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Certain polynomials for knots with integral representations

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In this paper we define a polynomial with integer coefficients for knots with certain integral representations of the knot group. This polynomial is determined up to power by the conjugacy class of the representation and is related to Dehn surgery on knots as follows: if $(1/m)$-surgery on such a knot yields a homotopy 3-sphere, then the absolute value of that polynomial at m must be equal to 1. The degree of the polynomial is related to the class number of the algebraic number field of the representation. We thus have a rough estimate on the number of Dehn surgeries of a given knot yielding simply-connected manifolds from the class number of the algebraic number field. As an example, we will calculate this polynomial for some knots and show that Property P actually follows from the polynomial for those knots, i.e. the polynomials do not take the value ± 1 at non-zero integers for those knots.

The knots for which we will define the polynomial form a large class, and we will call them integral knots. In fact, if a nontrivial knot has no essential closed surfaces in the complement, then that knot is an integral knot. The definition of integral knots we shall use in this paper is the following.

DEFINITION. Let J be a smooth knot in the 3-sphere S^3. We say J is an integral knot if there is an algebraic number field k, i.e., a finite extension of the rationals $\boldsymbol{Q}$ in $\boldsymbol{C}$, and a representation

$$\rho : \pi_1(S^3 - J) \longrightarrow PSL_2(\mathcal{O}_k),$$

where $\mathcal{O}_k$ is the ring of algebraic integers in k, such that:

(1) ρ is a parabolic representation in the sence that if $\gamma \in \pi_1(S^3 - J)$ is a peripheral element, then $\rho(\gamma) = \pm(\textit{unipotent matrix})$, and

(2) ρ is irreducible over the complex numbers, i.e., ρ is not conjugate in $PSL_2(\boldsymbol{C})$ to a group of upper-triangular matrices.

We will call (J, ρ) a k-integral knot to specify the algebraic number field k.

Integral knots form a large class. In fact we have the following fact which

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is an immediate consequence of Thurston's uniformization theorem and Bass-Serre arboreal theory.

PROPOSITION. *If a nontrivial knot J in S^3 has no incompressible non-boundary-parallel closed surface in its complement, then J is an integral knot.*

We briefly outline the argument of the proof for completeness.

First note that S^3-J is atoroidal. Hence if S^3-J is not a Seifert fiber space, then, by Thurston, J is a hyperbolic knot, i.e., there is a faithful representation ρ_0 of the knot group into $SL_2(F)$, with $F\subset \boldsymbol{C}$ a finitely generated extension of $\boldsymbol{Q}$, such that ρ_0 maps peripheral elements to unipotent matrices. Since J is knotted, the knot group contains a rank two free group, and since ρ_0 is faithful, $\mathrm{Im}\,\rho_0$ cannot be conjugated in $SL_2(\boldsymbol{C})$ into a group of uppertriangular matrices, which is a solvable subgroup. Since the complement of J contains no closed incompressible non-boundary-parallel surfaces, the knot group $\pi_1(S^3-J)$ cannot be decomposed as a nontrivial free product with amalgamation $\Gamma_0 *_A \Gamma_1$ in such a way that every peripheral subgroup is conjugate to a subgroup of Γ_0 or Γ_1 (by Stallings, Epstein, and Waldhausen, cf. [**4**] p. 32). Then, by Bass-Serre arboreal theory, for every valuation v of F, the action of $\pi_1(S^3-J)$ on the tree of $SL_2(F)$ relative to v has a fixed point, which implies trace$(\mathrm{Im}\,\rho_0)$ $\subset A=\bigcap_v A_v$. Using Burnside lemma, one can conclude $\mathrm{Im}(\rho_0)$ is conjugated into $SL_2(\mathcal{O}_k)$, where k is a finite extension of the algebraic number field k_0 containing all algebraic integers appearing in the matrices in $\mathrm{Im}\,\rho_0$. (This is Bass' SL_2-subgroup theorem [**1**], [**4**].)

