J. Math. Soc. Japan
Vol. 44, No. 1, 1992

On the order of growth of the Kloosterman zeta function

By Eiji YOSHIDA

(Received Dec. 6, 1990)

§1. Introduction.

Let $H=\{z=x+iy\in \boldsymbol{C} \mid \mathrm{Im} z=y>0\}$ be the complex upper half plane given the Riemann structure

$$ds^2 = y^{-2}(dx^2+dy^2), \tag{1.1}$$

and let $G=PSL(2, \boldsymbol{R})=SL(2, \boldsymbol{R})/\{\pm 1\}$. Then the group G acts on H as linear fractional transformation:

$$\gamma z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

and moreover the metric (1.1) gives rise to a G-invariant measure and the Laplace operator whose explicit forms are

$$d\mu(z) = y^{-2}dxdy \tag{1.2}$$

and

$$D = y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big). \tag{1.3}$$

Throughout this paper, we will suppose that $\Gamma(\subset G)$ is a congruence subgroup, though there is no need to make this restriction. In fact, all results given in this material can be generalized to any Fuchsian group of the first kind with a cusp ∞ by slight modifications. We further denote by $\mathcal{D}_\Gamma(=\Gamma\backslash H)$ the fundamental domain of Γ, which is always noncompact.

Let now $L^2(\mathcal{D}_\Gamma)$ be the Hilbert space consisting of all functions which are automorphic with respect to Γ and square integrable on $\mathcal{D}_\Gamma$, i.e.,

$$L^2(\mathcal{D}_\Gamma) = \Big\{f \mid f(\gamma z) = f(z) \ \text{ for } \gamma\in\Gamma, \int_{\mathcal{D}_\Gamma} |f(z)|^2 d\mu(z) < \infty\Big\}.$$

Then, the space $L^2(\mathcal{D}_\Gamma)$ has a spectral decomposition in accordance with the operation of D:

$$L^2(\mathcal{D}_\Gamma) = L^2_0(\mathcal{D}_\Gamma)\oplus \boldsymbol{C}\oplus L^2_c(\mathcal{D}_\Gamma),$$

where each term on the right hand side denotes respectively the space of cusp forms, the space of constant functions and continuous part of the spectrum. Furthermore the space $L_0^2(\mathcal{D}_\Gamma)$ has an orthogonal basis $\{f_j\}_{j\geq 1}$ consisting of eigenforms of D, which are called Maass wave forms (see [**9**: Theorem 5.2.4] and [**6**: Remark 9.7]). We write $Df_j=-\lambda_j f_j$ and $\lambda_j=s_j(1-s_j)$, $s_j\in\boldsymbol{C}$ for each $j\geqq 1$. If we plot the points $s_j(j\geqq 1)$ on the complex plane, which are determined correspondingly by the value of λ_j, since all λ_j are positive, infinite number of s_j lie in the critical line and a finite number of s_j in the real segment (0, 1) if there exists. The eigenvalue λ_j which produces a real point s_j is called the exceptional eigenvalue. Whether or not exceptional eigenvalues are in existence is an important problem. Non-existence of such eigenvalues for congruence subgroups Γ has been conjectured by Selberg [**15**], which is equivalent to the assertion $\lambda_j\geqq 1/4$ for all $j\geqq 1$.

REMARK 1. This conjecture was proved for $\Gamma=PSL(2, \boldsymbol{Z})$ by Maass [**11**] or Roelcke [**13**], and recently proved by Huxley [**7**] for Hecke congruence groups $\Gamma_0(N)$ with $N\leqq 17$(cf. [**8**: P. 173]). For more examples which satisfy $\lambda_j\geqq 1/4$, see also Sarnak [**14**] or Hejhal [**5**: Notes for chapter eleven].

Let q be the smallest positive integer such that $\begin{pmatrix}1 & q\\ & 1\end{pmatrix}\in\Gamma$. Then the Kloosterman sum is defined by

$$S(m, n, c, \Gamma)=\sum_{\substack{0\leq a<qc\\ 0\leq d<qc}} e\Big(\frac{1}{qc}(ma+nd)\Big), \qquad \begin{pmatrix}a & *\\ c & d\end{pmatrix}\in\Gamma$$

for $c>0$, where $m, n\in\boldsymbol{Z}_{\neq 0}$ and $e(x)=\exp(2\pi i x)$. Moreover we denote Selberg's Kloosterman zeta function by

$$Z_{m,n}(s, \Gamma)=\sum_{c>0}\frac{S(m, n, c, \Gamma)}{c^{2s}}. \tag{1.4}$$

Since it is clear that $|S(m, n, c, \Gamma)|\leqq qc$, this series converges absolutely for $\operatorname{Re}(s)>1$. If we then use Weil's estimate for Kloosterman sums (cf. [**8**: P. 178]), it follows that the range of absolute convergence can be taken as $\operatorname{Re}(s)>3/4$. The analytic properties of the Kloosterman zeta function after continued to $\operatorname{Re}(s)>1/2$ are closely connected with, through the estimation of sums of Kloosterman sums, a problem for Fourier coefficients of holomorphic automorphic forms or several kinds of ones in analytic number theory (see [**1**] or [**8**]).

