

## On Noether's inequality for threefolds

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### Introduction.

Let  $S$  be a complex minimal algebraic surface of general type. Let  $K_S$  be the canonical bundle of  $S$  and  $p_g(S)$  be the geometric genus of  $S$ . Then in general, we have a classical inequality:  $K_S^2 \geq 2p_g(S) - 4$ , which is Noether's inequality.

In this paper, we will study a three-dimensional analogue. Since we have Noether's inequality for minimal surfaces (and also canonical models of surfaces) we expect some inequalities between the geometric genus and the cube of the first Chern class for three dimensional canonical models, which may be singular and not factorial. Very optimistically, we might expect that: for any canonical model  $X$  of a threefold of general type, we should have  $K_X^3 \geq 2p_g(X) - 6$ . But that is not the case in general.

**MAIN THEOREM** (Theorems 2.4, 3.1, 4.1). *Let  $X$  be a three-dimensional algebraic variety defined over  $\mathbf{C}$ . Assume that  $X$  has at most canonical singularities and that a canonical divisor  $K_X$  is nef and big. Let  $d = \dim \Phi_{K_X}(X)$ .*

- (1) *If  $d=3$ , then  $K_X^3 \geq 2p_g - 6$ .*
- (2) *If  $d=2$  and  $K_X$  is Cartier, then either*
  - 2a)  $K_X^3 \geq 2p_g(X) - 4$  or
  - 2b)  $\Phi_{K_X}$  is birationally equivalent to a fibration of curves of genus two with a rational section over a birationally ruled surface.
- (3) *If  $d=1$ ,  $K_X$  is ample and  $X$  is factorial, then either*
  - 3a)  $K_X^3 \geq 2p_g(X) - 2$ ,
  - 3b)  $X$  is singular, the image is a rational curve, all the fibers are connected,  $K_X^3 = 1$  and  $p_g(X) = 2$  or
  - 3c) the rational map  $\Phi_{K_X}$  is a morphism and the general fibers of  $\Phi_{K_X}$  are normal algebraic irreducible surfaces with only canonical singularities which have ample canonical divisors,  $c_1^2 = 1$ ,  $q = 0$  and  $1 \leq p_g \leq 2$ .

Moreover, the case 2b) really occurs. In (3.2), we will construct a smooth projective variety  $X$  with an ample canonical divisor  $K_X$  such that  $\dim \Phi_{K_X}(X)$

$=2$  and  $K_X^3=(4p_g(X)-10)/3$  ( $p_g(X)=7, 10, 13, \dots$ ). We do not know yet whether the case 3b) or 3c) should occur or not.

In Section 1, we shall recall some classical results. We shall prove the theorem in the case  $d=3, 2$  and  $1$  in Sections 2, 3 and 4, respectively. In Section 2, results are valid for arbitrary dimension  $\geq 2$ . In the case where the equality holds, a detailed classification of  $X$  is systematically done by Fujita ([F2]).

The author was very inspired by Horikawa's results [H2], [H3], where Horikawa studied threefolds with trivial canonical bundle.

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### Notation

Throughout the paper, everything will be assumed to be defined over  $\mathcal{C}$ . We refer to [KMM] as for the definition of  $\mathbf{Q}$ -divisor, canonical singularity, terminal singularity, nef, big and so forth.

Let  $X$  be an  $n$ -dimensional normal complete variety. For a divisor  $D$  on  $X$ ,  $\Phi_D$  denotes the rational map associated to the complete linear system  $|D|$ . When  $D$  is a canonical divisor  $K_X$ , rational map  $\Phi_{K_X}$  is called the *canonical map* of  $X$ . Let  $Y$  be a smooth model of  $X$ .

$\text{Bs}|D|$ : the base locus of  $|D|$ , namely the set-theoretical intersection of all the members of  $|D|$ .

$|D|_{red}$ : the reduced part of  $|D|$ , namely  $|D| - (\text{the fixed component of } |D|)$ .

$H^i(X, \mathcal{O}(D))$  (or simply  $H^i(X, D)$  or  $H^i(D)$ ): the  $i$ -th cohomology group of the sheaf of module associated to  $D$ .

$h^i(D) = \dim_{\mathcal{C}} H^i(D)$ .

$\chi(X, D) = \sum_i (-1)^i h^i(D)$ : the Euler characteristic of  $\mathcal{O}(D)$ .

$\Delta(X, D) = n + D^n - h^0(D)$ : the  $\Delta$ -genus of a prepolarized variety  $(X, D)$  ([F1]).

$p_g(X) = h^0(K_Y) = H^2(X, \mathcal{O}_X)$ : the geometric genus of  $X$ .

$q(X) = H^1(X, \mathcal{O}_X)$ : the irregularity of  $X$ .

$c_i(X)$ : the  $i$ -th Chern class of  $X$ .

$\sim$ : algebraic equivalence.

$\cong$ : numerical equivalence.

$=_{\mathbf{Q}}$ :  $\mathbf{Q}$ -linear equivalence.

