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On Nikulin's theorem on fixed components of linear systems on K3 surfaces

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

By Tohsuke URABE

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§ 0. Introduction.

In this article we would like to give more natural verification to Nikulin's theorems in Nikulin [4] and Nikulin [5]. (Nikulin [4] is a report distributed at the symposium held at Tokyo Metropolitan University in August 1990.)

First we explain Nikulin's results.

Let X be a K3 surface (i.e., a smooth projective algebraic surface over algebraically closed field such that the canonical line bundle K_X is trivial and $H^1(X, \mathcal{O}_X)=0$). A line bundle L (resp. a divisor D) on X is *nef*, if $L \cdot C \geqq 0$ (resp. $D \cdot C \geqq 0$) holds for every irreducible algebraic curve C on X. Needless to say, the self-intersection number $C \cdot C = C^2$ is an even integer with $\geqq -2$ for every irreducible curve C. $C^2=-2$ if and only if C is a smooth rational curve. Moreover, $h^0(\mathcal{O}_X(C))=\dim H^0(\mathcal{O}_X(C))=\dim|C|+1=C^2/2+2$ by Riemann-Roch theorem.

Here we quote the following proposition contained in [4] and [5]. It follows easily from Saint-Donat [6].

PROPOSITION 0.1. *Let* $H \in \mathrm{Pic}(X)$ *be a nef line bundle. One of the following cases* (1)-(4) *holds.*

(1) $H^2>0$. *The complete linear system* $|H|$ *contains an irreducible curve and has no fixed point.* $\dim|H|=H^2/2+1>0$.

(2) $H^2=0$, $|H|=m|E|$, $m>0$, *where* $|E|$ *is an elliptic pencil.* ($|H|$ *contains an irreducible curve for* $m=1$ *only.*)

(3) $H^2>0$, $|H|=m|E|+\Gamma$, $m \geqq 2$, *where* $|E|$ *is an elliptic pencil and* Γ *is an irreducible curve with* $\Gamma^2=-2$ *and* $E \cdot \Gamma=1$. *Here also* $m=\dim|H|=H^2/2+1$. Γ *is the fixed part of* $|H|$.

(4) $H \cong \mathcal{O}_X$, $|H|=\{\emptyset\}$.

Let $\Delta=\sum n_i \Delta_i$ be an effective divisor on X. By $G(\Delta)$ we denote the dual graph of intersections of the components Δ_i of Δ, the weight of the vertex corresponding to Δ_i being the multiplicity n_i, the number of edges connecting two vertices corresponding to Δ_i and Δ_j being the intersection number $\Delta_i \cdot \Delta_j$. If a distinguished component Δ_0 of Δ is given, $G(\Delta, \Delta_0)$ denotes the dual graph $G(\Delta)$ with the distinguished vertex corresponding to Δ_0. (In this article the word "component" is used for a divisor with the meaning "irreducible component" for simplicity. If a connected component is considered, we use the words "connected component" adding "connected".)

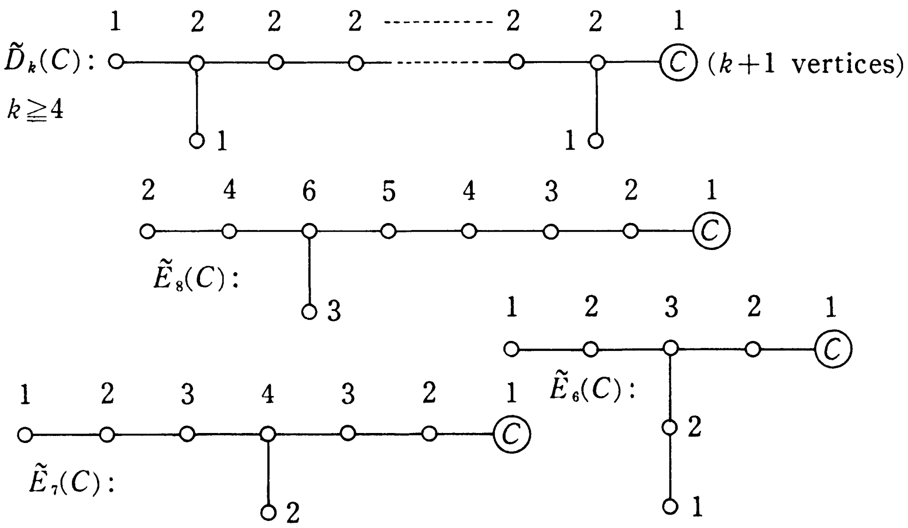
Note that any linear system can be written $|D|+\Delta$ where $|D|$ is the moving part and Δ is the fixed part. The divisor D is necessarily nef, and one of the conditions (1), (2) and (4) in Proposition 0.1 holds for the line bundle $\mathcal{O}_X(D)$.

We would like to consider one of Nikulin's theorems in [**4**] and [**5**]. We divide it into three pieces corresponding to the cases (1), (2) and (4) in Proposition 0.1.