If S^3-J is a Seifert fiber space, then J is a torus knot. In this case, it is easy to find a representation satisfying both conditions (1) and (2) of Definition. Note that our definition doesn't assume the faithfullness of the representation. (See Example 1.)

Let (J, ρ) be an integral knot. We shall readily define a polynomial $N_{(J,\rho)}(x)$ with integer coefficients $\in \boldsymbol{Z}[x]$ which satisfies the following properties:

THEOREM. (1) *The constant term of $N_{(J,\rho)}(x)$ is* 1, *and $N_{(J,\rho)}(-x)=N_{(\bar{J},\rho)}(x)$, where $(\bar{J}, \rho)$ is the mirror image of J with the same representation ρ in S^3.*

(2) *If $(1/m)$-Dehn surgery on J yields a simply-connected* 3*-manifold, then $|N(m)|$, the absolute value of this polynomial at m, is equal to* 1.

(3) *If (J, ρ) is a k-integral knot, then the degree of the polynomial* $\deg N(x)$ *is less than or equal to $C_k[k, \boldsymbol{Q}]$, where C_k is the class number of the algebraic number field k, and $[k, \boldsymbol{Q}]=$the extension degree of k.*

More precisely, the polynomial $N_{(J,\rho)}(x)$ is determined up to power by the equivalence class of the $PSL_2(\boldsymbol{C})$-representation ρ. Here we say two polynomials $N_i(x)\in\boldsymbol{Z}[x]$, $i=1, 2$, are equal *up to power* when there is a polynomial $N_0(x)\in\boldsymbol{Z}[x]$ such that $N_i(x)=(N_0(x))^{n_i}$ for some non-negative integers n_i.

Statements (1) and (2) of the Theorem are valid independently of specific choice of the polynomial $N_{(J,\rho)}(x)$, whereas Statement (3) should be understood to be valid under the assumption that we do not add any redundant power to the polynomial $N_{(J,\rho)}(x)$.

Recall that an n/m-Dehn surgery on a knot J in the 3-sphere S^3 is, by definition, performed by removing a tubular neighborhood V of J and plugging it back in such a way that an n/m-curve on the boundary of S^3-V (with respect to a fixed orientation of S^3) bounds a 2-disc in V. Hence n/m-Dehn surgery on a knot yields a homology 3-sphere if and only if $n=1$, and the Property P Conjecture is the conjecture that m must be equal to 0 if the original knot is nontrivial and the resulting homology 3-sphere is simply-connected.

Recall also that C_k the class number of k is, by definition, the order of the ideal class group of k, which is always finite. Although it is known that the number of Dehn surgeries which yield simply-connected manifolds is finite for a fixed nontrivial knot, the Theorem explains this fact for integral knots from the finiteness of the class number of the algebraic number field.

The definition of the polynomial $N_{(J,\rho)}(x)$ for an integral knot (J, ρ) is natural and elementary except that we need to use the principal ideal theorem of classical class field theory. All we need to use here is the following: for any algebraic number field k and any ideal $\mathfrak{a}$, there exists a field extension K/k such that the extension $\mathfrak{A}$ of $\mathfrak{a}$ is principal in K, i.e., there exists an element $d\in K$ and every element of $\mathfrak{A}$ is a multiple of d by an algebraic integer of K. The field K is an intermediate field of the Hilbert absolute class field L over k, where every ideal of k becomes principal. The Galois group $\mathrm{Gal}(L/k)$ is isomorphic to the ideal class group of k. Hence, in particular, the extension degree $[K, k]$ is less than or equal to the class number of k.

Definition of the polynomial.

Let (J, ρ) be a k-integral knot. Let μ and $\lambda\in\pi_1(S^3-J)$ be represented respectively by a meridian and a preferred longitude of J, i.e., they are commuting peripheral elements such that μ is null-homotopic in a regular neighborhood of J in S^3, generating the homology group $H_1(S^3-J)\cong\boldsymbol{Z}$, and λ is null-homologous in S^3-J. We fix the orientations of μ and λ in such a way that the orientation (μ, λ) of the boundary torus $\partial(S^3-\mathrm{nbd}^\circ(J))$ is compatible with the given orientation of S^3-J.