### § 1. Preliminaries.

We will recall here classical results concerned with linear systems without proof.

(1.1) DEFINITION. Let  $C$  be a curve and  $D$  be a divisor on  $C$ . We say  $D$  is a *special divisor* if and only if  $h^0(D) > 0$  and  $h^1(D) > 0$ .

(1.2) CLIFFORD'S THEOREM ([Ha, IV, 5.4]). Let  $C$  be a curve and  $D$  be a special divisor on  $C$ . Then  $\deg D \geq 2h^0(D) - 2$ . Moreover, the equality holds if and only if either  $D = 0$ ,  $K_C$  or a multiple of  $g^1_2$  (in the last case  $C$  is hyperelliptic).

We frequently use the following lemma.

(1.3) LEMMA. Let  $Z \subset \mathbf{P}^N$  be a variety which is not contained in any hyperplane. Then  $\deg Z \geq N - \dim Z + 1$ .

In the following, a *minimal surface* means a smooth surface which has no  $(-1)$ -curves.

(1.4) NOETHER'S INEQUALITY ([BPV, VII, (3.1)]). Let  $S$  be a minimal algebraic surface of general type. Then  $K_S^2 \geq 2p_g(S) - 4$ .

We remark that even if  $S$  is a normal surface with canonical singularities with a nef and big canonical divisor, we have the same inequality.

(1.5) CASTELNUOVO'S INEQUALITY ([C], [E, p. 297], [Be, Theorem 5.5]). Let  $S$  be a minimal surface of general type. Suppose that  $K_S^2 < 3p_g(S) - 7$  holds. Then  $\Phi_{K_S}$  is a rational map of degree 2 onto a ruled surface.

(1.6) In particular, if  $\Phi_{K_S}$  is birational or composed with a pencil, we have  $K_S^2 \geq 3p_g(S) - 7$ .

## § 2. Case $\dim \Phi_{K_X}(X) = \dim X$ .

In this section, we will treat an  $n$ -dimensional algebraic variety ( $n \geq 2$ ).

(2.1) PROPOSITION ([F1]). Let  $X$  be an  $n$ -dimensional complete normal algebraic variety and let  $H$  be a nef and big Cartier divisor such that  $\dim \Phi_H(X) = n$ . Assume  $p_g(X) > 0$ . Then  $H^n \geq 2h^0(X, H) - 2n$ .

PROOF. By taking a resolution of singularities, we may assume that  $X$  is nonsingular.

STEP 1. First of all, we treat the case  $\text{Bs}|H| = \emptyset$ . By Bertini's theorem, we can take general divisors  $H_1, H_2, \dots, H_{n-1} \in |H|$  such that the intersections  $X_k := H_1 \cap H_2 \cap \dots \cap H_k$  are smooth irreducible varieties for  $k \leq n-1$ . We denote  $X_{n-1}$  by  $C$ . We shall show that  $H|_C$  is a special divisor. From the exact sequence:

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \mathcal{O}_{X_k}(H) \longrightarrow \mathcal{O}_{X_{k+1}}(H) \longrightarrow 0,$$

we have  $h^0(X_k, H|_{X_k}) - 1 \leq h^0(X_{k+1}, H|_{X_{k+1}})$  for  $k \leq n-2$ . Thus we have  $h^0(H|_C)$

$\geq h^0(H) - (n-1) \geq 2$ . On the other hand, we have:

$$0 \longrightarrow H^0(K_{X_{n-2}} - H|_{X_{n-2}}) \longrightarrow H^0(K_{X_{n-2}}) \longrightarrow H^0(K_C - H|_C).$$

Since  $K_{X_{n-2}} = (K_X + (n-2)H)|_{X_{n-2}}$  and  $p_g(X) > 0$ , we have  $p_g(X_{n-2}) > 0$ . Since  $h^0(H|_{X_{n-2}}) \geq 3 > 2$ ,  $H|_{X_{n-2}}$  is not contained in  $\text{Bs}|K_{X_{n-2}}|$ .

Thus we have  $h^1(H|_C) = h^0(K_C - H|_C) > 0$ . We can apply Clifford's theorem for  $H|_C$  and we have  $H^n \geq 2h^0(H|_C) - 2 \geq 2h^0(H) - 2n$ .

STEP 2. Next we shall treat general cases. Let  $\mu: Y \rightarrow X$  be a modification such that  $\Phi_{\mu^*H}$  is a morphism. Let  $Z$  be the image of  $\Phi_{\mu^*H}$ . We write  $\mu^*H = M + F$  where  $F$  and  $M$  are the fixed part and the movable part of  $\mu^*H$ , respectively. By the definition of  $\mu$ ,  $|M|$  is free from base points, and in particular,  $M$  is nef. And we have  $(\mu^*H)^{n-1} \cdot F \geq 0$  since  $H$  is nef. Thus we have:

$$\begin{aligned} H^n &= (\mu^*H)^n = (\mu^*H)^{n-1} \cdot (M+F) \geq (\mu^*H)^{n-1} \cdot M \\ &= (\mu^*H)^{n-2} \cdot (M+F) \cdot M \geq (\mu^*H)^{n-2} \cdot M^2 \geq \dots \geq M^n. \end{aligned}$$

By Step 1,  $M^n \geq 2h^0(M) - 2n$ . ■

Next we shall study the case where the equality holds (cf. [F2]).

