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On the decomposition of conformally flat manifolds

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1. Introduction.

Let M be a smooth n-manifold and C a conformal class on M. (M, C) is *conformally flat* if for any point p of M, there exists a metric g contained in C such that g is flat on some neighborhood of p. A conformal class C is called a *flat conformal structure* if (M, C) is conformally flat. A manifold M is said to be *conformally flat* if M admits a flat conformal structure. In this paper, we always assume a manifold M to be smooth, compact and connected with $\dim M=n\geqq 3$, unless otherwise stated. For an orientable manifold M, we also assume that M is oriented.

DEFINITION 1.1. An n-manifold M is said to be *nontrivial* if M is not diffeomorphic to the standard n-sphere S^n. And M is *C-prime* if

(1) M is non-trivial and conformally flat, and

(2) there is no decomposition $M=M_1\# M_2$ (a connected sum of M_1 and M_2), where each of M_1 and M_2 has the property (1).

A well-known theorem of Kulkarni [12] states that a connected sum of conformally flat manifolds is also conformally flat. Thus, connected sums of C-prime manifolds are conformally flat. On the other hand, a simple observation gives the following proposition.

PROPOSITION 2.1. *Every non-trivial conformally flat manifold is diffeomorphic to a connected sum of a finite number of C-prime manifolds.*

Thus the classification problem of conformally flat manifolds is reduced to the classification of C-prime manifolds. A decomposition $M=P_1\#\cdots\# P_k$, where each P_i is C-prime, is called a *C-prime decomposition* of M in this paper.

The purpose of this paper is to show several results concerning the C-prime decomposition of conformally flat manifolds. In section 2 we prove Proposition 2.1 above and some sufficient conditions for a manifold to be C-prime. We also discuss the Yamabe invariant $\mu(M, C)$ (see Definition 2.4) of a conformally flat manifold (M, C). And we see that, for some M, there exists a sequence of flat conformal structures on M, which maximizes the Yamabe invariant, such that the limit of this sequence gives a decomposition of M.

In section 3 and section 4, we devote our attention to a certain class of conformally flat manifolds. Let Ω be an open subset of S^n, and Γ a discrete subgroup of the conformal transformation group Conf(S^n, C_0) of (S^n, C_0), where C_0 denotes the conformal class containing the standard metric on S^n. If Γ leaves Ω invariant and acts freely and properly discontinuously on Ω, then the quotient space Ω/Γ is Hausdorff. In the case Ω/Γ is a manifold, Ω/Γ has a natural conformal structure and it is conformally flat. A conformally flat manifold (M, C) is called *Kleinian* if (M, C) is conformal to Ω/Γ for some Ω and Γ. A conformal class C on M is called a *Kleinian structure* if (M, C) is Kleinian. And a manifold M is said to be *Kleinian* if M admits a Kleinian structure. Another theorem of Kulkarni and Pinkall [**13**] says that a connected sum of Kleinian manifolds is also Kleinian. The following theorem says that the converse of this theorem is true in a weak sense.

THEOREM 3.2. *Let M be Kleinian. Suppose M is diffeomorphic to a connected sum $M_1\#M_2$, where M_1 and M_2 are not necessarily conformally flat. Then there exists Kleinian manifolds M_i' $(i=1, 2)$ such that M_i' is homeomorphic to M_i $(i=1, 2)$ and $M=M_1'\#M_2'$.*

We cannot expect M_i' to be diffeomorphic to M_i. Because an exotic n-sphere Σ^n $(n\geqq 7)$ satisfies $\Sigma^n\#(-\Sigma^n)=S^n$ but Σ^n does not admit a flat conformal structure, where $-\Sigma^n$ is Σ^n with the opposite orientation. As a corollary, we see that if a Kleinian manifold M is C-prime, then M is topologically prime (see Definition 3.3 and Corollary 3.4). In section 4, we discuss the Yamabe invariant $\mu(M, C)$ of a Kleinian manifold (M, C). Main results are the following.

THEOREM 4.3. *Let M be an oriented Kleinian 3-manifold. And let $P_1\#\cdots\#P_k$ be the C-prime decomposition of M, if M is non-trivial. Then M admits a Kleinian structure with positive Yamabe invariant if and only if M is diffeomorphic to S^3 or each P_i is diffeomorphic to either a spherical space form or $S^1\times S^2$.*

THEOREM 4.6. *Suppose M admits a Kleinian structure C with non-negative Yamabe invariant, and is diffeomorphic to a connected sum $M_1\#M_2$, where M_1 and M_2 are not necessarily conformally flat. Then there exists M_i' $(i=1, 2)$ as in Theorem 3.2 and each M_i' admits a Kleinian structure with non-negative Yamabe invariant.*

In [**19**], Schoen and Yau proved that if (M, C) is conformally flat with positive Yamabe invariant, then the developing map of (M, C) is injective, and therefore (M, C) is Kleinian. Thus, Theorem 4.3 gives the classification of oriented 3-manifolds admitting a flat conformal structure with positive Yamabe invariant.

2. The C-prime decomposition of conformally flat manifolds.

In this section, manifolds under consideration are assumed to be conformally flat.

PROPOSITION 2.1. *Every non-trivial manifold M is diffeomorphic to a connected sum $P_1\#\cdots\#P_k$ of C-prime manifolds.*

PROOF. If M is not C-prime, then there exist nontrivial manifolds M_1 and M_2 such that $M=M_1\#M_2$. If either M_1 or M_2 is not C-prime, then we can decompose it again. All we have to do is to show this process stops in a finite number of steps. We denote by $d(M)$ the smallest number of generators of the fundamental group $\pi_1(M)$ of M. Then, by the van Kampen theorem and the Grushko-Neumann theorem, $d(M_1\#M_2)=d(M_1)+d(M_2)$ holds. Thus, if $M=M_1\#\cdots\#M_k$ with $d(M)<k$, then $d(M_i)=0$ for some i. By a theorem of Kuiper [**11**] M_i must be diffeomorphic to S^n. This completes the proof. q.e.d.

