(5.12)\quad \int_{\gamma'} f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)d\rho &= \int_{\gamma'} f_n(\zeta-\rho t)\partial_\rho^n w_n^{-n}(t, z, \zeta, \rho, \lambda)d\rho \\ &= \int_{\gamma'} t^n f_0(\zeta-\rho t)w_n^{-n}(t, z, \zeta, \rho, \lambda)d\rho+I_n(t, z, \zeta, \rho, \lambda)|_{\partial\gamma'},\end{aligned}$$

where

$$(5.13)\qquad I_n(t, z, \zeta, \rho, \lambda)=\textstyle\sum_{k=0}^{n-1} t^k f_{n-k}(\zeta-\rho t)w_n^{-k-1}(t, z, \zeta, \rho, \lambda).$$

LEMMA 5.3. *It holds that for $(t, z, \zeta)\in U(r)$ $(0<r<R)$*

$$(5.14)\qquad |I_n(t, z, \zeta, \rho, \lambda)|\leqq CM(\lambda)|\log(\zeta-\rho t)||\rho|^{c+2}\alpha^{n+1}A^n\Big(|t|+\frac{|\zeta-\rho t|}{|\rho|}\Big)^n.$$

PROOF. We have

$$|I_n(t, z, \zeta, \rho, \lambda)|\leqq CM(\lambda)|\log(\zeta-\rho t)|\Big(\textstyle\sum_{k=0}^{n-1}\dfrac{A^n|t|^k|C(\zeta-\rho t)|^{n-k}(n+1)!}{(n-k)!k!|\rho|^{n-k-2-c}}\Big)\alpha^{n+1}.$$

This means (5.14).

LEMMA 5.4. *Suppose that $\gamma_i\subset S_0$ or $\gamma_i\subset S_1$. Let K be a compact set in $U(r)-\cup_{\rho\in\gamma_i}\{(t, z, \zeta)\,;\ \zeta-\rho t=0\}$. Then there exist A_{K,γ_i} and L such that*

$$(5.15)\qquad |W_{n,\gamma_i}(t, z, \zeta, \lambda)|\leqq A_{K,\gamma_i}M(\lambda)(L(|\zeta|+|t|))^{n+1}\quad \text{on } K,$$

where $L=L(\delta)$ is independent of K, γ_i and n.

PROOF. Suppose $\gamma_i\subset S_0$. After changing the right hand side of (5.8) into the form (5.12), estimate (5.12), using Lemmas 5.2 and 5.3. Then we have (5.15). When $\gamma_i\subset S_1$, the statement follows from the same arguments as $\gamma_i\subset S_0$.

Suppose $\gamma_i\subset S_\infty$. Then, by Proposition 5.1, we have for $(t, z, \zeta)\in U(r)$

$$(5.16)\qquad |f_n(\zeta-\rho t)w_n(t, z, \zeta, \rho, \lambda)|\leqq A_{\gamma_i}M(\lambda)|\log(\zeta-\rho t)||\zeta-\rho t|^n\alpha^{n+1}$$

on $\rho\in\gamma_i$. Hence

LEMMA 5.5. *Let K be a compact set in $U(r)-(\cup_{\rho\in\gamma_i}\{(t, z, \zeta)\,;\ \zeta-\rho t=0\}\cup\{t=0\})$. Suppose that $\gamma_i\subset S_\infty$ and $|\pi(\gamma_i)|\leqq\varepsilon/|t|$ for any $(t, z, \zeta)\in K$. Then there exist $A_{K,\gamma,\varepsilon}$ and L such that*

$$(5.17)\qquad |W_{n,\gamma_i}(t, z, \zeta, \lambda)|\leqq A_{K,\gamma_i,\varepsilon}M(\lambda)(L(|\zeta|+\varepsilon))^{n+1}\quad \text{on } K,$$

where L is independent of K, γ_i and n.

Lemmas 5.4 and 5.5 mean

PROPOSITION 5.6. *Let K be a compact set in $\tilde{X}$ and γ be a path in $\tilde{Z}$ such that $K\subset\widetilde{U(r)-(\cup_{\rho\in\gamma}\{(t, z, \zeta)\,;\,\zeta-\rho t=0\}\cup\{t=0\})}$ and $|\pi(\gamma)|\leqq\varepsilon/|t|$ for any $(t, z, \zeta)\in K$. Then there exist $A_{K,\gamma,\varepsilon}$ and L such that*

$$(5.18)\qquad |W_{n,\gamma}(t, z, \zeta, \lambda)|\leqq A_{K,\gamma,\varepsilon}M(\lambda)(L(|\zeta|+|t|+\varepsilon))^{n+1}\qquad on\ \ K,$$

where $L=L(\delta)$ is independent of K, γ, ε and n.

