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Unirational elliptic surfaces in characteristic 2

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§ 0. Introduction.

Let X be a non-singular projective surface defined over an algebraically closed field k of characteristic p. X is called unirational if there exists a generically surjective rational mapping from the projective space $\boldsymbol{P}^2$ to X. In characteristic 0, the unirationality of an algebraic surface is equivalent to the rationality. In general, however, there exist irrational unirational algebraic surfaces, so, it is interesting to characterize unirational surfaces over an algebraically closed field of characteristic $p>0$, a problem which many people have been concerned with. From this point of view, T. Katsura [**4**] has completely determined irrational elliptic surfaces with sections which are unirational and of base change type (for definition, see Definition 1.1) in the case where the characteristic p is more than two. The objective of this paper is to give a similar result in characteristic two.

THEOREM 0.1. *Let k be an algebraically closed field of characteristic* 2, *and $\boldsymbol{P}^1$ the projective line with the function field $k(\boldsymbol{P}^1)=k(t)$. Any minimal Weierstrass normal form* (*for definition, see e.g. p.* 171 [**17**]) *of an irrational unirational elliptic surface $f: X\to\boldsymbol{P}^1$ of base change type with sections over k is given by one of the following:*

$(b_j,\ c_j,\ d_j\in k)$

(1) $y^2+t^6y=x^3+t(b_0t^3+b_1t^2+b_2t+b_3)t^3+c_0t^6x+d_0t^{11}, \quad (b_3,\ c_0\neq 0),$

(2) $y^2+t^6y=x^3+t(b_0t^3+b_1t^2+b_2t+b_3)x^2, \quad (b_3\neq 0),$

(3) $y^2+t^2y=x^3+t(b_0t+b_1)x^2+t^3(c_0t^4+c_1t^2+c_2), \quad (c_0,\ c_2\neq 0),$

(4) $y^2+t^2y=x^3+t(b_0t^2+b_1t+b_2)x^2+c_0t^4x+t^3(d_0t^2+d_1), \quad (b_0,\ c_0,\ d_1\neq 0),$

(5) $y^2+t^2y=x^3+t(b_0t^2+b_1t+b_2)x^2+c_0t^2x+d_0t^3, \quad (b_0,\ b_2c_0+d_0\neq 0),$

(6) $y^2+t^2y=x^3+b_0t^2x^2+t^5(c_0t+c_1), \quad (c_0\neq 0),$

(7) $y^2+t^2y=x^3+t^2(b_0t+b_1)x^2+c_0t^4x+d_0t^5, \quad (b_0,\ c_0\neq 0),$

$$(8)\qquad y^2+t^2y=x^3+t^2(b_0t+b_1)x^2, \qquad (b_0\neq 0),$$

$$(9)\qquad y^2+t^2y=x^3+t(b_0t+b_1)x^2+t^6(c_0t+c_1), \qquad (b_1,\ c_0\neq 0),$$

$$(10)\qquad y^2+t^2y=x^3+t(b_0t^2+b_1t+b_2)x^2+c_0t^4x+d_0t^5, \qquad (b_0,\ b_2,\ c_0\neq 0),$$

$$(11)\qquad y^2+t^2y=x^3+t(b_0t^2+b_1t+b_2)x^2, \qquad (b_0,\ b_2\neq 0),$$

$$(12)\qquad y^2+t^2(t+1)^2y=x^3+t^2(t+1)^2(c_0t+c_1)x+t^3(t+1)^3(d_0t+d_1), \ (d_0,\ d_0+d_1\neq 0),$$

$$(13)\qquad y^2+t^2(t+1)^2y=x^3+c_0t^3(t+1)^3x+d_0t^5(t+1)^4, \qquad (d_0\neq 0),$$

$$(14)\qquad y^2+t^2(t+1)^2y=x^3+c_0t^3(t+1)^3x, \qquad (c_0\neq 0),$$

$$(15)\qquad y^2+t^2(t+1)^2y=x^3+b_0t(t+1)x^2, \qquad (b_0\neq 0),$$

$$(16)\qquad y^2+t^2(t+1)^2y=x^3+b_0t^2(t+1)x^2+t^2(t+1)^2(c_0t+c_1)x+t^3(t+1)^3(d_0t+d_1),$$
$$(b_0,\ d_1,\ b_0(c_0+c_1)+d_0+d_1\neq 0),$$

$$(17)\qquad y^2+t^2(t+1)^2y=x^3+t(t+1)(b_0t+b_1)x^2+c_0t^3(t+1)^3x+d_0t^5(t+1)^4,$$
$$(b_0c_0+d_0,\ b_1,\ b_0+b_1\neq 0),$$

$$(18)\qquad y^2+t^2(t+1)^2y=x^3+t^2(t+1)^2(c_0t^2+c_1t+c_2)x+t^3(t+1)^3(d_0t^3+d_1t+d_2),$$
$$(d_0,\ d_1,\ d_0+d_1+d_2\neq 0),$$

$$(19)\qquad y^2+t^2(t+1)^2y=x^3,$$

$$(20)\qquad y^2+t^2(t+1)^2y=x^3+t(t+1)(b_0t+b_1)x^2, \qquad (b_0,\ b_1,\ b_0+b_1\neq 0),$$

$$(21)\qquad y^2+t^2(t+1)^2y=x^3+b_0t^2(t+1)x^2+c_0t^3(t+1)^2x+d_0t^4(t+1)^3, \ (b_0, b_0c_0+d_0\neq 0).$$

A non-singular projective algebraic surface is called supersingular if the second Betti number is equal to the Picard number. Shioda ([**14**]) proved that a unirational surface is supersingular. By counting the second Chern number $c_2(X)$, we have:

COROLLARY 0.1.1. *Let k be an algebraically closed field of characteristic* 2. *Then, an irrational unirational elliptic surface X over k of base change type with sections is a supersingular $K3$ surface.*

In characteristic more than two, there exists irrational unirational elliptic surfaces of base change type which are not K3 surfaces ([**4**]). We can also find examples of unirational K3 surfaces in characteristic 2 in Artin [**1**]. Furthermore, in the course of the proof of Theorem 0.1, we have:

COROLLARY 0.1.2. *Let k, X be as above. Then, the generic fiber of X is a supersingular elliptic curve.*

We note that also in characteristic more than two, an irrational unirational elliptic surface of base change type has the generic fiber which is also a supersingular elliptic curve. Moreover, for each elliptic surfaces listed in Theorem 0.1, we can determine the Mordell-Weil group of the generic fiber of the elliptic surface, because we know the numbers of the components of the singular fibers (cf. Proposition 3.4). The key point of the proof of Theorem 0.1 is the local analysis of the behaviour of singular fibers under a specific purely inseparable base change, which can be summarized as follows:

THEOREM 0.2. (cf. Theorem 2.1) *Fix the following notation:*
k: *an algebraically closed field of characteristic* 2,
$K:=k((t))$: *the field of formal power series in one variable over* k,
$K':=k((s))$ $(s^2=t)$: *the purely inseparable extension of* K *of degree* 2,

Let E *be an elliptic curve over* K, *and let* E_0 *be the singular fiber of the Kodaira-Néron model* ([8]) *of* E. *(In what follows, by abuse of language, we call* E_0 *the singular fiber of* E.) *By the base change* Spec $k[[s]]\to$Spec $k[[t]]$, *the type of the singular fiber of* $E'=E\times_K K'$ *is given as in the following table where we use Néron's symbol (cf.* [7] :)

type over K	*type over* K'
A	A
B_ν	$B_{2\nu}$
C_1	C_1, C_2, C_4, $C_{5,\mu}$, C_7, C_8
C_2	$C_{5,\mu}$
C_3	C_6
C_4	C_1, C_2, C_4, $C_{5,\mu}$, C_7, C_8
$C_{5,\nu}$	C_1, C_2, C_4, $C_{5,\mu}$, C_7, C_8
C_6	C_3
C_7	$C_{5,\mu}$
C_8	C_1, C_2, C_4, $C_{5,\mu}$, C_7, C_8

The proof of Theorem 0.2 is based on an algorithm of Tate for the determination of the type of a singular fiber ([**18**]) and on a formula discribing the relation between the conductor and the discriminant (Ogg [**10**]; see also Sect. 1). Theorem 0.1 is proved after the determination of the types of the singular fibers of $f: X \to \boldsymbol{P}^1$.

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§ 1. Preliminaries and lemmas.

Let C be a non-singular complete algebraic curve defined over an algebraically closed field k of characteristic p, and let $f: X \to C$ be a relatively minimal elliptic surface defined over k.

DEFINITION 1.1. Let $f: X \to C$ be an elliptic surface. X is said to be a unirational elliptic surface of base change type if there exist a curve C' and a surjective morphism $g: C' \to C$ such that the fiber product $X \times_C C'$ is rational.

LEMMA 1.2. (Katsura [4], p. 524) *Suppose that $f: X \to C$ is a unirational elliptic surface of base change type. Then C is rational, and X is transformed into a rational surface by a purely inseparable base change.*

By the above lemma, it is important to determine whether the transformed surface is rational or not, and the following facts are useful for it:

LEMMA 1.3. (e.g. Katsura [4], p. 525) *Let $f: X \to C$ be a relatively minimal elliptic surface with a section. Then, X is rational if and only if the second Chern number $c_2(X)$ is equal to 12 and $C \cong \boldsymbol{P}^1$.*

LEMMA 1.4. (Ogg [10]) *Let $f: X \to C$ be as in Lemma 1.3. Then we have*

$$c_2(X) = \sum_{P \in G} \mathrm{ord}_P \Delta_P ,$$

where Δ_P is the discriminant of the minimal Weierstrass normal form of $f: X \to C$ at $P \in C$.

The latter lemma is a global version of the following fact, which is well-known and useful for our study, too.

LEMMA 1.5. (Ogg, loc. cit.) *Let K be a complete discrete valuation field with algebraically closed residue field k, of characteristic $p \geqq 0$, $G = \mathrm{Gal}(\overline{K}^{sep}/K)$ its absolute Galois group, E/K an elliptic curve defined over K, and E_ℓ the group of ℓ-torsion points on E for each prime $\ell \neq p$. Then we have Serre's measure $\delta = \delta(K, E_\ell)$ (for details, see Ogg [10]) and,*

$$\mathrm{ord}\,\Delta = N + \varepsilon + \delta - 1 ,$$

where Δ is the minimal discriminant of E, $\varepsilon = 0, 1, 2$, as the reduced curve over

k is elliptic, has a node, or has a cusp, and N denotes the number of irreducible components of the singular fiber of E/K.

We remark that by definition, δ is invariant under a purely inseparable base field extension. We formulate here the results of Tate [**18**] and Néron [**8**] on singular fibres of an elliptic surface in characteristic 2 for the readers' convenience.

LEMMA 1.6. *Let R be a complete valuation ring and v (resp. K, resp. k) its normal valuation, (resp. quotient field, resp. residue field). Assume that k is algebraically closed and of characteristic 2. Let E/K be an elliptic curve over K with a K-rational point O, defined by the minimal Weierstrass normal form*

$$y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6\,, \qquad (a_i\in R)\,.$$

Then the type of the singular fiber of E is given in Table 1.1. Here it is provided that in types B_ν, $v(a_3)$, $v(a_4)$, $v(a_6)\geqq 1$, and in type $C_1, \cdots, C_8$, $v(a_i)\geqq 1$.