(2.2) PROPOSITION. *Let  $X$  be an  $n$ -dimensional normal complete algebraic variety,  $H$  a nef Cartier divisor on  $X$  with  $\dim \Phi_H(X) = n$ . Suppose that  $H^n = 2h^0(H) - 2n$  holds. Then  $\text{Bs}|H| = \emptyset$  and one of the following conditions is satisfied:*

- a)  $\Phi_H$  is birational,
- b)  $\Phi_H$  is a generically finite double cover onto a normal variety  $Z \subset \mathbf{P}^{h^0(H)-1}$ , where  $\Delta$ -genus  $\Delta(Z, \mathcal{O}_Z(1)) = 0$ .

PROOF. By Step 2 of the proof of (2.1), we see that  $(\mu^*H)^k M^{n-k-1} F = 0$  for  $0 \leq k \leq n-1$ . These yield  $M^{n-l} F^l = 0$  for  $1 \leq l \leq n$ . In particular, when  $l=1$  we see that the codimension of  $\Phi_M(F)$  in  $\Phi_M(Y)$  is greater than one. Note that  $F = \mu^*H - M$  is  $\Phi_M$ -nef.

LEMMA. *Let  $f: X \rightarrow Y$  be a projective surjective morphism between normal projective  $n$ -dimensional varieties. Suppose a divisor  $E$  in  $X$  is contracted to a subscheme  $Z$  whose codimension in  $Y$  is greater than one. Then  $E$  is not  $f$ -nef.*

PROOF. Take a hyperplane section  $H$  of  $Y$ . Choose general sections  $L_1, \dots, L_{n-2}$  from  $|f^*H|$  such that the intersection  $S = L_1 \cap \dots \cap L_{n-2}$  is a smooth irreducible surface. Then  $E|_S$  is collapsed to points, so we have  $(E \cdot E|_S) = (E|_S)^2 < 0$ .

PROOF OF 2.2 CONTINUED. Using this lemma, we come to a contradiction. Thus, we get  $F = 0$ .  $\Phi_H$  is in fact a morphism, and we have  $\text{Bs}|H| = \emptyset$ .

Hence we have  $\deg Z \cdot \deg \Phi_H = H^n = 2h^0(H) - 2n$ . On the other hand, we have  $\deg Z \geq h^0(H) - n$  by (1.3). Thus we have  $\deg \Phi_H = 1$  or  $2$ . In the case  $\deg \Phi_H = 2$ , we have  $\Delta(Z, \mathcal{O}_Z(1)) = \deg Z + n - h^0(H) = 0$ . ■

Fujita gave a classification of polarized varieties with  $\Delta$ -genus 0 in his paper ([F1]).

(2.3) EXAMPLE ([F2]):

- a-1) Let  $C$  be a canonical curve of genus  $\geq 2$ . Then  $X=C$ ,  $H=K_C$  satisfies the condition of (2.2).
- a-2) Let  $S$  be a K3-surface embedded in some projective space. Then  $X=S$ ,  $H=\mathcal{O}_S(1)$  satisfies the condition.
- b-1) Let  $X$  be a double covering of  $\mathbf{P}^n$  branched along a smooth hypersurface of degree  $(2n+4)$ . Then  $X$ ,  $H=K_X$  satisfies the condition.
- b-2) Let  $X$  be a double covering of a smooth quadric  $\mathbf{Q}^n \subset \mathbf{P}^{n+1}$  branched along a smooth member of  $|\mathcal{O}_{\mathbf{Q}^n}(2n+2)|$ . Then  $X$ ,  $H=K_X$  satisfies the condition.

Next we will study canonical models.

(2.4) THEOREM. *Let  $X$  be a normal complete variety of general type of dimension  $n$  with at most canonical singularities. Suppose that  $K_X$  is nef and that  $\dim \Phi_{K_X}(X) = n$ . Then  $K_X^n \geq 2p_g(X) - 2n$ .*

PROOF. Note that  $K_X$  is a  $\mathbf{Q}$ -divisor. Let  $\mu: Y \rightarrow X$  is a modification such that  $Y$  is a smooth projective variety and  $|K_Y|_{red}$  is free from base points. We write:

$$K_Y =_{\mathbf{Q}} \mu^* K_X + \sum a_j E_j \quad (a_j \in \mathbf{Q}, a_j \geq 0) =_{\mathbf{Q}} M + F,$$

where  $F$  and  $M$  are the fixed part and the movable part of  $K_Y$ , respectively, and  $E_j$  are exceptional divisors. Then we have:

$$\begin{aligned} K_X^n &= (\mu^* K_X)^n = (\mu^* K_X)^{n-1} (M + F - \sum a_j E_j) \geq (\mu^* K_X)^{n-1} M \\ &= (\mu^* K_X)^{n-2} (M + F - \sum a_j E_j) M \geq (\mu^* K_X)^{n-2} M^2 \geq \dots \geq M^n \end{aligned}$$

as in the proof of (2.1). Since  $|M|$  is base-point-free and  $\dim \Phi_M(Y) = n$ , the inequality follows. ■

