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Rigidity and boundary behavior of holomorphic mappings

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1. Introduction.

Fatou's theorem, which asserts that every bounded holomorphic function in the unit disc Δ has non-tangential limits almost everywhere in the unit circle $\partial\Delta$, plays an important role in the theory of bounded holomorphic functions. Of course, this theorem does not hold on all holomorphic mappings of the unit disc into general complex manifolds. One of the most typical examples is given by universal covering maps. Namely, let S be a hyperbolic compact Riemann surface, and let $\pi:\Delta\rightarrow S$ be a holomorphic universal covering map. Then, as is well-known (cf. [**Ts**] Chapter XI), every point in $\partial\Delta$ is a point of approximation, that is, for every point $e^{i\theta}\in\partial\Delta$ and for every point $p\in S$ there exists a sequence of points $\{a_n\}$ in $\pi^{-1}(p)\subset\Delta$ converging to $e^{i\theta}$ in a Stolz sector. Accordingly, the boundary behavior of $\pi:\Delta\rightarrow S$ is quite complicated.

However, for a certain holomorphic mapping $f:\Delta\rightarrow S$, the boundary behavior remains to be simple. Assume that there exists a non-trivial deformation of f with a complex analytic parameter (see Section 2 for an exact definition). Then it is easy to see that the holomorphic mapping f has radial limits almost everywhere in $\partial\Delta$ (cf. Section 2). This example indicates that the rigidity of a holomorphic mapping of Δ into a complex manifold is closely connected with the boundary behavior. The boundary behavior is related to analytic properties of the target manifold.

In this paper, we shall investigate holomorphic mappings of Δ into C-hyperbolic manifolds with some additional conditions. In Section 2, we shall give a sufficient condition of a domain $M\subset \boldsymbol{C}^m$ under which every holomorphic proper mapping of Δ into M is rigid, or, under which every non-rigid holomorphic mapping of Δ into M behaves tamely near $\partial\Delta$. Some examples are shown in Section 3. In Section 4, we shall investigate non-rigid mappings. In Section 5, we shall investigate a manifold on which the Carathéodory pseudodistance is a distance and study holomorphic mappings of Δ into such a manifold. There, we introduce an ideal boundary of a manifold following the manner of compactification of a Riemann surface.

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2. Rigidity of holomorphic mappings of the unit disc.

In this paper Δ denotes the unit disc $\{z\in \boldsymbol{C};\ |z|<1\}$. A subset R of a complex manifold M is pluripolar (resp. analytic) if for every point $p\in R$ there exist a neighborhood U of p in M and a plurisubharmonic function s on U (resp. finite set of holomorphic functions $\varphi_1, \cdots, \varphi_n$ on U) such that $U\cap R=\{q\in U;\ s(q)=-\infty\}$ (resp. $U\cap R=\{q\in U;\ \varphi_1(q)=\cdots=\varphi_n(q)=0\}$).

We begin with the definition of rigidity.

DEFINITION. Let M be a complex manifold. A holomorphic mapping $f:\Delta\to M$ is said to be *rigid* if there exists no holomorphic mapping $\hat{f}:\Delta\times\Delta\to M$ such that $\hat{f}(\cdot, 0)=f(\cdot)$ on Δ and that $\hat{f}(\cdot, 0)\not\equiv\hat{f}(\cdot, \zeta)$ for some $\zeta\in\Delta$.

Namely, a holomorphic mapping $f:\Delta\to M$ is rigid if there exists no non-trivial deformation with a complex analytic parameter.

Let M be a bounded domain in $\boldsymbol{C}^m$. Then every holomorphic mapping $f:\Delta\to M$ has non-tangential limits almost everywhere in $\partial\Delta$. If a holomorphic proper map is deformed with a complex analytic parameter, then the non-tangential limits will be also deformed with the complex analytic parameter. In order to get a domain M such that every holomorphic proper mapping $f:\Delta\to M$ is rigid, therefore, it is natural to consider conditions about analytic subsets in $\overline{M}$. However, as we shall see in an example in the next section (Example 3.4), there exists a rigid holomorphic proper mapping f whose boundary values are surrounded by analytic sets in ∂M. Hence we give another type of condition beside that.

THEOREM 2.1. *Let M be a bounded domain in $\boldsymbol{C}^m$. Assume that there exists a countable set of pluripolar sets $\{R_k\}_{k=1}^{\infty}$ in $\boldsymbol{C}^m$ with $R_k\cap M=\emptyset$ for all natural number k such that for each point $p\in\partial M\setminus\bigcup_{k=1}^{\infty}R_k$ one of the followings takes place:*

(i) *every holomorphic mapping $h:\Delta\to\overline{M}$ with $h(0)=p$ is a constant map.*

(ii) *for every sequence $\{p_i\}\subset M$ with $\lim_{i\to\infty}p_i=p$ and for every positive number α, the Euclidean diameters of the hyperbolic balls $\{q\in M;\ d_M(p_i, q)<\alpha\}$ converge to 0 as $i\to\infty$, where d_M is the Kobayashi distance.*

Then every holomorphic proper mapping $f:\Delta\to M$ is rigid. In fact, every non-rigid holomorphic proper mapping has radial limits in M (not in ∂M) almost everywhere in $\partial\Delta$.

Before proving this theorem we note the following fact: if for any two points p and q in M the Kobayashi distance $d_M(p, q)$ is realized by one holomorphic mapping $\varphi:\Delta\to M$, then the condition (i) implies the condition (ii).