THEOREM 0.2. *Let C be an irreducible curve on X with $C^2>0$. Let Δ be an effective divisor on X. The two conditions below are equivalent.*

(A) $|C+\Delta|=|C|+\Delta$.

(B) *Every component Δ_i of Δ satisfies $\Delta_i^2=-2$. The graph $G(C+\Delta, C)$ is a tree (particularly all components of $C+\Delta$ intersect transversally at at most one point), and $G(C+\Delta, C)$ has no subtree of type $\tilde{D}_k(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$ or $\tilde{E}_8(C)$ below.*

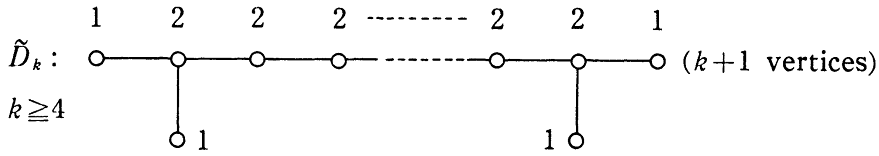


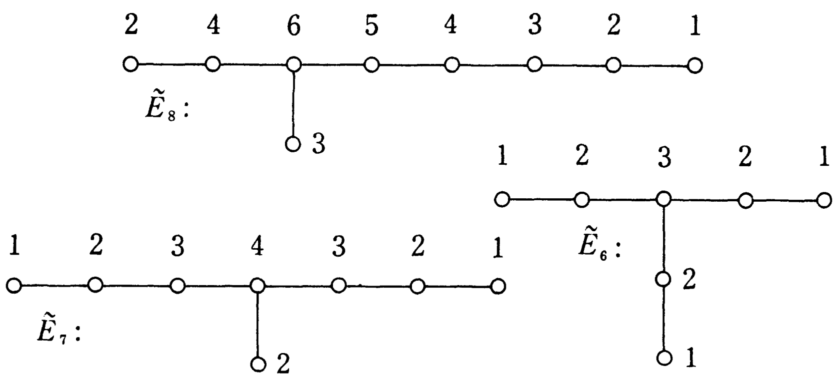
If we treat a curve E on X with $E^2=0$, we cannot assume that the multiplicity of E in the divisor under consideration is equal to 1. (See Proposition 0.1.) The corresponding theorem can be stated as follows.

THEOREM 0.3. *Let E be an irreducible curve on X with $E^2=0$. Let Δ be* *an effective divisor on X. Let m be a positive integer. The two conditions below are equivalent.*

(A) $|mE+\Delta|=|mE|+\Delta$.

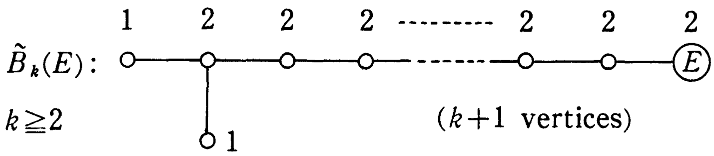
(B) *Every component* Δ_i *of* Δ *satisfies* $\Delta_i^2=-2$. *The graph* $G(mE+\Delta, E)$ *is a tree without a subtree of type* $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$ *or* $\tilde{E}_8$ *below and one of the following conditions* (1)-(4) *holds.*



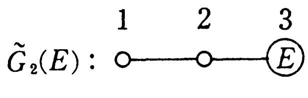


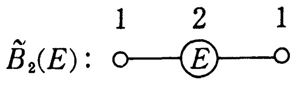
(1) $m=1$ *and* $G(mE+\Delta, E)$ *has no subtree of type* $\tilde{D}_k(E)$, $\tilde{E}_6(E)$, $\tilde{E}_7(E)$ *or* $\tilde{E}_8(E)$.

(2) $m=2$ *and* $G(mE+\Delta, E)$ *has no subtree of the type* $\tilde{B}_k(E)$ *below.*

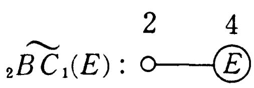


(3) $m=3$ *and* $G(mE+\Delta, E)$ *has no subtree of the type* $\tilde{G}_2(E)$ *or of the type* $\tilde{B}_2(E)$ *below.*





(4) $m\geqq 4$ *and* $G(mE+\Delta, E)$ *has no subtree of type* ${}_2\widetilde{BC}_1(E)$ *below or of type* $\tilde{B}_2(E)$.



REMARK. The above four conditions have the following relation. Let $(n)'$ $(1 \leqq n \leqq 4)$ denote the latter half of the above corresponding condition (n). $(n)'$ is a statement on $G(mE+\Delta, E)$ free from the value of m. Under this notation, if $m \geqq n \geqq 2$ and if $G(mE+\Delta, E)$ is a tree satisfying $(n)'$, then $(n-1)'$ also holds.

THEOREM 0.4. *Let Δ be an effective divisor on X. The following two conditions are equivalent.*

(A) $\dim|\Delta|=0$.

