J. Math. Soc. Japan
Vol. 44, No. 4, 1992

Pushing up by 2′-automorphisms of a Sylow 2-subgroup

By Yasuhiko TANAKA

(Received Nov. 8, 1991)

Section 1. Introduction.

In this paper, we study a certain "pushing up" problem of finite groups. So-called pushing up problems have been born in the study to classify the finite simple groups. Among them, one of the most interesting thing is to determine the isomorphism classes of amalgams of finite groups. Here, by an amalgam, we mean a triple (X, B, Y) of finite groups satisfying the following conditions: B is a common subgroup of X and Y; no nonidentity subgroup of B is normal both in X and in Y.

Many group theorists have been investigating those problems, and obtaining several results. Some of them have been applied to revising the classification of the finite simple groups of characteristic 2 type, where, by definition, a finite group G of even order is of characteristic 2 type if every 2-local subgroup L of G satisfies the condition that $C_L(O_2(L)) \subseteq O_2(L)$. For example, we have determined the isomorphism classes of amalgams of 2-irreducible solvable groups [**7**], [**13**], [**14**], [**8**]; the results are applied to the alternative proof [**1**], [**4**], [**5**] of the theorem of the combined work of Janko [**9**], Smith [**10**], Gorenstein and Lyons [**6**], which classifies the finite simple groups with solvable 2-local subgroups.

A finite group G is said to be 2-irreducible if a Sylow 2-subgroup of G is contained in a unique maximal subgroup of G. In the course of the analysis in [**14**], [**8**], we needed the following theorem in order to delete the case that either X or Y is 2-closed in an amalgam (X, B, Y).

THEOREM A ((2.3) of [**14**]). *Let G be a 2-irreducible solvable group with $C_G(O_2(G)) \subseteq O_2(G)$. Let $S \in \mathrm{Syl}_2(G)$, and let A be a group of automorphisms of S of odd order. Then some nonidentity A-invariant subgroup of S is normal in G.*

The purpose of this paper is to generalize Theorem A for nonsolvable groups. We obtain the following theorem.

Supported in part by the Grant-in-Aid for Scientific Research from Oita University (No. 03-1-05).

THEOREM B. *Let G be a 2-irreducible group with $C_G(O_2(G))\subseteqq O_2(G)$. Let $S\in \mathrm{Syl}_2(G)$, and let A be a group of automorphisms of S of odd order. Then some nonidentity A-invariant subgroup of S is normal in G.*

In his papers [**11**], [**12**], Stellmacher already proved the similar theorem under the additional hypothesis that $G/O_2(G)\cong SL_2(2^n)$ $(n\geqq 1)$. Thus we can say that Theorem B is a generalization of his result as well.

Since our aim is to analyze the 2-local structure of finite simple groups of characteristic 2 type, we will prove Theorem B for $\mathcal{K}$-groups, that is, finite groups all of whose simple sections are isomorphic to known simple groups. Considering that Theorem A already contributes to revising the classification of the finite simple groups, we can hope that Theorem B also does in the near future.

The proof of Theorem B (including the proof of Theorem A) is given in Section 3. If Theorem B is false, then no nonidentity characteristic subgroup of S is normal in G. A discription of such a $\mathcal{K}$-group G is already given in [**2**], which asserts that $J(G)\equiv T\times L\,(\mathrm{mod}\,Z(J(G)))$, where T is a 2-group and L is a direct product of copies of $SL_2(2^m)\boldsymbol{E}_{2^m}\,(m\geqq 1)$ or copies of $\Sigma_{2m+1}\boldsymbol{E}_{2^m}\,(m=2^l\geqq 1)$. With the help of the properties of $GF(2)$-representations of $SL_2(2^m)$ and Σ_{2m+1} collected in Section 2, we conclude that $\langle O_2(L)^A\rangle \lhd G$, thereby proving Theorem B.

Throughout the remainder of this paper, all groups are finite and all $GF(2)$-representations are finite dimensional. We close this section with some definitions.

Let G be a group. We denote by $\mathcal{A}(G)$ the set of elementary abelian 2-subgroups of G of maximal order. Define $J(G)=\langle \mathcal{A}(G)\rangle$, the Thompson subgroup of G. Let V be a $GF(2)G$-module. Suppose that V is faithful. An offending subgroup of G with respect to V is an elementary abelian 2-subgroup B of G with $|V: C_V(B)|\leqq |B|$. We denote by $\mathcal{O}(G, V)$ the set of offending subgroups of G with respect to V. Let H be another group, and let W be a $GF(2)H$-module. Suppose that there exists a homomorphism $\sigma: G\to H$. We say that V is induced by W through σ if there exists a linear isomorphism $f: V\to W$ such that $(vg)^f=v^f g^\sigma$ for all $v\in V$ and all $g\in G$.