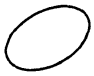
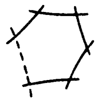
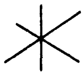
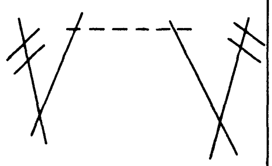
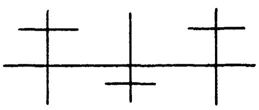
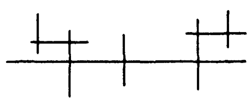
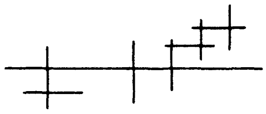
LEMMA 1.7. *Let R, v, K, k, E, a_i be as above. (Unlike Lemma 1.6, the Weierstrass normal form need not be minimal.) Then the type of the singular fiber of E is determined by the following algorithm.*

(1) $v(\Delta)=0 \Rightarrow$ type A.
(2) *Assume $v(\Delta)\neq 0$. Then we can change coordinates so that $v(a_3)$, $v(a_4)$, $v(a_6)\geqq 1$. Do so. Then $v(a_1)=0 \Rightarrow$* type *B_ν with $\nu=v(\Delta)$.*
(3) *Assume $v(a_1)\geqq 1$. Then $v(a_6)=1 \Rightarrow$* type *C_1.*
(4) *Assume $v(a_6)\geqq 2$. Then $v(a_2a_3^2+a_4^2)=2 \Rightarrow$* type *$C_2$.*
(5) *Assume $v(a_2a_3^2+a_4^2)\geqq 3$. Then $v(a_3)=1 \Rightarrow$* type *C_3.*
(6) *Assume $v(a_3)\geqq 2$. Then we can change coordinates so that $v(a_2)\geqq 1$, $v(a_4)\geqq 2$, $v(a_6)\geqq 3$. Then $v(a_2a_4+a_6)=3 \Rightarrow$* type *$C_4$.*
(7) *Assume $v(a_2a_4+a_6)\geqq 4$. Then $v(a_2^2+a_4)=2 \Rightarrow$* type *$C_{5,\nu}$.*
(8) *Assume $v(a_2^2+a_4)\geqq 3$. Then we can change coordinates so that $v(a_2)\geqq 2$, $v(a_4)\geqq 3$, $v(a_6)\geqq 4$. Do so. Then $v(a_3)=2 \Rightarrow$* type *C_6.*
(9) *Assume $v(a_3)\geqq 3$. Then we can also assume $v(a_6)\geqq 5$. Then $v(a_4)=3 \Rightarrow$* type *C_7.*
(10) *Assume $v(a_4)\geqq 4$. Then $v(a_6)=5 \Rightarrow$* type *C_8.*
(11) *If $v(a_6)\geqq 6$, then the original equation is not minimal. Restart from* (1) *with a_i/π^i (π is a prime element of R) instead of a_i.*

Finally, we summarize the well-known results concerning with the Mordell-Weil rank of unirational elliptic surfaces.

LEMMA 1.8. (Ogg [**9**], Shafarevich [**12**]) *Let $f: X\to C$ be a relatively minimal elliptic surface with sections. Assume that the $k(C)/k$-trade* (cf. Lang [**7**])

Table 1.1.

Néron symbol	picture	number of irred. components	necessary and sufficient condition of a_i
A		1	$\Delta \in R^*$.
B_ν $(\nu>0)$		ν	$v(a_1)=0,\ v(a_3)>\nu/2,\ v(a_4)>\nu/2,$ $v(a_6)=\nu$.
C_1		1	$v(a_6)=1$.
C_2		2	$v(a_4)=1,\ v(a_6)\geqq 2$.
C_3		3	$v(a_4)\geqq 2,\ v(a_6)\geqq 2,\ v(a_3)=1$.
C_4		5	$v(a_3)\geqq 2,\ v(a_4)\geqq 2,\ v(a_6)\geqq 3,$ $v(a_2a_4+a_6)=3$.
$C_{5,\nu}$ $(\nu>0)$		$5+\nu$	$v(a_2)=1,$ if $\nu=2n-1,$ $v(a_3)=n+1,\ v(a_4)\geqq n+2,\ v(a_6)\geqq 2n+2,$ if $\nu=2n,$ $v(a_4)=n+2,\ v(a_3)\geqq n+2,\ v(a_6)\geqq 2n+3.$
C_6		7	$v(a_3)=2,\ v(a_2)\geqq 2,\ v(a_4)\geqq 3,$ $v(a_6)\geqq 4$.
C_7		8	$v(a_4)=3,\ v(a_2)\geqq 2,$ $v(a_3)\geqq 3,\ v(a_6)\geqq 5$.
C_8		9	$v(a_3)\geqq 3,\ v(a_2)\geqq 2,$ $v(a_4)\geqq 4,\ v(a_6)=5$.

of the generic fiber E is one point. Then, the Picard number $\rho(X)$ of X is given by

$$\rho(X) = \mathrm{rk}(E)+2+\sum_{v\in\Sigma}(m_v-1),$$

where $\mathrm{rk}(E)$ *denotes the Mordell-Weil rank of E over $k(C)$, where $\sum$ denotes the finite set of points v of C for which $f^{-1}(v)$'s are the singular fibers, and where for $v\in\sum$, m_v is the number of the irreducible components of $f^{-1}(v)$.*

LEMMA 1.9. (e.g. Katsura [**5**]) *Let $f: X\to C$ be a relatively minimal elliptic surface with sections. If f has at least one singular fiber composed of rational curves, then the $k(C)/k$-trace of the generic fiber is one point.*

As for the second Chern number $c_2(X)$ and the Picard number $\rho(X)$ of X, we know the following result.

LEMMA 1.10. *If a smooth projective surface X is unirational, then*

$$\rho(X) = c_2(X)-2.$$

PROOF. Since X is unirational, the second betti number $b_2(X)$ of X is equal to $\rho(X)$ (cf. Shioda [**14**]), and its Albanese variety is trivial. The result follows from these facts. □

§2. Singular fibres and base change.

In this section, we study how the types of singular fibers of an elliptic surface in characteristic 2 are transformed by a purely inseparable base change of degree 2.

We use the following notation in this section:

k : an algebraically closed field in characteristic 2,

$K=k((t))$: the field of power series in one variable over k,

$K'=k((s))(t=s^2)$: the purely inseparable extension of K of degree 2.

Let E/K be an elliptic curve over K defined by

$$y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6 \qquad (a_i(t)\in k[[t]]).$$

We assume that this equation is a minimal Weierstrass normal form and that $a_i(t)$'s are normalized as in Table 1.1 of Lemma 1.6. The minimal discriminant of E/K (see e.g. p. 172 [**17**]) is given by

$$\begin{aligned}\Delta &= a_1^6a_6+a_1^5a_3a_4+a_1^4a_2a_3^2+a_1^4a_4^2+a_3^4+a_1^3a_3^3\\ &= a_1^6a_6+a_1^3a_3(a_3^2+a_4a_1^2)+a_1^4a_3^2a_2+(a_3^2+a_4a_1^2)^2\end{aligned}$$

and the j-invariant of E (see e.g. p. 46 [17]) is given by $j=a_1^{12}/\Delta$. We set $E'=E\times_K K'$, then, the (not necessary minimal) Weierstrass normal form of E' is given by

$$y^2+a_1(s^2)xy+a_3(s^2)y = x^3+a_2(s^2)x^2+a_4(s^2)x+a_6(s^2).$$

Δ': a minimal discriminant of E'/K'.

We also denote by $\mathrm{ord}_t(\ )$ (resp. $\mathrm{ord}_s(\ )$) the normalized discrete valuation over K (resp. K') such that $\mathrm{ord}_t(t)=1$ (resp. $\mathrm{ord}_s(s)=1$).

For $a_4=\sum_{i=0}^{\infty}a_{4,i}t^i (a_{4,i}\in k)$, we set

$$\text{(2.1)}\qquad \begin{cases} a_{4,o} = \sum_{i=0}^{\infty} a_{4,2i+1}t^{2i+1} & \text{(odd degree terms)}, \\ a_{4,e} = \sum_{i=0}^{\infty} a_{4,2i}t^{2i} & \text{(even degree terms)}. \end{cases}$$

Considering the change of coordinates of E' over K' defined by

$$\text{(2.2)}\qquad \begin{cases} x = U^2x'+R, \\ y = U^3y'+SU^2x'+T \qquad (R, S, T, U\in K' \quad (U\neq 0)), \end{cases}$$

we have the new Weierstrass normal form

$$\text{(2.3)}\qquad y'^2+a_1'x'y'+a_3'y' = x'^3+a_2'x'^2+a_4'x'+a_6',$$

where

$$\text{(2.4)}\qquad \begin{cases} Ua_1' = a_1, \\ U^2a_2' = a_2+Sa_1+R+S^2, \\ U^3a_3' = a_3+Ra_1, \\ U^4a_4' = a_4+Sa_3+(T+RS)a_1+R^2, \\ U^6a_6' = a_6+Ra_4+R^2a_2+R^3+Ta_3+T^2+RTa_1, \end{cases}$$

The new discriminant of this Weierstrass normal form is given by Δ/U^{12}. We set $\delta=\delta(K, E_\ell)$ (See Lemma 1.5.) ($\ell\neq 2$), and we denote by N (resp. N') the number of irreducible components of the special fiber of E (resp. E').

THEOREM 2.1. *The type of the singular fiber of E' is given as follows.*

[A] *Assume E/K is of* type A. *Then E'/K' is of* type A, *and*

$$\mathrm{ord}_t(\Delta)=0, \qquad \mathrm{ord}_s(\Delta')=0, \qquad \mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1).$$

[B_ν] ($\nu>0$) *Assume E/K is of* type B_ν. *Then E'/K' is of* type $B_{2\nu}$, *and*

$$\mathrm{ord}_t(\Delta)=\nu, \qquad \mathrm{ord}_s(\Delta')=2\nu, \qquad \mathrm{ord}_t(j)=-\nu.$$

[C_1] *Assume E/K is of* type C_1. *Let $l\geq 0$ be the largest non-negative* *integer such that*

$$\mathrm{ord}_t(a_1) \geqq 2l, \quad \mathrm{ord}_t(a_3) \geqq 2l+1, \quad \mathrm{ord}_t(a_{4,o}) \geqq 2l+1, \quad \mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 6l.$$

$[C_1-C_1]_l$ $(l\geqq 1)$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l$, *then* E'/K' *is of* type C_1, and

$$\mathrm{ord}_t(\Delta)=12l, \quad \mathrm{ord}_s(\Delta')=12l, \quad \mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l \geqq 12l \geqq 12.$$

$[C_1-C_2]_l$ $(l\geqq 1)$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+1$ *and* $\mathrm{ord}_t(a_1)=2l$, *then* E'/K' *is of* type C_2, *and*

$$\mathrm{ord}_t(\Delta)=12l+1, \qquad \mathrm{ord}_s(\Delta')=12l+2, \qquad \mathrm{ord}_t(j)=12l-1 \geqq 11.$$

$[C_1-C_4]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+2$ *and* $\mathrm{ord}_t(a_1)\geqq 2l+1$, *then* E'/K' *is of* type C_4, *and*

$$\mathrm{ord}_t(\Delta)=12l+4, \qquad \mathrm{ord}_s(\Delta')=12l+8,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l-4 \geqq 12l+8 \geqq 8.$$

$[C_1-C_7]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+4$, $\mathrm{ord}_t(a_1)=2l+1$, $\mathrm{ord}_t(a_3)\geqq 2l+2$, *and* $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *then* E'/K' *is of* type C_7, *and*

$$\mathrm{ord}_t(\Delta)=12l+7, \qquad \mathrm{ord}_s(\Delta')=12l+14, \qquad \mathrm{ord}_t(j)=12l+5 \geqq 5.$$

$[C_1-C_8]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+4$, $\mathrm{ord}_t(a_1)\geqq 2l+2$, $\mathrm{ord}_t(a_3)\geqq 2l+2$, *and* $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *then* E'/K' *is of* type C_8, *and*

$$\mathrm{ord}_t(\Delta)=12l+8, \qquad \mathrm{ord}_s(\Delta')=12l+16,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l-8 \geqq 12l+16 \geqq 16.$$

$[C_1-C_5]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+3$, $\mathrm{ord}_t(a_1)\geqq 2l+1$, $\mathrm{ord}_t(a_3)\geqq 2l+2$, *and* $\mathrm{ord}_t(a_{4,o})=2l+1$, *then let* $l'(\geqq l\geqq 0)$ *be the largest non-negative integer such that*

$$\mathrm{ord}_t(a_1) \geqq 2l'+1, \qquad \mathrm{ord}_t(a_3) \geqq 2l'+2, \qquad \mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 6l'+3.$$

$[C_1-C_{5,4\mu-2}]_{l,l'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4)=\begin{cases} 6l'+3 \\ 6l'+5 \\ 6l'+7, \end{cases} \text{ \textit{then} } E'/K' \text{ \textit{is of} type } C_{5,4\mu-2} \text{ \textit{with} } \mu=\begin{cases} 3l'-3l+1 \\ 3l'-3l+2 \\ 3l'-3l+3, \end{cases}$$

and

$$\mathrm{ord}_t(\Delta)=\begin{cases} 12l'+6 \\ 12l'+10 \\ 12l'+14, \end{cases} \quad \mathrm{ord}_s(\Delta')=\begin{cases} 24l'+12-12l \\ 24l'+20-12l \\ 24l'+28-12l, \end{cases} \quad \mathrm{ord}_t(j)=\begin{cases} 24l'-12l+6 \geqq 6 \\ 24l'-12l+14 \geqq 14 \\ 24l'-12l+22 \geqq 22. \end{cases}$$