Now let us recall the definition of the sets X_0 and X (see (1.12))

$$(5.19)\ \begin{cases} X_0=\{(t, z, \zeta)\in \boldsymbol{C}^{n+2};\ |t|<r,\ |z|<r,\ |\zeta|<r \text{ and } c|t|<|\zeta|\}, \\ X=\{(t, z, \zeta)\in \boldsymbol{C}^{n+2};\ |t|<r,\ |z|<r,\ |\zeta|<r\}-\{t=0\}\cup\{\zeta=0\}\cup\{t=\zeta\}. \end{cases}$$

We have

THEOREM 5.7. *Let $W(t, z, \zeta, \lambda)\in\mathcal{O}(X_0)$ be a solution defined by* (3.24) *of the equation*

$$(5.20)\qquad \mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)W(t, z, \zeta, \lambda)=\frac{g(t, z, \zeta, \lambda)}{2\pi i\zeta}$$

(*see Theorem* 3.7). *Then $W(t, z, \zeta, \lambda)$ is holomorphically extensible to $\tilde{X}$ as a function of (t, z, ζ), that is, $W(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})$.*

PROOF. Put $\varepsilon=1/2L$ and $r=1/6L$. Then $|\zeta/t|\leqq r/|t|<\varepsilon/|t|$ and $\varepsilon/|t|>3$ for $(t, z, \zeta)\in\tilde{X}$. Let $\tilde{x}\in\tilde{X}$ and $\sigma=\{\sigma(s)\,;\ 0\leqq s\leqq 1\}$ be a continuous path in $\tilde{X}$ such that $\sigma(0)\in X_0$ and $\sigma(1)=\tilde{x}$. Then we can choose a compact set K in $\tilde{X}$ and a homotopic deformation $\gamma=\gamma_\sigma$ of the integration path such that $\sigma\subset K\subset\widetilde{U(r)-(\cup_{\rho\in\gamma}\{\zeta-\rho t=0\}\cup\{t=0\})}$ and $|\pi(\gamma)|\leqq\varepsilon/|t|$ for $(t, z, \zeta)\in K$ (see Fig. 5.1). Then, by Proposition 5.6, we have $|W_{n,\gamma}(t, z, \zeta, \lambda)|\leqq M(\lambda)A_{K,\gamma}(5/6)^{n+1}$. Hence $W_\gamma(t, z, \zeta, \lambda)=\sum_{n=-1}^{+\infty}W_{n,\gamma}(t, z, \zeta, \lambda)$ converges on K, which is the holomorphic prolongation of $W(t, z, \zeta, \lambda)$ along σ. Thus we have $W(t, z, \zeta, \lambda)\in\mathcal{O}_0(\tilde{X})\subset\mathcal{O}(\tilde{X})$.

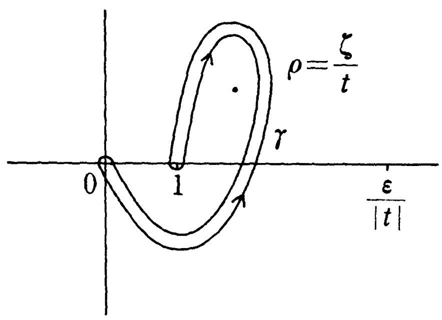

Fig. 5.1.

§6. Integral representation.

In §6 we complete the proof of Theorem 1.4 which gives a representation of singular solutions of (1.2). $W(t, z, \zeta, \lambda)$ contains a holomorphic parameter λ. We integrate functions of (t, z, ζ, λ) with respect to λ and obtain a formula of singular solutions. For this purpose we give a representation of $f(t, z) \in \mathcal{O}(\widetilde{\Omega - K_1})$, where $K_1 = \{\varphi_1(t, z) = 0\}$ with $\varphi_1(t, z)$ given by (1.5).

PROPOSITION 6.1. *Let $f(t, z) \in \mathcal{O}(\widetilde{\Omega - K_1})$. Then $f(t, z)$ is represented in the following form in a neighbourhood Ω' ($\Omega' \subset \Omega$) of the origin.*

Let $\theta \in \boldsymbol{R}$ and let $(t, z) \in \Omega'$ with $|\arg \varphi_1(t, z) - \pi - \theta| < \pi$. Then there exist functions $\hat{f}_\theta(t, z, \lambda) \in \mathcal{O}(\Omega' \times \boldsymbol{C})$ and $h_\theta(t, z) \in \mathcal{O}(\Omega')$ such that

$$(6.1) \qquad f(t, z) = \int_0^{\infty e^{i\phi}} \exp(\lambda \varphi_1(t, z)) \hat{f}_\theta(t, z, \lambda) d\lambda + h_\theta(t, z), \qquad |\phi + \theta| < \pi/2,$$

holds. Moreover $\hat{f}_\theta(t, z, \lambda)$ has the following properties.