(B) *Every component Δ_i of Δ satisfies $\Delta_i^2=-2$, and the graph $G(\Delta)$ is a tree without a subtree of type $\tilde{D}_k$, $\tilde{E}_6$, $\hat{E}_7$ or $\tilde{E}_8$.*

In Nikulin [4] only Theorem 0.2 was given. In Nikulin [5] we can find one theorem containing all the contents of our three theorems Theorem 0.2, 0.3 and 0.4. Here we divide it into three in order to make the statements more precise. Besides, one notices that the verification in [4] or [5] seems slightly artificial. We give more natural verification in this article.

This article is a result of discussions with Professor V. V. Nikulin. I express very deep thanks to him. Also I thank the referee for his nice comments.

REMARK. Perhaps you can understand the meaning of the names $\check{B}_k(E)$ and $\tilde{G}_2(E)$ of the graphs in Theorem 0.3, because obviously they correspond to the extended Dynkin graphs of type B_k and G_2 respectively. However, some explanation is necessary for ${}_2\widetilde{BC}_1(E)$.

Root systems are finite sets of vectors satisfying certain axioms related to symmetries. We have four infinite series A_k, B_k, C_k, D_k and five exceptional ones E_6, E_7, E_8, F_4, G_2 of irreducible root systems. It is known that another series BC_k of irreducible root systems appears, if we ignore one non-essential axiom called the axiom of the reduced property. (Bourbaki [2].) Also for BC_k we can define the Weyl group, the Dynkin graph and the extended Dynkin graph.

If E has a virtual fractional self-intersection number $E^2=-1/2$, then the graph 1 o——(E) 2 can be regarded as the extended Dynkin graph of type BC_1. (Urabe [7].) Thus when E^2 is arbitrary, this graph can be called $\widetilde{BC}_1(E)$. The above graph in (4) is 2 times this graph. With the lower left subscript indicating the multiplicity 2 the graph in (4) is called ${}_2\widetilde{BC}_1(E)$.

§ 1. The verification.

Our verification of the above three theorems Theorem 0.2-4 is very similar. After giving the exact verification of the most complicated Theorem 0.3, we give comments on the different points in the verification of Theorem 0.2 and Theorem 0.4. We consider Theorem 0.3 below.

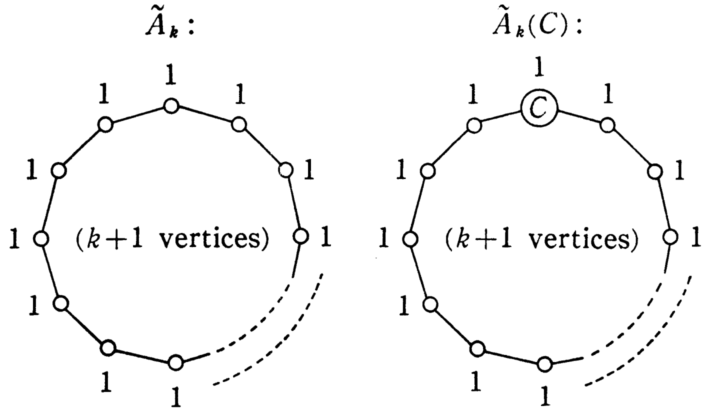
We have very simple verification due to Nikulin for the implication (A)⇒(B). (Nikulin [4], [5].)

First, by Riemann-Roch theorem every component Δ_i of Δ satisfies $\Delta_i^2=-2$ under the condition (A) and is isomorphic to a smooth rational curve.

Next we can check easily the following three facts.

(1) For every two irreducible curves Δ_1 and Δ_2 on X, the sum $\Delta_1+\Delta_2$ is a nef divisor, if $\Delta_1\cdot\Delta_2\geqq 2$.

(2) If $\tilde{\Delta}$ is a divisor on X giving the following graph $\tilde{A}_k$ or $\tilde{A}_k(C)$ with $k\geqq 1$, then $\tilde{\Delta}$ is nef. (In $\tilde{A}_k(C)$ C is an irreducible curve on X with $C^2\geqq 0$.)



(3) If $\tilde{\Delta}$ is a divisor on X giving the graph $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_k(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, $\tilde{B}_k(E)$, $\tilde{G}_2(E)$, or ${}_2\widetilde{BC}_1(E)$, then $\tilde{\Delta}$ is nef. (C is the same as above. E is an irreducible curve with $E^2=0$.)

Under (A) the condition (B) follows easily from Proposition 0.1 and the above facts.

We give our own verification to the implication (B)⇒(A).

Assume that (A) does not hold. We will deduce that (B) does not hold.

First we introduce a partial order $\geqq$ among divisors on X by the following condition.

$$D\geqq D' \Longleftrightarrow D-D' \quad \text{is effective.}$$

(Note that the zero divisor is effective by definition.) Set

$$S=\{\Delta' \mid \Delta\geqq\Delta'\geqq 0,\ |mE+\Delta'|\neq|mE|+\Delta'\}\ .$$

By assumption $\Delta\in S\neq\emptyset$. S is a finite partially ordered set with respect to the order $\geqq$. Let $\bar{\Delta}\in S$ be a minimal element.