By Definition of integral knots, $\rho(\mu)\in PSL_2(\mathcal{O}_k)$ has trace ±2, and is represented by a unipotent matrix A of the form

$$A=\begin{pmatrix}1-\alpha & \beta \\ \gamma & 1+\alpha\end{pmatrix}, \quad \text{with } \alpha, \beta, \gamma \in \mathcal{O}_k \text{ such that } \alpha^2+\beta\gamma=0.$$

Let $\mathfrak{a}$ be the ideal of k generated by α and β, $\mathfrak{a}=(\alpha, \beta)$, and K be the finite extension of k in which $\mathfrak{a}$ becomes a principal ideal $\mathfrak{A}=(d)$ for some $d\in K$ as described above. If $\alpha=\alpha_0 d$ and $\beta=\beta_0 d$, then from our choice of d, α_0 and β_0 are coprime elements of $\mathcal{O}_K$. Hence there exists a matrix $P\in SL_2(\mathcal{O}_K)$ of the form

$$P=\begin{pmatrix}\beta_0 & \varepsilon \\ \alpha_0 & \delta\end{pmatrix},$$

and then A is uppertriangulated by P:

$$P^{-1}AP=\begin{pmatrix}1 & \zeta \\ 0 & 1\end{pmatrix}, \quad \text{with } \zeta\neq 0, \in \mathcal{O}_K .$$

Since a peripheral element $\nu=n\mu+m\lambda$ commutes with μ, $\rho(\nu)$ is simultaneously uppertriangulated by the matrix P. Hence we can define a character ϕ of the peripheral subgroup into $\mathcal{O}_K$, by the relation

$$P^{-1}\rho(\nu)P=\pm\begin{pmatrix}1 & \phi(\nu) \\ 0 & 1\end{pmatrix}\in PSL_2(\mathcal{O}_K).$$

We define the polynomial $N_{(J,\rho,K)}(m)\in \boldsymbol{Z}[m]$ by taking the norm $N_{K/\boldsymbol{Q}}$ of the element $\phi(\mu+m\lambda)$:

$$N_{(J,\rho,K)}(m)=\pm N_{K/\boldsymbol{Q}}(\phi(\mu+m\lambda))=\pm\prod_{\sigma}\phi(\mu+m\lambda)^{\sigma},$$

where σ runs through all isomorphisms of K into $\boldsymbol{C}$, and the sign $\pm$ is fixed by imposing the condition:

$$N_{(J,\rho,K)}(0)=1 .$$

This condition is possible by the proof of Statement (2) of Theorem. In fact if $\prod_{\sigma}\phi(\mu+0\lambda)^{\sigma}$ were not ± 1, then we would get a nontrivial representation of the fundamental group of S^3.

Note that for any matrix $\tilde{P}\in SL_2(\mathcal{O}_K)$ which uppertriangulates $\rho(\nu)$, the upperright entry of $\tilde{P}^{-1}\rho(\nu)\tilde{P}$ is $u^2\phi(\nu)$ for some unit u of $\mathcal{O}_K$. This shows that the definition of the polynomial $N_{(J,\rho,K)}(m)$ is independent of the choice of the uppertriangulation, since $N_{K/\boldsymbol{Q}}(u^2\phi(\mu+m\lambda))=(N_{K/\boldsymbol{Q}}(u))^2 N_{K/\boldsymbol{Q}}(\phi(\mu+m\lambda))=N_{K/\boldsymbol{Q}}(\phi(\mu+m\lambda))$.