(1) *For any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for $|\arg \lambda + \theta| < \pi/2$*

$$(6.2) \qquad |\hat{f}_\theta(t, z, \lambda)| \leqq C_\varepsilon \exp(\varepsilon |\lambda|).$$

(2) *Let $|\theta - \theta'| < \pi - 2\delta$ $(0 < 2\delta < \pi)$ and put $\hat{f}_{\theta, \theta'}(t, z, \lambda) = \hat{f}_\theta(t, z, \lambda) - \hat{f}_{\theta'}(t, z, \lambda)$. Then there is a constant $c_\delta > 0$ independent of θ and θ' such that*

$$(6.3) \qquad |\hat{f}_{\theta, \theta'}(t, z, \lambda)| \leqq C_{\theta, \theta'} \exp(-c_\delta |\lambda|)$$

for λ with $\max(-\theta, -\theta') - \pi/2 + \delta < \arg \lambda < \min(-\theta, -\theta') + \pi/2 - \delta$.

We have Proposition 6.1 from Propositions given in Appendix. Now let $W_\theta(t, z, \zeta, \lambda)$ be a solution of (2.12) where we put $\hat{f}(t, z, \lambda) = \hat{f}_\theta(t, z, \lambda)$. Then it follows from (6.2) and Theorem 5.7 that for any $\varepsilon > 0$ and any compact set K in $\tilde{X}$

$$(6.4) \qquad |W_\theta(t, z, \zeta, \lambda)| \leqq C_{K, \varepsilon} \exp(\varepsilon |\lambda|)$$

holds for λ with $|\arg \lambda + \theta| < \pi/2$.

Put $S_\theta = \{s \neq 0;\ |\arg s - \pi - \theta| < \pi\}$ and define

$$(6.5) \qquad U_\theta^*(t, z, \zeta, s) = \int_0^{\infty e^{i\phi}} \exp(\lambda s) W_\theta(t, z, \zeta, \lambda) d\lambda, \qquad |\phi + \theta| < \pi/2.$$

$U_\theta^*(t, z, \zeta, s)$ is holomorphic as a function of s in S_θ by (6.4) and $U_\theta^*(t, z, \zeta, s) \in \mathcal{O}_0(\tilde{X})$ as a function of (t, z, ζ). Moreover

PROPOSITION 6.2. *$U_\theta^*(t, z, \zeta, s)$ is holomorphic in Y_0 with $Y_0 = X_0 \times \{0 < |s| < R_1\}$ for some R_1 and extensible to $\tilde{Y}$ with $Y = X \times \{0 < |s| < R_1\}$.*

PROOF. Let θ and θ' be $|\theta - \theta'| < \pi/2$. We have

$$(6.6)\qquad \mathcal{L}(W_\theta(t, z, \zeta, \lambda)-W_{\theta'}(t, z, \zeta, \lambda))=\frac{\hat{g}_{\theta,\theta'}(t, z, \zeta, \lambda)}{2\pi i\zeta},$$

where $\hat{g}_{\theta,\theta'}(t, z, \zeta, \lambda)=\dfrac{\tilde{f}_{\theta,\theta'}(t, z, \lambda)}{(1-t\chi_0(t, z, \zeta))}$. Let $W_{\theta,\theta'}(t, z, \zeta, \lambda)\in\mathcal{O}(X_0\times\boldsymbol{C}^1)\cap\mathcal{O}(\tilde{X}\times\boldsymbol{C}^1)$ be a unique solution of

$$(6.7)\qquad \mathcal{L}W_{\theta,\theta'}(t, z, \zeta, \lambda)=\frac{\hat{g}_{\theta,\theta'}(t, z, \zeta, \lambda)}{2\pi i\zeta}.$$

It follows from (6.3) that for any compact set K in $\tilde{X}$

$$(6.8)\qquad |W_{\theta,\theta'}(t, z, \zeta, \lambda)|\leqq C_{K,\theta,\theta'}\exp(-C|\lambda|)\qquad (C>0)$$

for λ with $\max(-\theta, -\theta')-\pi/4<\arg\lambda<\min(-\theta, -\theta')+\pi/4$. We have, by the uniqueness in $\mathcal{O}_0(\tilde{X})$, $W_\theta(t, z, \zeta, \lambda)=W_{\theta'}(t, z, \zeta, \lambda)+W_{\theta,\theta'}(t, z, \zeta, \lambda)$. Put

$$(6.9)\qquad U^*_{\theta,\theta'}(t, z, \zeta, s)=\int_0^{\infty e^{i\phi}}\exp(\lambda s)W_{\theta,\theta'}(t, z, \zeta, \lambda)d\lambda,$$