(2.5) PROPOSITION. *Let  $X$  be the same as in the assumption of (2.4). Assume moreover that the equality holds and  $n \geq 2$ . Then  $K_X$  is a Cartier divisor and  $\Phi_{K_X}$  is a double covering onto a normal variety  $Z \subset \mathbf{P}^{p_g(X)-1}$ , where  $\Delta$ -genus  $\Delta(Z, \mathcal{O}_Z(1)) = 0$ .*

PROOF. Similarly as the proof of (2.2), one can show that  $K_X = \mu_* M$  is Cartier and that  $|K_X|$  is free from base points. Take general members  $H_1, \dots$ ,

$H_{n-1} \in |K_X|$  such that  $C = H_1 \cap \dots \cap H_{n-1}$  is a smooth curve. We put  $D = K_{X|C}$ . Then we have  $\text{deg} D = 2h^0(D) - 2$  from the proof of (2.1), and by adjunction formula,  $K_C = nK_{X|C} = nD \neq D \neq 0$ . Thus by Clifford's theorem,  $D$  is a multiple of  $g_{\frac{1}{2}}$  and  $\Phi_{K_{X|C}}$  is a double covering of  $\mathbf{P}^1$ . This completes the proof. ■

§ 3. Case  $\dim \Phi_{K_X}(X) = 2$ .

We shall prove here the following :

(3.1) THEOREM. *Let  $X$  be a complete canonical Gorenstein algebraic threefold of general type. Suppose that a canonical divisor  $K_X$  is nef and that  $\dim \Phi_{K_X}(X) = 2$ . Then  $K_X^3 \geq 2p_g(X) - 4$ , unless  $\Phi_{K_X}$  is birationally equivalent to a fibration of curves of genus two with a rational section over a birationally ruled surface.*

PROOF. First we may assume that  $X$  is  $\mathbf{Q}$ -factorial terminal Gorenstein by [R], [K]. It follows that  $X$  is factorial. We note that a terminal Gorenstein singularity is an isolated hypersurface singularity of multiplicity at most two ([R]). Let

$$\mu: Y = X_m \longrightarrow \dots \longrightarrow X_j \xrightarrow{\mu_j} X_{j-1} \longrightarrow \dots \longrightarrow X_0 = X$$

$$\cup \qquad \qquad \cup \qquad \qquad \cup$$

$$D_j \qquad \qquad E_j \qquad \qquad Z_j$$

be a successive blowings-up such that: ([Hi])

- a) the center  $Z_j$  of the blowing-up  $\mu_j$  is a smooth subvariety,
- b)  $\bigoplus_{m=0}^{\infty} \mathcal{I}_{Z_j}^m / \mathcal{I}_{Z_j}^{m+1}$  is  $\mathcal{O}_{X_{j-1}} / \mathcal{I}_{Z_j}$ -flat,
- c) every point in  $Z_j$  is a point of indeterminacy,
- d)  $\Phi_{\mu^* K_X}$  is a morphism.

We note that in every  $X_j$ , the multiplicity of each singular locus is at most two, and that the singularities are hypersurface singularities. Let  $F_0$  and  $M_0$  be the fixed and movable part of  $|K_X|$ , respectively. Since  $X$  is factorial,  $M_0$  is a Cartier divisor. Thus we have Cartier divisors  $\mu_j^* \dots \mu_1^* M_0$ . Let  $E_j$  be the exceptional Cartier divisor of  $\mu_j$  and  $D_j$  the total transform of  $E_j$  in  $Y$ . We write:

$$\mu^* M_0 = M + \sum_{j=1}^m r_j D_j (r_j \in \mathbf{Z}),$$

$$K_Y = \mu^* K_X + \sum_{j=1}^m a_j D_j (a_j \in \mathbf{Z}),$$

where  $M$  is the movable part of  $\mu^* K_X$ . We note that each  $r_j$  is a positive integer by the condition c). The discrepancies  $a_j$  are as follows:

- i)  $a_j = 2$  if  $Z_j$  is a point of multiplicity one on  $X_{j-1}$ ,
- ii)  $a_j = 1$  if  $Z_j$  is a point of multiplicity two or if  $Z_j$  is a curve of mul-

tiplicity one,

iii)  $a_j=0$  if  $Z_j$  is a curve of multiplicity two,

iv)  $a_j=-1$  if  $Z_j$  is a surface of multiplicity two.