In order to prove Theorem 2.1, we need the following lemma.

LEMMA 2.2. *Let M and $\{R_k\}$ be as in Theorem 2.1. Then for every holomorphic mapping $f: \boldsymbol{H} \to M$ there exists a subset $E \subset \boldsymbol{R}$ with $mes(E)=0$ such that the mapping f has a non-tangential limit outside $\bigcup_{k=1}^{\infty} R_k$ at every point $x \in \boldsymbol{R}-E$. Here, $\boldsymbol{H}$ is the upper half plane in $\boldsymbol{C}$, and $mes(\cdot)$ stands for the one dimensional Lebesgue measure.*

PROOF. For a holomorphic mapping $f: \boldsymbol{H} \to M$ and a point $x \in \boldsymbol{R}$ at which f has a non-tangential limit, let $f_*(x)$ denote the non-tangential limit at x. By Fatou's theorem $f_*(x)$ exists at almost every point $x \in \boldsymbol{R}$. Hence it is sufficient to show that for each natural number k the set

$$E_k = \{x \in \boldsymbol{R};\ f_*(x) \in R_k\}$$

has measure 0. (Note that E_k is measurable, since f_* is measurable.)

Assume that $mes(E_k)>0$ for some natural number k, and fix one of such numbers, say k. We may assume that R_k is represented by a negative plurisubharmonic function defined in an open set U in $\boldsymbol{C}^m$; $R_k = \{p \in U;\ s(p) = -\infty\}$.

For each $x \in \boldsymbol{R}$ and $y>0$, put

$$S_x^y = \{z \in \boldsymbol{H};\ \pi/4 < arg(z-x) < 3\pi/4,\ \mathrm{Im}\, z < y\}.$$

Then, from the definition of E_k, for each point $x \in E_k$ there exists a positive number y such that $f(S_x^y) \subset U$. For each $x \in E_k$, put

$$y(x) = \sup\{y;\ f(S_x^y) \subset U\}.$$

Then it is easy to see that the assignment $x \mapsto y(x)$ is an upper semicontinuous function on E_k. Therefore, the subset $E^n \subset E_k$ defined by

$$E^n = \{x \in E_k;\ y(x) > 1/n\}$$

is a measurable set for each natural number n. Since

$$E_k = \bigcup_{n=1}^{\infty} E^n$$

and E_k is of positive measure, there exists a natural number m such that $mes(E^m)>0$. Fix such a number m. Since $mes(E^m)>0$, there exists a compact set K in E^m such that $mes(K)>0$. Set

$$\Omega = \bigcup_{x \in K} S_x^{1/m}.$$

Then $\partial\Omega \cap \boldsymbol{R} = K$, since K is compact. Choose a component Ω_0 of Ω such that $mes(\partial\Omega_0 \cap \boldsymbol{R})>0$. Note that $\partial\Omega_0 \cap \boldsymbol{R}$ is a compact subset of K, since different components of Ω are away from each other near $\boldsymbol{R}$. Therefore, $\boldsymbol{R} \setminus \partial\Omega_0 \cap \boldsymbol{R}$ is a disjoint union of open intervals;

$$\boldsymbol{R}\setminus\partial\Omega_0\cap\boldsymbol{R}=\bigcup_{n=1}^{\infty}I_n,\qquad I_n=(a_n,\,b_n),\qquad n=1,\,2,\,\cdots.$$

Let w_n denote the harmonic measure of I_n with respect to the upper half plane $\boldsymbol{H}$, that is, the Poisson integral of the characteristic function of I_n $(n=1, 2, \cdots)$. In the same way, let w denote the harmonic measure of $\boldsymbol{R}\setminus\partial\Omega_0\cap\boldsymbol{R}=\bigcup_{n=1}^{\infty}I_n$. Then

$$w(z)=\sum_{n=1}^{\infty}w_n(z),\qquad z\in\boldsymbol{H}.$$

As is well-known in the theory of bounded harmonic functions, w has radial limits $w_*(x)$ at almost all points $x\in\boldsymbol{R}$ and $w_*(x)=1$ almost everywhere in $\boldsymbol{R}\setminus\partial\Omega_0\cap\boldsymbol{R}$ and $w_*(x)=0$ almost everywhere in $\partial\Omega_0\cap\boldsymbol{R}$. Hence by the assumption $mes(\partial\Omega_0\cap\boldsymbol{R})>0$, w is not a constant function.

Now we show that w is bounded away from 0 uniformly on $\partial\Omega_0\cap\boldsymbol{H}$. Indeed, $\partial\Omega_0\cap\boldsymbol{H}$ is decomposed as follows:

$$\partial\Omega_0\cap\boldsymbol{H}=\Big(\bigcup_{n=1}^{\infty}l_n\Big)\cup(\partial\Omega_0\cap\{\operatorname{Im}z=1/m\}),$$

where l_n is the union of two sides of the triangle with base I_n:

$$l_n=\{z\in\boldsymbol{H};\ arg(z-b_n)=3\pi/4 \text{ or } arg(z-a_n)=\pi/4,\ \operatorname{Im}z\leqq(b_n-a_n)/2\}$$

if $a_n\neq-\infty$ and if $b_n\neq\infty$, and

$$l_n=\{z\in\boldsymbol{H};\ arg(z-b_n)=3\pi/4,\ \operatorname{Im}z\leqq1/m\},\qquad \text{if } a_n=-\infty\,,$$

$$l_n=\{z\in\boldsymbol{H};\ arg(z-a_n)=\pi/4,\ \operatorname{Im}z\leqq1/m\},\qquad \text{if } b_n=\infty\,.$$