where $\max(-\theta, -\theta')-\pi/4<\phi<\min(-\theta, -\theta')+\pi/4$. Then, by (6.8), $U^*_{\theta,\theta'}(t, z, \zeta, s)\in\mathcal{O}(X_0\times\{|s|<C\})$ and it is extensible to $\tilde{X}\times\{|s|<C\}$. We have $U^*_\theta(t, z, \zeta, s)=U^*_{\theta'}(t, z, \zeta, s)+U^*_{\theta,\theta'}(t, z, \zeta, s)$ in $X_0\times(S_\theta\cap S_{\theta'}\cap\{|s|<C\})$ and in $\tilde{X}\times(S_\theta\cap S_{\theta'}\cap\{|s|<C\})$. So $U^*_\theta(t, z, \zeta, s)$ is holomorphic in $X_0\times((S_\theta\cup S_{\theta'})\cap\{|s|<C\})$ and in $\tilde{X}\times((S_\theta\cup S_{\theta'})\cap\{|s|<C\})$. By continuing this process, it is shown that $U^*_\theta(t, z, \zeta, s)$ is holomorphic in Y_0 and is extensible holomorphically to $\tilde{Y}$.

Put $T=\{(t, z, \zeta)\in U(r);\ \Phi(t, z, \zeta)=0\}$. We have

Lemma 6.3. *There is a holomorphic function $\psi(t, z)$ in a neighbourhood of $t=z=0$ such that $T=\{\zeta=\psi(t, z)\}$, $K_1=\{\psi(t, z)=0\}$, $K_2=\{\psi(t, z)=t\}$ and*

$$(6.10)\qquad \psi(t, z)=(z_1+H_1(0, z, \hat{\xi})t)/(H_1(0, z, \hat{\xi})-H_2(0, z, \hat{\xi}))+O(|t|^2+|z|^2),$$

Proof. We have

$$\Phi(t, z, \zeta)=z_1+H_1(0, z, \hat{\xi})t+(H_2(0, z, \hat{\xi})-H_1(0, z, \hat{\xi}))\zeta+O(|t|^2+|\zeta|^2).$$

Since $H_2(0, z, \hat{\xi})-H_1(0, z, \hat{\xi})\neq 0$, it follows from the implicit function theorem that there is a holomorphic function $\psi(t, z)$ with (6.10) in a neighbourhood of the origin such that $T=\{\zeta=\psi(t, z)\}$. Since $\Phi(t, z, 0)=\varphi_1(t, z)$ and $\Phi(t, z, t)=\varphi_2(t, z)$, we have $K_1=\{\varphi_1(t, z)=0\}=\{\psi(t, z)=0\}$ and $K_2=\{\varphi_2(t, z)=0\}=\{\psi(t, z)=t\}$.

Proof of Theorem 1.4. We denote by $U^*(t, z, \zeta, s)$ the prolongation of $U^*_\theta(t, z, \zeta, s)$ defined on S_θ for some θ. It follows from (6.10) in Lemma 6.3 that there is a positive constant k such that if $|z_1|>k|t|$, $|\psi(t, z)|>c|t|$ holds for some $c>1$. Put $\Omega_0'=\{(t, z);\ |t|<r,\ |z|<r$ and $|z_1|>k|t|\}$ for a small $r>0$. For $(t, z)\in\Omega_0'$ we can choose a path Γ in ζ-space so that it is closed and surrounding once $\{|\zeta|\leqq|t|\}$ in X_0 and $\{\zeta;\ \Phi(t, z, \zeta)=0\}$ is outside of Γ (see Fig.

1.1). Since $U^*(t, z, \zeta, s)\in\mathcal{O}_0(\tilde{Y})$, we can define for $(t, z)\in\Omega_0'$

$$(6.11)\qquad v(t, z)=\int_\Gamma U^*(t, z, \zeta, \Phi(t, z, \zeta))d\zeta$$
$$=\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))W_\theta(t, z, \zeta, \lambda)d\lambda,\quad |\phi+\theta|<\pi/2,$$

which is holomorphic in $\tilde{\Omega}_0'$. We show (6.11) is a desired integral representation. From the method of construction of $W_\theta(t, z, \zeta, \lambda)$, we have

$$(6.12)\qquad L(t, z, \partial_t, \partial_z)v(t, z)=\int_\Gamma L(t, z, \partial_t, \partial_z)U^*(t, z, \zeta, \Phi(t, z, \zeta))d\zeta$$
$$=\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\lambda\mathcal{L}_1(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda$$
$$+\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))L(t, z, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda.$$