Let  $S$  be the image of  $\Phi_M$ . We have the Stein factorization of  $\Phi_M=\psi\circ\varphi$ , where  $\varphi: Y\rightarrow Z$  has connected fibers and  $\psi: Z\rightarrow S$  is finite. We denote the general fiber of  $\varphi$  by  $f$ . We have  $M^2\sim(\deg\psi)(\deg S)f$ . We have  $K_X^3=(\mu^*K_X)^3=(\mu^*K_X)^2\cdot(M+\mu^*F_0+\sum_{j=1}^m r_j D_j)\geq(\mu^*K_X)^2\cdot M$ , since  $K_X$  is nef and  $D_j$  is exceptional. We have moreover  $(\mu^*K_X)^2\cdot M=(\mu^*K_X)\cdot(M+\mu^*F_0+\sum_{j=1}^m r_j D_j)\cdot M\geq(\mu^*K_X)\cdot M^2$ , since  $\mu^*K_X$  and  $M$  are nef. Also we have  $\deg S\geq p_g(X)-2$  by (1.3). Consequently we have  $K_X^3\geq(\deg\psi)(p_g(X)-2)(\mu^*K_X)\cdot f$ . Thus if either  $\deg\psi\geq 2$  or  $\mu^*K_X\cdot f\geq 2$  then we are done.

In what follows, we assume that  $\deg\psi=\mu^*K_X\cdot f=1$ . We define a set  $\mathcal{J}$  to be  $\{j \mid D_j\cdot f\neq 0\}$ . And we have

$$(A) \quad 1 = \mu^*K_X\cdot f = \left(M + \mu^*F_0 + \sum_{j=1}^m r_j D_j\right)\cdot f = (\mu^*F_0)\cdot f + \sum_{j=1}^m r_j(D_j\cdot f),$$

where  $\mu^*F_0\cdot f\geq 0$  and  $D_j\cdot f\geq 0$ . On the other hand, by adjunction, we have:

$$(B) \quad 2p_a(f)-2 = K_Y\cdot f = \mu^*K_X\cdot f + \sum_{j=1}^m a_j(D_j\cdot f) = 1 + \sum_{j=1}^m a_j(D_j\cdot f).$$

Since the left-hand side of (B) is an even integer, the set  $\mathcal{J}$  is not empty. From (A), we have exactly one  $i$  such that  $\mathcal{J}=\{i\}$ ,  $D_i\cdot f=r_i=1$  and  $D_j\cdot f=0$  for  $j\neq i$ . If the center  $Z_i$  is contained in some exceptional divisor  $E_{i'}$  ( $i'<i$ ), then we have  $D_{i'}\cdot f>0$  since  $f$  is nef and  $D_i\leq D_{i'}$ , which is a contradiction. Thus the morphism  $\mu_i$  is a blowing-up of a curve or a point. It follows that  $0\leq a_i\leq 2$  and in fact  $a_i=1$  by (B). We have moreover  $\mu^*F_0\cdot f=0$  and  $p_a(f)=2$ . Taking a resolution of  $Y$ , we have a fibration of curves with genus  $p_g(f)$ . Since  $X$  is of general type,  $p_g(f)=2$ . Since  $a_i=1$  and  $Z_i$  is not contained in any exceptional locus,  $Z_i$  is a terminal point or a smooth curve. Hence the component(s) of  $E_i$  is a birationally ruled surface and its strict transform  $E'_i$  in  $Y$  is a birationally ruled surface. We have also that  $E'_i\cdot f=1$ , which means  $E'_i$  is a rational section. ■

There really exist varieties  $X$  which do not satisfy the inequality  $K_X^3\geq 2p_g(X)-4$ , even if we restrict ourselves to projective complex manifolds. We will show a construction of examples of such varieties.

(3.2) PROPOSITION. *There is a smooth projective threefold  $X$  of general type such that  $K_X$  is ample,  $\dim\Phi_{K_X}(X)=2$  and  $K_X^3=(4p_g(X)-10)/3$  ( $p_g(X)=7, 10, 13, \dots$ ).*

PROOF. We construct such  $X$  as the image of a rational morphism from the double covering of  $\mathbf{P}^1$ -bundle over a Hirzebruch surface.