Also note that if we orient μ and λ both oppositely (hence still compatible with the orientation of S^3-J), we get the character $-\phi(n\mu+m\lambda)$ instead of

$\phi(n\mu+m\lambda)$. This ambiguity in sign is eliminated by the sign fixing condition $N_{(J,\rho,K)}(0)=1$. Hence the polynomial $N_{(J,\rho,K)}(x)$ is unambiguously defined for unoriented J.

Suppose we have two k_i-integral knots (J, ρ_i), $i=1, 2$, with the same knot J, and equivalent $PSL_2(\boldsymbol{C})$-representations ρ_i, i.e., there is an intertwiner $U\in PSL_2(\boldsymbol{C})$ such that $\rho_2=U^{-1}\rho_1 U$. Choosing some field extensions K_i/k_i, we can define the polynomials $N_{(J,\rho_i,K_i)}(x)$ as above. Next Lemma shows that these polynomials are equal up to power. Here recall we say two polynomials are equal up to power when they are some powers of a same polynomial $\in\boldsymbol{Z}[x]$.

Hence, finally, we denote by $N_{(J,\rho)}(x)$ this (power-)equivalence class of the polynomials $N_{(J,\rho,K)}(x)$, which is determined solely by the equivalence class of the $PSL_2(\boldsymbol{C})$-representation ρ. We also write $N_{(J,\rho)}(x)$, by abuse of notation, to denote individual polynomial $N_{(J,\rho,K)}(x)$ when the ambiguity in the power is inessential.

LEMMA. *Let (J, ρ_i) be k_i-integral knots, $i=1, 2$. Suppose that ρ_i are two equivalent $PSL_2(\boldsymbol{C})$-representations of the knot group, each of whose images sits in $PSL_2(\mathcal{O}_{K_i})$ for a finite extension K_i/k_i, $i=1, 2$, respectively, and that the peripheral subgroup $\rho_i(\langle\mu, \lambda\rangle)$ is uppertriangulable in $PSL_2(\mathcal{O}_{K_i})$. Then the two polynomials $N_{(J,\rho_i,K_i)}(x)\in\boldsymbol{Z}[x]$, $i=1, 2$, are equal up to power.*

PROOF. We may assume, after conjugation, that each peripheral subgroup $\rho_i(\langle\mu, \lambda\rangle)$ is already uppertriangulated in $PSL_2(\mathcal{O}_{K_i})$, $i=1, 2$. Then if $U\in PSL_2(\boldsymbol{C})$ is an intertwiner from ρ_1 to ρ_2, U must be an uppertriangular matrix, since conjugation by U preserves unipotent uppertriangular subgroups $\rho_i(\langle\mu, \lambda\rangle)$. Then if $\phi_i: \langle\mu, \lambda\rangle\to\mathcal{O}_{K_i}$ are the characters defined by ρ_i, we see $\phi_2(\mu+m\lambda)=u^2\phi_1(\mu+m\lambda)$, where $u\in\boldsymbol{C}$ is an eigenvalue of U.

Now fix any finite normal extension L of $\boldsymbol{Q}$ in $\boldsymbol{C}$ which contaınes both K_i. Note that $u^2\in L$. For each i, we can define the polynomial $N_{(J,\rho_i,K)}(x)\in\boldsymbol{Z}[x]$ by taking the norm $N_{L/\boldsymbol{Q}}(\phi_i(\mu+x\lambda))$, since $N_{(J,\rho_i,L)}(x)$ is independent of the uppertriangulation in $PSL_2(\mathcal{O}_L)$. Thus, as $\phi_i(\mu+x\lambda)$ is in K_i, we have $N_{L/\boldsymbol{Q}}(\phi_i(\mu+x\lambda)) = N_{K_i/\boldsymbol{Q}}(N_{L/K_i}(\phi_i(\mu+x\lambda))) = N_{K_i/\boldsymbol{Q}}(\phi_i(\mu+x\lambda))^{[L:K_i]}$. Hence $(N_{(J,\rho_2,K_2)}(x))^{[L:K_2]} = \pm N_{L/\boldsymbol{Q}}(\phi_2(\mu+x\lambda)) = \pm N_{L/\boldsymbol{Q}}(u^2\phi_1(\mu+x\lambda)) = \pm N_{L/\boldsymbol{Q}}(u^2)(N_{(J,\rho_1,K_1)}(x))^{[L:K_1]}$. Combining this with $N_{(J,\rho_2,K_2)}(0)=N_{(J,\rho_1,K_1)}(0)=1$, we get the conclusion of Lemma.