If a point $z\in\partial\Omega_0\cap\boldsymbol{H}$ belongs to l_n, then

$$w(z)\geqq w_n(z)=\frac{1}{\pi}\,arg\,\frac{z-b_n}{z-a_n}>1/2\,.$$

Since $\partial\Omega_0\cap\{\operatorname{Im}z=1/m\}$ is a compact set in $\boldsymbol{H}$

$$\inf\{w(z);\,z\in\partial\Omega_0,\ \operatorname{Im}z=1/m\}>0\,.$$

Hence we have

$$\inf\{w(z);\,z\in\partial\Omega_0\cap\boldsymbol{H}\}=\alpha>0\,.$$

Now, $s\circ f$ is a negative subharmonic function on Ω_0, and that for each $z\in\partial\Omega_0\cap\boldsymbol{R}$, $\lim_{\Omega_0\ni\zeta\to z}s\circ f(\zeta)=-\infty$. Note that $s\circ f\not\equiv-\infty$, since $R_k\cap M=\emptyset$. Therefore for each positive number ε,

$$w(z)-\varepsilon s\circ f(z)>\alpha\,,\qquad \text{on } \partial\Omega_0\,.$$

Hence by the minimal principle for superharmonic functions,

$$w(z)-\varepsilon s\circ f(z)>\alpha\,, \qquad \text{on } \Omega_0\,.$$

Since ε is arbitrary,

$$w>\alpha \qquad \text{on } \Omega_0\,.$$

On the other hand, as noted above, there exists a point $x\in\partial\Omega_0\cap\boldsymbol{R}$ with $\lim_{y\to 0}w(x+iy)=0$. This is the desired contradiction. □

PROOF OF THEOREM 2.1. Let $f:\Delta\to M$ be a holomorphic proper mapping. Let $\hat{f}:\Delta\times\Delta\to M$ be a holomorphic mapping with $\hat{f}(\cdot, 0)=f(\cdot)$. We shall show that $\hat{f}(\cdot, \zeta)=\hat{f}(\cdot, 0)=f(\cdot)$ for every $\zeta\in\Delta$. Fix a point $\zeta\in\Delta$ arbitrarily. From Lemma 2.2, for almost every $e^{i\theta}\in\partial\Delta$ the non-tangential limit $f_*(e^{i\theta})$ at $e^{i\theta}$ exists and satisfies the condition (i) or (ii).

Assume that $f_*(e^{i\theta})$ satisfies (i). Let $\{z_n\}\subset\Delta$ be a sequence converging to $e^{i\theta}$ in a Stolz sector. We may assume, taking a subsequence if necessary, that the sequence of mappings $\{\hat{f}(z_n, \cdot):\Delta\to M\}$ converges to a holomorphic mapping $h:\Delta\to\overline{M}$. Then we have

$$h(0)=\lim_{n\to\infty}\hat{f}(z_n, 0)=\lim_{n\to\infty}f(z_n)=f_*(e^{i\theta})\,.$$

Since $f_*(e^{i\theta})$ satisfies (i), h is a constant map. It follows that

$$\lim_{n\to\infty}\hat{f}(z_n, \zeta)=h(\zeta)=f_*(e^{i\theta})\,.$$

Since $\{z_n\}$ is an arbitrary sequence converging to $e^{i\theta}$ in a Stolz sector, $\hat{f}(\cdot, \zeta)$ has a non-tangential limit $\hat{f}_*(e^{i\theta}, \zeta)$ at $e^{i\theta}$ and that $\hat{f}_*(e^{i\theta}, \zeta)=f_*(e^{i\theta})$.

Assume that $f_*(e^{i\theta})$ satisfies (ii). Let $\{z_n\}\subset\Delta$ be a sequence converging to $e^{i\theta}$ in a Stolz sector. Then

$$d_M(\hat{f}(z_n, \zeta), \hat{f}(z_n, 0))\leqq d_{\Delta\times\Delta}((z_n, \zeta), (z_n, 0))=d_\Delta(\zeta, 0)\,.$$

Since $\lim_{n\to\infty}\hat{f}(z_n, 0)=\lim_{n\to\infty}f(z_n)=f_*(e^{i\theta})$ satisfies (ii), $\lim_{n\to\infty}\hat{f}(z_n, \zeta)=f_*(e^{i\theta})$. It follows that $\hat{f}(\cdot, \zeta)$ has a non-tangential limit $\hat{f}_*(e^{i\theta}, \zeta)$ at $e^{i\theta}$ and that $\hat{f}_*(e^{i\theta}, \zeta)=f^*(e^{i\theta})$.

Now, the holomorphic mappings $\hat{f}(\cdot, \zeta):\Delta\to M$ and $f:\Delta\to M$ have the same non-tangential limits almost everywhere in $\partial\Delta$. Therefore, by Riesz' theorem

$$\hat{f}(\cdot, \zeta)\equiv f(\cdot) \qquad \text{on } \Delta\,.$$

Hence f is rigid.