By (2.5), we have

$$(6.13)\qquad \int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\lambda\mathcal{L}_1(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda$$
$$=\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\lambda(-\partial_\zeta\Phi(t, z, \zeta))\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda$$
$$=-\partial_\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda$$
$$+\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\partial_\zeta\{\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)\}\,d\lambda.$$

Hence, we have, by (2.7) and (2.12),

$$(6.14)\qquad L(t, z, \partial_t, \partial_z)v(t, z)$$
$$=\int_\Gamma d\zeta\Big\{-\partial_\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda\Big\}$$
$$+\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\partial_\zeta\{\chi(t, z, \zeta)\mathcal{M}(t, z, \zeta, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)\}\,d\lambda$$
$$+\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))L(t, z, \partial_t, \partial_z)W_\theta(t, z, \zeta, \lambda)d\lambda$$
$$=\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))(1-t\chi_0(t, z, \zeta))\{\mathcal{L}(t, z, \zeta, \partial_t, \partial_z, \partial_\zeta)W(t, z, \zeta, \lambda)\}\,d\lambda$$
$$=\int_\Gamma d\zeta\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, \zeta))\frac{\hat{f}_\theta(t, z, \lambda)}{2\pi i\zeta}\,d\lambda=\int_0^{\infty e^{i\phi}}\exp(\lambda\Phi(t, z, 0))\hat{f}_\theta(t, z, \lambda)d\lambda$$
$$=f_\theta(t, z)=f(t, z)+h(t, z),$$

where $h(t, z)$ is holomorphic at the origin. Put $w(t, z)=u(t, z)-v(t, z)$. Then, since $w(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$ and $L(t, z, \partial_t, \partial_z)w(t, z)=h(t, z)$, $w(t, z)$ is holomorphic at the origin by (1.11). This completes the proof of Theorem 1.4.

PROOF OF THEOREM 1.5. Let us proceed to analysis of $v(t, z)$ defined by

(6.11). By Proposition 6.2 $U_*(t, z, \zeta, s)\in\mathcal{O}_0(\tilde{Y})$. So we can deform the integration path Γ if $\phi(t, z)\neq 0$, t (see Fig. 6.1.) and have the holomorphic prolongation of $v(t, z)$ from $\tilde{\Omega}_0'$ to a wider set. It follows from this deformation and Lemma 6.3 that $v(t, z)$ is holomorphic except the set $\{t=0\}\cup\{\phi(t, z)=0\}\cup\{\phi(t, z)=t\}=K_0\cup K_1\cup K_2$. Thus we have $u(t, z)\in\mathcal{O}(\widetilde{\Omega'-K_0\cup K_1\cup K_2})$ for a neighbourhood Ω' of $(t, z)=(0, 0)$.

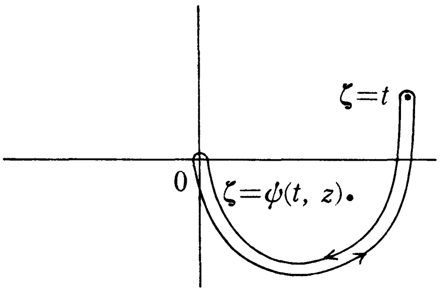

Fig. 6.1.

§7. Proof of Theorems.

In §7 we show Theorems 1.6-1.8. Let $L_i(t, z, \partial_t, \partial_z)$ $(i=1, 2)$ be those in §1. First of all we give a lemma.

LEMMA 7.1. *The following identities hold:*

$$L_2(t, z, \partial_t, \partial_z)t^k w(t, z)$$
$$= \{t^k L_2(t, z, \partial_t, \partial_z)+kt^{k-1}(2\partial_t+A(t, z, \partial_z))+k(k-1)t^{k-2}\}\, w(t, z), \tag{7.1}$$

$$L_1(t, z, \partial_t, \partial_z)t^k w(t, z) = \{t^k L_1(t, z, \partial_t, \partial_z)+kt^{k-1}a_0(t, z)\}\, w(t, z). \tag{7.2}$$

It is not difficult to show Lemma 7.1. So we omit the proof. Let $L(t, z, \partial_t, \partial_z) =t^2L_2(t, z, \partial_t, \partial_z)+tL_1(t, z, \partial_t, \partial_z)+c(t, z)$. By Lemma 7.1 we have

$$L(t, z, \partial_t, \partial_z)t^k w(t, z) = t^k M^k(t, z, \partial_t, \partial_z)w(t, z), \tag{7.3}$$

where

$$M^k(t, z, \partial_t, \partial_z) = t^2L_2(t, z, \partial_t, \partial_z)+tM_1^k(t, z, \partial_t, \partial_z)+M_0^k(t, z), \tag{7.4}$$

$$M_1^k(t, z, \partial_t, \partial_z) = 2k\partial_t+L_1(t, z, \partial_t, \partial_z)+kA(t, z, \partial_z), \tag{7.5}$$

$$M_0^k(t, z) = k(k-1)+ka_0(t, z)+c(t, z). \tag{7.6}$$

The indicial polynomial $\ell(M^k\,;\mu)$ of the Fuchsian operator $M^k(t, z, \partial_t, \partial_z)$ is

$$\ell(M^k\,;\mu) = \mu(\mu-1)+2k\mu+a_0(0, z)\mu+k(k-1)+ka_0(0, z)+c(0, z)$$
$$= \ell(L\,;\mu+k). \tag{7.7}$$