Section 2. Linear groups and symmetric groups.

In this section, we review $GF(2)$-representations of $SL_2(2^m)$ $(m\geqq 1)$ and $\Sigma_{2m+1}\,(m=2^l\geqq 1)$.

First, we define the natural modules for $SL_2(2^n)$ and Σ_n.

Let $n\geqq 1$. Let N_n be the 2-dimensional vector space of row vectors with coefficients in $GF(2^n)$. Then $SL_2(2^n)$ acts on N_n by right multiplication, and so N_n becomes a $GF(2)SL_2(2^n)$-module. In the following, we will call N_n the natural module for $SL_2(2^n)$.

Let $n\geqq 3$. Let Ω_n be a set of n letters, and denote by $\mathcal{P}(\Omega_n)$ the power set of Ω_n. Then Σ_n acts on $\mathcal{P}(\Omega_n)$. We can regard $\mathcal{P}(\Omega_n)$ as a $GF(2)\Sigma_n$-module in the following addition:

$$X+Y=(X\cup Y)-(X\cap Y) \qquad \text{for all} \quad X, Y\in\mathcal{P}(\Omega_n).$$

The $GF(2)\Sigma_n$-module $\mathcal{P}(\Omega_n)$ has two nontrivial submodules $\mathcal{P}_0$ and $\mathcal{P}_1$: $\mathcal{P}_0$ is the submodule generated by Ω_n; $\mathcal{P}_1$ is the submodule generated by the subsets of Ω_n of even order. Put $M_n=(\mathcal{P}_0+\mathcal{P}_1)/\mathcal{P}_0$. In the following, we will call M_n the natural module for Σ_n. Note that $\mathcal{P}(\Omega_n)=\mathcal{P}_0\oplus\mathcal{P}_1$ if n is odd. Therefore we have $M_n\cong\mathcal{P}_1$ if n is odd.

Now, we record the necessary facts on the natural module N_m of $SL_2(2^m)$ $(m\geqq 1)$, and on the natural module M_{2m+1} of $\Sigma_{2m+1}(m=2^l\geqq 1)$.

(2.1) LEMMA. *Let $G=SL_2(2^m)\,(m\geqq 1)$ or $\Sigma_{2m+1}\,(m=2^l\geqq 1)$, and $S\in\mathrm{Syl}_2(G)$. Let V be the natural module for G, and put $\mathcal{O}=\mathcal{O}(S, V)$. Then the following holds.*

(1) $|V: C_V(z)|=2^m$ *for all* $z\in Z(S)-\{1\}$.

(2) $m_2(S)=m$.

(3) *Let $B\in\mathcal{O}$. Then $|V: C_V(B)|=|B|$, and the following holds.*

 (i) *If $G=SL_2(2^m)$, then $B=S$ or 1.*

 (ii) *If $G=\Sigma_{2m+1}$, then B is generated by a subset of $\mathcal{T}_S$, where $\mathcal{T}_S$ is the set of transpositions contained in S.*

(4) *Let $B, C\in\mathcal{O}$.*

 (i) *If $[V, B]\supseteqq[V, Z(S)]$, then $|B|=2^m$.*

 (ii) *If $|B|=2^m$, then $[V, B]=[V, Z(S)]$ and $B\supseteqq C$.*

 (iii) *If $|C|=2^m$ and $[C_V(B), C]\supseteqq[V, Z(S)]$, then $B=1$.*

PROOF. First, let $G=SL_2(2^m)$. Then $S=Z(S)\cong\boldsymbol{E}_{2^m}$. Let $1\neq z\in S$, and $1\neq B\in\mathcal{O}$. Then $|V: C_V(z)|=|V: C_V(B)|=2^m$ because $C_V(x)=C_V(S)$ for all $x\in S-\{1\}$. This proves (1), (2), and (3). Part (4) is a consequence of (3).

Next, let $G=\Sigma_{2m+1}$. Let $\Omega=\{0, 1, 2, \cdots, 2m-1, 2m\}$, and regard $\mathcal{P}(\Omega)$ as a $GF(2)G$-module. We may assume that V is the submodule of $\mathcal{P}(\Omega)$ generated by the subsets of Ω of even order.