The above argument implies that if there exists a positive measure set $E\subset\partial\Delta$ such that $f_*(e^{i\theta})\in\partial M$ for all $e^{i\theta}\in E$ then f is rigid. It follows that for a non-rigid mapping $f:\Delta\to M$ the non-tangential limits $f_*(e^{i\theta})$ at almost all $e^{i\theta}\in\partial\Delta$ belong to M. □

COROLLARY 2.3. *Let N be a complex manifold with a covering space M*

satisfying the assumption of Theorem 2.1. *Then every non-rigid holomorphic mapping of* Δ *into* N *has non-tangential limits almost everywhere in* $\partial\Delta$.

3. Examples.

EXAMPLE 3.1. Let S be a hyperbolic Riemann surface of finite analytic type. Then the Teichmüller space $T(S)$ of S is identified with a bounded domain in $\boldsymbol{C}^m$ via the Bers embedding, and satisfies the assumption of Theorem 2.1. See [**Ta**] for details and applications. In this case, the sets $\{R_k\}_{k=1}^{\infty}$ are analytic sets defined by holomorphic functions which are defined globally on $\boldsymbol{C}^m$. For this reason, the proof for $T(S)$ is much simpler than that of Theorem 2.1, in which each R_k is defined only locally.

The following example shows that Theorem 2.1 can not be extended to all manifolds M.

EXAMPLE 3.2. Let l_1, l_2, l_3 and l_4 be complex lines in general position in $\boldsymbol{P}_2(\boldsymbol{C})$, and let l_0 be the line through $l_1\cap l_2$ and $l_3\cap l_4$. Put $M=\boldsymbol{P}_2(\boldsymbol{C})\setminus\bigcup_{i=0}^4 l_i$. Then M is complete hyperbolic and hyperbolically embedded in $\boldsymbol{P}_2(\boldsymbol{C})$ (cf. [**K**] Chapter VI). The boundary of M is of course the finite union of analytic sets $\bigcup_{i=0}^4 l_i$. However, all non-rigid holomorphic mappings do not have radial limits almost everywhere in $\partial\Delta$. Note that M is biholomorphic to the direct product of thrice punctured spheres. Let $\pi:\Delta\to\boldsymbol{C}\setminus\{0, 1\}$ denote a holomorphic universal covering map. Then the holomorphic mapping $f:\Delta\to(\boldsymbol{C}\setminus\{0, 1\})\times(\boldsymbol{C}\setminus\{0, 1\})$ defined by

$$f(z)=(\pi(z), 1/2), \qquad z\in\Delta$$

is non-rigid, since $\hat{f}:\Delta\times\Delta\to(\boldsymbol{C}\setminus\{0, 1\})\times(\boldsymbol{C}\setminus\{0, 1\})$ defined by

$$\hat{f}(z, \zeta)=\left(\pi(z), \frac{\zeta+1}{2}\right) \qquad (z, \zeta)\in\Delta\times\Delta$$

gives a deformation of f with a complex analytic parameter. However, f has radial limits nowhere in $\partial\Delta$ except for a null set, since π has radial limits nowhere except for a null set.

More generally, let S_1 and S_2 be hyperbolic Riemann surfaces of finite analytic type and let M be the product $S_1\times S_2$. Then the product of a universal covering map and a constant map is non-rigid holomorphic mapping which has radial limits nowhere except for a null set in $\partial\Delta$. On the other hand, the product $S_1\times S_1$ minus a suitable analytic set is a manifold to which Corollary 2.3 is applied;

EXAMPLE 3.3. Let S be a hyperbolic Riemann surface of finite analytic type with no non-trivial conformal automorphism. Set

$$M = S\times S\setminus\{(p,\,q)\in S\times S\,;\ p=q\}.$$

Then every non-rigid holomorphic mapping of Δ into M has radial limits almost everywhere in $\partial\Delta$. In fact, the universal covering space is a subvariety of $T(S)$ as follows (cf. [**BR**] p. 269). Let x_1 and x_2 be distinct points in S and set $\dot{S}=S-\{x_1\}$, $\ddot{S}=S-\{x_1,\,x_2\}$. Let $\pi_2: T(\ddot{S})\to T(\dot{S})$ and $\pi_1: T(\dot{S})\to T(S)$ be the Bers fiber spaces over $T(\dot{S})$ and $T(S)$ respectively (see [**B**]). Put $(\pi_2\circ\pi_1)^{-1}(0)=V$, where 0 is the base point of $T(S)$. Each point $[\mu]$ in V determines a quasiconformal mapping $w^\mu: S\to S$. Now set

$$\varphi([\mu]) = (w^\mu(x_1),\, w^\mu(x_2))\,, \qquad [\mu]\in V\,.$$

Then φ is a well-defined holomorphic mapping of V onto M. Let Γ be a Fuchsian group representing the Riemann surface S. The group Γ is identified with a subgroup of the modular group $Mod(\ddot{S})$ and acts on V. Let Γ' be the subgroup of $\Gamma<Mod(\ddot{S})$ consisting of all modular transformations $[\omega]_*$ induced by quasicoformal self-mappings ω of S fixing x_1 and x_2 point-wise. Then, by the same argument as in [**BR**], the holomorphic mapping $\varphi: V\to M$ is a covering map with covering group Γ'. It is easy to see that V is simply connected, hence is the universal covering space. It is also easy to see, from the definition of the Bers fiber space, the inclusion map $V\hookrightarrow T(\ddot{S})$ is proper. Therefore Corollary 2.3 is applied to M.