Let  $S$  be a Hirzebruch surface  $\Sigma_e = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ ,  $s$  the negative section which corresponds to the tautological line bundle  $\mathcal{O}(1)$ ,  $l$  the fiber. We take a divisor  $L = s + el$  and a line bundle  $\mathcal{L} = \mathcal{O}(L)$ . We put  $\mathbf{P} = \mathbf{P}_S(\mathcal{O} \oplus \mathcal{L}^{-2})$ , and let  $\pi: \mathbf{P} \rightarrow S$  be its structure morphism,  $\Sigma$  the section corresponding to  $\mathcal{O}_{\mathbf{P}}(1)$ . We take a divisor  $M = 5(\Sigma + 2\pi^*L)$ , which is free from base points. Let  $T \in |M|$  be a general smooth member. Since  $\Sigma_{1\Sigma} \cong -2L$ , we see that  $M_{1\Sigma}$  is trivial. Thus we may assume that  $\Sigma$  and  $T$  are disjoint. Since  $\Sigma + T \in |2(3\Sigma + 5\pi^*L)|$ , we can take the double covering  $\tau: Y \rightarrow \mathbf{P}$  branched along  $\Sigma + T$ . We can write  $\tau^*\Sigma = 2\Sigma_0$  and  $\tau^*T = 2T_0$ . Then we have  $\Sigma_0 + T_0 = \tau^*(3\Sigma + 5\pi^*L)$ . Since  $\Sigma_0 \cap T_0 = \emptyset$ , we have  $\Sigma_{01\Sigma_0} = (3\Sigma + 5\pi^*L)_{1\Sigma} \cong -L$ . We also have  $K_Y = \tau^*(K_{\mathbf{P}} + 3\Sigma + 5\pi^*L) = \tau^*(\pi^*(K_S - 2L) - 2\Sigma + 3\Sigma + 5\pi^*L) = \tau^*(\pi^*(s + (2e-2)l) + \Sigma)$ . We put  $N = s + (2e-2)l$  and take a divisor  $H = K_Y - \Sigma_0 = \Sigma_0 + \tau^*\pi^*(N)$ .

In what follows, we assume that  $e \geq 3$ .

(3.3) LEMMA.

- a)  $H$  is nef and big.
- b)  $3H - K_Y$  is nef and big.
- c) for an irreducible curve  $\Gamma$  in  $Y$ ,  $H \cdot \Gamma = 0$  holds if and only if  $\Gamma$  is a fiber of the ruling of  $\Sigma_0 \cong S$ .

PROOF. a) We first note that  $H_{1\Sigma_0} \cong \Sigma_{01\Sigma_0} - N \cong (e-2)l$ , which is nef. Suppose that there exists a curve  $\Gamma$  such that  $H \cdot \Gamma < 0$ . Since  $\tau^*\pi^*N$  is nef, we have  $\Sigma_0 \cdot \Gamma < 0$ , which means  $\Gamma \subset \Sigma_0$ . But  $H_{1\Sigma_0} \cong \Sigma_{01\Sigma_0} - N \cong (e-2)l$ , which is nef. Contradiction.

On the other hand, we have  $H^3 = H^2 \cdot \Sigma_0 + H^2 \cdot \tau^*\pi^*N = 0 + H \cdot \tau^*\pi^*N \cdot \Sigma_0 + H \cdot (\tau^*\pi^*N)^2 = (e-2)l \cdot N + (\Sigma_0 + \tau^*\pi^*N) \cdot (\tau^*\pi^*N) = e-2 + N^2 = 4e-6 > 0$ , which shows that  $H$  is big.

b) Obvious from  $3H - K_Y = H + \tau^*\pi^*N$ .

c) We suppose  $H \cdot \Gamma = 0$ . If  $\pi(\tau(\Gamma))$  is a point, then  $H \cdot \Gamma = \Sigma_0 \cdot \Gamma = (1/2)\tau^*\Sigma \cdot \Gamma > 0$ , which is a contradiction. Thus  $\pi(\tau(\Gamma))$  is a curve. Since  $N = s + (2e-2)l$  is ample, we have  $\Sigma_0 \cdot \Gamma < 0$  and  $\Gamma \subset \Sigma_0$ . Since  $H_{1\Sigma_0} = (e-2)l$ ,  $\Gamma$  is a fiber. ■

(3.4) We will go back to the proof of (3.2). By the Base Point Free Theorem ([KMM, Theorem 3-1-1]), we have  $\text{Bs}|mH| = \emptyset$  for  $m \gg 0$ . Thus we have a morphism  $\varphi := \Phi_{mH}: Y \rightarrow X \subset \mathbf{P}(H^0(mH))$ . By the lemma above,  $\varphi$  is the blowing-down along the ruling of  $\Sigma_0$ . Since  $\Sigma_{01\Sigma_0} = -L = -s - 2l$ ,  $X$  is smooth. We have  $\varphi^*K_X \cong H$  by the Base Point Free Theorem. The rest of the proposition follows from the following lemma.

(3.5) LEMMA.  $p_g(X) = 3e - 2$ .

PROOF. We have  $p_g(X) = h^0(H) = h^0(\Sigma_0 + \tau^*\pi^*N)$ . Here we consider the following exact sequence on  $Y$ :



$$0 \longrightarrow \mathcal{O}_Y(\Sigma_0) \longrightarrow \mathcal{O}_Y(\Sigma_0+T_0) \longrightarrow \mathcal{O}_{T_0}(\Sigma_0+T_0) \longrightarrow 0.$$