REMARK. In [**16**], Milnor pointed out that if the Poincaré conjecture is true, then the proof of the topological prime decomposition theorem for 3-manifolds becomes easier. The proof of Proposition 2.1 is exactly the same as the proof of the topological prime decomposition theorem for 3-manifolds, which was suggested by Milnor, if we replace the phrase "by a theorem of Kuiper" with "if the Poincaré conjecture is true".

By the proof of Proposition 2.1, we obtain the following sufficient condition for M to be C-prime.

COROLLARY 2.2. *If $d(M)=1$, then M is C-prime.*

Another sufficient condition is given by

PROPOSITION 2.3. *If the universal covering space $\tilde{M}$ of M is diffeomorphic to $\boldsymbol{R}^n$, then M is C-prime.*

PROOF. Suppose M is not C-prime. Then there exist non-trivial manifolds M_1 and M_2 such that $M=M_1\#M_2$. So we can take a subset S of M, which is an embedded S^{n-1}, such that $M\setminus S$ has two connected components, say L_1 and L_2, where L_i is diffeomorphic to $M_i\setminus(n\text{-disk})$ $(i=1, 2)$. We can also take a subset A of M, which is diffeomorphic to $(-1, 1)\times S^{n-1}$ and $\{0\}\times S^{n-1}$ corresponds to S. Since $\tilde{M}$ is diffeomorphic to $\boldsymbol{R}^n$, a lift $\tilde{S}$ of S separates $\tilde{M}$ into two connected components F_1 and F_2, where $F_1\cup\tilde{S}$ is compact and $F_2\cup\tilde{S}$ is non-compact. Let $\pi:\tilde{M}\to M$ be the covering projection. If $\pi(F_1)\cap L_1\neq\emptyset$ and $\pi(F_1)\cap L_2\neq\emptyset$, then we can take a lift $\tilde{S}'$ of S, which is contained in F_1. Let us denote two connected components of $\tilde{M}\setminus\tilde{S}'$ by F_1' and F_2', where F_1' is contained in F_1. If $\pi(F_1')\cap L_1\neq\emptyset$ and $\pi(F_1')\cap L_2\neq\emptyset$, then again we can take a lift $\tilde{S}''$ of S, where $\tilde{S}''$ is contained in F_1'. Since $F_1\cup\tilde{S}$ is compact, this process stops in a finite number of steps, and we can take a lift $\tilde{S}_0$ of S so that $\pi(F)$ is contained in either L_1 or L_2, where F is a bounded connected component of $\tilde{M}\setminus\tilde{S}_0$. It is easy to see that $\pi|F$ is a covering map onto either L_1 or L_2, where $\pi|F$ denotes the restriction of π to F. Moreover, it is injective, since π is injective on $\tilde{S}_0$. Thus either L_1 or L_2 is diffeomorphic to F. Since there exists a lift $\tilde{A}_0$ of A, which contains $\tilde{S}_0$, we see that F is homeomorphic to an open n-disk by the generalized Schoenflies theorem (see for example [3]). Hence either M_1 or M_2 is homeomorphic to S^n. By Kuiper's theorem [**11**], either M_1 or M_2 is diffeomorphic to S^n. This contradicts our assumption that M_1 and M_2 are non-trivial. q.e.d.

By Corollary 2.2 and Proposition 2.3, we see that lens spaces, $S^1\times S^{n-1}$, flat manifolds, hyperbolic manifolds and products of S^1 and hyperbolic manifolds are C-prime.

Next we discuss the Yamabe invariant $\mu(M, C)$ of (M, C), which is a conformal invariant concerning scalar curvature. Though we are considering only conformally flat manifolds in this section, manifolds and conformal classes in Definition 2.4, Fact 2.5, Fact 2.6 and Fact 2.7 are not necessarily conformally flat.

DEFINITION 2.4. The functional $I\colon C\to\boldsymbol{R}$, which is defined by

$$I(g)=\frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{(n-2)/n}},$$

is called the *Yamabe functional*, where R_g and dV_g denote the scalar curvature and the volume element of a metric g, respectively. And

$$\mu(M, C)=\inf_{g\in C} I(g)$$

is called the *Yamabe invariant* of (M, C).

The well-known Yamabe problem (see for example [**10**] or [**15**]) asked that, for any (M, C), whether there exists a metric g contained in C such that g satisfies $I(g)=\mu(M, C)$. This problem was answered affirmatively by the works of Yamabe [**21**], Trudinger [**20**], Aubin [**1**] and Schoen [**17**]. Some of the basic known facts on the Yamabe problem are the following.

FACT 2.5. *For any (M, C), there exists a metric g contained in C such that g satisfies $I(g)=\mu(M, C)$. And the scalar curvature of g is constant.*

FACT 2.6. *For any (M, C), $\mu(M, C)\leqq\mu(S^n, C_0)=I(g_0)$ holds, where g_0 denotes the standard metric of S^n, and the equality occurs if and only if (M, C) is conformal to (S^n, C_0).*

FACT 2.7. *A conformal class C on M contains a metric with positive (resp. zero, resp. negative) scalar curvature if and only if the Yamabe invariant $\mu(M, C)$ is positive (resp. zero, resp. negative).*

DEFINITION 2.8. An invariant $\mu_C(M)$ of M is defined by $\mu_C(M)=\sup\mu(M, C)$, where the supremum is taken over all flat conformal structures on M.