This example indicates that removing a countable union of analytic sets in a manifold may change the property of holomorphic mappings of Δ into the manifold. In fact, by the above argument, every holomorphic proper mapping of Δ into $\Delta\times\Delta\setminus\bigcup_{\gamma\in\Gamma}\{(p,\,\gamma(p));\ p\in\Delta\}$ is rigid, although there are non-rigid proper mappings of Δ into $\Delta\times\Delta$. Note that the domain $\Delta\times\Delta\setminus\bigcup_{\gamma\in\Gamma}\{(p,\,\gamma(p));\ p\in\Delta\}$ itself does not satisfies the assumption of Theorem 2.1.

EXAMPLE 3.4. Let M be the product

$$M = (\Delta\setminus[-1/2,\,1/2])\times\Delta\,.$$

Let $\pi: \Delta\to(\Delta_{3/4}\setminus[-1/2,\,1/2])$ be a holomorphic universal covering map, where $\Delta_{3/4}=\{z\in\Delta;\ |z|<3/4\}$. Then the holomorphic mapping $f: \Delta\to M$ defined by

$$f(z) = (\pi(z),\, z)\,, \qquad z\in\Delta$$

has radial limits $f_*(e^{i\theta})$ at almost all points $e^{i\theta}\in\partial\Delta$. Note that for each radial limit $f_*(e^{i\theta})$ there exists a holomorphic injection $h: \Delta\to\partial M$ with $h(0)=f_*(e^{i\theta})$. Hence no $f_*(e^{i\theta})$ satisfies the condition (i). However, it is easy to see that f is rigid. The rigidity of f is due to the following fact: there exists a positive measure set $F\subset\partial\Delta$ such that $f_*(e^{i\theta})\in[-1/2,\,1/2]\times\partial\Delta$ for each $e^{i\theta}\in F$ and every point in $[-1/2,\,1/2]\times\partial\Delta$ satisfies the condition (ii).

4. Rank of deformation.

In this section we investigate non-rigid mappings.

DEFINITION. Let $f:\Delta\to M$ be a holomorphic mapping. We shall say f has *deformation of rank* r if there exists a holomorphic mapping $\hat{f}:\Delta\times\Delta^r=\Delta\times\underbrace{\Delta\times\cdots\times\Delta}_{r\text{ times}}\to M$ with $\hat{f}(\cdot,\underbrace{0,\cdots,0}_{r})=f(\cdot)$ on Δ such that for some $z\in\Delta$ the Jacobian matrix of $\hat{f}(z,\cdot):\Delta^r\to M$ at some $\zeta\in\Delta^r$ is of rank r.

Let M be a bounded domain such that the maximal dimension of analytic subsets contained in M is l. Professor Y. Imayoshi showed that the rank of deformation of each proper holomorphic mapping of Δ into M is at most l using Jacobian matrices (oral communication). Note that we have shown in Lemma 2.2 that a countable union of pluripolar sets is negligible. Combining Lemma 2.2 with his idea, we shall extend that result.

Another rigidity property of certain holomorphic mappings on condition about *boundary components* appears in Sunada [**S**].

DEFINITION. Let M be a bounded domain in C^m. Assume that the union of analytic sets of dimension greater than l contained in ∂M is covered by a countable union $\bigcup_{k=1}^{\infty}R_k$ of pluripolar sets with $R_k\cap M=\emptyset$ $(k=1, 2, \cdots)$ and that the union of analytic sets of dimension l contained in ∂M is not covered by such a countable union of pluripolar sets. Then we shall say that the *essential maximal dimension* of analytic sets contained in ∂M is l.

THEOREM 4.1. *Let M be a complete hyperbolic bounded domain in C^m and assume that the essential maximal dimension of analytic sets contained in ∂M is l. Then for each proper holomorphic mapping $f:\Delta\to M$ the rank of deformation is at most l.*

PROOF. Let $f:\Delta\to M$ be a holomorphic proper mapping, and let $\hat{f}:\Delta\times\Delta^r\to M$ be a holomorphic mapping with $\hat{f}(\cdot, 0)=f$ on Δ. Assume that $r>l$. We shall show that each $r\times r$ minor of Jacobian matrix of $\hat{f}(z,\cdot)$ vanishes identically on Δ^r for every $z\in\Delta$.

Note that for all $\zeta\in\Delta^r$, $\hat{f}(\cdot,\zeta):\Delta\to M$ is proper. In fact, for each sequence $\{z_n\}\subset\Delta$ with no accumulation points in Δ, $\{\hat{f}(z_n, 0)\}$ has no accumulation points in M, and that

$$d_M(\hat{f}(z_n, 0), \hat{f}(z_n,\zeta))\leqq d_{\Delta^r}(0,\zeta).$$

Hence the sequence $\{\hat{f}(z_n,\zeta)\}\subset M$ has no accumulation points in the complete hyperbolic domain M.

Let $\{\zeta_k\}_{k=1}^{\infty}$ be a countable dense subset in Δ^r. Let $\{R_h\}_{h=1}^{\infty}$ be a sequence of pluripolar sets which covers the union of analytic sets of dimension greater than l in ∂M. Since each $\hat{f}(\cdot, \zeta_k)$ is proper, there exists a subset E_k in $\partial\Delta$ with measure 0 such that $\hat{f}(\cdot, \zeta_k)$ has radial limits in $\partial M \setminus \bigcup_{h=1}^{\infty} R_h$ everywhere in $\partial\Delta \setminus E_k$. Set $E = \bigcup_{k=1}^{\infty} E_k$.