LEMMA 1.1. $\bar{\Delta}\neq 0$.

PROOF. Obvious.

LEMMA 1.2. *For any component* Γ *of* $\bar{\Delta}$

$$|mE+\bar{\Delta}-\Gamma| = |mE|+\bar{\Delta}-\Gamma.$$

PROOF. It follows from the minimality.

LEMMA 1.3. $|mE+\bar{\Delta}|$ *has no fixed component.*

PROOF. Assume that an irreducible curve Γ is a fixed component. If $\Gamma=E$, then we have $-2=\Gamma^2=E^2\geqq 0$ a contradiction. Thus Γ is a component of $\bar{\Delta}$. We have

$$\begin{aligned} |mE+\bar{\Delta}| &= |(mE+\bar{\Delta}-\Gamma)+\Gamma| \\ &= |mE+\bar{\Delta}-\Gamma|+\Gamma \qquad (\Gamma \text{ is a fixed component}) \\ &= |mE|+(\bar{\Delta}-\Gamma)+\Gamma \qquad (\text{Lemma 1.2}) \\ &= |mE|+\bar{\Delta} \\ &\neq |mE+\bar{\Delta}| \qquad (\text{By the choice of } \bar{\Delta}), \end{aligned}$$

a contradiction. Q.E.D.

By Lemma 1.3 the line bundle $H=\mathcal{O}_X(mE+\bar{\Delta})$ is nef and by Proposition 0.1 it defines a morphism $\varphi=\varphi_H: X\to \boldsymbol{P}^{\dim|H|}=P$. Here we divide the argument into three cases.

CASE 1. $(mE+\bar{\Delta})\cdot\Gamma_0=0$ for some component Γ_0 of $\bar{\Delta}$ and $(mE+\bar{\Delta})^2=0$.

By the above condition the image $\varphi(\Gamma_0)$ is a point, say $x_0\in P$. Note that by Lemma 1.2 for every component Γ of $\bar{\Delta}-\Gamma_0$, we have $\varphi(\Gamma)=\varphi(\Gamma_0)=\{x_0\}$. Thus $\varphi(\operatorname{Supp}\bar{\Delta})=\{x_0\}$. (Supp D denotes the support of a divisor D.)

On the other hand by Proposition 0.1, φ is decomposed into the composition of $\pi: X\to B$, $B\cong \boldsymbol{P}^1$ and the Veronese embedding $B\hookrightarrow P$ of degree $n=\dim|H|$. π is a morphism associated with an elliptic pencil $|E'|$, and has no multiple fiber. Therefore an equality of divisors $mE+\bar{\Delta}=\pi^*(x_0+x_1+\cdots+x_{n-1})$ holds for some points $x_0, x_1, \cdots, x_{n-1}\in B$. Assume $\pi(E)=\{x_0\}$. E is a component of $\pi^{-1}(x_0)$. Since $\pi^{-1}(x_0)\supset\operatorname{Supp}\bar{\Delta}$, $\pi^{-1}(x_0)$ is reducible. We have $-2=E^2\geqq 0$, a contradiction. We can conclude $E=\pi^*(x_1)$ for some point $x_1\in B$ with $x_1\neq x_0$. We have $\bar{\Delta}=(n-m)\pi^*(x_0)$. Thus the graph $G(\pi^*(x_0))$ is a subgraph of $G(mE+\bar{\Delta}, E)$ and a subgraph of $G(mE+\Delta, E)$. By Kodaira's classification of singular fibers of elliptic surfaces (Kodaira [3]), either $G(\pi^*(x_0))$ is not a tree, or of type $\tilde{D}_k$ or $\tilde{E}_l$ for some $k\geqq 4$, $l=6, 7, 8$. Thus the condition (B) never holds.

CASE 2. $(mE+\bar{\Delta})\cdot\Gamma_0=0$ for some component Γ_0 of $\bar{\Delta}$ and $(mE+\bar{\Delta})^2>0$.

By the same reason as in Case 1, $\varphi(\operatorname{Supp}\bar{\Delta})=\{x_0\}$ for some point $x_0\in P$.

By definition of φ we have a hyperplane $L_0 \subset P$ passing through x_0 with $mE+\bar{\Delta}=\varphi^* L_0$. Thus $(mE+\bar{\Delta})\cdot\Gamma=\varphi^*L_0\cdot\Gamma=0$ for every component Γ of $\bar{\Delta}$. Moreover, $\operatorname{Supp}\bar{\Delta}=\varphi^{-1}(x_0)$ holds, since $\varphi(\operatorname{Supp}(mE+\bar{\Delta}))\not\subset\{x_0\}$.

Let $T=\bigcup\{\Theta \mid \Theta$ is an irreducible curve on X with $\Theta\cdot(mE+\bar{\Delta})=0\}$. Let $\rho: X\to\bar{X}$ be the contraction morphism sending each connected component of T to a rational double point. (Saint-Donat [**6**].) The morphism φ factors through ρ and the induced morphism $\bar{\varphi}: \bar{X}\to P$ is finite.