$[C_1-C_{5,4\mu}]_{l,l'}$ *If*

$$\operatorname{ord}_t(a_3^2+a_1^2a_4)=\begin{cases}6l'+4\\6l'+6\\6l'+8,\end{cases}\quad \textit{then } E'/K' \textit{ is of } \text{type } C_{5,4\mu} \textit{ with } \mu=\begin{cases}3l'-3l+1\\3l'-3l+2\\3l'-3l+3,\end{cases}$$

and

$$\operatorname{ord}_t(\Delta)=\begin{cases}12l'+8\\12l'+12\\12l'+16,\end{cases}\quad \operatorname{ord}_s(\Delta')=\begin{cases}24l'+16-12l\\24l'+24-12l\\24l'+32-12l,\end{cases}$$

$$\operatorname{ord}_t(j)=\begin{cases}12\operatorname{ord}_t(a_1)-12l'-8\geqq 12l'+16\geqq 16\\12\operatorname{ord}_t(a_1)-12l'-12\geqq 12l'+12\geqq 12\\12\operatorname{ord}_t(a_1)-12l'-16\geqq 12l'+20\geqq 20.\end{cases}$$

$[C_2-C_{5,4\mu-1}]$ *Assume E/K is of* type C_2 *and* $\operatorname{ord}_t(a_1)<\operatorname{ord}_t(a_3)$, *then E'/K' is of* type $C_{5,4\mu-1}$ *with* $\mu=\operatorname{ord}_t(a_1)$, *and*

$$\operatorname{ord}_t(\Delta)=4\operatorname{ord}_t(a_1)+2,\quad \operatorname{ord}_s(\Delta')=8\operatorname{ord}_t(a_1)+4,\quad \operatorname{ord}_t(j)=8\operatorname{ord}_t(a_1)-2\geqq 6.$$

$[C_2-C_{5,4\mu-3}]$ *Assume E/K is of* type C_2 *and* $\operatorname{ord}_t(a_1)\geqq\operatorname{ord}_t(a_3)$, *then E'/K' is of* type $C_{5,4\mu-3}$ *with* $\mu=\operatorname{ord}_t(a_3)$, *and*

$$\operatorname{ord}_t(\Delta)=4\operatorname{ord}_t(a_3),\quad \operatorname{ord}_s(\Delta')=8\operatorname{ord}_t(a_3),\quad \operatorname{ord}_t(j)=12\operatorname{ord}_t(a_1)-4\operatorname{ord}_t(a_3)\geqq 8.$$

$[C_3]$ *If E/K is of* type C_3, *then E'/K' is of* type C_6, *and*

$$\operatorname{ord}_t(\Delta)=4,\quad \operatorname{ord}_s(\Delta')=8,\quad \operatorname{ord}_t(j)=12\operatorname{ord}_t(a_1)-4\geqq 8.$$

$[C_4]$ *Assume E/K is of* type C_4. *Let l be the largest positive integer such that*

$$\operatorname{ord}_t(a_1)\geqq 2l-1,\quad \operatorname{ord}_t(a_3)\geqq 2l,\quad \operatorname{ord}_t(a_{4,o})\geqq 2l+1,\quad \operatorname{ord}_t(a_3^2+a_1^2a_4)\geqq 6l-2.$$

$[C_4-C_1]_l$ *If* $\operatorname{ord}_t(a_3^2+a_4a_1^2)=6l-2$, *then E'/K' is of* type C_1, *and*

$$\operatorname{ord}_t(\Delta)=12l-4,\quad \operatorname{ord}_s(\Delta')=12l-8,\quad \operatorname{ord}_t(j)=12\operatorname{ord}_t(a_1)-12l+4\geqq 12l-8\geqq 4.$$

$[C_4-C_2]_l$ *If* $\operatorname{ord}_t(a_3^2+a_4a_1^2)\geqq 6l-1$ *and* $\operatorname{ord}_t(a_1)=2l-1$, *then E'/K' is of* type C_2, *and*

$$\operatorname{ord}_t(\Delta)=12l-3,\quad \operatorname{ord}_s(\Delta')=12l-6,\quad \operatorname{ord}_t(j)=12l-9\geqq 3.$$

$[C_4-C_4]_l$ *If* $\operatorname{ord}_t(a_3^2+a_4a_1^2)=6l$, $\operatorname{ord}_t(a_1)\geqq 2l$, *and* $\operatorname{ord}_t(a_3)\geqq 2l+1$, *then E'/K' is of* type C_4, *and*

$$\operatorname{ord}_t(\Delta)=12l,\quad \operatorname{ord}_s(\Delta')=12l,\quad \operatorname{ord}_t(j)=12\operatorname{ord}_t(a_1)-12l\geqq 12l\geqq 12.$$

$[C_4-C_7]_l$ *If* $\operatorname{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+2$, $\operatorname{ord}_t(a_1)=2l$, $\operatorname{ord}_t(a_3)\geqq 2l+1$, *and* $\operatorname{ord}_t(a_{4,o})$

$\geqq 2l+3$, *then* E'/K' *is of* type C_7, *and*

$$\mathrm{ord}_t(\Delta)=12l+3, \qquad \mathrm{ord}_s(\Delta')=12l+6, \qquad \mathrm{ord}_t(j)=12l-3\geqq 9.$$

$[C_4-C_8]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+2$, $\mathrm{ord}_t(a_1)\geqq 2l+1$, $\mathrm{ord}_t(a_3)\geqq 2l+2$, *and* $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *then* E'/K' *is of* type C_8, *and*

$$\mathrm{ord}_t(\Delta)=12l+4, \quad \mathrm{ord}_s(\Delta')=12l+8, \quad \mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l-4\geqq 12l+8\geqq 20.$$

$[C_4-C_5]_l$ *Assume* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+1$, $\mathrm{ord}_t(a_1)\geqq 2l$, $\mathrm{ord}_t(a_3)\geqq 2l+1$, *and* $\mathrm{ord}_t(a_{4,o})=2l+1$, *and let* $l'(\geqq l\geqq 1)$ *be the largest positive integer such that*

$$\mathrm{ord}_t(a_1)\geqq 2l', \qquad \mathrm{ord}_t(a_3)\geqq 2l'+1, \qquad \mathrm{ord}_t(a_3^2+a_1^2a_4)\geqq 6l'+1.$$

$[C_4-C_{5,4\mu-2}]_{l,l'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4)=\begin{cases}6l'+1\\ 6l'+3\\ 6l'+5,\end{cases} \quad \textit{then } E'/K' \textit{ is of } \text{type } C_{5,4\mu-2} \textit{ with } \mu=\begin{cases}3l'-3l+1\\ 3l'-3l+2\\ 3l'-3l+3,\end{cases}$$

and

$$\mathrm{ord}_t(\Delta)=\begin{cases}12l'+2\\ 12l'+6\\ 12l'+10,\end{cases} \quad \mathrm{ord}_s(\Delta')=\begin{cases}24l'+4-12l\\ 24l'+12-12l\\ 24l'+20-12l,\end{cases} \quad \mathrm{ord}_t(j)=\begin{cases}24l'-12l-2\geqq 10\\ 24l'-12l+6\geqq 18\\ 24l'-12l+14\geqq 26.\end{cases}$$

$[C_4-C_{5,4\mu}]_{l,l'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4)=\begin{cases}6l'+2\\ 6l'+4\\ 6l'+6,\end{cases} \quad \textit{then } E'/K' \textit{ is of } \text{type } C_{5,4\mu} \textit{ with } \mu=\begin{cases}3l'-3l+1\\ 3l'-3l+2\\ 3l'-3l+3,\end{cases}$$

and

$$\mathrm{ord}_t(\Delta)=\begin{cases}12l'+4\\ 12l'+8\\ 12l'+12,\end{cases} \quad \mathrm{ord}_s(\Delta')=\begin{cases}24l'+8-12l\\ 24l'+16-12l\\ 24l'+24-12l,\end{cases}$$

$$\mathrm{ord}_t(j)=\begin{cases}12\,\mathrm{ord}_t(a_1)-12l'-4\geqq 12l'+8\geqq 20\\ 12\,\mathrm{ord}_t(a_1)-12l'-8\geqq 12l'+4\geqq 16\\ 12\,\mathrm{ord}_t(a_1)-12l'-12\geqq 12l'+12\geqq 24.\end{cases}$$

$[C_{5,4m-3}-C_2]$ *Assume* E/K *is of* type $C_{5,4m-3}$, *and* $\mathrm{ord}_t(a_1)\geqq m$. *Then* E'/K' *is of* type C_2, *and*

$$\mathrm{ord}_t(\Delta)=8m, \qquad \mathrm{ord}_s(\Delta')=4m,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-8m\geqq 4m\geqq 4.$$

[$C_{5,4m-3}-C_{5,4\mu-2}$] *Assume* E/K *is of* type $C_{5,4m-3}$, *and* $\mathrm{ord}_t(a_1)<m$. *Then* E'/K' *is of* type $C_{5,4\mu-2}$ *with* $\mu=2(m-\mathrm{ord}_t(a_1))$, *and*

$$\mathrm{ord}_t(\Delta)=4\,\mathrm{ord}_t(a_1)+4m+1,\qquad \mathrm{ord}_s(\Delta')=8m-4\,\mathrm{ord}_t(a_1)+2,$$

$$\mathrm{ord}_t(j)=8\,\mathrm{ord}_t(a_1)-4m-1.$$

[$C_{5,4m-1}-C_7$] *Assume* E/K *is of* type $C_{5,4m-1}$, *and* $\mathrm{ord}_t(a_1)>m$. *Then* E'/K' *is of* type C_7, *and*

$$\mathrm{ord}_t(\Delta)=8m+4,\qquad \mathrm{ord}_s(\Delta')=4m+8,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-8m-4\geqq 4m+8\geqq 12.$$

[$C_{5,4m-1}-C_{5,4\mu-2}$] *Assume* E/K *is of* type $C_{5,4m-1}$, *and* $\mathrm{ord}_t(a_1)\leqq m$. *Then* E'/K' *is of* type $C_{5,4\mu-2}$ *with* $\mu=2m-2\,\mathrm{ord}_t(a_1)+1$, *and*

$$\mathrm{ord}_t(\Delta)=4\,\mathrm{ord}_t(a_1)+4m+3,\qquad \mathrm{ord}_s(\Delta')=8m-4\,\mathrm{ord}_t(a_1)+6,$$

$$\mathrm{ord}_t(j)=8\,\mathrm{ord}_t(a_1)-4m-3.$$

[$C_{5,4m-2}-C_4$] *Assume* E/K *is of* type $C_{5,4m-2}$, *and* $\mathrm{ord}_t(a_1)=m$. *Then* E'/K' *is of* type C_4, *and*

$$\mathrm{ord}_t(\Delta)=8m+2,\qquad \mathrm{ord}_s(\Delta')=4m+4,\qquad \mathrm{ord}_t(j)=4m-2\geqq 2.$$

[$C_{5,4m-2}-C_{5,4\mu-2}$] *Assume* E/K *is of* type $C_{5,4m-2}$, $\mathrm{ord}_t(a_1)<m$, *and* $\mathrm{ord}_t(a_3)\leqq m+\mathrm{ord}_t(a_1)$. *Then* E'/K' *is of* type $C_{5,4\mu-2}$ *with* $\mu=\mathrm{ord}_t(a_3)-2m$, *and*

$$\mathrm{ord}_t(\Delta)=4\,\mathrm{ord}_t(a_3)\geqq 8m+4,\qquad \mathrm{ord}_s(\Delta')=8\,\mathrm{ord}_t(a_3)-12m\geqq 4m+8,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-4\,\mathrm{ord}_t(a_3)\gneqq 0.$$