PROOF OF THEOREM 1.6. First we assume $n_0=0$. So the conditions of the theorem mean that $c(0, z)=z_1^p c_0(z)$, $c_0(0)\neq 0$ and $\ell(M^1; \mu)\neq 0$ for all $\mu\in \boldsymbol{Z}_+$. Let us expand $f(t, z)\in\mathcal{O}(\Omega): f(t, z)=\sum_{n=0}^{+\infty} f_n^*(z)t^n$. Define

$$v(t, z) = f_0^*(z)/\varphi_1(t, z)^p c_0(z). \tag{7.8}$$

Then there is a function $g(t, z)$ with a pole on K_1 such that

$$L(t, z, \partial_t, \partial_z)v(t, z) = tg(t, z)+f_0^*(z)(z_1/\varphi_1(t, z))^p. \tag{7.9}$$

Put $u(t, z)=v(t, z)+tw(t, z)$. Since $u(t, z)$, $v(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$ and $u(0, z)=v(0, z)$, $w(t, z)$ also belongs to $\mathcal{O}_{\tilde{\omega}_0}$ (see Definition 1.2 and (1.9)). Then we have

$$\begin{aligned} tM^1(t, z, \partial_t, \partial_z)w(t, z) &= L(t, z, \partial_t, \partial_z)u(t, z)-L(t, z, \partial_t, \partial_z)v(t, z) \\ &= -tg(t, z)+f(t, z)-f_0^*(z)(z_1/\varphi_1(t, z))^p \\ &= -tg(t, z)+\textstyle\sum_{n=1}^{+\infty} f_n^*(z)t^n+f_0^*(z)(\varphi_1(t, z)^p-z_1^p)/\varphi_1(t, z)^p. \end{aligned} \tag{7.10}$$

Since $\varphi_1(0, z)=z_1$, there is a function $h(t, z)\in\mathcal{O}(\Omega-K_1)$ with a pole on K_1 such that

$$M^1(t, z, \partial_t, \partial_z)w(t, z) = h(t, z). \tag{7.11}$$

In the above formula we have $w(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$, $h(t, z)\in\mathcal{O}(\Omega-K_1)$ and $\ell(M^1; \mu)\neq 0$ for $\mu\in\boldsymbol{Z}_+$. Consequently, when $n_0=0$, Theorem 1.6 follows from Theorem 1.5. Let $n_0>0$. Put $k=n_0$. By the assumption there exists a polynomial of $tv(t, z)=\sum_{n=0}^{k-1} t^n v_n(z)$ such that $L(t, z, \partial_t, \partial_z)v(t, z)-f(t, z)=t^k g(t, z)$, where $g(t,z)$ is holomorphic at the origin. Put $u(t, z)=v(t, z)+t^k w(t, z)$. Then we have

$$\begin{aligned} &L(t, z, \partial_t, \partial_z)u(t, z) \\ &\quad = L(t, z, \partial_t, \partial_z)v(t, z)+t^k M^k(t, z, \partial_t, \partial_z)w(t, z) = f(t, z). \end{aligned} \tag{7.12}$$

Hence $M^k(t, z, \partial_t, \partial_z)w(t, z)=-g(t, z)$. Since $\ell(M^k; \mu)=\ell(M; \mu+k)$, the roots of $\ell(M^k; \mu)=0$ are $\mu_i(z)-k$ $(i=1, 2)$. Hence we have Theorem 1.6 for $n_0>0$ by the preceding result for $n_0=0$.