Proof of Theorem.

(1) $N_{(J,\rho)}(0)=N_{(J,\rho,K)}(0)=1$ is the sign fixing condition itself which logically depends on the proof of (2) below. The mirror image $\bar{J}$ of J is obtained by

reversing the orientation of S^3. Then this amounts to choosing the orientation $\pm(\mu, -\lambda)$ instead of (μ, λ). Then the character becomes $\pm\phi(n\mu-m\lambda)$ and from this we have $N_{(J,\rho)}(x)=N_{(J,\rho)}(-x)$.

(2) Suppose the absolute value of the polynomial $N_{(J,\rho)}(m)$ is not equal to 1 for an integer $m\in\boldsymbol{Z}$. Since the norm $N_{N/\boldsymbol{Q}}(\phi(\mu+m\lambda))$ is the product of the Archimedean valuations of $K/\boldsymbol{Q}$ up to ±1, there is, due to the product formula, a discrete valuation v of $K/\boldsymbol{Q}$ such that $v(\phi(\mu+m\lambda))$ is strictly positive. We can now construct a nontrivial representation of the fundamental group of the Dehn surgered manifold by an argument parallel to that of Thurston in pp. 25-27 of [**M-B**] as follows. Let l be the smallest non-negative value of the valuation v of the elements which appear in the lower left entry of the matrix representation $P^{-1}\rho P\colon \pi_1(S^3-J)\to PSL_2(\mathcal{O}_K)$. Note that the existence of l is assured by the hypothesis of irreducibility over the complex numbers of the representation ρ in the definition of the integral knots. Let $Q\in PSL_2(K)$ be the diagonal matrix

$$Q=\pm\begin{pmatrix}1/c & 0\\ 0 & 1\end{pmatrix},$$

where c is an element of $\mathcal{O}_K$ whose valuation $v(c)$ is equal to l. From the choice of l, the representation $\tilde{\rho}=Q^{-1}P^{-1}\rho PQ$ maps the whole knot group $\pi_1(S^3-J)$ into $PSL_2(\mathcal{O}_v)$, and the peripheral element $\mu+m\lambda$ is represented by the uppertriangular unipotent matrix whose upper right entry is $c\phi(\mu+m\lambda)$. Note that the valuation v of that element is strictly positive. Hence if we set the residue field $\Lambda=\mathcal{O}_v/\mathfrak{m}_v$, then $\tilde{\rho}$ induces the representation

$$\bar{\rho}:\pi_1(J(1/m))\longrightarrow PSL_2(\Lambda)$$

where $J(1/m)$ denotes the manifold obtained from S^3 by the $(1/m)$ Dehn surgery along the knot J and $\pi_1(J(1/m))=\pi_1(S^3-J)/\langle\mu+m\lambda\rangle$. If the minimum value l of the valuation is attained by the lower left entry of $P^{-1}\rho(\gamma)P$ for an element $\gamma\in\pi_1(S^3-J)$, then the lower left entry of the matrix $\tilde{\rho}(\gamma)$ will have valuation $=0$, i.e. a unit of $\mathcal{O}_v$, hence $\bar{\rho}(\gamma)\in PSL_2(\Lambda)$ is non-trivial. Hence the representation $\bar{\rho}$ provides a nontrivial representation of the fundamental group of the Dehn surgered manifold $J(1/m)$, which proves (2) of Theorem.