[$C_{5,4m-2}-C_{5,4\mu}$]-1 *Assume* E/K *is of* type $C_{5,4m-2}$, *and* $\mathrm{ord}_t(a_1)>m$, $\mathrm{ord}_t(a_3)>m+\mathrm{ord}_t(a_1)$. *Then* E'/K' *is of* type $C_{5,4\mu}$, $\mu=\mathrm{ord}_t(a_1)$, *and*

$$\mathrm{ord}_t(\Delta)=4m+4\,\mathrm{ord}_t(a_1)+2,\qquad \mathrm{ord}_s(\Delta')=8\,\mathrm{ord}_t(a_1)-4m+4,$$

$$\mathrm{ord}_t(j)=8\,\mathrm{ord}_t(a_1)-4m-2\gneqq 0.$$

[$C_{5,4m-2}-C_{5,4\mu}$]-2 *Assume* E/K *is of* type $C_{5,4m-2}$ *and* $\mathrm{ord}_t(a_1)<m$. *Then* E'/K' *is of* type $C_{5,4\mu}$, $\mu=2m-2\,\mathrm{ord}_t(a_1)$, *and*

$$\mathrm{ord}_t(\Delta)=4m+4\,\mathrm{ord}_t(a_1)+2,\qquad \mathrm{ord}_s(\Delta')=8m-4\,\mathrm{ord}_t(a_1)+4,$$

$$\mathrm{ord}_t(j)=8\,\mathrm{ord}_t(a_1)-4m-2.$$

[$C_{5,4m}-C_{5,4\mu}$]-0 *Assume* E/K *is of* type $C_{5,4m}$ *and* $\mathrm{ord}_t(a_1)\leqq m$. *Then* E'/K' *is of* type $C_{5,4\mu}$ *with* $\mu=2m-2\,\mathrm{ord}_t(a_1)+1$, *and*

$$\mathrm{ord}_t(\Delta) = 4\,\mathrm{ord}_t(a_1)+4m+4, \qquad \mathrm{ord}_s(\Delta') = 8m-4\,\mathrm{ord}_t(a_1)+8,$$

$$\mathrm{ord}_t(j) = 8\,\mathrm{ord}_t(a_1)-4m-4.$$

$[C_{5,4m}-C]$ *Assume E/K is of* type $C_{5,4m}$ *and* $\mathrm{ord}_t(a_1)>m$. *Let i be the largest positive integer such that*

$$\mathrm{ord}_t(a_1) \geqq m+2i-1, \qquad \mathrm{ord}_t(a_3) \geqq 2m+2i,$$

$$\mathrm{ord}_t(a_{4,0}) \geqq 2m+2i+1, \qquad \mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 4m+6i-2.$$

$[C_{5,4m}-C_1]_i$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=4m+6i-2$, *then E'/K' is of* type C_1, *and*

$$\mathrm{ord}_t(\Delta) = 8m+12i-4, \qquad \mathrm{ord}_s(\Delta') = 4m+12i-8,$$

$$\mathrm{ord}_t(j) = 12\,\mathrm{ord}_t(a_1)-8m-12i+4 \geqq 4m+12i-8 > 0.$$

$[C_{5,4m}-C_2]_i$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 4m+6i-1$, $\mathrm{ord}_t(a_3)=2m+2i$, *and* $\mathrm{ord}_t(a_1)=m+2i-1$, *then E'/K' is of* type C_2, *and*

$$\mathrm{ord}_t(\Delta) = 8m+12i-3, \qquad \mathrm{ord}_s(\Delta') = 4m+12i-6,$$

$$\mathrm{ord}_t(j) = 4m+12i-9 > 0.$$

$[C_{5,4m}-C_4]_i$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=4m+6i$, $\mathrm{ord}_t(a_1)\geqq m+2i$, *and* $\mathrm{ord}_t(a_3)\geqq 2m+2i+1$, *then E'/K' is of* type C_4, *and*

$$\mathrm{ord}_t(\Delta) = 8m+12i, \qquad \mathrm{ord}_s(\Delta') = 4m+12i,$$

$$\mathrm{ord}_t(j) = 12\,\mathrm{ord}_t(a_1)-8m-12i \geqq 4m+12i > 0.$$

$[C_{5,4m}-C_7]_i$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 4m+6i+2$, $\mathrm{ord}_t(a_1)=m+2i$, $\mathrm{ord}_t(a_3)\geqq 2m+2i+1$, *and* $\mathrm{ord}_t(a_{4,0})\geqq 2m+2i+3$, *then E'/K' is of* type C_7, *and*

$$\mathrm{ord}_t(\Delta) = 8m+12i+3, \qquad \mathrm{ord}_s(\Delta') = 4m+12i+6,$$

$$\mathrm{ord}_t(j) = 4m+12i-3 > 0.$$

$[C_{5,4m}-C_8]_i$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=4m+6i+2$, $\mathrm{ord}_t(a_1)\geqq m+2i+1$, $\mathrm{ord}_t(a_3)\geqq 2m+2i+2$, *and* $\mathrm{ord}_t(a_{4,0})\geqq 2m+2i+3$, *then E'/K' is of* type C_8, *and*

$$\mathrm{ord}_t(\Delta) = 8m+12i+4, \qquad \mathrm{ord}_s(\Delta') = 4m+12i+8,$$

$$\mathrm{ord}_t(j) = 12\,\mathrm{ord}_t(a_1)-8m-12i-4 \geqq 4m+12i+8 > 0.$$

$[C_{5,4m}-C_5]_i$ *Assume* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 4m+6i+1$, $\mathrm{ord}_t(a_1)\geqq m+2i$, $\mathrm{ord}_t(a_3)\geqq 2m+2i+1$, *and* $\mathrm{ord}_t(a_{4,0})=2m+2i+1$, *and let $i'(\geqq i\geqq 1)$ be the largest positive integer such that*

$$\mathrm{ord}_t(a_1) \geqq m+2i', \quad \mathrm{ord}_t(a_3) \geqq 2m+2i'+1, \quad \mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 4m+6i'+1.$$

$[C_{5,4m}-C_{5,4\mu-2}]_{i,i'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4)=\begin{cases}6i'+4m+1\\6i'+4m+3\\6i'+4m+5,\end{cases}$$

then E'/K' *is of* type $C_{5,4\mu-2}$ *with* $\mu=\begin{cases}3i'-3i+1\\3i'-3i+2\\3i'-3i+3,\end{cases}$

and

$$\mathrm{ord}_t(\Delta)=\begin{cases}12i'+8m+2\\12i'+8m+6\\12i'+8m+10,\end{cases}\qquad \mathrm{ord}_s(\Delta')=\begin{cases}24i'+4m+4-12i\\24i'+4m+12-12i\\24i'+4m+20-12i,\end{cases}$$

$$\mathrm{ord}_t(j)=\begin{cases}24i'-12i+4m-2\geqq 14\\24i'-12i+4m+6\geqq 22\\24i'-12i+4m+14\geqq 30.\end{cases}$$

$[C_{5,4m}-C_{5,4\mu}]_{i,i'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4)=\begin{cases}6i'+4m+2\\6i'+4m+4\\6i'+4m+6,\end{cases}$$

then E'/K' *is of* type $C_{5,4\mu}$ *with* $\mu=\begin{cases}3i'-3i+1\\3i'-3i+2\\3i'-3i+3,\end{cases}$

and

$$\mathrm{ord}_t(\Delta)=\begin{cases}12i'+8m+4\\12i'+8m+8\\12i'+8m+12,\end{cases}\qquad \mathrm{ord}_s(\Delta')=\begin{cases}24i'+4m+8-12i\\24i'+4m+16-12i\\24i'+4m+24-12i,\end{cases}$$

$$\mathrm{ord}_t(j)=\begin{cases}12\,\mathrm{ord}_t(a_1)-12i'-8m-4\geqq 12i'+4m+8>0\\12\,\mathrm{ord}_t(a_1)-12i'-8m-8\geqq 12i'+4m+4>0\\12\,\mathrm{ord}_t(a_1)-12i'-8m-12\geqq 12i'+4m+12>0.\end{cases}$$

$[C_6]$ *Assume* E/K *is of* type C_6. *Then* E'/K' *is of* type C_3, *and*

$$\mathrm{ord}_t(\Delta)=8,\qquad \mathrm{ord}_s(\Delta')=4,\qquad \mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-8\geqq 4.$$

$[C_7-C_{5,4\mu-3}]$ *Assume* E/K *is of* type C_7 *and* $\mathrm{ord}_t(a_1)+1<\mathrm{ord}_t(a_3)$. *Then* E'/K' *is of* type $C_{5,4\mu-3}$ *with* $\mu=\mathrm{ord}_t(a_1)$, *and*

$$\mathrm{ord}_t(\Delta)=4\,\mathrm{ord}_t(a_1)+6, \qquad \mathrm{ord}_s(\Delta')=8\,\mathrm{ord}_t(a_1),$$

$$\mathrm{ord}_t(j)=8\,\mathrm{ord}_t(a_1)-6\geqq 2.$$

$[C_7-C_{5,4\mu-1}]$ *Assume* E/K *is of* type C_7 *and* $\mathrm{ord}_t(a_1)+1\geqq\mathrm{ord}_t(a_3)$. *Then* E'/K' *is of* type $C_{5,4\mu-1}$ *with* $\mu=\mathrm{ord}_t(a_3)-2$, *and*

$$\mathrm{ord}_t(\Delta)=4\,\mathrm{ord}_t(a_3), \qquad \mathrm{ord}_s(\Delta')=8\,\mathrm{ord}_t(a_3)-12,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-4\,\mathrm{ord}_t(a_3)>0.$$

$[C_8]$ *Assume* E/K *is of* type C_8. *Let* l *be the largest positive integer such that*

$$\mathrm{ord}_t(a_1)\geqq 2l-1, \qquad \mathrm{ord}_t(a_3)\geqq 2l+1,$$

$$\mathrm{ord}_t(a_{4,o})\geqq 2l+1, \qquad \mathrm{ord}_t(a_3^2+a_1^2a_4)\geqq 6l-1.$$

$[C_8-C_7]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l$, $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *and* $\mathrm{ord}_t(a_1)=2l-1$, *then* E'/K' *is of* type C_7, *and*

$$\mathrm{ord}_t(\Delta)=12l-1, \qquad \mathrm{ord}_s(\Delta')=12l-2, \qquad \mathrm{ord}_t(j)=12l-11>0.$$

$[C_8-C_8]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l$, $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *and* $\mathrm{ord}_t(a_1)\geqq 2l$, *then* E'/K' *is of* C_8, *and*

$$\mathrm{ord}_t(\Delta)=12l, \qquad \mathrm{ord}_s(\Delta')=12l, \qquad \mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l\geqq 12l.$$

$[C_8-C_1]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+2$, $\mathrm{ord}_t(a_1)\geqq 2l$, $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *and* $\mathrm{ord}_t(a_3)\geqq 2l+2$, *then* E'/K' *is of* type C_1, *and*

$$\mathrm{ord}_t(\Delta)=12l+4, \qquad \mathrm{ord}_s(\Delta')=12l-4,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l-4\geqq 12l-4>0.$$

$[C_8-C_2]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+3$, $\mathrm{ord}_t(a_1)=2l$, $\mathrm{ord}_t(a_3)\geqq 2l+2$, *and* $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *then* E'/K' *is of* type C_2, *and*

$$\mathrm{ord}_t(\Delta)=12l+5, \qquad \mathrm{ord}_s(\Delta')=12l-2, \qquad \mathrm{ord}_t(j)=12l-5>0.$$

$[C_8-C_4]_l$ *If* $\mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+4$, $\mathrm{ord}_t(a_1)\geqq 2l+1$, $\mathrm{ord}_t(a_3)\geqq 2l+3$, $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$, *and then* E'/K' *is of* type C_4, *and*

$$\mathrm{ord}_t(\Delta)=12l+8, \qquad \mathrm{ord}_s(\Delta')=12l+4,$$

$$\mathrm{ord}_t(j)=12\,\mathrm{ord}_t(a_1)-12l-8\geqq 12l+4>0.$$

$[C_8-C_5]_l$ $(l\geqq 2)$ *Assume* $\mathrm{ord}_t(a_{4,o})=2l+1$, *and let* $l'(\geqq l\geqq 2)$ *be the largest positive integer such that*

$$\mathrm{ord}_t(a_1) \geqq 2l'-1, \qquad \mathrm{ord}_t(a_3) \geqq 2l'+1, \qquad \mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 6l'-1.$$