Take an $r \times r$ minor g_z of the Jacobian matrix of $\hat{f}(z, \cdot): \Delta \to M$, where $z \in \Delta$. Then the minor $g_z(\zeta)$ at ζ depends on $z \in \Delta$ and $\zeta \in \Delta^r$ holomorphically, hence $g_z(\zeta)$ is a holomorphic function on $\Delta \times \Delta^r$. Let $e^{i\theta}$ be a point of $\partial\Delta \setminus E$ and let $\{z_n\}_{n=1}^{\infty} \subset \Delta$ be a sequence converging to $e^{i\theta}$ non-tangentially. Then, taking a subsequence if necessary, we may assume that $\{\hat{f}(z_n, \cdot)\}_{n=1}^{\infty}$ converges to a holomorphic mapping of Δ^r into ∂M. For each k, the sequence $\{g_{z_n}(\zeta_k)\}_{n=1}^{\infty}$ converges to an $r \times r$ minor of the Jacobian matrix of this mapping at ζ_k. Now, $\hat{f}_*(e^{i\theta}, \zeta_k) = \lim_{n \to \infty} \hat{f}(z_n, \zeta_k)$ belongs to $\partial M \setminus \bigcup_{h=1}^{\infty} R_h$. Hence by the assumption $r > l$, it follows that the $r \times r$ minor $\lim_{n \to \infty} g_{z_n}(\zeta_k)$ is equal to 0 for each k. Note that $g.(\zeta_k)$ is a bounded holomorphic function on Δ. Hence $g_z(\zeta_k) = 0$ for all $z \in \Delta$ and for each k. Since $\{\zeta_k\}$ is dense in Δ^r, $g_z(\cdot) \equiv 0$ on Δ^r for all $z \in \Delta$. □

5. Manifolds with Carathéodory distances.

First we recall the following fact (cf. [CC] pp. 96-98). Let M be a complex manifold and let Q be a class of continuous functions on M. Then there exists a compact topological space $M_Q^* \supset M$, called a Q-compactification of M, with the following properties:

(i) M is dense in M_Q^*.

(ii) The topology induced from M_Q^* coincides with the topology of M.

(iii) Every function in Q is continuously extended to M_Q^*.

(iv) Any two distinct points a, b in $M_Q^* \setminus M$ are separated by Q. Namely, there exists a function $f \in Q$ such that $f(a) \neq f(b)$.

We sketch the construction of M_Q^*. See [CC] for details. For each $f \in Q \cup C_0$, $\hat{\boldsymbol{C}}_f$ denote a copy of $\hat{\boldsymbol{C}}$. Here, $C_0 = C_0(M)$ is the class of all continuous functions on M with compact support. Put

$$\hat{\boldsymbol{C}}^{Q \cup C_0} = \prod_{f \in Q \cup C_0} \hat{\boldsymbol{C}}_f,$$

equipped with the product topology. Define a mapping $\phi: M \to \hat{\boldsymbol{C}}^{Q \cup C_0}$ by

$$\phi(a) = \{f(a)\}_{f \in Q \cup C_0},$$

where $a \in M$. Then it is easy to see that ϕ is injective and continuous from the definition of the topology of $\hat{\boldsymbol{C}}^{Q \cup C_0}$. Put $M^* = \overline{\phi(M)}$. Then $\phi: M \to \phi(M)$ is an open map, and under the identification of M with $\phi(M)$, M^* has the property (i) and (ii). For each $f \in Q \cup C_0$, let $\pi_f: \hat{\boldsymbol{C}}^{Q \cup C_0} \to \hat{\boldsymbol{C}}_f$ be a projection. Then, under the identification of M with $\phi(M)$, π_f is a continuous extension of f, and for any two distinct points $a, b\in M^*\setminus M$, there exists a function $f\in Q$ with $\pi_f(a)\neq\pi_f(b)$. Hence M^* has the desired properties. Note that such a compactification is unique up to homeomorphisms which are the identity on M.

Now, assume that M is a complex manifold on which the Carathéodory pseudo-distance c_M is a distance. Take a countable dense subset $L=\{(a_n, b_n)\}_{n=1}^{\infty}$ in $M\times M$ and choose a holomorphic function $g_n: M\rightarrow\Delta$ realizing the Carathéodory distance $c_M(a_n, b_n)$ for each n. Put $Q=\{g_n\}_{n=1}^{\infty}$. Then we have the Q-compactification M_Q^* of M. Note that M_Q^* may depend on the choice of $Q=\{g_n\}$.