Let $\mathcal{J}$ be the ideal sheaf of the finite set $\bar{\varphi}^{-1}(x_0)\subset\bar{X}$. $\mathcal{J}$ is a subsheaf of the structure sheaf $\mathcal{O}_{\bar{X}}$ of $\bar{X}$. Note that every point in $\bar{\varphi}^{-1}(x_0)$ is a singular point of $\bar{X}$, since $\varphi^{-1}(x_0)=\rho^{-1}(\bar{\varphi}^{-1}(x_0))=\operatorname{Supp}\bar{\Delta}$ has pure codimension 1. Since $\bar{X}$ has rational double points as singularities, the pull-back $\mathcal{J}\mathcal{O}_X$ of $\mathcal{J}$ on X is locally free and we can write $\mathcal{J}\mathcal{O}_X=\mathcal{O}_X(-\bar{\Delta}')$ for some effective divisor $\bar{\Delta}'$ with $\operatorname{Supp}\bar{\Delta}'=\varphi^{-1}(x_0)$. One knows that $mE+\bar{\Delta}-\bar{\Delta}'$ is effective, since $\bar{\varphi}^*L_0$ passes through $\bar{\varphi}^{-1}(x_0)$. Since $\varphi(E)\not\subset\{x_0\}$, E is not a component of $\bar{\Delta}'$ and $\bar{\Delta}-\bar{\Delta}'$ is effective, too.

Since $\bar{\Delta}'\neq 0$, by the minimality of $\bar{\Delta}$ we have

$$|mE+\bar{\Delta}-\bar{\Delta}'| = |mE|+\bar{\Delta}-\bar{\Delta}'. \qquad *$$

On the other hand it is known that for every component Γ of $\operatorname{Supp}\bar{\Delta}'=\operatorname{Supp}\bar{\Delta}=\varphi^{-1}(x_0)$, the inequality $\Gamma\cdot\bar{\Delta}'\leqq 0$ holds. (Artin [**1**].)

REMARK. Assume that a rational double point $\bar{x}_0\in\bar{\varphi}^{-1}(x_0)$ is of type U. $U=A_k\,(k\geqq 1)$, $D_l(l\geqq 4)$, E_6, E_7 or E_8. Let Δ_U be the connected component of $\bar{\Delta}'$ with $\operatorname{Supp}\Delta_U\subset\rho^{-1}(\bar{x}_0)$. The divisor Δ_U is the maximal ideal divisor of the singularity $(\bar{X}, \bar{x}_0)$. The graph $G(\Delta_U)$ is the Dynkin graph of the type U. Besides, if there is an irreducible curve F such that $F\cdot\Gamma=-\Delta_U\cdot\Gamma$ for every component Γ of Δ_U, then $G(\Delta_U+F)$ coincides with the extended Dynkin graph $\tilde{U}$ (Note the tilde over U.) in Theorem 0.3 or in the beginning of this section.

Thus $(mE+\bar{\Delta}-\bar{\Delta}')\cdot\Gamma=-\bar{\Delta}'\cdot\Gamma\geqq 0$. It implies that $mE+\bar{\Delta}-\bar{\Delta}'$ is nef. One knows that one of the cases (1)-(4) in Proposition 0.1 holds for $H'=\mathcal{O}_X(mE+\bar{\Delta}-\bar{\Delta}')$.

Assume that case (1) or case (2) holds. Then $|mE+\bar{\Delta}-\bar{\Delta}'|$ has no fixed component. By * we have $\bar{\Delta}=\bar{\Delta}'$.

Assume that case (4) holds. By * we can conclude $\bar{\Delta}=\bar{\Delta}'$ also in this case.

Therefore in what follows for the time being we assume that case (3) takes place. By * one knows that $m\geqq 2$, and $\bar{\Delta}-\bar{\Delta}'=\bar{\Gamma}$ for some irreducible curve $\bar{\Gamma}$ with $\bar{\Gamma}^2=-2$ and $\bar{\Gamma}\cdot E=1$.

Assume that $\operatorname{Supp}\bar{\Delta}=\operatorname{Supp}\bar{\Delta}'$ is not connected. We have a component Γ of $\bar{\Delta}$ with $\bar{\Gamma}\cdot\Gamma=0$, $\bar{\Delta}'\cdot\Gamma=-1$ or -2. Thus $0=(mE+\bar{\Delta})\cdot\Gamma=(mE+\bar{\Delta}'+\bar{\Gamma})\cdot\Gamma=mE\cdot\Gamma+\bar{\Delta}'\cdot\Gamma$. Since $m\geqq 2$, we have $m=2$, $E\cdot\Gamma=1$, and $\bar{\Delta}'\cdot\Gamma=-2$. We can see easily $|2E+\bar{\Gamma}+\Gamma|\neq|2E|+\bar{\Gamma}+\Gamma$. Thus $\bar{\Delta}=\bar{\Gamma}+\Gamma$ by the minimality of $\bar{\Delta}$. Then, we have $\bar{\Delta}'=\Gamma$ and $\mathrm{Supp}\,\bar{\Delta}\neq\mathrm{Supp}\,\bar{\Delta}'$, a contradiction. Thus $\mathrm{Supp}\,\bar{\Delta}=\mathrm{Supp}\,\bar{\Delta}'$ is connected.