$[C_8-C_{5,4\mu-2}]_{l,l'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4) = \begin{cases} 6l'-1 \\ 6l'+1 \\ 6l'+3, \end{cases} \textit{then } E'/K' \textit{ is of } \text{type } C_{5,4\mu-2} \textit{ with } \mu = \begin{cases} 3l'-3l+1 \\ 3l'-3l+2 \\ 3l'-3l+3, \end{cases}$$

and

$$\mathrm{ord}_t(\Delta) = \begin{cases} 12l'-2 \\ 12l'+2 \\ 12l'+6, \end{cases} \quad \mathrm{ord}_s(\Delta') = \begin{cases} 24l'-4-12l \\ 24l'+4-12l \\ 24l'+12-12l, \end{cases}$$

$$\mathrm{ord}_t(j) = \begin{cases} 24l'-12l-10 \geqq 14 \\ 24l'-12l-2 \geqq 22 \\ 24l'-12l+6 \geqq 30. \end{cases}$$

$[C_8-C_{5,4\mu}]_{l,l'}$ *If*

$$\mathrm{ord}_t(a_3^2+a_1^2a_4) = \begin{cases} 6l' \\ 6l'+2 \\ 6l'+4, \end{cases} \textit{then } E'/K' \textit{ is of } \text{type } C_{5,4\mu} \textit{ with } \mu = \begin{cases} 3l'-3l+1 \\ 3l'-3l+2 \\ 3l'-3l+3, \end{cases}$$

and

$$\mathrm{ord}_t(\Delta) = \begin{cases} 12l' \\ 12l'+4 \\ 12l'+8, \end{cases} \quad \mathrm{ord}_s(\Delta') = \begin{cases} 24l'-12l \\ 24l'+8-12l \\ 24l'+16-12l, \end{cases}$$

$$\mathrm{ord}_t(j) = \begin{cases} 12\,\mathrm{ord}_t(a_1)-12l' \geqq 12l' > 0 \\ 12\,\mathrm{ord}_t(a_1)-12l'-4 \geqq 12l'-4 > 0 \\ 12\,\mathrm{ord}_t(a_1)-12l'-8 \geqq 12l'+4 > 0. \end{cases}$$

PROOF. Now, we prove Theorem 2.1.

[Case 1.] If E/K is of type A, B_ν, C_2, C_3, C_6 or C_7, then the proof is relatively easy and guessed from the way of the following steps. So, we omit the proof.

[Case 2.] The case where E/K is of type $C_{5,\nu}$.

We set $S=\sqrt{a_2}$, $U=1$, $R=T=0$ in $k[[s]]$. Then, by the change of coordinates of E' in (2.2), we have $a_1'=a_1$, $a_2'=\sqrt{a_2}a_1$, $a_3'=a_3$, $a_4'=a_4+\sqrt{a_2}a_3$, and $a_6'=a_6$. Set $l=\mathrm{ord}_t(a_1)$.

(1) Assume that $\nu=4m-3$.

By Lemma 1.6, the Weierstrass normal form satisfies $t|a_1$, $t\|a_2$, $t^{2m}\|a_3$, $t^{2m+1}|a_4$, and $t^{4m}|a_6$. Here, $t^j|a_i$, means a_i is divisible by t^j, and $t^j\|a_i$ means a_i is divisible by t^j and is not divisible by t^{j+1}.

(1-1) If $m\leqq l$, we apply the algorithm of Tate (Lemma 1.7) to (a_i'/s^{im})'s. Then, a_1'/s^m, a_3'/s^{3m}, and a_6'/s^{6m} are divisible by s and $s\|a_4'/s^{4m}$, hence E'/K' is of type C_2, and

$$\begin{aligned}\mathrm{ord}_t(\Delta) &= \mathrm{ord}_t(a_1^6a_6+a_1^5a_3a_4+a_1^4a_2a_3^2+a_1^4a_4^2+a_3^4+a_1^3a_3^3)\\ &= \mathrm{ord}_t(a_3^4) = 8m,\\ \mathrm{ord}_s(\Delta') &= 16m-12m = 4m,\\ \mathrm{ord}_t(j) &= 12l-8m.\end{aligned}$$

(1-2) If $m>l$, we apply the algorithm to (a_i'/s^{il})'s. Then, we have $s|a_1'/s^l$, $s^2|a_3'/s^{3l}$, $s^3|a_4'/s^{4l}$, $s^4|a_6'/s^{6l}$, and $s\|a_2'/s^{2l}$, hence E'/K' is of type $C_{5,\mu}$, and

$$\begin{aligned}\mathrm{ord}_t(\Delta) &= \mathrm{ord}_t(a_1^4a_2a_3^2) = 4l+4m+1,\\ \mathrm{ord}_s(\Delta') &= 8l+8m+2-12l = 8m-4l+2,\\ \mathrm{ord}_t(j) &= 12l-4l-4m-1 = 8l-4m-1\,.\end{aligned}$$

By the formula of Ogg (Lemma 1.5), we have

$$\begin{aligned}\delta &= (4l+4m+1)-(5+4m-3)-2+1\\ &= (8l+4m+2)-(5+\mu)-2+1,\end{aligned}$$

so $\mu=8m-8l+2$.

In the case of type $C_{5,\nu}$, the following subcases are similar to (1).

(2) Assume $\nu=4m-1$.

In this case we have $t|a_1$, $t\|a_2$, $t^{2m+1}\|a_3$, $t^{2m+2}|a_4$, and $t^{4m+2}|a_6$. Apply the algorithm to a_i'/s^{im} if $l>m$, to a_i'/s^{il} otherwise. We get $[C_{5,4m-1}-C_7]$ and $[C_{5,4m-1}-C_{5,4\mu-2}]$.

(3) Assume $\nu=4m-2$.

In this case we have $t|a_1$, $t\|a_2$, $t^{2m+1}|a_3$, $t^{2m+1}\|a_4$, and $t^{4m+1}|a_6$. Apply the algorithm to a_i'/s^{im} if $l>m$, to a_i'/s^{il} otherwise. We get $[C_{5,4m-2}-C_4]$, $[C_{5,4m-2}-C_{5,4\mu-2}]$, $[C_{5,4m-2}-C_{5,4\mu}]_{-1}$, and $[C_{5,4m-2}-C_{5,4\mu}]_{-2}$.

(4) Assume $\nu=4m$.

In this case we have $t|a_1$, $t\|a_2$, $t^{2m+2}|a_3$, $t^{2m+2}\|a_4$, and $t^{4m+3}|a_6$. If $l\leqq m$, apply the algorithm to a_i'/s^{il}. We get $[C_{5,4m}-C_{5,4\mu}]_{-0}$. We will treat the case when $l>m$ in Case 3.

[Case 3.] The case where E/K is of type $C_{5,4m}(\mathrm{ord}_t(a_1)>m)$, C_1, C_4, or C_8.

(1) Assume E/K is of type C_1. Then, a_1, a_2, a_3, and a_4 are divisible by t, and $t\|a_6$.

As described in the statement of Theorem 2.1, let $l>0$ be the largest integer such that

$$\mathrm{ord}_t(a_1) \geqq 2l, \qquad \mathrm{ord}_t(a_3) \geqq 2l+1,$$

$$\mathrm{ord}_t(a_{4,o}) \geqq 2l+1, \qquad \mathrm{ord}_t(a_3^2+a_4a_1^2) \geqq 6l.$$

By the change of coordinate with $U=1$, $R=\sqrt{a_4}$, $S=\sqrt{a_2}+(a_{4,e})^{1/4}$, and $T=\sqrt{a_2a_4+a_6}$ we have

$$s^2,\ s^{4l} \mid a_1' = a_1,$$

$$s^{2l+1} \mid a_2' = (\sqrt{a_2}+(a_{4,e})^{1/4})a_1+(a_{4,o})^{1/2},$$

$$s^2,\ s^{6l} \mid a_3' = a_3+\sqrt{a_4}a_1,$$

$$s^3,\ s^{4l+1} \mid a_4' = \sqrt{a_2}a_3+\sqrt{a_6}a_1+(a_{4,e})^{1/4}(a_3+\sqrt{a_4}a_1),$$

$$s^3,\ s^{6l+1} \mid a_6' = (\sqrt{a_6}+\sqrt{a_2a_4})(a_3+\sqrt{a_4}a_1).$$

Now, we apply the algorithm of Tate to a_i'/s^{il}.

Since $s^{l+1}|a_1'$, we see that

$$\text{if } s^{6l+1} \parallel a_6' (\Leftrightarrow \mathrm{ord}_t(a_3^2+a_4a_1^2)=6l), \text{ then } C_1\ (l\geqq 1).$$

If so, we have

$$\begin{aligned}\mathrm{ord}_t(\Delta) &= \mathrm{ord}_t(a_1^6a_6+a_1^3a_3(a_3^2+a_4a_1^2)+a_1^4a_2a_3^2+(a_3^2+a_4a_1^2)^2)\\ &= \mathrm{ord}_t((a_3^2+a_4a_1^2)^2) = 12l,\end{aligned}$$

$$\mathrm{ord}_s(\Delta') = 2\,\mathrm{ord}_t(\Delta)-12l = 24l-12l = 12l,$$

$$\mathrm{ord}_t(j) = 12\,\mathrm{ord}_t(a_1)-\mathrm{ord}_t(\Delta) \geqq 24l-12l = 12l \geqq 12.$$

Assume that $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+1$. Then, we have

$$\text{if } s^{4l+1} \parallel a_4' (\Leftrightarrow \mathrm{ord}_t(a_1)=2l) \text{ then } C_2(l\geqq 1).$$

If so, we have

$$\mathrm{ord}_t(\Delta) = \mathrm{ord}_t(a_1^6a_6) = 12l+1,$$

$$\mathrm{ord}_s(\Delta') = 12l+2,$$

$$\mathrm{ord}_j(j) = 12l-1 \geqq 11.$$

Assume that $\mathrm{ord}_t(a_1)\geqq 2l+1$.

REMARK 2.1.1. Under the above assumptions, (i.e. $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+1$, $\mathrm{ord}_t(a_1)\geqq 2l+1$, $\mathrm{ord}_t(a_3)\geqq 2l+1$, $\mathrm{ord}_t(a_{4,o})\geqq 2l+1$), we have

$$\mathrm{ord}_t(a_3^2+a_4a_1^2) \geqq 6l+2, \quad \text{and} \quad \text{if } l\geqq 1, \qquad \mathrm{ord}_t(a_3) \geqq 2l+2.$$

PROOF. This follows from the calculation of the valuation of the odd degree terms in $a_3^2+a_4a_1^2$. $\square$

Now, since $s^{3l+2}|a_3'$, $s^{2l+1}|a_2'$, $s^{4l+3}|a_4'$, and $s^{6l+3}|a_6'$, we see that

$$\text{if } s^{6l+3} \| a_2'a_4'+a_6' \ (\Leftrightarrow s^{6l+3} \| a_6' \Leftrightarrow \mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+2), \text{ then } C_4.$$

From now on, we omit the calculus of $\mathrm{ord}_t(\Delta)$, $\mathrm{ord}_s(\Delta')$, and $\mathrm{ord}_t(j)$.
Assume that $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+3$.
Then we have $\mathrm{ord}_t(a_3)\geqq 2$ if $l=0$.

If so, we have $s^{4l+3}|a_4'$ and $s^{6l+4}|a_6'$, therefore, we see that

$$\text{if } s^{2l+1} \| a_2'(\Leftrightarrow \mathrm{ord}_t(a_{4,o})=2l+1), \text{ then } C_{5,\mu}.$$

(μ is determined in the following subcase (2).)
Assume that $\mathrm{ord}_t(a_{4,o})\geqq 2l+3$. Then $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+4$. Therefore, we see that

$$\text{if } s^{4l+3} \| a_4'(\Leftrightarrow \mathrm{ord}_t(a_1)=2l+1), \text{ then } C_7.$$

Assume that $\mathrm{ord}_t(a_1)\geqq 2l+2$. Then we have $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+4$. If $l\geqq 1$, then we have $\mathrm{ord}_t(a_3)\geqq 2l+3$. Therefore, we see that

$$\text{if } s^{6l+5} \| a_6'(\Leftrightarrow \mathrm{ord}_t(a_3^2+a_4a_1^2)=6l+4), \text{ then } C_8.$$

Assume that $\mathrm{ord}_t(a_3^2+a_1^2a_4)\geqq 6l+5$, then $\mathrm{ord}_t(a_3^2+a_1^2a_4)\geqq 6l+6$, and $\mathrm{ord}_t(a_3)\geqq 2l+3$ if $l=0$. This contradicts the maximality of l.