By taking the direct image by  $\tau$ , we have:

$$\begin{array}{ccccc} 0 \longrightarrow \tau_*\mathcal{O}_Y(\Sigma_0) & \longrightarrow & \tau_*\mathcal{O}_Y(\Sigma_0+T_0) & \longrightarrow & \tau_*\mathcal{O}_Y(\Sigma_0+T_0)|_T \\ & & \parallel & & \parallel \\ & & \mathcal{O}_P \oplus \mathcal{O}(3\Sigma+5\pi^*L) & \xrightarrow{\text{restr.}} & \mathcal{O}_P \oplus \mathcal{O}(3\Sigma+5\pi^*L)|_T, \end{array}$$

since  $T$  is contained in the branch locus of  $\tau$ . Thus we have  $\tau_*\mathcal{O}(\Sigma_0) \cong \mathcal{O} \oplus \mathcal{O}(3\Sigma+5\pi^*L-T) \cong \mathcal{O} \oplus \mathcal{O}(-2\Sigma-5\pi^*L)$ . We have:

$$\begin{aligned} h^0(\Sigma_0+\tau^*\pi^*N) &= h^0(\tau_*\Sigma_0+\pi^*N) \\ &= h^0((\mathcal{O} \oplus \mathcal{O}(-2\Sigma-5\pi^*(s+el))) \otimes \pi^*(s+(2e-2)l)) \\ &= h^0(S, s+(2e-2)l) \\ &= h^0(\mathbf{P}^1, (\mathcal{O} \oplus \mathcal{O}(-e)) \otimes \mathcal{O}(2e-2)) \\ &= 3e-2. \quad \blacksquare \end{aligned}$$

We also see that in the case  $\dim \Phi_{K_X}(X)=2$  we can not expect a generalization of Beauville's result (1.6 pencil case), namely, a general inequality of the form  $K_X^3 \geq 3$  (or 4 etc.)  $p_g(X)+const$ .

**§ 4. Case  $\dim \Phi_{K_X}(X)=1$ .**

(4.1) THEOREM. *Let  $X$  be a three dimensional canonical model which is factorial. Assume that  $\dim \Phi_{K_X}(X)=1$ . Then*

- a)  $K_X^3 \geq 2p_g(X)-2$ ,
- b)  $X$  is singular, the image is a rational curve, all the fibers are connected,  $K_X^3=1$  and  $p_g(X)=2$  or
- c) the canonical map  $\Phi_{K_X}$  is a morphism onto a rational curve and the general fibers of  $\Phi_{K_X}$  are surfaces with only canonical singularities, ample canonical divisors,  $c_1^2=1, q=0$  and  $p_g=1$  or 2.

PROOF. Let  $\mu_0: X_0 \rightarrow X$  be a crepant blowing-up such that  $X_0$  is  $\mathbf{Q}$ -factorial Gorenstein terminal.  $X_0$  is automatically factorial. We use the similar notation  $\mu_j, \mu, X_j, E_j, D_j, Z_j$ , and so on as in (3.1) except that  $X_0$  is possibly different from  $X$ . Let  $C$  be the image of  $\Phi_{K_Y}$  and  $Y \xrightarrow{\phi} Z \xrightarrow{\psi} C$  be the Stein factorization of  $\Phi_{K_Y}$ . We write:

$$\begin{aligned} n &= \deg \phi \cdot \deg C, \\ \mu^*K_X &= M+F \sim nS+F, \\ K_X &= M_0+F_0 \sim nS_0+F_0, \end{aligned}$$

$$F = \mu^*F_0 + \sum_{j=0}^m r_j D_j (r_j \in \mathbf{Z}),$$

$$K_Y = \mu^*K_X + \sum_{j=0}^m a_j D_j (a_j \in \mathbf{Z}, a_j \leq 2),$$

where  $F$  (resp.  $F_0$ ) is the fixed part of  $\mu^*K_X$  (resp.  $K_X$ ) and  $S$  is the general fiber of  $\varphi$ . We take  $r_0$  to be 1. We note that  $r_j > 0$  for all  $j$  and that  $a_0 = 0$ . We have  $K_X^3 = (\mu^*K_X)^3 = (\mu^*K_X)^2 \cdot (nS + \mu^*F_0 + \sum r_j D_j) = n(\mu^*K_X)^2 \cdot S + (\mu^*K_X)^2 \cdot (\mu^*F_0) \geq n(\mu^*K_X)^2 \cdot S > 0$  since  $K_X$  is nef and big. By (1.3), we have  $n \geq (p_g(X) - 1) \deg \phi$ . Thus if  $(\mu^*K_X)^2 \cdot S \geq 2$  or  $\deg \phi \geq 2$  then the inequality holds.

In what follows, we assume that  $(\mu^*K_X)^2 \cdot S = \deg \phi = 1$ . Then we have  $1 = (\mu^*K_X)^2 \cdot S = \mu^*K_X \cdot (\mu^*F_0 + \sum r_j D_j) \cdot S$ .

CASE 1.  $|K_X|_{red}$  has base points.

In this case, since the exceptional divisors arising from the resolution of base points dominates the curve  $C$ ,  $C$  is rational. Since  $\mu^*K_X$  and  $S$  are nef, one of the following conditions is satisfied:

- 1)  $(\mu^*K_X) \cdot (\sum r_j D_j) \cdot S = 1$  and  $(\mu^*K_X) \cdot (\mu^*F_0) \cdot S = 0$ ,
- 2)  $(\mu^*K_X) \cdot (\sum r_j D_j) \cdot S = 0$  and  $(\mu^*K_X) \cdot (\mu^*F_0) \cdot S = 1$ .