Let $E=\langle(12), (34), \cdots, (2m-1, 2m)\rangle$ and $t=(1,3,5,\cdots,2m-1)\,(2,4,6,\cdots,2m)$. Then $E\langle t\rangle$ is a 2-subgroup of G, and so we may assume that $E\langle t\rangle\subseteqq S$, renumbering if necessary. Then $\mathcal{T}_S=\{(12), (34), \cdots, (2m-1, 2m)\}$. Thus S acts on $\mathcal{T}_S$, and so $C_S(E)=E$. Since S is transitive on $\mathcal{T}_S$, we have $Z(S)=C_E(S)=\langle(12)\,(34)\cdots(2m-1, 2m)\rangle$. Therefore $C_V(Z(S))=[V, Z(S)]=\langle\{1, 2\}, \{3, 4\}, \cdots, \{2m-1, 2m\}\rangle$, so (1) holds. Parts (2) and (3) follow from 3.1 of [2] and 3.1 of [3], respectively. For the proof of (4), let $B, C\in\mathcal{O}$. Put $B=\langle(2i_1-1, 2i_1), (2i_2-1, 2i_2), \cdots,$

$(2i_r-1, 2i_r)\rangle$, where $|B|=2^r$, and $1\leqq i_1<i_2<\cdots<i_r\leqq m$. Then we have $[V, B]=\langle\{2i_1-1, 2i_1\}, \{2i_2-1, 2i_2\}, \cdots, \{2i_r-1, 2i_r\}\rangle$. This shows that if $[V, B]\supseteqq[V, Z(S)]$ then $r=m$, and that if $r=m$ then $[V, B]=[V, Z(S)]$ and $B=\langle\mathcal{T}_S\rangle\supseteqq C$. Hence (i) and (ii) hold. Let $\{j_1, j_2, \cdots, j_s\}=\{1, 2, \cdots, m\}-\{i_1, i_2, \cdots, i_r\}$, where $s=m-r$. Then $[C_V(B), \langle\mathcal{T}_S\rangle]=\langle\{2j_1-1, 2j_1\}, \{2j_2-1, 2j_2\}, \cdots, \{2j_s-1, 2j_s\}\rangle$. This shows that if $|C|=2^m$ and $[C_V(B), C]\supseteqq[V, Z(S)]$ then $s=m$. Hence (iii) holds.

We need some more definitions.

Let $n\geqq 1$. We denote by Q_n the semidirect product of N_n by $SL_2(2^n)$, and by Δ_n the semidirect product of M_n by Σ_n.

Let V_n be the 3-dimensional vector space of row vectors with coefficients in $GF(2^n)$. We define the action of $SL_2(2^n)$ on V_n as follows:

$$(x, y, z)\begin{pmatrix} a & b \\ c & d \end{pmatrix}=(ax+cy, bx+dy, \sqrt{ab}+\sqrt{cd}+z).$$

Let $\hat{S}L_2(2^n)$ be a representation group of $SL_2(2^n)$. Then V_n becomes a $GF(2)\hat{S}L_2(2^n)$-module, which is induced through the natural homomorphism $\hat{S}L_2(2^n)\to SL_2(2^n)$. We denote by R_n the semidirect product of V_n by $\hat{S}L_2(2^n)$.

Let $\hat{\Sigma}_n$ be a representation group of Σ_n. Then M_n becomes a $GF(2)\hat{\Sigma}_n$-module, which is induced through the natural homomorphism $\hat{\Sigma}_n\to\Sigma_n$. We denote by Γ_n the semidirect product of M_n by $\hat{\Sigma}_n$.

The necessary facts concerning those groups are listed in the following lemma.

(2.2) LEMMA. *The following holds.*

(1) *The groups* $R_m(m\geqq 2)$ *and* $\Gamma_{2m+1}(m=2^l\geqq 1)$ *are central extensions of the groups* Q_m *and* Δ_{2m+1}, *respectively.*

(2) *Let* $K=R_m(m\geqq 2)$ *or* $\Gamma_{2m+1}(m=2^l\geqq 1)$. *Then* $O_2(K)$ *is elementary abelian, and* $O_2(K)=[O_2(K), K]Z(K)$, *which is a direct product if* $K=\Gamma_{2m+1}$.