(2) The case where E/K is of type C_1, and E'/K' is of type $C_{5,\mu}$.

From the consideration in (1) of case 3, we know that $l(\geqq 0)$ satisfies $\mathrm{ord}_t(a_1)\geqq 2l+1$, $\mathrm{ord}_t(a_2)\geqq 1$, $\mathrm{ord}_t(a_3)\geqq 2l+2$, $\mathrm{ord}_t(a_{4,o})=2l+1$, $\mathrm{ord}_t(a_6)=1$, and $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l+3$.

Let $l'(\geqq l\geqq 0)$ be the largest integer such that $\mathrm{ord}_t(a_1)\geqq 2l'+1$, $\mathrm{ord}_t(a_3)\geqq 2l'+2$, and $\mathrm{ord}_t(a_3^2+a_4a_1^2)\geqq 6l'+3$.

Assume that $\mathrm{ord}_t(a_3^2+a_1^2a_4)=6l'+3$. Then, we have

$$\begin{aligned}
\mathrm{ord}_t(\Delta) &= \mathrm{ord}_t(a_1^6a_6+a_1^3a_3(a_3^2+a_4a_1^2)+a_1^4a_2a_3^2+(a_3^2+a_1^2a_4)^2) \\
&= \mathrm{ord}_t((a_3^2+a_1^2a_4)^2)=12l'+6, \\
\mathrm{ord}_s(\Delta') &= 24l'-12l+12, \\
\mathrm{ord}_t(a_1) &= \frac{1}{2}(\mathrm{ord}_t(a_{4,o}a_1^2)-2l-1)=\frac{1}{2}(\mathrm{ord}_t(a_3^2+a_1^2a_4)-2l-1) \\
&= \frac{1}{2}(6l'+3-2l-1)=3l'-l+1,
\end{aligned}$$

so,

$$\begin{aligned}\mathrm{ord}_t(j) &= 36l'-12l+12-12l'-6\\ &= 12(l'-l)+12l'+6 \geqq 6.\end{aligned}$$

Finally we obtain μ from the formula of Ogg:

$$\delta = 12l'+6-1-2+1 = (24l'+12-12l)-(5+\mu)-2+1,$$

$$\mu = 12l'-12l+2 = 4(3l'-3l+1)-2.$$

Now assume that $\mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 6l'+4$.

REMARK 2.1.2. Under the above condition,

$$\mathrm{ord}_t(a_1) \geqq 2l'+2.$$

PROOF. We can prove this in a similar way to Remark 2.1.1. □

Since $\mathrm{ord}_t(a_3^2+a_1^2a_4)=6l'+4$, we have

$$\mathrm{ord}_t(\Delta) = 2\,\mathrm{ord}_t(a_3^2+a_1^2a_4) = 12l'+8,$$

$$\mathrm{ord}_s(\Delta') = 24l'-12l+16,$$

$$\mathrm{ord}_t(j) = 12\,\mathrm{ord}_t(a_1)-12l'-8 \geqq 24l'+24-12l'-8 \geqq 16,$$

$$\mu = 4(3l'-3l+1).$$

In the case where $\mathrm{ord}_t(a_3^2+a_1^2a_4)=6l'+5$, $6l'+6$, it is quite similar.

Assume that $\mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 6l'+7$. Then we know that $\mathrm{ord}_t(a_1) \geqq 2l'+3$, $\mathrm{ord}_t(a_4) \geqq 2l'+4$ by a similar way to Remark 2.1.2. Therefore, we have

$$\mathrm{ord}_t(a_3^2+a_1^2a_4) = \begin{cases} 6l'+7 \\ 6l'+8 \end{cases} \Longrightarrow \mathrm{ord}_t(\Delta) = \begin{cases} 12l'+14 \\ 12l'+16, \end{cases}$$

and $\mathrm{ord}_s(\Delta')$, $\mathrm{ord}_t(j)$, and μ are also easily calculated.

If $\mathrm{ord}_t(a_3^2+a_1^2a_4) \geqq 6l'+9$, it contradicts the maximality of l', so, this subcase is proved.

(3) The case where E/K is of type C_4, C_8, or $C_{5,4m}(\mathrm{ord}_t(a_1)>m)$ is treated in a similar way to the subcases (1) and (2).

Hence, we complete the proof of Theorem 2.1.

§ 3. The Weierstrass normal form of the irrational unirational elliptic surfaces of base change type with sections.

Here, we prove Theorem 0.1.

PROPOSITION 3.1. *Let k be an algebraically closed field of characteristic 2, and let E be an elliptic curve over $K=k((t))$. Let us put $K^{(i)}:=K(t^{1/2^i})$ $(i \geqq 0)$,*

$E^{(i)}:=E\times_K K^{(i)}$, and $\Delta^{(i)}:=$the minimal discriminant of $E^{(i)}$. Assume that $\operatorname{ord}\Delta^{(i)}\leqq 12$ *for some $i>0$, and that if* $\operatorname{ord}\Delta=12$, *then* $\operatorname{ord}\Delta^{(i)}<12$ *for some $i>0$. Then, the singular fiber of E is given as in Table* 3.1, *where* $v_1:=\operatorname{ord}_t(a_1)$ *for* $a_1\neq 0$.

Note that if $f: X\to \boldsymbol{P}^1$ is an irrational unirational elliptic surface of base change type with sections, then any singular fiber of f must be formally isomorphic to one of the above [1]—[34] in Table 3.1 by Lemma (1.3). In Table 3.1, shown are the types of the special fibers and the valuations of the j-invariants over the original field K and the extension $K^{(i)}$.

REMARK 3.1.1. In Table 3.1, we have, $\operatorname{ord}\Delta^{(i)}=2^i\operatorname{ord}\Delta$ in [1], $\cdots$, [6], $\operatorname{ord}\Delta^{(i+2)}=\operatorname{ord}\Delta^{(i)}$ $(i\geqq 2)$ in [7], $\cdots$, [20], $\operatorname{ord}\Delta^{(i+2)}=\operatorname{ord}\Delta^{(i)}$ $(i\geqq 0)$ in [21], $\cdots$, [34].

PROOF OF PROPOSITION 3.1. In the following, I, J, M denote positive integers.

Step 1. Easier cases.

(1) If E/K is of type C_3, then the type over $K^{(2I-1)}$ is C_6, and the type over $K^{(2I)}$ is C_3, so we get [29] in Table 3.1.

(2) If E/K is of type C_6, then we get [28] as in (1).

(3) If E/K is of type $[C_2-C_{5,4\mu-3}]$ with $\operatorname{ord}(\Delta^{(0)})=4J$, then the type over $K^{(2I-1)}$ is $[C_{5,4J-3}-C_2]$ with $\operatorname{ord}(\Delta^{(2I-1)})=8J$, and the type over $K^{(2I)}$ is $[C_2-C_{5,4J-3}]$, with $\operatorname{ord}(\Delta^{(2I)})=4J$. We consider only the case where $\operatorname{ord}(\Delta^{(2I-1)})$ or $\operatorname{ord}(\Delta^{(2I)})\leqq 12$, so, putting $J=1, 2, 3$, we get, respectively, [30], [32], and

[96] $$[C_2-C_{5,9}]-[C_{5,9}-C_2]-[C_2-C_{5,9}]-\cdots$$
$$\operatorname{ord}(\Delta^{(0)})=12,\qquad \operatorname{ord}(\Delta^{(1)})=24,\ \cdots.$$

By assumption, the case [96] is excluded.

(4) If E/K is of type $[C_{5,4m-3}-C_2]$, $[C_7-C_{5,4\mu-1}]$, $[C_{5,4m-1}-C_7]$, or $[C_{5,4m-2}, C_{5,4\mu-2}]$, as in (3), we get [31], [33], [34], and

[97] $$[C_7-C_{5,3}]-[C_{5,3}-C_7]-[C_7-C_{5,3}]-\cdots$$
$$\operatorname{ord}(\Delta^{(0)})=12,\qquad \operatorname{ord}(\Delta^{(1)})=12,\ \cdots,$$

[98] $$[C_{5,2}-C_{5,2}]-[C_{5,2}-C_{5,2}]-[C_{5,2}-C_{5,2}]-\cdots$$
$$\operatorname{ord}(\Delta^{(0)})=12,\qquad \operatorname{ord}(\Delta^{(1)})=12,\ \cdots.$$

By assumption, the cases [97] and [98] are excluded.

(5) If E/K is of type $[C_2-C_{5,4\mu-1}]$ with $\operatorname{ord}(\Delta^{(0)})=4J+2$, then the type over $K^{(2I-1)}$ is $[C_{5,4J-1}-C_7]$ with $\operatorname{ord}(\Delta^{(2I-1)})=8J+4$, and the type over $K^{(2I)}$ is $[C_7-C_{5,4J-1}]$ with $\operatorname{ord}(\Delta^{(2I)})=16J-4$. So, putting $J=1$, we get [12].

Table 3.1(1/2)

type of fiber/K	$\mathrm{ord}_t(j)/K$	type over $K^{(0)}$ /ord$(\Delta^{(0)})$	type over $K^{(1)}$ /ord$(\Delta^{(1)})$	type over $K^{(2)}$ /ord$(\Delta^{(2)})$	type over $K^{(3)}$ /ord$(\Delta^{(3)})$	type over $K^{(4)}$ /ord$(\Delta^{(4)})$	…
[1]	-1	$B_1/1$	$B_2/2$	$B_4/4$	$B_8/8$	…	…
[2]	-2	$B_2/2$	$B_4/4$	$B_6/8$	…	…	…
[3]	-3	$B_3/3$	$B_6/6$	$B_{12}/12$	…	…	…
[4]	-4	$B_4/4$	$B_8/8$	…	…	…	…
[5]	-5	$B_5/5$	$B_{10}/10$	…	…	…	…
[6]	-6	$B_6/6$	$B_{12}/12$	…	…	…	…
[7]	5	$[C_1-C_7]_0/7$	$[C_7-C_{5,5}]/14$	$[C_{5,5}-C_2]/16$	$[C_2-C_{5,5}]/8$	$[C_{5,5}-C_2]/16$	…
[8]	9	$[C_4-C_7]_1/15$	$[C_7-C_{5,9}]/18$	$[C_{5,9}-C_2]/24$	$[C_2-C_{5,9}]/12$	$[C_{5,9}-C_2]/24$	…
[9]	1	$[C_8-C_7]_1/11$	$[C_7-C_{5,1}]/10$	$[C_{5,1}-C_2]/8$	$[C_2-C_{5,1}]/4$	$[C_{5,1}-C_2]/8$	…
[10]	3	$[C_4-C_2]_1/9$	$[C_2-C_{5,3}]/6$	$[C_{5,3}-C_7]/12$	$[C_7-C_{5,3}]/12$	$[C_{5,3}-C_7]/12$	…
[11]	6	$[C_1-C_{5,2}]_{0,0}/6$	$[C_{5,2}-C_{5,2}]/12$	$[C_{5,2}-C_{5,2}]/12$	$[C_{5,2}-C_{5,2}]/12$	…	…
[12]	6	$[C_2-C_{5,3}]/6$	$[C_{5,3}-C_7]/12$	$[C_7-C_{5,3}]/12$	$[C_{5,3}-C_7]/12$	…	…
[13]	2	$[C_7-C_{5,1}]/10$	$[C_{5,1}-C_2]/8$	$[C_2-C_{5,1}]/4$	$[C_{5,1}-C_2]/8$	…	…
[14]	10	$[C_7-C_{5,5}]/14$	$[C_{5,5}-C_2]/16$	$[C_2-C_{5,5}]/8$	$[C_{5,5}-C_2]/16$	…	…
[15]	18	$[C_7-C_{5,9}]/18$	$[C_{5,9}-C_2]/24$	$[C_2-C_{5,9}]/12$	$[C_{5,9}-C_2]/24$	…	…
[16]	4	$[C_{5,8}-C_{5,4}]$-0/20	$[C_{5,4}-C_1]_1/16$	$[C_1-C_{5,4}]_{0,0}/8$	$[C_{5,4}-C_1]_1/16$	…	…
[17]	2	$[C_{5,2}-C_4]/10$	$[C_4-C_1]_1/8$	$[C_1-C_4]_0/4$	$[C_4-C_1]_1/8$	…	…