This invariant $\mu_C(M)$ is well-defined since $\mu(M, C)\leqq\mu(S^n, C_0)$ holds for any (M, C). Note that $\mu_C(M)$ is positive if and only if there exists a flat conformal structure C on M such that $\mu(M, C)$ is positive.

EXAMPLES. (1) If the fundamental group $\pi_1(M)$ of M is finite, then M is diffeomorphic to a spherical space form and $\mu_C(M)=\mu(M, C_0')=|\pi_1(M)|^{-2/n}\mu(S^n, C_0)$, where C_0' denotes the conformal class containing the constant curvature metric on M and $|\pi_1(M)|$ denotes the order of $\pi_1(M)$.

(2) $\mu_C(S^1\times S^{n-1})=\mu(S^n, C_0)$ (see the remark following the proof of Theorem 2.9).

(3) If M admits a flat metric, then $\mu_C(M)=0$. And $\mu(M, C)=\mu_C(M)$ holds if and only if C contains a flat metric (this follows from [**5**, Corollary C]).

(4) If a 4-manifold M admits a metric g with negative constant curvature, then $\mu_C(M)=\mu(M, C_0)$, where C_0 denotes the conformal class containing g. And $\mu(M, C)=\mu_C(M)$ holds if and only if C is conformal to C_0 (see [**7**] and [**8**]).

(5) $\mu_C(S^1\times N^{n-1})=\lim_{r\to 0}\mu(S^1\times N^{n-1}, C_r)=0$, where N^{n-1} is an $(n-1)$-manifold admitting a negative constant curvature metric and C_r denotes the conformal class containing the product of the metric of S^1 with radius r and the metric with constant curvature -1 on N^{n-1} (this follows from [**5**, Corollary C]). Note that if $r=1/k$ with k positive integer, then C_r is *a* Kleinian structure.

(6) $\mu_C(S^m\times N^m)=0$, where N^m is as in (5). In this case, $\mu_C(S^m\times N^m)=\mu(S^m\times N^m, C)$ holds if and only if C contains the product of the metric with constant curvature 1 on S^m and the metric with constant curvature -1 on N^m (see [**14**]).

A certain modification of [**9**, Theorem 2] and [**9**, Corollary 1.11] gives the following theorem. We denote by $M_1\amalg M_2$ the disjoint union of M_1 and M_2. Note that the Yamabe invariant can be defined for a compact and disconnected manifold (see [**9**, Lemma 1.10]), though Fact 2.5, Fact 2.6 and Fact 2.7 turn out to be false.

THEOREM 2.9. (1) *If* $\mu_C(M_1)\leqq 0$ *and* $\mu_C(M_2)\leqq 0$, *then* $\mu_C(M_1\amalg M_2)=-(|\mu_C(M_1)|^{n/2}+|\mu_C(M_2)|^{n/2})^{2/n}$.

(2) *Otherwise,* $\mu_C(M_1\amalg M_2)=\min\{\mu_C(M_1), \mu_C(M_2)\}$.

(3) $\mu_C(M_1\# M_2)\geqq \mu_C(M_1\amalg M_2)$.

(4) *Suppose the equality in* (3) *holds for* M_1 *and* M_2 *and suppose that there exists a flat conformal structure* C_0 *on* $M_1\amalg M_2$ *such that* $\mu(M_1\amalg M_2, C_0)=\mu_C(M_1\amalg M_2)$. *Then there exists a sequence* $\{C_\varepsilon'\}$ *of flat conformal structures on* $M_1\# M_2$, *which satisfies* $\lim_{\varepsilon\to 0}\mu(M_1\# M_2, C_\varepsilon')=\mu_C(M_1\# M_2)$, *such that a suitable choice of a metric* g_ε' *contained in* C_ε' *gives a sequence* $\{g_\varepsilon'\}$ *satisfying*

$$\lim_{\varepsilon\to 0}(M_1\# M_2, g_\varepsilon')=(M_1\setminus\{p_1\}, g|M_1\setminus\{p_1\})\bigcup_{p_1=p_2}(M_2\setminus\{p_2\}, g|M_2\setminus\{p_2\}) \tag{2.10}$$

for some metric g *contained in* C_0 *and for some point* p_i *of* M_i $(i=1, 2)$. *The union in the right hand side of* (2.10) *is given by the formal identification of* p_1 *and* p_2.

PROOF. (1) and (2) are just the same as [**9**, Corollary 1.11].

(3) Let M be a compact but not necessarily connected manifold, and p_1 and p_2 two distinct points of M. Remove two disks around p_1 and p_2 and attach $I\times S^{n-1}$ by identifying each boundary, where I denotes a closed interval of $\boldsymbol{R}$. Then we obtain a compact manifold M'. If $M=M_1\amalg M_2$ and p_α is a point of M_α $(\alpha=1, 2)$, then $M'=M_1\# M_2$.