PROOF. This follows immediately from the definition, noting that $|Z(\hat{S}L_2(2^m))|\leqq 2$, $|Z(\hat{\Sigma}_{2m+1})|\leqq 2$ because the Schur multipliers of $SL_2(2^m)$ and Σ_{2m+1} have order at most 2.

Section 3. Proof of Theorem B.

We will prove Theorem B for $\mathcal{K}$-groups by way of contradiction. Suppose that Theorem B is false. Then no nonidentity characteristic subgroup of S is normal in G. Hence, by the main result of [2], G is described as follows:

$G=SJ(G)$;

$J(G)=TH_1\cdots H_k$, the central product of a 2-subgroup T and the S-conjugates $H_1, \cdots, H_k$ of a subgroup H, where H is a central factor group of $R_m(m\geqq 2)$ or $\Gamma_{2m+1}(m=2^l\geqq 1)$.

Define $N=J(G)$, $Q=O_2(N)$, $R=S\cap N$, $N_i=H_iQ$, $V_i=H_i\cap Q$, $R_i=S\cap N_i$, $(1\leqq i\leqq k)$, $L=H_1\cdots H_k$, $V=V_1\cdots V_k$.

Then we have $V_i=O_2(H_i)$ $(1\leqq i\leqq k)$, $V=O_2(L)$, and $Q=TV$.

(1) *$Z(L)\subseteqq V$, V is elementary abelian, and $C_N(V)=C_N(V/Z(L))=Q$.*

PROOF. Since $Z(H_i)\subseteqq V_i$ and V_i is elementary abelian $(1\leqq i\leqq k)$ by (2.2) (2), we have $Z(L)=Z(H_1)\cdots Z(H_k)\subseteqq V_1\cdots V_k=V$, and V is elementary abelian. Moreover, we have $Q=TV\subseteqq C_N(V)\subseteqq C_N(V/Z(L))=TC_L(V/Z(L))=TC_{H_1}(V_1/Z(H_1))\cdots C_{H_k}(V_k/Z(H_k))$. Since $H_i/Z(H_i)\cong Q_m$ or $\Delta_{2m+1}(1\leqq i\leqq k)$, we have $C_{H_i}(V_i/Z(H_i))=V_i(1\leqq i\leqq k)$, so $Q\subseteqq C_N(V/Z(L))=TV_1\cdots V_k=Q$.

Let $^{\#}$ and $^{-}$ denote the natural homomorphisms of V onto $V/Z(L)$ and of N onto N/Q, respectively. Then $V^{\#}=V_1^{\#}\times\cdots\times V_k^{\#}$ and $\bar{N}=\bar{N}_1\times\cdots\times\bar{N}_k$. We will regard $V^{\#}$ as a faithful $GF(2)\bar{N}$-module. Let $1\leqq i\leqq k$. Then $\bar{N}_i\cong H_i/V_i$, $V_i^{\#}\cong V_i/Z(H_i)$ as $GF(2)H_i/V_i$-modules, and $H_i/Z(H_i)\cong Q_m$ or Δ_{2m+1}. Thus, regarded as a $GF(2)\bar{N}_i$-module, $V_i^{\#}$ is induced either by the natural module for $SL_2(2^m)$ through an isomorphism $\bar{N}_i\rightarrow SL_2(2^m)$, or by the natural module for Σ_{2m+1} through an isomorphism $\bar{N}_i\rightarrow\Sigma_{2m+1}$.

(2) *Let $B\subseteqq R$, and suppose $\bar{B}\in\mathcal{O}(\bar{R}, V^{\#})$. Then $|V^{\#}: C_{V^{\#}}(\bar{B})|=|\bar{B}|$, and there exist subgroups $B_i\subseteqq B\cap R_i$ $(1\leqq i\leqq k)$ such that $\bar{B}=\bar{B}_1\times\cdots\times\bar{B}_k$ and $\bar{B}_i\in\mathcal{O}(\bar{R}_i, V_i^{\#})$ $(1\leqq i\leqq k)$.*