Table 3.1(2/2)

type of fiber/K	$\mathrm{ord}_t(j)/K$	type over $K^{(0)}$ /ord$(\Delta^{(0)})$	type over $K^{(1)}$ /ord$(\Delta^{(1)})$	type over $K^{(2)}$ /ord$(\Delta^{(2)})$	type over $K^{(3)}$ /ord$(\Delta^{(3)})$	type over $K^{(4)}$ /ord$(\Delta^{(4)})$	$\cdots$
[18]	6	$[C_{5,6}-C_4]/18$	$[C_4-C_4]_1/12$	$[C_4-C_4]_1/12$	$[C_4-C_4]_1/12$	$\cdots$	$\cdots$
[19]	1	$[C_{5,3}-C_{5,2}]/11$	$[C_{5,2}-C_4]/10$	$[C_4-C_1]_1/8$	$[C_1-C_4]_0/4$	$[C_4-C_1]_1/8$	$\cdots$
[20]	3	$[C_{5,9}-C_{5,6}]/21$	$[C_{5,6}-C_4]/18$	$[C_4-C_4]_1/12$	$[C_4-C_4]_1/12$	$\cdots$	$\cdots$
[21]	$12v_1-8\geqq 4$	$[C_4-C_1]_1/8$	$[C_1-C_4]_0/4$	$[C_4-C_1]_1/8$	$[C_1-C_4]_0/4$	$\cdots$	$\cdots$
[22]	$12v_1-4\geqq 8$	$[C_1-C_4]_0/4$	$[C_4-C_1]_1/8$	$[C_1-C_4]_0/4$	$[C_4-C_1]_1/8$	$\cdots$	$\cdots$
[23]	$12v_1-8\geqq 16$	$[C_1-C_8]_0/8$	$[C_8-C_1]_1/16$	$[C_1-C_8]_0/8$	$[C_8-C_1]_1/16$	$\cdots$	$\cdots$
[24]	$12v_1-16\geqq 8$	$[C_8-C_1]_1/16$	$[C_1-C_8]_0/8$	$[C_8-C_1]_1/16$	$[C_1-C_8]_0/8$	$\cdots$	$\cdots$
[25]	$12v_1-16\geqq 8$	$[C_{5,4}-C_1]_1/16$	$[C_1-C_{5,4}]_{0,0}/8$	$[C_{5,4}-C_1]_1/16$	$[C_1-C_{5,4}]_{0,0}/8$	$\cdots$	$\cdots$
[26]	$12v_1-8\geqq 16$	$[C_1-C_{5,4}]_{0,0}/8$	$[C_{5,4}-C_1]_1/16$	$[C_1-C_{5,4}]_{0,0}/8$	$[C_{5,4}-C_1]_1/16$	$\cdots$	$\cdots$
[27]	$12v_1-24\geqq 12$	$[C_{5,8}-C_1]_1/24$	$[C_1-C_{5,8}]_{0,0}/12$	$[C_{5,8}-C_1]_1/24$	$[C_1-C_{5,8}]_{0,0}/12$	$\cdots$	$\cdots$
[28]	$12v_1-8\geqq 4$	$C_6/8$	$C_3/4$	$C_6/8$	$C_3/4$	$\cdots$	$\cdots$
[29]	$12v_1-4\geqq 8$	$C_3/4$	$C_6/8$	$C_3/4$	$C_6/8$	$\cdots$	$\cdots$
[30]	$12v_1-8\geqq 4$	$[C_2-C_{5,1}]/4$	$[C_{5,1}-C_2]/8$	$[C_2-C_{5,1}]/4$	$[C_{5,1}-C_2]/8$	$\cdots$	$\cdots$
[31]	$12v_1-8\geqq 4$	$[C_{5,1}-C_2]/8$	$[C_2-C_{5,1}]/4$	$[C_{5,1}-C_2]/8$	$[C_2-C_{5,1}]/4$	$\cdots$	$\cdots$
[32]	$12v_1-8\geqq 16$	$[C_2-C_{5,5}]/8$	$[C_{5,5}-C_2]/16$	$[C_2-C_{5,5}]/8$	$[C_{5,5}-C_2]/16$	$\cdots$	$\cdots$
[33]	$12v_1-16\geqq 8$	$[C_{5,5}-C_2]/16$	$[C_2-C_{5,5}]/8$	$[C_{5,5}-C_2]/16$	$[C_2-C_{5,5}]/8$	$\cdots$	$\cdots$
[34]	$12v_1-24\geqq 12$	$[C_{5,9}-C_2]/24$	$[C_2-C_{5,9}]/12$	$[C_{5,9}-C_2]/24$	$[C_2-C_{5,9}]/12$	$\cdots$	$\cdots$

(6) If E/K is of type $[C_7-C_{5,4\mu-3}]$, as in (5), we get [13], [14], and [15].

(7) If E/K is of type $[C_1-C_{5,4\mu-2}]_{l,l'}$, with $\mathrm{ord}(\Delta^{(0)})=12l'+6$, then the type over $K^{(2I-1)}$ is $[C_{5,4(3l'-3l+1)-2}-C_{5,4(3l+1)-2}]$ with $\mathrm{ord}(\Delta^{(2I-1)})=24l'-12l+12$, and the type over $K^{(2I)}$ is $[C_{5,4(3l+1)-2}-C_{5,4(3l'-3l+1)-2}]$ with $\mathrm{ord}(\Delta^{(2I)})=12l'+12$. So, putting $l'=l=0$, we get [11].

(8) If E/K is of type $[C_1-C_{5,4\mu-2}]$, $[C_4-C_{5,4\mu-2}]$, $[C_{5,4m}-C_{5,4\mu-2}]$, or $[C_8-C_{5,4\mu-2}]$, we can treat as above. Since $\mathrm{ord}(\Delta^{(I)})>12$ in any cases, we can exclude these cases.

(9) If E/K is of type $[C_4-C_2]_l$ with $\mathrm{ord}(\Delta^{(0)})=12l-3$, then the type over $K^{(1)}$ is $[C_2-C_{5,4(3l-2)-1}]$ with $\mathrm{ord}(\Delta^{(1)})=12l-6$, the type over $K^{(2I)}$ is $[C_{5,4(3l-2)-1}-C_7]$ with $\mathrm{ord}(\Delta^{(2I)})=24l-12$, and the type over $K^{(2I+1)}$ is $[C_7-C_{5,4(3l-2)-1}]$ with $\mathrm{ord}(\Delta^{(2I+1)})=12l$. Putting $l=1$, we get [10].

(10) If E/K is of type $[C_4-C_7]$, $[C_1-C_2]_l$, $[C_1-C_7]_l$, $[C_{5,4m}-C_2]_i$, $[C_{5,4m}-C_7]_i$, $[C_8-C_2]_l$, or $[C_8-C_7]_l$, as in (9), we get [7], [8], [9].

(11) If E/K is of type B_ν, then it is clear that $\nu\leqq 6$ and we get [1], $\cdots$, [6] in Table 3.1.

Step 2. Relatively stable cases.

(12) Assume E/K is of type $[C_4-C_1]_l$, $[C_4-C_{5,4(3l'-3l+2)}]_{l,l'}$, $[C_{5,4(3M)}-C_1]_i$, $[C_{5,4(3M)}-C_{5,4(3i'-3i+2)}]_{i,i'}$, $[C_1-C_4]_l$, $[C_1-C_{5,4(3l'-3l+3)}]_{l,l'}$, $[C_{5,4(3M-1)}-C_4]_i$, or $[C_{5,4(3M-1)}-C_{5,4(3i'-3i+3)}]_{i,i'}$. Temporarily, we call the 8 types, (a), (b), (c), (d), (e), (f), (g), (h) in the above order.

LEMMA 3.1.2. *If E/K is of type* (a), (b), (c), (d) (*resp.* (e), (f), (g), (h)), *then the type over $K^{(2I-1)}$ is* (e), (f), (g), (h) (*resp.* (a), (b), (c), (d)), *and the type over $K^{(2I)}$ is* (a), (b), (c), (d) (*resp.* (e), (f), (g), (h)) *respectively.*

PROOF. It suffices to prove this lemma only for $K^{(1)}$ and $K^{(2)}$. Let $N^{(0)}$, $N^{(1)}$, $N^{(2)}$ be the number of irreducible components of the special fiber of E, $E\times_K K^{(1)}$, $E\times_K K^{(2)}$, respectively.

Recall that Serre's mesure $\delta(E)$ is invariant under a purely inseparable base change. By Theorem 2.1, if E is of additive reduction type (see [**17**], p. 179), then so are $E\times_K K^{(1)}$ and $E\times_K K^{(2)}$. Therefore, by the formula of Ogg (Lemma 1.5), we have

$$\mathrm{ord}(\Delta^{(0)})-N^{(0)}=\mathrm{ord}(\Delta^{(1)})-N^{(1)}=\mathrm{ord}(\Delta^{(2)})-N^{(2)}.$$

In particular, we have

$$\mathrm{ord}(\Delta^{(0)})-N^{(0)}\equiv\mathrm{ord}(\Delta^{(1)})-N^{(1)}\equiv\mathrm{ord}(\Delta^{(2)})-N^{(2)}\qquad(\mathrm{mod}\ 12).$$

Hence, if E/K is of type (a), (b), (c), or (d), by Table 1.2 and Theorem 2.1,

$$N^{(1)}\equiv 1,\qquad N^{(2)}\equiv 5,\qquad(\mathrm{mod}\ 12).$$

So, again by Table 1.2, $E\times K^{(1)}$ is of type C_1 or $C_{5,4(3M-1)}$, and $E\times K^{(2)}$ is of type C_4 or $C_{5,4(3M)}$. The expected results are acquired from the calculus of $\mathrm{ord}(\Delta^{(1)})$, $\mathrm{ord}(\Delta^{(2)})$ and Theorem 2.1.

If E/K is of type (e), (f), (g), or (h), then we get the result in a similar way. □

By the above lemma, if E/K is of type (a), (b), $\cdots$, (h), then so is $E\times_K K^{(i)}$ for all $I\geqq 0$.

It is clear that $\mathrm{ord}(\Delta^{(0)})\leqq 12$ if and only if E/K is either of type (a) and $l=1$, or of type (e) and $l=0$.

Suppose E/K is of type (b), (c), (d), (f), (g), or (h). Then, we have $\mathrm{ord}(\Delta^{(1)})\geqq 16$. Therefore, by Lemma 3.1.2, we have $\mathrm{ord}(\Delta^{(I)})\geqq 16$ for all $I>0$ in these cases. Hence, by assumption, these cases are excluded. Suppose E/K is of type (a) with $l=1$ (resp. of type (e) with $l=0$). Then, we have $\mathrm{ord}\,\Delta^{(0)}=8$ resp. $\mathrm{ord}\,\Delta^{(0)}=4$). Using these facts, we get [21] and [22] in Table 3.1.

(13) Assume E/K is of type $[C_4-C_4]$, $[C_4-C_{5,4\mu}]$, $[C_4-C_8]$, $[C_1-C_4]$, $[C_1-C_{5,4\mu}]$, $[C_1-C_8]$, $[C_{5,4m}-C_1]$, $[C_{5,4m}-C_4]$, $[C_{5,4m}-C_{5,4\mu}]$, $[C_{5,4m}-C_8]$, $[C_8-C_1]$, $[C_8-C_4]$, $[C_8-C_{5,4\mu}]$, or $[C_8-C_8]$. We get easily [23], [24], [25], [26], and [27] in a similar way.

Step 3. Remaining cases.

(14) Assume E/K is of type $[C_{5,4m}-C_{5,4\mu}]$-0.