For any positive real number ρ, there exists a flat conformal structure C on M such that $\mu(M, C)+\rho>\mu_C(M)$. Since C is a flat conformal structure, there exists a metric g contained in C such that g is flat on some neighborhood of each p_α. With respect to the normal coordinates $(x_\alpha{}^1, \cdots, x_\alpha{}^n)$ around p_α, for some positive real number δ, g can be written as $g=\delta_{ij}dx_\alpha{}^i dx_\alpha{}^j$ for $x_\alpha=(x_\alpha{}^1, \cdots, x_\alpha{}^n)$ with $|x_\alpha|=(\sum|x_\alpha{}^i|^2)^{1/2}\leqq 2\delta$. Take a smooth function $0\leqq w_\delta\leqq 1$ defined on $\boldsymbol{R}$ as $w_\delta(r)=0$ if $|r|\geqq\delta$, and $w_\delta(r)=1$ if $|r|\leqq\delta_0$ for $\delta_0<\delta$, and define a metric g_ε on $M\setminus\{p_1, p_2\}$ by

$$g_\varepsilon=\exp\{\log(\varepsilon^2|x_\alpha|^{-2})w_\delta(\varepsilon^{-1}|x_\alpha|)\}g$$

for $0<\varepsilon\leqq 1$. Then, around p_α, g_ε can be written as

$$g_\varepsilon=\begin{cases}\varepsilon^2|x_\alpha|^{-2}\delta_{ij}dx_\alpha{}^i dx_\alpha{}^j & \text{if } |x_\alpha|\leqq\varepsilon\delta_0\\ \delta_{ij}dx_\alpha{}^i dx_\alpha{}^j & \text{if } \varepsilon\delta\leqq|x_\alpha|\leqq 2\delta\end{cases}$$

and on $M\setminus\{B(p_1, 2\delta)\cup B(p_2, 2\delta)\}$, g_ε coincides with g, where $B(p_\alpha, 2\delta)$ denotes the set of all points with $|x_\alpha|\leqq 2\delta$. With respect to new coordinates $(y_\alpha{}^1, \cdots, y_\alpha{}^n)$, where $y_\alpha{}^i=\varepsilon^{-1}x_\alpha{}^i$, g_ε is written as

$$g_\varepsilon=\varepsilon^2\exp\{\log(|y_\alpha|^{-2})w_\delta(|y_\alpha|)\}\delta_{ij}dy_\alpha{}^i dy_\alpha{}^j$$

for $|y_\alpha| \leqq 2\delta\varepsilon^{-1}$. In particular,

$$g_\varepsilon = \begin{cases} \varepsilon^2 |y_\alpha|^{-2} \delta_{ij} dy_\alpha^i dy_\alpha^j & \text{if } |y_\alpha| \leqq \delta_0 \\ \varepsilon^2 \delta_{ij} dy_\alpha^i dy_\alpha^j & \text{if } \delta \leqq |y_\alpha| \leqq 2\delta\varepsilon^{-1} \end{cases}$$

holds. Note that $(B(p_\alpha, \varepsilon\delta_0)\setminus\{p_\alpha\}, g_\varepsilon | B(p_\alpha, \varepsilon\delta_0)\setminus\{p_\alpha\})$ is isometric to a half infinite cylinder $[0, \infty)\times S^{n-1}(\varepsilon)$ of radius ε. Put $\delta' = \exp(\log\delta_0 - \varepsilon^{-4}) < \delta_0$, and let $N = M\setminus\{B(p_1, \varepsilon\delta')\cup B(p_2, \varepsilon\delta')\}$. Identifying two boundary components of $(N, g_\varepsilon | N)$ by an isometry, we obtain a conformally flat Riemannian manifold (M', g_ε'). It is easy to see that (M', g_ε') contains a subset isometric to $(0, 2\varepsilon^{-3})\times S^{n-1}(\varepsilon)$. Then $\mu_C(M')\geqq\mu_C(M)$ follows from a slight modification of the proof of [**9**, Theorem 2].

(4) Take C_0 as C in the proof of (3). Then, it is clear that g_ε' in the proof of (3) satisfies (2.10). And flat conformal structures C_ε' containing g_ε' satisfy $\lim_{\varepsilon\to 0}\mu(M', C_\varepsilon') = \mu(M', C_0) = \mu_C(M')$ by the proof of (3). q.e.d.

REMARK. If both $\mu_C(M_1)$ and $\mu_C(M_2)$ are positive, then $\mu_C(M_1 \# M_2)$ is also positive by Theorem 2.9. That is, a connected sum of two manifolds admitting a conformally flat metric with positive scalar curvature also admits a conformally flat metric with positive scalar curvature. This fact was proved by Schoen and Yau ([**18**, Corollary 5], see also [**9**]).

REMARK. If we put $M = S^n$, then M' is an S^{n-1} bundle over S^1. Since we have seen that $\mu_C(M')\geqq\mu_C(M)$ holds and since, by Fact 2.6, $\mu_C(M)\leqq\mu(S^n, C_0) = \mu_C(S^n)$ holds for any M, we get (2) in the examples following Definition 2.8. See also the remark following [**9**, Lemma 6.2].

REMARK. Theorem 2.9 suggests that it may be possible to get a C-prime decomposition of M as the limit of a suitable sequence of flat conformal structures on M, which maximizes the Yamabe invariant.

Known examples satisfying the assumption of (4) are, for instance, the following.

(a) $M_1 = T^n$ and $M_2 = T^n$.

(b) $M_1 = S^m \times N^m$, where N^m is an m-manifold admitting a negative constant curvature metric and M_2 is T^{2m}, $S^m\times N^m$ or $2m$-manifold with $\mu_C(M_2) > 0$.

(c) $M_1 = T^{2m}$ and M_2 is a $2m$-manifold with $\mu_C(M_2) > 0$.