Now we proceed to show Theorems 1.7 and 1.8. Let $L(t, z, \partial_t, \partial_z)= tL_2(t, z, \partial_t, \partial_z)+L_1(t, z, \partial_t, \partial_z)$. By Lemma 7.1 we have

$$L(t, z, \partial_t, \partial_z)t^k w(t, z) = t^{k-1}\hat{M}^k(t, z, \partial_t, \partial_z)w(t, z), \tag{7.13}$$

where

$$\hat{M}^k(t, z, \partial_t, \partial_z) = t^2 L_2(t, z, \partial_t, \partial_z)+t\hat{M}_1^k(t, z, \partial_t, \partial_z)+\hat{M}_0^k(t, z), \tag{7.14}$$

$$\hat{M}_1^k(t, z, \partial_t, \partial_z) = 2k\partial_t+L_1(t, z, \partial_t, \partial_z)+kA(t, z, \partial_z), \tag{7.15}$$

$$\hat{M}_0^k(t, z) = k(k-1)+ka_0(t, z). \tag{7.16}$$

The indicial polynomial $\ell(\hat{M}^k; \mu)$ of the Fuchsian operator $\hat{M}^k(t, z, \partial_t, \partial_z)$ is

(7.17) $$\ell(\hat{M}^k; \mu) = \mu(\mu-1)+2k\mu+a_0(0, z)\mu+k(k-1)+ka_0(0, z) = \ell(L; \mu+k).$$

Proof of Theorem 1.7. Let $v(t, z)\in\mathcal{O}(\Omega'-K_1)$, $\Omega'\subset\Omega$, be a solution of

(7.18) $$\begin{cases} L_2(t, z, \partial_t, \partial_z)v(t, z) = 0, \\ v(0, z) = u_0(z). \end{cases}$$

The existence of $v(t, z)$ is assured by Hamada [**3**], Hamada, Leray and Wagschal [**4**] and Wagschal [**10**]. Put $u(t, z)=v(t, z)+tw(t, z)$. Then we have

(7.19) $$\begin{aligned} \hat{M}^1(t, z, \partial_t, \partial_z)w(t, z) &= L(t, z, \partial_t, \partial_z)u(t, z)-L(t, z, \partial_t, \partial_z)v(t, z) \\ &= f(t, z)-L_1(t, z, \partial_t, \partial_z)v(t, z) = h(t, z) \in \mathcal{O}(\widetilde{\Omega'-K_1}). \end{aligned}$$

Since $\ell(\hat{M}^1; \mu)=(\mu+1)(\mu+a_0(0, z))\neq 0$ for $\mu\in\boldsymbol{Z}_+$ from the assumption, we have Theorem 1.7 from Theorem 1.5.

Proof of Theorem 1.8. Put $u(t, z)=u_0(z)+tw(t, z)$. Then

(7.20) $$\begin{aligned} &L(t, z, \partial_t, \partial_z)u(t, z) \\ &\quad = L(t, z, \partial_t, \partial_z)u_0(z)+M^1(t, z, \partial_t, \partial_z)w(t, z) = f(t, z). \end{aligned}$$

We have $M^1(t, z, \partial_t, \partial_z)w(t, z) = f(t, z)-L(t, z, \partial_t, \partial_z)u_0(z) = g(t, z)\in\mathcal{O}(\Omega)$ and $w(t, z)\in\mathcal{O}_{\tilde{\omega}_0}$. Since $\ell(M^1; \mu)=\ell(M; \mu+1)$, the roots of $\ell(M^1; \mu)=0$ satisfy the condition of Theorem 1.6. So we have Theorem 1.8.

§ 8. Appendix.

Let $z=(z_0, z_1, \cdots, z_N)=(z_0, z')$ be the coordinate of $\boldsymbol{C}^{N+1}$, $W=\{z\in\boldsymbol{C}^{N+1}; |z|\leqq R\}$, $K=\{z_0=0\}$, $W(a, b)=\{z\in\widetilde{W-K}; a<\arg z_0<b\}$ and $\mathring{W}=\{z\in\boldsymbol{C}^{N+1}; |z|<R\}$. We give a representation of $f(z)\in\mathcal{O}(\widetilde{W-K})$. For a given $\theta\in\boldsymbol{R}$ define

(A.1) $$\hat{f}_\theta(z', \lambda) = \frac{1}{2\pi i}\int_{T(\theta)}\exp(-\lambda t_0)f(t_0, z')dt_0.$$

$T(\theta)$ is a path starting at $Re^{i(\theta+2\pi)}$, going to $\varepsilon e^{i(\theta+2\pi)}$ $(0<\varepsilon<R)$, rounding the origin once on $|t_0|=\varepsilon$ and ending at $Re^{i\theta}$ (see Fig. A.1).

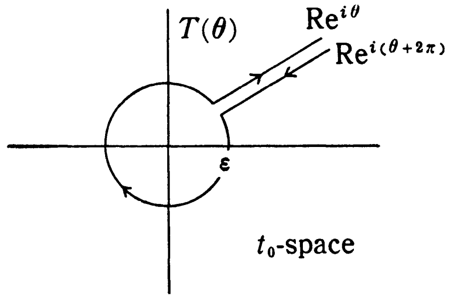

Fig. A.1.