Since $0=(mE+\bar{\Delta}'+\bar{\Gamma})\cdot\bar{\Gamma}=(m-2)+\bar{\Delta}'\cdot\bar{\Gamma}$ and since $\bar{\Delta}'\cdot\bar{\Gamma}=-2$, -1 or 0, we have three possibilities.

(a) $m=4$, $\bar{\Delta}'\cdot\bar{\Gamma}=-2$.

(b) $m=3$, $\bar{\Delta}'\cdot\bar{\Gamma}=-1$.

(c) $m=2$, $\bar{\Delta}'\cdot\bar{\Gamma}=0$.

In the case (a) $\bar{\Delta}'$ is of type A_1 and $mE+\bar{\Delta}=4E+2\bar{\Gamma}$. This implies that $G(mE+\bar{\Delta}, E)$ is of type ${}_2\widetilde{BC}_1(E)$ and the condition (B) does not hold.

In the case (b) $\bar{\Delta}'$ is not of type A_1. If it is not of type A_2, then we have a component Γ of $\bar{\Delta}'$ with $\Gamma\cdot\bar{\Gamma}=1$, $\bar{\Delta}'\cdot\Gamma=0$. We have $0=(3E+\bar{\Delta}'+\bar{\Gamma})\cdot\Gamma=3E\cdot\Gamma+1$, a contradiction. Thus $\bar{\Delta}'$ is of type A_2 and we can write $mE+\bar{\Delta}=3E+2\bar{\Gamma}+\Gamma$ with $\bar{\Gamma}^2=\Gamma^2=-2$, $E\cdot\bar{\Gamma}=\bar{\Gamma}\cdot\Gamma=1$, and $E\cdot\Gamma=0$. Thus $G(mE+\bar{\Delta}, E)$ is of type $\tilde{G}_2(E)$, and (B) never holds.

We consider the case (c). In this case $\bar{\Delta}'$ is neither of type A_1 nor of type A_2. Assume that a component Γ of $\bar{\Delta}'$ satisfies $\bar{\Delta}'\cdot\Gamma=-1$ and $\Gamma\cdot\bar{\Gamma}=0$. Thus $0=(2E+\bar{\Delta}'+\bar{\Gamma})\cdot\Gamma=2E\cdot\Gamma-1$, a contradiction. It implies that $\bar{\Delta}'$ is not of type A_k for $k\geqq 4$. Moreover, if $\bar{\Delta}'\cdot\Gamma=-1$ for a component Γ of $\bar{\Delta}'$, then $\Gamma\cdot\bar{\Gamma}=1$. Assume conversely that we have a component Γ of $\bar{\Delta}'$ with $\Gamma\cdot\bar{\Gamma}=1$. We have $0=(2E+\bar{\Delta}'+\bar{\Gamma})\cdot\Gamma=2E\cdot\Gamma+\bar{\Delta}'\cdot\Gamma+1$. Thus $\bar{\Delta}'\cdot\Gamma=-1$. In conclusion we have $\bar{\Delta}'\cdot\Gamma=-1\Leftrightarrow\bar{\Gamma}\cdot\Gamma=1$ for a component Γ of $\bar{\Delta}'$. By the classification of rational double points this implies that $\bar{\Delta}'$ is of type D_k for some $k\geqq 4$ or of type A_3. $\bar{\Gamma}$ corresponds to the end of the longest arm of the graph D_k, if $\bar{\Delta}'$ is of type D_k. $\bar{\Gamma}$ corresponds to the middle vertex of the graph A_3, if $\bar{\Delta}'$ is of type A_3. It implies that $G(mE+\bar{\Delta}, E)$ is of type $\tilde{B}_k(E)$ for some $k\geqq 3$. The condition (B) never holds. Here we conclude the case where the case (3) in Proposition 0.1 takes place for H'.

We can continue the verification assuming $\bar{\Delta}=\bar{\Delta}'$. Here note that

$$mE\cdot\Gamma=-\bar{\Delta}\cdot\Gamma=-\bar{\Delta}'\cdot\Gamma \qquad **$$

for every component Γ of $\varphi^{-1}(x_0)$. We have three cases.

(a) A point in $\bar{\varphi}^{-1}(x_0)$ is of type D_k or E_l for some k, l.

(b) A point in $\bar{\varphi}^{-1}(x_0)$ is of type A_k with $k\geqq 2$.

(c) Every point in $\bar{\varphi}^{-1}(x_0)$ is of type A_1.