(3) As we can choose K as an intermediate field of the absolute class field L/k, $[K:k]\leqq[L:k]=C_k$. Since the degree of $N_{(J,\rho,K)}(x)$ is equal to the extension degree of $K/\boldsymbol{Q}$, we have $\deg N_{(J,\rho,K)}(x)=[K:\boldsymbol{Q}]=[K:k][k:\boldsymbol{Q}]\leqq C_k[k:\boldsymbol{Q}]$. This completes the proof of Theorem.

Some examples.

To demonstrate that the polynomial $N_{(J,\rho)}(x)$ actually verifies Property P for some knots, we will here calculate the polynomial $N_{(J,\rho)}(x)$ for torus knots (Example 1) and the figure-eight knot (Example 2).

EXAMPLE 1 (torus knots). Let J be a torus knot of type (p, q), where p and q are relatively prime integers greater than 1. We first find a representation ρ and show that (J, ρ) is a k_{2pq}-integral knot with $k_{2pq}=\boldsymbol{Q}(e^{i\pi/pq})$, the $2pq$-th cyclotomic field. Then we calculate the polynomial $N_{(J,\rho)}(x)=N_{(J,\rho,K)}$ for $K=k_{2pq}$ and show that $N_{(J,\rho)}(x)=(1-pqx)^{\varphi(2pq)}$, where $\varphi(2pq)=\#\{1\leqq m< 2pq; (m, 2pq)=1\}$, the Euler function. Obviously $N_{(J,\rho)}(m)\neq\pm1$ for non-zero integers m.

Calculation. The knot group has a presentation

$$\langle c, d; c^p=d^q\rangle$$

whose two generators are represented by the cores of two solid tori forming S^3 glued together along the boundary torus containing J. If we take integers r, s such that $pr+qs=1$, then $\mu=d^r c^s$ is a meridian and the preferred longitude commuting with μ is $\lambda=c^p\mu^{-pq}$. Let ξ and ζ be primitive $2p$-th and $2q$-th roots of unity respectively. We define a representation ρ of the knot group in $PSL_2(\mathcal{O}_{k_{2pq}})$ by setting

$$\rho(c)=\pm\begin{pmatrix}\xi & 1\\ 0 & \bar{\xi}\end{pmatrix}^q,\qquad \rho(d)=\pm\begin{pmatrix}\bar{\zeta} & 0\\ t & \zeta\end{pmatrix}^p,$$

where $t=2-\xi\bar{\zeta}-\bar{\xi}\zeta$. Since $\rho(c)^p=\pm1$, and $\rho(d)^q=\pm1$, the representation is well defined. We fined

$$\rho(\mu)=\pm\begin{pmatrix}\xi\bar{\zeta} & \zeta\\ \xi t & t+\bar{\xi}\zeta\end{pmatrix},$$

and trace $\rho(\mu)=\pm(t+\xi\bar{\zeta}+\bar{\xi}\zeta)=\pm2$, which shows this representation is parabolic. To check this representation is irreducible over the complex numbers, it suffices to see that $\rho(c)$ and $\rho(d)$ have no common fixed points when regarded as Möbius transformations. The fixed point sets of $\rho(c)$ and $\rho(d)$ are respectively $\{1/(\bar{\xi}-\xi), \infty\}$ and $\{0, (\bar{\zeta}-\zeta)/t\}$, which are disjoint as $\xi\zeta\neq1$. Hence we see that (J, ρ) is a k_{2pq}-integral knot.

To find the polynomial $N_{(J,\rho)}(x)$, we need to upper-triangulate $\rho(\mu)$. As the ideal $\mathfrak{a}=(1-\xi\bar{\zeta}, \bar{\zeta})=(1)$ is principal, we may take $K=k_{2pq}$ and

$$P=\pm\begin{pmatrix}\bar{\zeta} & -1\\ 1-\xi\bar{\zeta} & \xi\end{pmatrix}\in PSL_2(\mathcal{O}_k)\quad k=k_{2pq},$$

and have

$$P^{-1}\rho(\mu)P=\pm\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix},\qquad P^{-1}\rho(\lambda)P=\pm\begin{pmatrix}1 & -pq\zeta\\ 0 & 1\end{pmatrix}.$$

Hence we have

$$N_{(J,\rho)}(x)=\prod_{\sigma\in\mathrm{Gal}(k/Q)}(1-pqx)\zeta^{\sigma}=(1-pqx)^{\varphi(2pq)}\prod_{(j,2pq)=1}\zeta^{j}=(1-pqx)^{\varphi(2pq)},$$

as claimed.