If M is a relatively compact subset of a complex manifold N on which the Carathéodory pseudo-distance c_N is a distance, then we have a compactification $\overline{M}$ in N beside the compactification M_Q^*. For the dense subset $L=\{(a_n, b_n)\}_{n=1}^{\infty}$ as above, choose a holomorphic function $h_n: N\rightarrow\Delta$ realizing the Carathéodory distance $c_N(a_n, b_n)$ for each n. Put $Q_0=\{h_n\}_{n=1}^{\infty}$. Then, the Q_0-compactification $M_{Q_0}^*$ is homeomorphic to $\overline{M}$. In fact, by the uniqueness of the Q_0-compactification it is sufficient to show that Q_0 separates distinct points on $\overline{M}$. Let $a, b\in\overline{M}$ be distinct points. Then there exists a subsequence $\{(a_{n_i}, b_{n_i})\}\subset L$ with $\lim_{i\rightarrow\infty}(a_{n_i}, b_{n_i})=(a, b)$. Since $c_N(a, b)>0$, we may assume that there exists a positive number α such that $c_N(a_{n_i}, b_{n_i})>\alpha$ for every i. Let ρ denote the Poincaré distance of Δ. If $h(a)=h(b)$ for every $h\in Q_0$,

$$\begin{aligned}0<\alpha<c_N(a_{n_i}, b_{n_i})&=\rho(h_{n_i}(a_{n_i}), h_{n_i}(b_{n_i}))\\ &\leqq\rho(h_{n_i}(a_{n_i}), h_{n_i}(a))+\rho(h_{n_i}(a), h_{n_i}(b))+\rho(h_{n_i}(b), h_{n_i}(b_{n_i}))\\ &=\rho(h_{n_i}(a_{n_i}), h_{n_i}(a))+\rho(h_{n_i}(b), h_{n_i}(b_{n_i}))\\ &\leqq d_N(a_{n_i}, a)+d_N(b_{n_i}, b)\,.\end{aligned}$$

Here, d_N is the Kobayashi distance on N. This is a contradiction since $\lim_{i\rightarrow\infty}d_N(a_{n_i}, a)=\lim_{i\rightarrow\infty}d_N(b_{n_i}, b)=0$. This argument also shows that $Q=\{g_n\}_{n=1}^{\infty}$ separates distinct points in M.

Note that $\overline{M}$ and M_Q^* are not necessarily the same compactification. For example, let M be a simply connected relatively compact region in Δ and let $f: M\rightarrow\Delta$ be a conformal mapping. Then, since the Carathéodory distance on M is the pull back of the Poincaré distance ρ via f, M_Q^* is realized by $\overline{f(M)}=\overline{\Delta}$ for any choice of Q. On the other hand, the mapping $f: M\rightarrow\Delta$ extends to a homeomorphism of $\overline{M}$ onto $\overline{f(M)}=\overline{\Delta}$ if and only if M is a Jordan domain. Hence $\overline{M}$ and M_Q^* are the same compactification if and only if M is a Jordan domain.

We also note that M_Q^* is metrizable in the above situation. Take a sequence of functions $\{j_k\}_{k=1}^{\infty}\in C_0(M)$ with $\|j_k\|_{\infty}<1$ for each k such that for each $a\in M$ there exists a function j_k with $i_k(a)\neq0$. Then $Q\cup\{j_k\}_{k=1}^{\infty}$ separates distinct points in M_Q^*. It follows that the function $d: M_Q^*\times M_Q^*\rightarrow\boldsymbol{R}$ defined by

$$d(a, b)=\sum_{k=1}^{\infty}\frac{1}{2^k}|g_k(a)-g_k(b)|+\sum_{k=1}^{\infty}\frac{1}{2^k}|j_k(a)-j_k(b)|$$

is a distance on M_Q^*, since Q separates M_Q^*. It follows that M_Q^* is separable.

LEMMA 5.1. *Let M be a complex manifold on which the Carathéodory pseudo-distance c_M is a distance. Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be a countable dense subset of $M\times M$, and let $Q=\{g_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on M such that $g_n: M\to\Delta$ realizes the Carathéodory distance $c_M(a_n, b_n)$ for each n. Then every holomorphic mapping $f:\Delta\to M$ has non-tangential limits in the Q-compactification M_Q^* almost everywhere in $\partial\Delta$. If M is a relatively compact region in a complex manifold N on which the Carathéodory pseudo-distance c_N is a distance, then every holomorphic mapping $f:\Delta\to M$ has non-tangential limits in $\bar{M}\subset N$ almost everywhere in $\partial\Delta$.*

In either case, if two holomorphic mappings $f_1:\Delta\to M$ and $f_2:\Delta\to M$ have the same non-tangential limits in a positive measure set in $\partial\Delta$, then $f_1\equiv f_2$ on Δ.

PROOF. For the sake of simplicity of notation, we shall give a proof with respect to radial limits. The statement for non-tangential limits is proved in the same way. Let $f:\Delta\to M$ be a holomorphic mapping. For each $e^{i\theta}\in\partial\Delta$, put

$$A_f(\theta)=\bigcap_{0<r<1}\overline{\{f(r'e^{i\theta});\, 1>r'>r\}},$$

where the closure is taken in M_Q^*.

For each $g_n\in Q$, there exists a subset $E_n\subset\partial\Delta$ with $mes(E_n)=0$ such that the holomorphic mapping $g_n\circ f:\Delta\to\Delta$ has a radial limit $(g_n\circ f)_*(e^{i\theta})$ at every point $e^{i\theta}\in\partial\Delta\setminus E_n$. Set $E=\bigcup_{n=1}^{\infty}E_n$, and fix a point $e^{i\theta}\in\partial\Delta\setminus E$ arbitrarily. For each point $p\in A_f(\theta)$, there exists a sequence $\{r_m\}_{m=1}^{\infty}\subset(0, 1)$ with $r_m\nearrow 1$ such that $\lim_{m\to\infty}f(r_m e^{i\theta})=p$. Since each g_n is continuous on M_Q^*, $(g_n\circ f)_*(e^{i\theta})=\lim_{m\to\infty}g_n\circ f(r_m e^{i\theta})=g_n(\lim_{m\to\infty}f(r_m e^{i\theta}))=g_n(p)$. Hence we have

$$A_f(\theta)\subset\bigcap_{n=1}^{\infty}\{p\in M_Q^*;\, g_n(p)=(g_n\circ f)_*(e^{i\theta})\}.$$