PROOF. Put $B^{(i)}=R_1\cdots R_{i-1}BR_{i+1}\cdots R_k\cap R_i\,(1\leqq i\leqq k)$. Then $\bar{B}\subseteqq\bar{B}^{(1)}\times\cdots\times\bar{B}^{(k)}$, and $C_{V^{\#}}(\bar{B})=C_{V_1^{\#}}(\bar{B}^{(1)})\times\cdots\times C_{V_k^{\#}}(\bar{B}^{(k)})$ because $[V_i^{\#}, \bar{N}_j]=1$ if $1\leqq i\neq j\leqq k$. Let $1\leqq i\leqq k$. Since $\bar{B}^{(i)}$ is elementary abelian, we have $|V_i^{\#}: C_{V_i^{\#}}(\bar{B}^{(i)})|\geqq|\bar{B}^{(i)}|$ by (2.1) (3). Therefore

$$|\bar{B}|\leqq\prod_{i=1}^{k}|\bar{B}^{(i)}|\leqq\prod_{i=1}^{k}|V_i^{\#}: C_{V_i^{\#}}(\bar{B}^{(i)})|=|V^{\#}: C_{V^{\#}}(\bar{B})|\leqq|\bar{B}|.$$

Thus $|V^{\#}: C_{V^{\#}}(\bar{B})|=|\bar{B}|$, $\bar{B}=\bar{B}^{(1)}\times\cdots\times\bar{B}^{(k)}$, and $|V_i^{\#}: C_{V_i^{\#}}(\bar{B}^{(i)})|=|\bar{B}^{(i)}|(1\leqq i\leqq k)$. Put $B_i=B\cap B^{(i)}(1\leqq i\leqq k)$. Then those subgroups have the desired properties.

(3) *Let $B\in\mathcal{A}(S)$. Then $\bar{B}\in\mathcal{O}(\bar{R}, V^{\#})$, $(B\cap Q)V\in\mathcal{A}(S)$, and if $B\subseteqq Q$, then $V\subseteqq B$.*

PROOF. Let $B\in\mathcal{A}(S)$. Then $B\subseteqq J(S)\subseteqq S\cap N=R$. Since $(B\cap Q)V$ is elementary abelian, the maximality of $|B|$ yields that $|(B\cap Q)V|\leqq|B|$. Thus we have $V\subseteqq B$ if $B\subseteqq Q$, and

$$|V^{\#}: C_{V^{\#}}(\bar{B})|\leqq|V: B\cap V|=|(B\cap Q)V: B\cap Q|\leqq|B: B\cap Q|=|\bar{B}|.$$

This shows that $\bar{B}\in\mathcal{O}(\bar{R}, V^{\#})$, and so $|V^{\#}: C_{V^{\#}}(\bar{B})|=|\bar{B}|$ by (2). Therefore $|(B\cap Q)V|=|B|$ by the above inequalities, and then $(B\cap Q)V\in\mathcal{A}(S)$.

Now, define $V_R=[V, Z(\bar{R})]$, and $(V_i)_{R_i}=[V_i, Z(\bar{R}_i)]$ $(1\leqq i\leqq k)$.

Since $[V_i, \bar{N}_j]=1$ $(1\leqq i\neq j\leqq k)$, we have

$$V_R=[V_1\cdots V_k, Z(\bar{R}_1)\cdots Z(\bar{R}_k)]=(V_1)_{R_1}\cdots(V_k)_{R_k}.$$

(4) *There exists an element $a\in A$ such that $[V, V^a]=V_R=[V, V^{a^{-1}}]$.*

PROOF. By (3), we have $\langle V^A\rangle\subseteqq J(S)\subseteqq R$. Suppose $\langle V^A\rangle\subseteqq Q$. Since $[Q, O^2(G)]\subseteqq[TV, L]\subseteqq V$, we have $\langle V^A\rangle\lhd SO^2(G)=G$, contrary to our hypothesis. Thus there exists an element $a\in A$ such that $V^a\nsubseteq Q$. Hence $[V^{a^{-1}}, V]=[V, V^a]^{a^{-1}}\neq 1$ by (1), and then $V^{a^{-1}}\nsubseteq Q$ also by (1).

Now, replacing a by a^{-1} if necessary, we may assume that

$$|V: V\cap Q^a|\leqq|V: V\cap Q^{a^{-1}}|=|V^a: V^a\cap Q|.$$

Then we have $\bar{V}^a\in\mathcal{O}(\bar{R}, V^{\#})$ by (1). Hence, by (2), there exist subgroups $B_i\subseteqq V^a\cap R_i$ such that $\bar{V}^a=\bar{B}_1\times\cdots\times\bar{B}_k$ and $\bar{B}_i\in\mathcal{O}(\bar{R}_i, V_i^{\#})$ $(1\leqq i\leqq k)$. Therefore we have