By [**2**, Corollary 8.8], [**5**] and (3) of Theorem 2.9, $\mu_C(M_1 \# M_2) = 0 = \mu_C(M_1 \amalg M_2)$ for these manifolds. And $\mu_C(M_1\amalg M_2) = \mu(M_1 \amalg M_2, C)$ holds for C satisfying $\mu(M_1, C|M_1) = \mu_C(M_1)$ and $\mu(M_2, C|M_2) > 0$ (if $\mu_C(M_2) > 0$) or $\mu(M_2, C|M_2) = \mu_C(M_2)$ (if $M_2 = T^n$, $S^m\times N^m$) by [**9**, Lemma 1.10].

3. The Kleinian case.

Let M be a Kleinian manifold. Then there exists an open subset Ω of S^n and a regular covering $\pi: \Omega \to M$, where its deck transformation is an element of $\mathrm{Conf}(S^n, C_0)$. On the other hand, if there exists a conformal structure C on M and a regular conformal covering $\pi: \Omega \to (M, C)$ for some Ω, then each element of the deck transformation group Γ of this covering can be uniquely extended to an element of $\mathrm{Conf}(S^n, C_0)$ by Liouville's theorem, and hence Γ is a discrete subgroup of $\mathrm{Conf}(S^n, C_0)$. That is, C is a Kleinian structure. Since we assume M to be connected and hence we can choose Ω to be connected, we assume that Ω is connected in the rest of this paper.

PROPOSITION 3.1. *Let M be a non-trivial Kleinian manifold. If the homotopy group $\pi_{n-1}(M)$ is trivial, then M is C-prime.*

PROOF. Suppose M is not C-prime. Then there exist non-trivial conformally flat manifolds M_1 and M_2 such that $M = M_1 \# M_2$. So we can take L_1, L_2, S and A as in the proof of Proposition 2.3. Since M is Kleinian, there exists a connected open subset Ω of S^n and a regular covering $\pi: \Omega \to M$. Take a lift $\tilde{S}$ of S and fix it. Then $\tilde{S}$ separates S^n $(\supset \Omega)$ into two open disks D_1 and D_2 by the generalized Schoenflies theorem. If Ω contains neither D_1 nor D_2, in other words, there exist points p_i contained in $D_i \setminus (D_i \cap \Omega)$ $(i=1, 2)$, then $\tilde{S}$ represents a non-trivial element of the homotopy group $\pi_{n-1}(\Omega)$. This contradicts our assumption $\pi_{n-1}(M)=0$. Thus, we may assume $\Omega \setminus \tilde{S}$ contains D_1. Hence, $\Omega \setminus \tilde{S} = (\Omega \cap D_1) \amalg (\Omega \cap D_2) = D_1 \amalg (\Omega \cap D_2)$, where $D_1 \amalg (\Omega \cap D_2)$ denotes the disjoint union of D_1 and $\Omega \cap D_2$. Since $D_1 \cup \tilde{S}$ is compact, we can derive a contradiction as in the proof of Proposition 2.3. q.e.d.

THEOREM 3.2. *Let M be Kleinian. Suppose M is diffeomorphic to a connected sum $M_1 \# M_2$, where M_1 and M_2 are not necessarily conformally flat. Then there exist Kleinian manifolds M_1' and M_2' such that M_i' is homeomorphic to M_i $(i=1, 2)$ and $M = M_1' \# M_2'$.*

PROOF. Let L_1, L_2, A and S be as in the proof of Proposition 2.3, and $\pi: \Omega \to M$ be a regular covering map, which induces a Kleinian structure on M. Take a lift $\tilde{A}$ of A, and let $\tilde{S}$ be a subset of $\tilde{A}$, which corresponds to S. Define A_+ and A_- by $A_+ = L_2 \cap A$ and $A_- = L_1 \cap A$, respectively. Let $\tilde{A}_+$ and $\tilde{A}_-$ be subsets of $\tilde{A}$, where $\tilde{A}_+$ and $\tilde{A}_-$ correspond to A_+ and A_-, respectively. By the generalized Schoenflies theorem, $S^n \setminus \tilde{S}$ has two connected components and both are homeomorphic to an open n-disk. Denote them by D_1 and D_2 so that $D_1 \supset \tilde{A}_+$ and $D_2 \supset \tilde{A}_-$. Moreover $D_i \cup \tilde{S} (i=1, 2)$ is diffeomorphic to a closed n-disk if $n \neq 4$. In the case $n=4$, using a result of [4], we see that there exists $\tilde{S} \subset A$

such that $D_i \cup \tilde{S}$ is diffeomorphic to a closed n-disk. Let M_1' be a manifold obtained by attaching D_1 to $L_1 \cup S \cup A_+$ by $\pi | \tilde{A}_+ : \tilde{A}_+ \to A_+$. Since π induces a flat conformal structure on $L_1 \cup S \cup A_+$, M_1' is conformally flat. It is easy to see that M_1' is homeomorphic to M_1 (but not diffeomorphic in general). We can construct a conformally flat manifold M_2' in the same way, and M_2' is homeomorphic to M_2. Clearly, $M = M_1' \# M_2'$. The above construction of M_i' defines a flat conformal structure on each of M_i'. In the rest of the proof, the word "conformal" means conformal with respect to this flat conformal structure. Note that there exist conformal embeddings $\xi_1 : D_1 \cup \tilde{S} \cup \tilde{A}_- \to M_1'$ and $\xi_2 : D_2 \cup \tilde{S} \cup \tilde{A}_+ \to M_2'$ by our construction.