Case 1-1). By the first equality, it holds that  $r_i = (\mu^*K_X) \cdot D_i \cdot S = 1$  and  $(\mu^*K_X) \cdot D_j \cdot S = 0$  ( $\forall j \neq i$ ) for some  $i$ . Moreover, together with  $\sum r_j D_j = n(\mu^*S_0 - S)$ , we get  $n(\mu^*K_X) \cdot (\mu^*S_0 - S) \cdot S = 1$ , from which  $n = 1$  follows. Thus  $p_g(X) \leq 2$  holds. And  $K_X^3 \geq 2p_g(X) - 2$  holds except that  $K_X^3 = 1$  and  $p_g(X) = 2$ . As in the proof of (3.1), we can show that  $Z_i$  is not contained in the exceptional locus in  $X_{i-1}$  (or possibly  $i = 0$ ). Suppose  $X$  is smooth. Then we have  $K_S \cdot (\mu^*K_X)_{1S} = (\mu^*K_X + \sum a_j D_j) \cdot \mu^*K_X \cdot S = 1 + (\sum a_j D_j) \cdot \mu^*K_X \cdot S$ . This value is odd, because by Riemann-Roch formula we see that  $\chi(\mu^*K_{X_{1S}}) - \chi(\mathcal{O}_S) = 1/2\{(\mu^*K_{X_{1S}})(\mu^*K_{X_{1S}} - K_S)\} = 1/2\{1 - \mu^*K_{X_{1S}} \cdot K_S\}$  is an integer. Thus  $(\sum a_j D_j) \cdot \mu^*K_X \cdot S = a_i$  (note that  $\mu_0$  is the identity map) is even, so we have  $a_i = 2$ . Hence  $D_i$  comes from a blowing-up of a point, which contradicts to  $(\mu^*K_X) \cdot D_i \cdot S > 0$ .

Case 1-2). We take a sufficiently large positive integer  $m$  and a general smooth member  $H \in |m\mu^*K_X|$ . Then we have  $0 = m\mu^*K_X \cdot (\sum r_j D_j) \cdot nS = m\mu^*K_X \cdot (\sum r_j D_j) \cdot \{\mu^*(nS_0) - \sum r_j D_j\} = -(\sum r_j D_j)_{1H}^2$ . This means that for each  $j$ ,  $\mu(D_j)$  is a point on  $X$ . Thus we have  $0 = \mu^*K_X \cdot (\sum r_j D_j) \cdot S = (nS + F) \cdot (\sum r_j D_j) \cdot S = F \cdot (\sum r_j D_j) \cdot S = (\mu^*F_0 + \sum r_j D_j) \cdot (\sum r_j D_j) \cdot S = (\sum r_j D_j)_{1S}^2$ . This means for each  $j$ ,  $D_j|_S \cong 0$  and  $D_j$  is mapped to a point on  $Z$ . Thus applying Zariski's Main Theorem between a normalization  $Y'$  of  $Y$  which is projective over  $Y$  and the normalization of  $Z$ , we have  $\text{Bs}|K_X|_{red} = \emptyset$  and we have come to a contradiction.

Case 2.  $|K_X|_{red}$  is base-point-free.

We note that if  $C$  is irrational,  $|K_X|_{red}$  is always base-point-free.

The general fiber  $S_0$  is a surface with at most canonical singularities without

(-1)- or (-2)-curves.  $K_{S_0}$  is ample. Since  $K_X \sim nS_0 + F_0$ , we have  $K_{S_0} = K_{X|S_0} = F_{0|S_0}$ , and  $K_{S_0}^2 = K_X^2 \cdot S_0 = 1$ . By Noether's inequality, we have  $p_g(S_0) \leq (1/2)K_{S_0}^2 + 2$ . That means  $p_g(S_0) \leq 2$ . On the other hand, consider a standard exact sequence:

$$0 \longrightarrow H^0(K_X - S_0) \longrightarrow H^0(K_X) \longrightarrow H^0(K_{S_0}).$$

Since the second arrow is not an isomorphism, it follows that  $H^0(K_{S_0}) \neq 0$ . Thus we have  $1 \leq p_g(S_0) \leq 2$ . The equality  $q(S_0) = 0$  automatically follows from  $K_{S_0}^2 = 1$  (Kodaira, see [Bo, Theorem 11]). ■

We note that the surfaces with small  $K^2$  are studied in detail (cf. [H1]).

(4.2) PROBLEM. We assume that  $\dim \Phi_{K_X}(X) = 1$ . Our assumption that  $X$  is factorial is a quite strong condition, which we cannot get rid of. We have not yet found an example which does not satisfy the inequality  $K_X^3 \geq 2p_g(X) - 2$ .

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