For $\hat{f}_\theta(z', \lambda)$ we have

PROPOSITION A.1. (1) *$\hat{f}_\theta(z', \lambda)$ is an entire function of λ.*

(2) *For any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that*

$$\text{(A.2)} \qquad \sup_{|z'|\leqq R} |\hat{f}_\theta(z', \lambda)| \leqq C_\varepsilon \exp(\varepsilon|\lambda|) \qquad \textit{for } \lambda \textit{ with } |\arg\lambda+\theta|<\pi/2\,.$$

(3) *Put $\hat{f}_{\theta,\theta'}(z', \lambda)=\hat{f}_\theta(z', \lambda)-\hat{f}_{\theta'}(z', \lambda)$. Suppose $|\theta-\theta'|<\pi-2\delta$ $(0<2\delta<\pi)$. Then there is a $c_\delta>0$ independent of θ and θ' such that*

$$\text{(A.3)} \qquad |\hat{f}_{\theta,\theta'}(z, \lambda)| \leqq C_{\theta,\theta'} \exp(-c_\delta|\lambda|)$$

for λ with $\max(-\theta, -\theta')-\pi/2+\delta<\arg\lambda<\min(-\theta, -\theta')+\pi/2-\delta$.

PROOF. The proof of assertions (1) and (2) is easy. So we omit it. We show (3). Define a path $T^*(\theta, \theta')$ on $|t_0|=R$: $T^*(\theta, \theta')=\{t_0=Re^{i((1-s)\theta+s\theta')}$; $0\leqq s\leqq 1\}$. We have

$$\text{(A.4)} \qquad \hat{f}_{\theta,\theta'}(z', \lambda) = \frac{1}{2\pi i}\left\{\int_{T(\theta)} \exp(-\lambda t_0)f(t_0, z')dt_0-\int_{T(\theta')} \exp(-\lambda t_0)f(t_0, z')dt_0\right\}$$

$$= \frac{1}{2\pi i}\left\{\int_{T^*(\theta+2\pi,\, \theta'+2\pi)} \exp(-\lambda t_0)f(t_0, z')dt_0-\int_{T^*(\theta,\, \theta')} \exp(-\lambda t_0)f(t_0, z')dt_0\right\}.$$

Put $\phi=\arg\lambda$. Since it holds that $|\exp(-\lambda t_0)|\leqq\exp(-|\lambda|R\cos(\phi+\arg t_0))$ on $T^*(\theta, \theta')$ and $T^*(\theta+2\pi, \theta'+2\pi)$, we have $\cos(\phi+\arg t_0)\geqq c_\delta>0$ on the same arcs for any λ with $\max(-\theta, -\theta')-\pi/2+\delta<\phi<\min(-\theta, -\theta')+\pi/2-\delta$. This means (3).

For the inversion formula we define

$$\text{(A.5)} \qquad f_\theta(z) = \int_a^{\infty e^{i\phi}} \exp(\lambda z_0)\hat{f}_\theta(z', \lambda)d\lambda\,,$$

where $|\phi+\theta|<\pi/2$ and a is a fixed constant. It follows from (A.2) that $f_\theta(z)$ is holomorphic in $W(\theta, \theta+2\pi)$.

PROPOSITION A.2. *There exists a function $h_\theta(z)\in\mathcal{O}\{|z_0|<R,\ |z'|\leqq R\}$ such*

that

$$f_\theta(z) = f(z) + h_\theta(z) \quad in\ W(\theta, \theta+2\pi), \tag{A.6}$$

which means that $f_\theta(z) \in \mathcal{O}(\mathring{W} - K)$.

Proof. We have for z_0 with $|\arg z_0 + \phi - \pi| < \pi/2$

$$f_\theta(z) = \frac{1}{2\pi i}\int_a^{\infty e^{i\phi}} \exp(\lambda z_0) d\lambda \int_{T(\theta)} \exp(-\lambda t_0) f(t_0, z') dt_0\,. \tag{A.7}$$

Put $\phi = -\theta$ and let $\arg z_0 = \theta + \pi$ and $|z_0| > 2\varepsilon$. Then we have

$$f_\theta(z) = \frac{1}{2\pi i}\int_{T(\theta)} \{f(t_0, z')\exp(-a(t_0 - z_0))/(t_0 - z_0)\}\, dt_0 = f(z) - h_\theta(z), \tag{A.8}$$

where $h_\theta(z) = \frac{1}{2\pi i}\int_{T(\theta)} \{f(t_0, z')\exp(-a(t_0-z_0))/(t_0-z_0)\}\, dt_0$ and $T^*(\theta)$ is a path starting at $Re^{i\theta}$ and going around anti-clockwise on the circle $|t| = R$ once. So $h_\theta(z)$ is holomorphic on $\{z\,;\, |z_0| < R,\ |z'| \leqq R\}$.

Proposition 6.1 easily follows from Propositions A.1 and A.2.

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