To see M_i' is Kleinian, we construct a regular conformal covering $\Omega_i \to M_i'$, where Ω_i is a connected open subset of S^n. Choose a connected component Ω_0 of $\Omega \setminus \pi^{-1}(S)$. Since $\pi(\Omega \setminus \pi^{-1}(S)) = L_1 \amalg L_2$, $\pi(\Omega_0)$ is contained in either L_1 of L_2. Assume $\pi(\Omega_0) \subset L_1$. Take $\tilde{x} \in \Omega_0$ and let $x = \pi(\tilde{x})$. For any point y of L_1, there exists a path γ contained in L_1, which starts at x and ends at y. Take a lift $\tilde{\gamma}$ of γ so that $\tilde{\gamma}$ starts at $\tilde{x}$. Since $\gamma \cap S = \phi$, $\tilde{\gamma}$ is entirely contained in Ω_0 and hence $y \in \pi(\Omega_0)$. Thus, $\pi(\Omega_0) = L_1$. It is easy to see that $\pi | \Omega_0 : \Omega_0 \to L_1$ is conformal covering. Moreover, since $\pi : \Omega \to M$ is regular, $\pi | \Omega_0 : \Omega_0 \to L_1$ is also regular. Let $\{\tilde{S}_\lambda / \lambda \in \Lambda\}$ be the set of all lifts of S. Since M is Kleinian, there is a discrete subgroup Γ of $\mathrm{Conf}(S^n, C_0)$ such that Γ acts on Ω as the deck transformation group of the covering $\pi : \Omega \to M$. Thus, there exists an unique element γ_λ of Γ, which satisfies $\gamma_\lambda(\tilde{S}) = \tilde{S}_\lambda$ for $\tilde{S}$ and $\tilde{S}_\lambda$, where $\tilde{S}$ is a lift of S which we took in the first part of the proof. Let Λ_0 be the set of all λ with $\tilde{S}_\lambda$ contained in the closure of Ω_0. And denote two connected components of $S^n \setminus \tilde{S}_\lambda$ by $D_{1,\lambda}$ and $D_{2,\lambda}$, respectively, so that $D_{2,\lambda}$ contains Ω_0. Then, $\gamma_\lambda(D_1) = D_{1,\lambda}$ and $D_{1,\lambda} \cap D_{1,\mu} = \emptyset$ for $\lambda, \mu \in \Lambda_0$ with $\lambda \neq \mu$. Let $\tilde{A}_\lambda$ be a lift of A, where $\tilde{A}_\lambda$ contains $\tilde{S}_\lambda$. And let $\tilde{A}_{-,\lambda}$ be a lift of A_-, where $\tilde{A}_{-,\lambda}$ is contained in $\tilde{A}_\lambda$. Clearly, for $\lambda \in \Lambda_0$, $\tilde{A}_{-,\lambda}$ is contained in Ω_0. Define Ω_1 by $\Omega_0 \cup \{\cup_{\lambda \in \Lambda_0} (\tilde{S}_\lambda \cup D_{1,\lambda})\}$. And define $\pi_1 : \Omega_1 \to M_1'$ as $\pi_1 = \pi$ on Ω_0 and $\pi_1 = \xi_1 \circ \gamma_\lambda^{-1}$ on $D_{1,\lambda} \cup \tilde{S}_\lambda \cup \tilde{A}_{-,\lambda}$ for $\lambda \in \Lambda_0$. Then it is easy to see that $\pi_1 : \Omega_1 \to M_1'$ is a well-defined regular conformal covering. Thus, M_1' is Kleinian (if $\pi(\Omega_0)$ is contained in L_2, then M_2' is Kleinian). Since, for each of L_1 and L_2, there exists a connected component of $\Omega \setminus \pi^{-1}(S)$, which covers L_i $(i=1, 2)$, we see that both M_1' and M_2' are Kleinian. q.e.d.

REMARK. It is easy to see that Ω_1 contains Ω by our construction. And the deck transformation group Γ_1 of the covering $\pi_1 : \Omega_1 \to M_1'$ is generated by $\{\gamma_\lambda \circ \gamma_{\lambda_0}^{-1} / \lambda \in \Lambda_0\}$, where λ_0 is some fixed element of Λ_0 (in fact, Γ_1 coincides with $\{\gamma_\lambda \circ \gamma_{\lambda_0}^{-1} / \lambda \in \Lambda_0\}$). Thus M_1' admits a Kleinian structure defined by $\pi_1 : \Omega_1 \to M_1'$, where Ω_1 contains Ω and the deck transformation group Γ_1 is a subgroup of Γ. The same is true for M_2'.

DEFINITION 3.3. A non-trivial manifold M is *topologically prime* if there is no decomposition $M=M_1\#M_2$, where each of M_1 and M_2 is not homeomorphic to S^n.

If we replace the condition "not homeomorphic to S^n" by "not diffeomorphic to S^n", then this definition makes no sense. In fact, if an exotic n-sphere Σ^n exists, then $M=M\#\Sigma^n\#(-\Sigma^n)$ ($n\geqq 7$) holds and hence every n-manifold cannot be topologically prime. In the case $n=3$, a topologically prime manifold is just a prime manifold in the sense of 3-dimensional topology (see [**16**]). Theorem 3.2 says that the C-prime decomposition is reasonably fine in some sense, and this can be stated as the following corollary.

COROLLARY 3.4. *Let M be a Kleinian manifold. Then M is C-prime if and only if M is topologically prime.*

Combining Theorem 3.2 and Corollary 3.4 with the proof of Proposition 2.1, we get the following.

COROLLARY 3.5. *Let M be a non-trivial Kleinian manifold. Then there exists a C-prime decomposition $P_1\#\cdots\#P_k$ of M such that each P_i is Kleinian and topologically prime.*

In particular, for a Kleinian 3-manifold M, M is C-prime if and only if M is prime in the sense of 3-dimensional topology. This fact gives the following corollaries.