EXAMPLE 2 (figure-eight knot). Let J be the figure-eight knot. If ρ is an irreducible representation (necessarily the representation of the hyperbolic structure on the complement), then $N_{(J,\rho)}(x)=1+12x^2$.

Calculation. This knot is a hyperbolic 2-bridge knot and its knot group has a presentation

$$\langle \mu, \nu\,;\, \mu^{-1}\nu\mu\nu^{-1}\mu\nu=\nu\mu^{-1}\nu\mu\rangle$$

with two generators μ and ν depicted in Figure below.

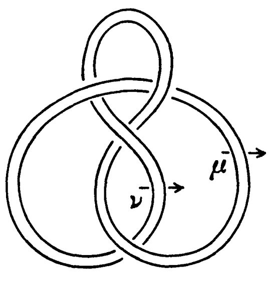

Figure (figure-eight knot).

The preferred longitude λ that commutes with the meridian μ is

$$\lambda=\nu\mu^{-1}\nu^{-1}\mu^{2}\nu^{-1}\mu^{-1}\nu\,.$$

We may assume, after conjugation if necessary, that the generators μ and ν are represented by the matrices of the form

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},\quad\text{and}\quad\begin{pmatrix}1 & 0\\ -z & 1\end{pmatrix},$$

respectively. Plugging these matrices into the relator, we find a single equation for z:

$$z^2+z+1=0\,.$$

Hence $z=\omega$ a primitive cubic root of unity, and the figure-eight knot is a $\boldsymbol{Q}(\sqrt{-3})$-integral knot. Since the integer ring $\boldsymbol{Z}[\omega]$ is a principal ideal domain, the absolute class field is $\boldsymbol{Q}(\sqrt{-3})$ itself. The longitude λ is represented by the matrix

$$\begin{pmatrix} 1 & -4\omega-2 \\ 0 & 1 \end{pmatrix}.$$

Hence the upper-right entry of the matrix representing the element $\mu\lambda^m$ equals $1+m(-4\omega-2)$. Since the Galois group is generated by the complex conjugation, we find $N_{(J,\rho)}(x)=(1+x(-4\omega-2))(1+x(-4\bar{\omega}-2))=1+12x^2$, as claimed.

Questions related to the Property P Conjecture.

As noted in Proposition, non-torus integral knots contain an important subclass of the class of hyperbolic knots. Namely, a non-torus knot is a hyperbolic knot if and only if it has no essential torus in its complement, and a non-torus knot is an integral knot if it has no essential closed surface in its complement. By examining essential tori in knot complements, C. Gordon [3] has shown that The Property P Conjecture for arbitrary knots follows from the conjecture for hyperbolic knots, Hence it is natural to ask whether the conjecture can be reduced to the conjecture for integral knots, by examining incompressible surfaces in knot comprements. Therefore The Property P Conjecture may be divided into the following two questions:

QUESTION 1. *Is The Property P Conjecture for arbitrary knots true if it is true for knots without essential closed surfaces in the complement?*

QUESTION 2. *Are there hyperbolic integral knots J for which $N_{(J,hyperbolic)}(x)=\pm 1$ has non-trivial integer solutions?*

This program is similar to the one in Culler-Gordon-Luecke-Shalen [2]. In this respect, the polynomial $N_{(J,\rho)}(x)$ may be regarded as an algebraic number field version of the argument for atoroidal case in [2], where the function field of an algebraic curve of $SL_2(\boldsymbol{C})$-representations is used.

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