In the case (a) in ** $-\bar{\Delta}'\cdot\Gamma=1$ for some component Γ. Thus $m=1$. By ** one knows that $G(mE+\Delta, E)$ has a subgraph of type $\tilde{D}_k(E)$ or $\tilde{E}_l(E)$, and the condition (B) does not hold.

In the case (b) by the same reason as above we know $m=1$. By ** $G(mE+\Delta, E)$ has a subgraph of type $\tilde{A}_k(E)$ with $k\geqq 2$. (B) does not hold.

In the third case (c) $mE\cdot\Gamma=2$ for every component Γ of $\varphi^{-1}(x_0)$. Thus $m=1$ or 2. If $m=1$, then $E\cdot\Gamma=2$ and $G(mE+\Delta, E)$ is not a tree. (B) does not hold. Assume $m=2$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be all the components of $\varphi^{-1}(x_0)$. We have $mE+\bar{\Delta}=2E+\Gamma_1+\Gamma_2+\cdots+\Gamma_n$, $E\cdot\Gamma_i=1$ for $1\leqq i\leqq n$, and $\Gamma_i\cdot\Gamma_j=0$ for $i\neq j$. Since $|mE+\bar{\Delta}|\neq|mE|+\bar{\Delta}$, we have $n\geqq 2$. By the minimality of $\bar{\Delta}$, one knows $n=2$. Thus $G(mE+\bar{\Delta}, E)$ is of type $\tilde{B}_2(E)$, and the condition (B) never holds in this case, either.

CASE 3. $(mE+\bar{\Delta})\cdot\Gamma>0$ for every component Γ of $\bar{\Delta}$.

In this case assuming moreover that $G(mE+\Delta, E)$ is a tree, we will show that either the following (1) or (2) holds.

(1) $m\geqq 3$ and $G(mE+\Delta, E)$ contains a subgraph $\tilde{B}_2(E)$.

(2) $m\geqq 5$ and $G(mE+\Delta, E)$ contains a subgraph ${}_2\widetilde{BC}_1(E)$.

If this can be shown, we can complete the verification of Theorem 1.3.

First note that $E\cdot\Gamma=0$ or 1 and $\Gamma\cdot\Gamma'=0$ or 1 for arbitrary two components Γ, Γ' of $\bar{\Delta}$ by the assumption of a tree.

LEMMA 1.4. *$\Delta'^2\leqq -2$ for any non-zero effective divisor Δ' on X with* $\dim|\Delta'|=0$.

PROOF. By Riemann-Roch $1=h^0(\mathcal{O}_X(\Delta'))+h^0(\mathcal{O}_X(-\Delta'))\geqq(\Delta'^2/2)+2$. Q.E.D.

Assume that $\bar{\Delta}$ is not connected. Let Δ' be an arbitrary connected component of $\bar{\Delta}$. By the minimality of $\bar{\Delta}$, $|mE+\Delta'|=|mE|+\Delta'$. Thus $\dim|\Delta'|=0$ and $\Delta'^2\leqq -2$ by Lemma 1.4. We have two cases.

$$\text{(a)}\quad m\geqq 2 \qquad \text{(b)}\quad m=1.$$

First we consider the case (a). Assume moreover that Δ' is disjoint from E. Then, we have $\Delta'\cdot\Gamma=\bar{\Delta}\cdot\Gamma=(mE+\bar{\Delta})\cdot\Gamma>0$ for every component Γ of Δ'. Let $\Delta'=\sum l_i\Gamma_i$ be the irreducible decomposition. l_i's are positive integers. We have $-2\geqq\Delta'^2=\sum l_i\Gamma_i\cdot\Delta'\geqq 0$, a contradiction. Thus every connected component of $\bar{\Delta}$ intersects E. Since $\bar{\Delta}$ is not connected, we have two components Γ_1 and Γ_2 of $\bar{\Delta}$ with $\Gamma_1\cdot E=\Gamma_2\cdot E=1$, and $\Gamma_1\cdot\Gamma_2=0$. Since $m\geqq 2$, we can check easily $|mE+\Gamma_1+\Gamma_2|\neq|mE|+\Gamma_1+\Gamma_2$ and $|mE+\Gamma_i|=|mE|+\Gamma_i$ for $i=1, 2$. By the minimality we have $\bar{\Delta}=\Gamma_1+\Gamma_2$. Since $0<(mE+\Gamma_1+\Gamma_2)\cdot\Gamma_1=m-2$, we have $m\geqq 3$. The above (1) holds.

We proceed to the case (b). In this case $\Delta'\cdot\Gamma=\bar{\Delta}\cdot\Gamma=(E+\bar{\Delta})\cdot\Gamma-E\cdot\Gamma\geqq 0$ for every component Γ of Δ', since $(E+\bar{\Delta})\cdot\Gamma=(mE+\bar{\Delta})\cdot\Gamma\geqq 1$ and $E\cdot\Gamma\leqq 1$. Writing $\Delta'=\sum l_i\Gamma_i$, we have $-2\geqq\Delta'^2=\sum l_i\Gamma_i\cdot\Delta'\geqq 0$, a contradiction.