COROLLARY 3.6. *Let M be an oriented non-trivial Kleinian 3-manifold. Then M is C-prime if and only if either M is diffeomorphic to $S^1\times S^2$ or the homotopy group $\pi_2(M)$ of M is trivial.*

PROOF. The "if" part follows from Corollary 2.2 and Proposition 3.1. The "only if" part follows from Corollary 3.4 and [**16**, Theorem 2]. q.e.d.

COROLLARY 3.7. *Let M be an oriented non-trivial Kleinian 3-manifold. The C-prime decomposition of M is unique up to permutation and each P_i is Kleinian.*

PROOF. Take a C-prime decomposition $M=P_1\#\cdots\#P_k$ of M. Then, by Theorem 3.2 and Corollary 3.4, each P_i is topologically prime and Kleinian. That is, any C-prime decomposition of M is a prime decomposition of M in the sense of 3-dimensional topology, and hence unique by [**16**, Theorem 1]. q.e.d.

4. The Yamabe invariants of Kleinian manifolds.

First, we introduce an invariant for Kleinian manifolds, which is defined by the same manner as $\mu_C(M)$ in section 2.

DEFINITION 4.1. For a Kleinian manifold M, $\mu_K(M)$ of M is defined by $\mu_K(M)=\sup\mu(M, C)$, where the supremum is taken over all Kleinian structures on M. We define $\mu_K(M)=-\infty$ for a non-Kleinian manifold M.

This invariant $\mu_K(M)$ is well-defined and $\mu_K(M)$ is positive if and only if there exists a Kleinian structure C on M such that $\mu(M, C)$ is positive. This follows from the same reason that $\mu_C(M)$ has such properties. Note that flat conformal structures discussed in the examples following Definition 2.8 are all Kleinian. Then we get the following examples of $\mu_K(M)$.

EXAMPLES. (1) If the fundamental group $\pi_1(M)$ of a Kleinian manifold M is finite, then M is diffeomorphic to a spherical space form and $\mu_K(M)=|\pi_1(M)|^{-2/n}\mu(S^n, C_0)$, where $|\pi_1(M)|$ denotes the order of $\pi_1(M)$.

(2) $\mu_K(S^1\times S^{n-1})=\mu(S^n, C_0)$.

(3) If M admits a flat metric, then $\mu_K(M)=0$.

(4) If a 4-manifold M admits a metric g with negative constant curvature, then $\mu_K(M)=\mu(M, C_0)$, where C_0 denotes the conformal class containing g.

(5) $\mu_K(S^1\times N^{n-1})=0$, where N^{n-1} is an $(n-1)$-manifold admitting a negative constant curvature metric.

(6) $\mu_K(S^m\times N^m)=0$, where N^m is as in (5).

Our main interest in this section is how $\mu_K(M)$ changes if we take a connected sum of Kleinian manifolds or if we decompose a Kleinian manifold into a connected sum of Kleinian manifolds. The following analogue of Theorem 2.9 holds for $\mu_K(M)$. Since we are interested in connected manifolds, as a matter of convenience, we say that a conformal structure C on $M_1\amalg M_2$ is Kleinian if both $C|M_1$ and $C|M_2$ are Kleinian structures on M_1 and M_2, respectively.

THEOREM 4.2. *Let M_1 and M_2 be Kleinian.*

(1) *If $\mu_K(M_1)\leqq 0$ and $\mu_K(M_2)\leqq 0$, then $\mu_K(M_1\amalg M_2)=-(|\mu_K(M_1)|^{n/2}+|\mu_K(M_2)|^{n/2})^{2/n}$.*

(2) *Otherwise, $\mu_K(M_1\amalg M_2)=\min\{\mu_K(M_1), \mu_K(M_2)\}$.*

(3) $\mu_K(M_1\# M_2)\geqq\mu_K(M_1\amalg M_2)$.

(4) *Suppose the equality in* (3) *holds for M_1 and M_2, and suppose there exists a Kleinian structure C_0 on $M_1\amalg M_2$ such that $\mu(M_1\amalg M_2, C_0)=\mu_K(M_1\amalg M_2)$. Then there exists a sequence $\{C_\varepsilon'\}$ of Kleinian structures on $M_1\# M_2$, which satisfies $\lim_{\varepsilon\to 0}\mu(M_1\# M_2, C_\varepsilon')=\mu_K(M_1\# M_2)$, such that a suitable choice of a metric g_ε' contained in C_ε' gives a sequence $\{g_\varepsilon'\}$ satisfying*

$$\lim_{\varepsilon\to 0}(M_1\# M_2, g_\varepsilon')=(M_1\setminus\{p_1\}, g|M_1\setminus\{p_1\})\bigcup_{p_1=p_2}(M_2\setminus\{p_2\}, g|M_2\setminus\{p_2\})$$

for some metric g contained in C_0 and for some point p_i of M_i $(i=1, 2)$.

Proof. We use the same notation as in the proof of Theorem 2.9. Let $M=M_1 \amalg M_2$. Note that if each of $C|M_i$ $(i=1, 2)$ is a Kleinian structure, then the flat conformal structure defined by g_ε' is also Kleinian (this follows from the proof of [**13**, Theorem 5.6]). Thus the proof is the same as that of Theorem 2.9. q.e.d.

As an application of results in section 3 and Theorem 4.2, we get the following.