$$[V, V^a]=[V, \bar{V}^a]=[V_1, \bar{B}_1]\cdots[V_k, \bar{B}_k].$$

We can choose $z\in V^a-Q$ so that $\bar{z}\in Z(\bar{S})$ because $V^a\lhd S$. Hence $\bar{z}\in C_{Z(\bar{R})}(\bar{S})$. Let $\bar{z}=\bar{z}_1\cdots\bar{z}_k$, where $\bar{z}_i\in Z(\bar{R}_i)$ $(1\leqq i\leqq k)$. Since $\bar{S}$ transitively permutes $Z(\bar{R}_1),\cdots, Z(\bar{R}_k)$, $\bar{S}$ transitively permutes $\bar{z}_1,\cdots,\bar{z}_k$ as well. Thus $\bar{z}_i\neq 1$ $(1\leqq i\leqq k)$ as $\bar{z}\neq 1$. Hence $|V_i^{\#}: C_{V_i^{\#}}(\bar{z}_i)|=2^m$ $(1\leqq i\leqq k)$ by (2.1) (1), and so

$$\begin{aligned}2^{mk} &= \prod_{i=1}^{k}|V_i^{\#}: C_{V_i^{\#}}(\bar{z}_i)| = |V^{\#}: C_{V^{\#}}(\bar{z})|\\ &= |V: V\cap Q^a| \leqq |V^a: V^a\cap Q| = |\bar{V}^a|.\end{aligned}$$

This shows that $|\bar{V}^a|=2^{mk}$ and that $|\bar{B}_i|=2^m$ $(1\leqq i\leqq k)$ because $m_2(\bar{R}_i)=m$ $(1\leqq i\leqq k)$ by (2.1)(2). Hence $|V: V\cap Q^a|=|V^a: V^a\cap Q|=|V: V\cap Q^{a^{-1}}|$ by the above inequalities. Therefore it suffices to prove that $[V, V^a]=V_R$ by the symmetry between a and a^{-1}.

Let $1\leqq i\leqq k$. If H is a central factor group of R_m, then $\bar{B}_i=Z(\bar{R}_i)$, so $[V_i, \bar{B}_i]=(V_i)_{R_i}$. If H is a central factor group of Γ_{2m+1}, then $[V_i^{\#}, \bar{B}_i]=(V_i)_{\bar{R}_i}^{\#}$ by (2.1) (3-ii), and hence $[V_i, \bar{B}_i]=(V_i)_{R_i}$ because $V_i=[V_i, H_i]\times Z(H_i)=[V_i, \bar{N}_i]\times C_{V_i}(\bar{N}_i)$ by (2.2)(2). This shows that $[V, V^a]=(V_1)_{R_1}\cdots(V_k)_{R_k}=V_R$.

Let $*$ denote the natural homomorphism of N onto $N/TZ(L)$. Then $Q^*=V^*=V_1^*\times\cdots\times V_k^*$, and $N^*/V^*=N_1^*/V^*\times\cdots\times N_k^*/V^*$. We will regard V^* as a faithful $GF(2)N^*/V^*$-module. Let $1\leqq i\leqq k$. Then $N_i^*/V^*\cong\bar{N}_i$, and $V_i^*\cong V_i^{\#}$ as $GF(2)\bar{N}_i$-modules. Thus, regarded as a $GF(2)N_i^*/V^*$-module, V_i^* is induced either by the natural module for $SL_2(2^m)$ through an isomorphism $N_i^*/V^*\rightarrow$

$SL_2(2^m)$, or by the natural module for Σ_{2m+1} through an isomorphism $N_i^*/V^* \to \Sigma_{2m+1}$. We should remark, moreover, that $B_i^*V^*/V^* \in \mathcal{O}(R_i^*/V^*, V_i^*)$ if (and only if) $\bar{B}_i \in \mathcal{O}(\bar{R}_i, V_i^\#)$ for all subgroups $B_i \subseteq R_i$. Finally, let M^* be a subgroup of N^* so that $N^* = M^*V^*$ and $M^* \cap V^* = 1$.

(5) *Let* $B \in \mathcal{A}(S)$. *Then* $B^* \subseteq C_{V^*}(B^*)(B^*V^* \cap M^*) \subseteq C_{V^*}(B^*)B^*$.

PROOF. By (2), (3), and the above remark, there exist subgroups $B_i \subseteq B \cap R_i \,(1 \leqq i \leqq k)$ such that $B^*V^*/V^* = B_1^*V^*/V^* \times \cdots \times B_k^*V^*/V^*$ and $B_i^*V^*/V^* \in \mathcal{O}(R_i^*/V^*, V_i^*)\,(1 \leqq i \leqq k)$.