Set $l=\mathrm{ord}_t(a_1)$. If $l<(2m+2)/3$, the type over $K^{(1)}$ is also $[C_{5,4(2m-2l+1)}-C_{5,4m}]$-0. Putting $l=m=1$, we get

[99] $$[C_{5,4}-C_{5,4}]_0-[C_{5,4}-C_{5,4}]_0-\cdots$$
$$\mathrm{ord}(\Delta^{(0)})=12,\qquad \mathrm{ord}(\Delta^{(1)})=12,\ \cdots.$$

If $l\geqq(2m+2)/3$, we see easily that it suffices to check the cases where the type over $K^{(1)}$ is either $[C_{5,4}-C_1]$ with $\mathrm{ord}(\Delta^{(1)})=16$([25]) or $[C_{5,8}-C_1]$ with $\mathrm{ord}(\Delta^{(1)})=24$ ([27]). The type over $K^{(1)}$ becomes [25] if and only if $m=l=2$, so we get [16]. Since $8\neq 4(2m-2l+1)$ for any m, l, the case where the type $[C_{5,4}-C_1]$ with $\mathrm{ord}(\Delta^{(1)})=16$ over $K^{(1)}$ is excluded.

(15) Assume E/K is of type $[C_{5,4m-2}-C_{5,4\mu}]$. As above, it suffices to choose the cases where it comes to [25], [27] by Step 2, or [16], [99] by (14). Since it is impossible, we have none in this case.

(16) Assume E/K is of type $[C_{5,4m-2}-C_4]$. Considering if $\mathrm{ord}(\Delta^{(1)})\leqq 12$, or if it comes to [21] in Step 2, we get [17], [18].

(17) If E/K is of type $[C_{5,4m-3}-C_{5,4\mu-2}]$ or $[C_{5,4m-1}-C_{5,4\mu-2}]$, we get [19] or [20] since it must come to [17] or [18] over $K^{(1)}$, respectively.

This completes the proof of Proposition 3.1.

PROPOSITION 3.2. *Let $f: X \to \boldsymbol{P}^1$ be an irrational unirational elliptic surface of base change type with sections in characteristic 2. Then, X does not contain fibers of type* [1], [2], $\cdots$, [20]. *So, the j-invariant of the generic fiber is* 0. *Moreover $a_1(t)$ in Weierstrass normal form is* 0.

PROOF. If j-invariant is 0, then we have $a_1(t)=0$ since $j=a_1^{12}/\Delta$ in characteristic 2.

By Lemma 1.2, there exist a purely inseparable regular map $g: \boldsymbol{P}^1 \to \boldsymbol{P}^1$ and a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \overset{\tilde{g}}{\dashrightarrow} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \boldsymbol{P}^1 & \xrightarrow[g]{} & \boldsymbol{P}^1, \end{array}$$

where $\tilde{X}$ denotes the rational surface with structure of a relatively minimal elliptic surface $\tilde{f}: \tilde{X} \to \boldsymbol{P}^1$ which is birationally equivalent to $X \times_{\boldsymbol{P}^1} \boldsymbol{P}^1$ and where $\tilde{g}$ is the reduced rational map. By Remark 3.1.1, we may assume that the degree of g is 2, 4, or 8. We can assume that X has a Weierstrass normal form which is minimal over every point of the affine line $\boldsymbol{A}^1$ (since $k[t]$ is a principal ideal domain, this can be proved in a similar way to Proposition 8.2 in [16]) and that X has a regular fiber over the point at infinity.

In this normal form, we denote by $\Delta(t)$ the minimal discriminant, and by $j(t)$ the j-invariant. We also select a minimal Weierstrass normal form of $\tilde{f}: \tilde{X} \to \boldsymbol{P}^1$, and we denote by $\tilde{\Delta}(t)$ the corresponding discriminant.

Step 1. If X contains some fibers of type [7], $\cdots$, [20] and none of type [1], $\cdots$, [6], then $j(t)$ has only zeros and no poles on $\boldsymbol{P}^1$. This is absurd.

Step 2. Assume X contains some fibers of type [1], $\cdots$, [6] and none of type [7], $\cdots$, [34]. Then $j(t)$ has poles of total order 3 (if g has degree 4) or of total order 6 (if g has degree 2). But since $j=a_1^{12}/\Delta$, under regular fibers, $j(t)$ has a zero of order multiple of 12. It is absurd, too.

Step 3. Now we prove that if $c_2(\tilde{X})=12$, then X cannot contain any fibers of type [1], $\cdots$, [20].

(1) Assume $g: \boldsymbol{P}^1 \to \boldsymbol{P}^1$ has degree 8. Since $c_2(\tilde{X})=12$, combinations of the singular fibers of f which do not contradict steps 1, 2 are given by {[1], [9]}, {[1], [19]}, {[1], [21]}, {[1], [28]}, or {[1], [31]}. If the combination of singular fibers of f is either {[1], [9]}, or {[1], [19]}, then we have $c_2(X)=12$, hence, X is a rational surface. Therefore, these cases are excluded. If the combination of singular fibers of f is one of {[1], [21]}, {[1], [28]}, {[1], [31]}, then $j(t)$ has only one pole of order 1 under the fiber of type [1], but a zero of order greater than 4 under the other singular fiber, which is impossible.

(2) If g has degree 4 or 2, the proof is similar to (1). □

PROPOSITION3.3. *Any combination of the singular fibers of an irrational unirational elliptic surface $f: X \to \boldsymbol{P}^1$ of base change type with sections in characteristic 2 is given by one of the following.*

(1) {[27]}, (2) {[34]},

(3) {[21], [24]}, (4) {[21], [25]}, (5) {[21], [33]},

(6) {[28], [24]}, (7) {[28], [25]}, (8) {[28], [33]},

(9) {[31], [24]}, (10) {[31], [25]}, (11) {[31], [33]},

(12) {[21], [21], [28]}, (13) {[21], [28], [28]},

(14) {[28], [28], [31]}, (15) {[31], [31], [28]},

(16) {[21], [21], [31]}, (17) {[31], [31], [21]},

(18) {[21], [21], [21]}, (19) {[28], [28], [28]},

(20) {[31], [31], [31]}, (21) {[21], [28], [31]}.

PROOF. Since the possibilities of types of singular fibers are [21], $\cdots$, [34], it suffices to consider the following situation:

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\tilde{g}"] & X \arrow[d, "f"] \\ \boldsymbol{P}^1 \arrow[r, "F"'] & \boldsymbol{P}^1, \end{tikzcd}$$

where F is a purely inseparable morphism of degree 2, $\tilde{X}$ is a rational surface which is birationally equivalent to $X \times_{\boldsymbol{P}^1} \boldsymbol{P}^1$, $\tilde{f}: \tilde{X} \to \boldsymbol{P}^1$ is a relatively minimal elliptic surface, $\tilde{g}$ is a rational map which makes the diagram generically commutative. We choose all combinations of singular fibers on X which satisfies $c_2(\tilde{X})=12$ and $c_2(X) \neq 12$. Then we have the list of combinations in this proposition. □

PROOF OF THEOREM 0.1. By Proposition 3.3, we see that the number of singular fibers of $f: X \to \boldsymbol{P}^1$ is at most 3. So, we take (1), (3), and (12) in Proposition 3.3 as examples of irrational unirational elliptic surfaces $f: X \to \boldsymbol{P}^1$ of base change type with one singular fibers, two singular fibers, and three singular fibers, respectively, and prove the theorem only in these cases. The remaining cases can be proved similarly.

We write a minimal Weierstrass normal form of $f: X \to \boldsymbol{P}^1$ as follows:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

with $a_i \in k[t]$.

Proof of (1).

We may assume the singular fiber is over the point $t=0$ in $\boldsymbol{P}^1$. Since f has a regular fiber over the point $t=\infty$, as in the proof of Theorem II in Katsura [4] (p. 541), we have

$$a_1=0,\quad \deg a_3=6,\quad \deg a_2\leqq 4,\quad \deg a_4\leqq 8,\quad \deg a_6\leqq 12,$$

Since the type of the singular fiber is $[C_{5,8}-C_1]_1$ in Theorem 2.1, we have

$$\mathrm{ord}_t(a_3)=6,\quad \mathrm{ord}_t(a_2)=1,\quad \mathrm{ord}_t(a_4)=6,\quad \mathrm{ord}_t(a_6)\geqq 11.$$

So, we can write

$$a_3=t^6,\quad a_2=t(b_0t^3+b_1t^2+b_2t+b_3),\quad (b_3\neq 0),$$
$$a_4=t^6(c_0t^2+c_1t+c_2),\quad (c_2\neq 0),\quad a_6=t^{11}(d_0t+d_1),$$

with b_i, c_i, $d_i\in k$. By a suitable coordinate change, we can make c_0 and c_1 to zero, so we get a minimal normal form of f as in Theorem 0.1.

Proof of (3).

Assume the fiber of the type $[C_4-C_1]$ is over $t=0$, and the fiber of type $[C_8-C_1]$ is over $t=\infty$. Since the order of a minimal discriminant over $t=\infty$ is equal to 16, we have as in the proof of (1),

$$a_1=0,\quad \deg a_3=2,\quad \deg a_2\leqq 4,\quad \deg a_4\leqq 8,\quad \deg a_6\leqq 12.$$

So, by Theorem 2.1, we can write

$$a_3=t^2,\quad a_2=t(b_0t+b_1),\quad a_4=c_0t^2,\quad a_6=t^3(d_0t^4+d_1t^3+d_2t^2+d_3t+d_4),$$

with

$$b_1c_0+d_4\neq 0,\qquad d_0\neq 0.$$

After a suitable coordinate change, we have a minimal normal form of f as in Theorem 0.1.

Proof of (12).

Assume the fibers of type $[C_4-C_1]$ are over $t=0$, 1, and the fiber of type $[C_6]$ is over $t=\infty$. Since the order of a minimal discriminant over $t=\infty$ is equal to 8, we have as in the proof of (1),

$$a_1=0,\quad \deg a_3=4,\quad \deg a_2\leqq 4,\quad \deg a_4\leqq 8,\quad \deg a_6\leqq 12.$$

So, from Theorem 2.1, we can write

$$a_3=t^2(t+1)^2,\quad a_2=b_0t(t+1),\quad a_4=t^2(t+1)^2(c_0t+c_1),$$
$$a_6=t^3(t+1)^3(d_0t^2+d_1t+d_2),$$

with

$$b_0c_1+d_2 \neq 0, \quad b_0(c_0+c_1) \neq d_0+d_1+d_2 .$$

After a suitable coordinate change, we have a minimal normal form of f as in Theorem 0.1. □

PROOF OF COROLLARY 0.1.1. By Proposition 3.3 and Lemma 1.4, we have $c_2(X)=24$, hence X is a K3 surface. □

PROOF OF COROLLARY 0.1.2. By Proposition 3.2, the j-invariant of the generic fiber is 0. Therefore, it is supersingular. □

Finally, we calculate the Mordell-Weil group.

PROPOSITION 3.4. *Let r_i (resp. t_i) $(1 \geqq i \geqq 21)$ be the Mordell-Weil rank (resp. the order of the torsion subgroup of the Mordell-Weil group) of $f_i: X_i \to \boldsymbol{P}^1$, where $f_i: X_i \to \boldsymbol{P}^1$ is the elliptic surface defined by the Weierstrass normal form of* (i) *in Theorem* 0.1. *Then,*

i	1	2	3	4	5	6	7	8	9	10
r_i	8	7	8	8	7	6	6	5	7	7
t_i	0	0	0	0	0	0	0	0	0	0

i	11	12	13	14	15	16	17	18	19	20	21
r_i	6	6	4	3	4	7	6	8	2	5	5
t_i	0	0	0	0	0	0	0	0	3	0	0

PROOF.
By Lemmas 1.8, 1.9, and 1.10.

$$22 = c_2(X_i)-2 = \rho(X_i) = r_i+2+\sum_{\alpha=0,1,\infty}(m_\alpha-1).$$

Here, m_α denotes the number of irreducible components of the fiber over $t=\alpha$. Hence,

$$r_i = 23-\sum_{\alpha=0,1,\infty} m_\alpha .$$

Since we know m_α's explicitly, we can calculate r_i form this formula.

For $\alpha=0, 1, \infty$, let G_α be the group of the non-singular part of the fiber over $t=\alpha$. Since we have $(t_i, 2)=1$ from Corollary 0.1.2, we see t_i is a divisor of the order of the torsion subgroup of G_α for $\alpha=0, 1, \infty$ (cf. p. 304, Proposition 5.3.4 [**2**]). It follows that $t_i=0$ except for the case of $i=19$, and in the case of $i=19$, $t_i \leqq 3$. Since $y=0$ gives points of order 3 of (19), we conclude that $t_i=3$ for (19). □

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