Theorem 4.3. *Let M be an oriented Kleinian 3-manifold. And let $M=P_1 \# \cdots \# P_k$ be the C-prime decomposition of M, if M is non-trivial. Then $\mu_K(M)$ is positive if and only if M is diffeomorphic to S^3 or each P_i is diffeomorphic to either a spherical space form or $S^1 \times S^2$.*

Proof. Clearly, $\mu_K(S^3)$ is positive. If M is nontrivial, then by Corollary 3.6, for each P_i, either P_i is diffeomorphic to $S^1 \times S^2$ or the homotopy group $\pi_2(P_i)$ of P_i is trivial. If $\pi_2(P_i)=0$ and the universal covering space is non-compact, then P_i is $K(\pi, 1)$ by the Hurewicz theorem. But if P_i is $K(\pi, 1)$, then M carries no metric with positive scalar curvature by [**6**]. Thus, if $\pi_2(P_i)=0$, then the universal covering space of P_i is compact and hence diffeomorphic to S^n by Kuiper's theorem. Therefore P_i is diffeomorphic to a spherical space form, if $\pi_2(P_i)$ is trivial. This shows the "only if" part of Theorem 4.3. Conversely, if each P_i is diffeomorphic to either a spherical space form or $S^1 \times S^2$, then $\mu_K(M)$ is positive by Theorem 4.2. q.e.d.

Corollary 4.4. *Let M be an oriented Kleinian 3-manifold. Suppose that M admits a Kleinian structure C with $\mu(M, C)=0$ and that $\mu_K(M)=0$. Then C contains a flat metric. That is, M is a flat manifold.*

Proof. Let $M=P_1 \# \cdots \# P_k$ be the C-prime decomposition of M (which is unique by Corollary 3.7). Then each P_i is a spherical space form, $S^1 \times S^2$, or $K(\pi, 1)$. By Theorem 4.3, P_i must be $K(\pi, 1)$ for some i. Thus, by [**6**], M cannot admit a metric with positive scalar curvature and in particular metric with non-negative scalar curvature on M is flat. By our assumption, C contains a metric with zero scalar curvature (see Fact 2.7) and it must be a flat metric. q.e.d.

In [**19**], Schoen and Yau studied a relation between the scalar curvature and the developing map of a conformally flat manifold, using the Green's function for the conformal Laplacian. In particular, they proved that if a flat conformal structure C on M contains a metric with positive scalar curvature (i.e., $\mu(M, C)$ is positive), then the developing map of (M, C) is injective and hence (M, C) is Kleinian. Moreover, for a Kleinian manifold $M=\Omega/\Gamma$ they obtained some results concerning the Hausdorff dimension of the complement of Ω. Denote the Hausdorff dimension of a set E by $\dim_H E$.

THEOREM 4.5 (cf. [**19**, Theorem 4.7]). *Suppose* (M, C) *is conformal to* Ω/Γ *for some* Ω *and* Γ. *Then Yamabe invariant* $\mu(M, C)$ *of* (M, C) *is non-negative if and only if* Ω *satisfies* $\dim_H(S^n \setminus \Omega) \leqq (n-2)/2$.

For the proof, see [**19**]. By Theorem 4.5, we obtain some information on the relation between $\mu_K(M)$ and a decomposition of M.

THEOREM 4.6. *Suppose that* M *admits a Kleinian structure* C *with non-negative Yamabe invariant and that* M *is diffeomorphic to a connected sum* $M_1 \# M_2$, *where* M_1 *and* M_2 *are not necessarily conformally flat. Then there exist* M_i' $(i=1, 2)$ *as in Theorem* 3.2 *and each* M_i' *admits a Kleinian structure with non-negative Yamabe invariant.*

PROOF. Suppose (M, C) is conformal to Ω/Γ. Then, by Theorem 4.5, $\dim_H(S_n \setminus \Omega) \leqq (n-2)/2$. By the remark following Theorem 3.2, each M_i' admits a Kleinian structure C_i induced by the covering $\Omega_i \to M_i'$, where Ω_i contains Ω. Since $\dim_H(S^n \setminus \Omega_i) \leqq \dim_H(S^n \setminus \Omega) \leqq (n-2)/2$ follows from $S^n \setminus \Omega_i \subset S_n \setminus \Omega$, $\mu(M_i', C_i)$ is non-negative by Theorem 4.5. q.e.d.

If either M_1' or M_2' is non-trivial and not C-prime, then we can proceed with the above decomposition and obtain a C-prime decomposition $P_1 \# \cdots \# P_k$ of M. Consequently, each P_i has a Kleinian structure defined by the covering $\Omega_i \to P_i$ and Ω_i contains Ω. Thus, we get the following.

COROLLARY 4.7. *If a non-trivial manifold* M *admits a Kleinian structure* C *with non-negative Yamabe invariant, then there exists a* C*-prime decomposition* $P_1 \# \cdots \# P_k$ *of* M *such that each* P_i *admits a Kleinian structure with non-negative Yamabe invariant.*

The author hopes that M_i' $(i=1, 2)$ in Theorem 4.6 admits a Kleinian structure with positive Yamabe invariant in the case $\mu_K(M)$ is positive. Then this gives the converse of a result of Schoen and Yau ([**18**, Corollary 5], see also the remark following the proof of Theorem 2.9), and in particular, $\mu_K(P_i)$ in Corollary 4.7 turns out to be positive for a Kleinian manifold with $\mu_K(M)>0$. Thus the classification of manifolds admitting a flat conformal structure with positive Yamabe invariant (then it is Kleinian) is reduced to the classification of C-prime manifolds with $\mu_K(M)>0$.

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Added in proof. Recently Nayatani proved that if $\mu_k(M)$ is positive in Theorem 4.6, then M_i' admits a Kleinian structure with positive Yamabe invariant (private communication).