Let $B^* = \langle B^* \cap V^*, x_1v_1, \cdots, x_rv_r \rangle$, where $|B^* : B^* \cap V^*| = |B^*V^*/V^*| = 2^r$, and if $1 \leqq s \leqq r$ then $v_s \in V^*$ and $V^* \not\ni x_s \in B_{i(s)}^*V^* \cap M^*$ for some $i(s)\,(1 \leqq i(s) \leqq k)$. Then

$$\text{(i)}\quad [x_s, v_s] = (x_sv_s)^2 = 1 \,(1 \leqq s \leqq r)$$

and

$$\text{(ii)}\quad [x_s, v_t][x_t, v_s]^{-1} = [x_sv_s, x_tv_t] = 1 \;\; (1 \leqq s \neq t \leqq r)$$

because $(B^*)^2 = 1 = [B^*V^* \cap M^*, B^*V^* \cap M^*]$ and $[B^*, V^*] \subseteq V^*$. Note that $[V^*, x_s] \subseteq [V_{i(s)}^*, x_s]$ $(1 \leqq s \leqq r)$ by the definition of $i(s)$.

Let $1 \leqq s \neq t \leqq r$. If $i(s) \neq i(t)$, then $[V^*, x_s] \cap [V^*, x_t] \subseteq V_{i(s)}^* \cap V_{i(t)}^* = 1$, so $[x_s, v_t] = 1$ by (ii). Let $i = i(s) = i(t)$. Then $x_s, x_t \in R_i^* - V^*$. If H is a central factor group of R_m, then $v_t \in C_{V^*}(x_t) = C_{V^*}(R_i^*)$ by (i), so $[x_s, v_t] = 1$. Let H be a central factor group of Γ_{2m+1}. Then both x_s and x_t correspond to transpositions under an isomorphism $N_i^* \cap M^* \to \Sigma_{2m+1}$ by (2.1) (3-ii). As in the proof of (2.1), let $\Omega = \{0, 1, 2, \cdots, 2m-1, 2m\}$, and regard V_i^* as the submodule of $\mathcal{P}(\Omega)$. Then $[V_i^*, (pq)] = \langle \{p, q\} \rangle$ for all transpositions (pq). This shows that $[V^*, x_s] \cap [V^*, x_t] = [V_i^*, x_s] \cap [V_i^*, x_t] = 1$. Thus, together with (i), $v_t \in C_{V^*}(B^*V^*/V^*) = C_{V^*}(B^*)$ $(1 \leqq t \leqq r)$. Hence $B^* \subseteq C_{V^*}(B^*)(B^*V^* \cap M^*)$ because $B^* \cap V^* \subseteq C_{V^*}(B^*)$, and so $B^*V^* \cap M^* \subseteq C_{V^*}(B^*)B^*$.

(6) *Let* $B, C, D \in \mathcal{A}(S)$ *with* $V_R \subseteq [B, C]$, $V_R \subseteq [C, D]$. *If* $V \subseteq B$, *then* $V \subseteq D$.

PROOF. As in the proof of (5), there exist subgroups $C_i \subseteq C \cap R_i$ and $D_i \subseteq D \cap R_i \,(1 \leqq i \leqq k)$ such that $C^*V^*/V^* = C_1^*V^*/V^* \times \cdots \times C_k^*V^*/V^*$, $D^*V^*/V^* = D_1^*V^*/V^* \times \cdots \times D_k^*V^*/V^*$, and $C_i^*V^*/V^*, D_i^*V^*/V^* \in \mathcal{O}(R_i^*V^*/V^*, V_i^*)$ $(1 \leqq i \leqq k)$. Note that $V_R^* = (V_1)_{R_1}^* \times \cdots \times (V_k)_{R_k}^*$.