We can assume that $\bar{\Delta}$ is connected in the sequel. If for two different components Γ_1 and Γ_2 of $\bar{\Delta}$, $E\cdot\Gamma_1>0$ and $E\cdot\Gamma_2>0$, then $G(mE+\bar{\Delta}, E)$ contains a cycle, because $\bar{\Delta}$ is connected. Thus $G(mE+\Delta, E)$ is not a tree, contradicting the assumption. A component Γ of $\bar{\Delta}$ with $E\cdot\Gamma>0$ is unique, if it exists. Let $\bar{\Delta}=\sum_{i=1}^{r} l_i\Gamma_i$ be the irreducible decomposition. We can assume moreover $E\cdot\Gamma_i=0$ for $i\geqq 2$.

If $\bar{\Delta}=\Gamma_1$, then we have $|mE+\bar{\Delta}|=|mE|+\bar{\Delta}$, which contradicts the definition of $\bar{\Delta}$. Thus $\bar{\Delta}-\Gamma_1\neq 0$. Note that $\bar{\Delta}\cdot\Gamma_i=(mE+\bar{\Delta})\cdot\Gamma_i\geqq 1$ for $i\geqq 2$ and $\bar{\Delta}\cdot\Gamma_1=(mE+\bar{\Delta})\cdot\Gamma_1-mE\cdot\Gamma_1\geqq 1-m$. Thus $\Gamma_i\cdot(\bar{\Delta}-\Gamma_1)=\Gamma_i\cdot\bar{\Delta}-\Gamma_i\cdot\Gamma_1\geqq 0$ for $i\geqq 2$, and $\Gamma_1\cdot(\bar{\Delta}-\Gamma_1)\geqq 1-m+2=3-m$. On the other hand by the minimality we have $|mE+\bar{\Delta}-\Gamma_1|=|mE|+\bar{\Delta}-\Gamma_1$. Thus $\dim|\bar{\Delta}-\Gamma_1|=0$ and $(\bar{\Delta}-\Gamma_1)^2\leqq -2$ by Lemma 1.4. We have

$$-2\geqq(\bar{\Delta}-\Gamma_1)^2=(l_1-1)\Gamma_1\cdot(\bar{\Delta}-\Gamma_1)+\sum_{i=2}^{r} l_i\Gamma_i\cdot(\bar{\Delta}-\Gamma_1)\geqq(l_1-1)(3-m).$$

Thus $l_1\geqq 2$ and $m\geqq 4$. If $E\cdot\Gamma_1=0$, a stronger estimate $-2\geqq 3(l_1-1)$ is obtained. It is a contradiction. Thus $E\cdot\Gamma_1=1$. Since $|mE+2\Gamma_1|\neq|mE|+2\Gamma_1$ and since $|mE+\Gamma_1|=|mE|+\Gamma_1$, one has $\bar{\Delta}=2\Gamma_1$ by the minimality of $\bar{\Delta}$. We have $1\leqq(mE+\bar{\Delta})\cdot\Gamma_1=m-4$. This implies that the above (2) holds. We complete the verification of Theorem 0.3.

Next we would like to consider Theorem 0.2. The implication (A)⇒(B) is due to Nikulin and can be shown by the same argument as in Theorem 0.3. (B)⇒(A) can be shown by replacing E by C and setting $m=1$ in the above verification (B)⇒(A) of Theorem 0.3. The case of Theorem 0.2 is rather easier. First, we can check that Case 1 above never takes place for Theorem 0.2. In Case 2 the subcase corresponding to (3) in Proposition 0.1 never takes place, either. The last part of Case 2 discussing the case of $m=2$ is not necessary, since we can assume $m=1$. In Case 3 the subsubcase (a) in the subcase where $\bar{\Delta}$ is not connected is not necessary. The subcase where $\bar{\Delta}$ is connected in Case 3 becomes simpler, since $m=1$.

As for Theorem 0.4, (A)⇒(B) follows from Nikulin's same method as above. Setting $E=0$ and $m=0$ in the above argument of (B)⇒(A) of Theorem 0.3, we obtain the verification (B)⇒(A) of Theorem 0.4.

References

[1] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math., **88** (1966), 129-136.
[2] N. Bourbaki, Groupes et algèbre de Lie, Chaps. 4-6, Paris, Hermann, 1968.
[3] K. Kodaira, On compact analytic surfaces II, Ann. of Math., 77 (1963), 563-626.
[4] V. V. Nikulin, Linear systems on singular K3 surfaces (Report in August 1990).
[5] V. V. Nikulin, Weil linear systems on singular K3 surfaces, Max Planck Institute Preprint, **90-92** (1990).
[6] B. Saint-Donat, Projective models of K3 surfaces, Amer. J. Math., **96** (1974), 602-639.

[7] T. Urabe, Dynkin graphs and quadrilateral singularities, preprint (1990).

Tohsuke URABE
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa, Hachioji
Tokyo 192-03
Japan
(e-mail: urabe@math. metro-u. ac. jp)