Assume that $A_f(\theta)$ contains two distinct points a and b. Then, since $A_f(\theta)$ is connected, we may assume that both of a and b belong to M or both of a and b belong to M_Q^*. In either case, there exists a function $g_n\in Q$ such that $g_n(a)\neq g_n(b)$. On the other hand, since $a, b\in A_f(\theta)$, $g_n(a)=g_n(b)=(g_n\circ f)_*(e^{i\theta})$. This is a contradiction. Hence $A_f(\theta)$ consists of exactly one point, say a_θ. It follows that f has a radial limit a_θ at every point $e^{i\theta}\in\partial\Delta\setminus E$.

If M is a relatively compact region in N, define $A_f(\theta)$ using the closures in $\bar{M}$ instead of closures in M_Q^*, and take a holomorphic function $h_n: N\to\Delta$ representing the Carathéodory distance $c_N(a_n, b_n)$, instead of g_n for each n.

Since, as noted before, the sequence $\{h_n\}_{n=1}^{\infty}$ separates distinct points of $\overline{M}$, an exactly parallel argument shows that each holomorphic mapping $f: \Delta \to M$ has non-tangential limits in $\overline{M}$ almost everywhere in $\partial\Delta$.

Now assume that two holomorphic mappings $f_1: \Delta \to M$ and $f_2: \Delta \to M$ have the same non-tangential limits in M_Q^* (resp. in $\overline{M}$). If $f_1 \not\equiv f_2$ on Δ, there exists a point $a \in \Delta$ such that $f_1(a) \neq f_2(a)$. Hence there exists a holomorphic function $g: M \to \Delta$ with $g(f_1(a)) \neq g(f_2(a))$ which extends continuously to M_Q^* (resp. $\overline{M}$). On the other hand since g is continuous on M_Q^* (resp. $\overline{M}$) and since the non-tangential limits of f_1 and f_2 coincide in a positive measure set, so do the non-tangential limits of $g \circ f_1$ and $g \circ f_2$. Hence $g \circ f_1 \equiv g \circ f_2$ on Δ. This is a contradiction. □

This lemma enable us to extend Theorem 2.1. Indeed, in the proof of Theorem 2.1, we used the fact that the target M is a bounded domain in $\boldsymbol{C}^m$ only to show a holomorphic mapping $f: \Delta \to M$ has non-tangential limits almost everywhere in $\partial\Delta$.

THEOREM 5.2. *Let M be a relatively compact domain in a complex manifold N on which the Carathéodory pseudo-distance c_N is a distance. Assume that there exists a countable set of pluripolar sets $\{R_k\}_{k=1}^{\infty}$ in N with $R_k \cap M = \emptyset$ for all natural number k such that for each point $p \in \partial M \setminus \bigcup_{k=1}^{\infty} R_k$ one of the followings takes place:*

(i) *every holomorphic mapping $h: \Delta \to \overline{M}$ with $h(0) = p$ is a constant map.*

(ii) *for any two sequences $\{p_i\}$ and $\{q_i\}$ such that $\lim_{i\to\infty} p_i = p$ and that $d_M(p_i, q_i) < \alpha$ for some positive number α independent of i, $\{q_i\}$ also converges to p, where d_M is the Kobayashi distance on M.*

Then every holomorphic proper mapping $f: \Delta \to M$ is rigid. In fact, every non-rigid holomorphic proper mapping has non-tangential limits in M (not in ∂M) almost everywhere in $\partial\Delta$.

Let M be a complex manifold on which the Carathéodory pseudo-distance c_M is a distance, and let $Q = \{g_n\}_{n=1}^{\infty}$ be as in Lemma 5.1. We shall say a subset $R \subset M_Q^*$ is *pluripolar* if for every point $p \in M_Q^*$ there exist a neighborhood U of p in M_Q^* and a plurisubharmonic function s on $U \cap M$ such that $U \cap R = \{q \in U; \lim_{M \cap U \ni q' \to q} s(q') = -\infty\}$. With this terminology, we have the following:

THEOREM 5.3. *Let M be a complex manifold on which the Carathéodory pseudo-distance is a distance. Let $Q = \{g_n\}$ be the sequence of holomorphic functions as in Lemma 5.1, and let M_Q^* be a Q-compactification of M. Assume that there exists a countable set of pluripolar sets $\{R_k\}_{k=1}^{\infty}$ in M_Q^* with $R_k \cap M = \emptyset$ for all natural number k such that each point $p \in \partial M \setminus \bigcup_{k=1}^{\infty} R_k$ satisfies the following: for any two sequences $\{p_i\}$ and $\{q_i\}$ such that $\lim_{i\to\infty} p_i = p$ and that $d_M(p_i, q_i) < \alpha$*

for some positive number α independent of i,, $\{q_i\}$ also converges to p.

Then every holomorphic proper mapping $f: \Delta \rightarrow M$ is rigid. In fact, every non-rigid holomorphic proper mapping has non-tangential limits in M (not in ∂M) almost everywhere in $\partial\Delta$.

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