Suppose that $V \subseteq B$. Then $B \subseteq Q$ by (1), so $B^* = V^*$. Thus $V_R^* \subseteq [B^*, C^*] = [V^*, C^*] = [V_1^*, C_1^*] \times \cdots \times [V_k^*, C_k^*]$, and so $(V_i)_{R_i}^* \subseteq [V_i^*, C_i^*] = [V_i^*, C_i^*V^*/V^*]$ $(1 \leqq i \leqq k)$. Therefore $|C_i^*V^*/V^*| = 2^m (1 \leqq i \leqq k)$ by (2.1) (4-i). Hence $C_i^*V^*/V^* \supseteq D_i^*V^*/V^* (1 \leqq i \leqq k)$ by (2.1) (4-ii), and so $C_{V^*}(C^*) \subseteq C_{V^*}(D^*)$ and $C^*V^* \cap M^* \supseteq D^*V^* \cap M^*$. Thus, together with (5), we have $D^* \subseteq C_{V^*}(D^*)(D^*V^* \cap M^*) \subseteq C_{V^*}(D^*)(C^*V^* \cap M^*) \subseteq C_{V^*}(D^*)C^*$, and so $V_R^* \subseteq [D^*, C^*] = [C_{V^*}(D^*)C^*, C^*] =$

$[C_{V_1^*}(D_1^*), C_1^*]\times\cdots\times[C_{V_k^*}(D_k^*), C_k^*]$. This shows that $(V_i)_{R_i}^*\subseteqq[C_{V_i^*}(D_i^*), C_i^*]=[C_{V_i^*}(D_i^*V^*/V^*), C_i^*V^*/V^*]$ $(1\leqq i\leqq k)$, and then $D_i^*V^*/V^*=1$ $(1\leqq i\leqq k)$ by (2.1) (4-iii). Hence $D^*\subseteqq V^*=Q^*$, and so $D\subseteqq Q$. This implies that $V\subseteqq D$ by (3).

(7) *A contradiction.*

PROOF. By (3), we can take $B\in\mathcal{A}(S)$ so that $V\subseteqq B$. Since $V_R^a=V_R$ by (4), we have $[B^{a^n}, B^{a^{n+1}}]=[B, B^a]^{a^n}\supseteqq[V, V^a]^{a^n}=V_R^{a^n}=V_R(n\in\boldsymbol{Z})$. Therefore (5) shows that $V\subseteqq B^{a^{2n}}(n\in\boldsymbol{Z})$ because $V\subseteqq B$. But then $V\subseteqq B^{a^{-1}}$ as a has odd order, and so $[V, V^a]\subseteqq[B, B]=1$, contrary to (4). This contradiction completes the proof of Theorem B.

References

[1] K. Gomi, A note on the thin finite simple groups, J. Algebra, **102** (1986), 194-198.

[2] K. Gomi, A pushing up theorem for groups of characteristic 2 type, Sci. Papers College Arts Sci. Univ. Tokyo, **37** (1987), 73-102.

[3] K. Gomi, On $GF(2)$-representations of finite groups, in Group Theory and Its Applications (ed. T. Kondo), College of General Education, University of Tokyo, Tokyo, 1983.

[4] K. Gomi and M. Hayashi, A pushing-up approach to the quasithin simple finite groups with solvable 2-local subgroups, 1989, preprint.

[5] K. Gomi and M. Hayashi, Finite simple groups all of whose 2-local subgroups are solvable, 1990, preprint.

[6] D. Gorenstein and R. Lyons, Nonsolvable finite groups with solvable 2-local subgroups, J. Algebra, **38** (1976), 453-522.

[7] K. Gomi and Y. Tanaka, On pairs of groups having a common 2-subgroup of odd indices I-II, Sci. Papers College Arts Sci. Univ. Tokyo, **35** (1985), 11-30; **40** (1990), 37-47.

[8] M. Hayashi and Y. Tanaka, Amalgams of solvable groups, to appear in J. Fac. Sci. Univ. Tokyo Sect. IA Math., **39** (1992).

[9] Z. Janko, Nonsolvable finite groups all of whose 2-local subgroups are solvable, J. Algebra, **21** (1972), 458-517.

[10] F. Smith, Finite simple groups all of whose 2-local subgroups are solvable, J. Algebra, **34** (1975), 481-520.

[11] B. Stellmacher, On groups with edge-transitive automorphism groups, Illinois J. Math., **28** (1984), 211-266.

[12] B. Stellmacher, Pushing up, Arch. Math., **46** (1986), 8-17.

[13] Y. Tanaka, Certain pairs of finite groups with a common 2-subgroup, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **33** (1986), 247-272.

[14] Y. Tanaka, Amalgams of quasithin solvable groups, doctoral dissertation, University of Tokyo, 1990, Japan. J. Math., **17** (1991), 203-266.

Yasuhiko TANAKA
Faculty of Engineering
Oita University
Dannohara, Oita 870-11
Japan