Next, let us denote by T the set of exceptional eigenvalues s_j which satisfy $1/2<s_j<1$. Then, for any positive ε chosen so that $(1/2, (1/2)+2\varepsilon)\cap T=\phi$, we define the domain U_ε in the complex s-plane to be

$$\Big\{\sigma \mid \frac{1}{2}<\sigma<\frac{1}{2}+\varepsilon\Big\}\times\{\tau \mid |\tau|\leqq 1\} \tag{1.5}$$

for $s=\sigma+i\tau$. Moreover put M to be any fixed positive number such as $M>1$. Under this setting, the purpose in the present article is to prove the following

THEOREM. *The Kloosterman zeta function* $Z_{m,n}(s, \Gamma)$ *defined by* (1.4) *can be continued meromorphically to* $\mathrm{Re}(s)>1/2$ *with at most a finite number of simple poles at* $s=s_j$ *lying in* $(1/2, 1)$, *and satisfies the following estimate:*

$$Z_{m,n}(s, \Gamma)=O\Big(q^{2\sigma}\frac{|mn|^{1/2}|\tau|^{1/2}}{(\sigma-(1/2))^3}\Big) \tag{1.6}$$

for $s=\sigma+i\tau$, $1/2<\sigma<M$ *as* $|\tau|\geqq 1$, *where the implied constant depends solely on* M, *and moreover*

$$Z_{m,n}(s, \Gamma)=O\Big(q^{2\sigma}\frac{|mn|^{1/2}}{(\sigma-(1/2))^3\sqrt{\tau^2+(\sigma-(1/2))^2}}\Big) \tag{1.7}$$

for $s\in U_\varepsilon$ *with an absolute constant in O-symbol.*

The first advances in this direction were made by Goldfeld-Sarnak [**4**: Theorem 1]. They obtained $O\Big(\frac{|mn|}{q^2}\cdot\frac{|\tau|^{1/2}}{\sigma-(1/2)}\mathrm{vol}(\mathscr{D}_\Gamma)\Big)$ as $|\tau|\geqq 1$, and Hejhal [**5**: P. 709] more refined formula, say, roughly speaking, $\min\{|m||n|^{1/2}, |m|^{1/2}|n|\}$ and $(\sigma-(1/2))^2$ instead of $|mn|$ and $(\sigma-(1/2))$. Hence our result is a slight improvement of them with respect to the growth of m and n. In addition, it should be noted that both Goldfeld-Sarnak and Hejhal observed such problem in more general situations.

In order to derive the growth condition of the Kloosterman zeta function, the usual way is to consider the inner product with respect to the non-holomorphic Poincaré series. Such inner product has already been calculated by several authors, and known up to now, to have two types of representations. The one is by Goldfeld-Sarnak or Hejhal, and the other is by Kuznetsov [**10**: Lemma in section 4] or Deshouillers-Iwaniec [**1**: Lemma 4.1 and 4.3]. On the other hand, in [**18**], we derived new formula for Fourier coefficients of the non-holomorphic Poincaré series. By making use of such formula, we can obtain the new type of representation for the inner product in the case of $mn>0$(see Proposition in section 2). From this, the assertion of Theorem follows naturally. Furthermore, if we apply Theorem to the estimation for sums of Kloosterman sums, by similar process of evaluation as in Hejhal [**5**: Appendix E], we can see, for example, the following

COROLLARY. *Let* Γ *be the Hecke congruence group* $\Gamma_0(N)$ *with* $N\leqq 17$. *Then, since Selberg's eigenvalue conjecture is true, and since* $q=1$, *we have*

$$\sum_{c<x}\frac{S(m, n, c, \Gamma)}{c}=O(|mn|^{1/2}x^{1/6}(\log x)^2)$$

with an absolute constant in O-symbol.

The above result is also a slight improvement of [**4**: Theorem 2] and [**5**: P. 694] when we restrict ourselves to such congruence groups.

§ 2. Inner product formula.

In this section, we will calculate the inner product of the non-holomorphic Poincaré series, and at that time, we use the formula obtained in [**18**] as a representation for Fourier coefficients of such series. We now start with introducing the non-holomorphic Poincaré series.

Let Γ_∞ be the stabilizer of a cusp ∞ in Γ, i.e., $\Gamma_\infty=\left\{\begin{pmatrix}1 & qn\\ & 1\end{pmatrix} \mid n\in\boldsymbol{Z}\right\}$, and let m be an arbitrary nonzero integer. Then the non-holomorphic Poincaré series is defined by

$$P_m(z, s, \Gamma)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\exp\left\{\frac{2\pi}{q}i(mx(\gamma z)-|m|y(\gamma z))\right\}y(\gamma z)^s \tag{2.1}$$

for $z\in H$ and $s\in\boldsymbol{C}$, where $\gamma z=x(\gamma z)+iy(\gamma z)$. This series converges absolutely for $\mathrm{Re}(s)>1$ and belongs to the Hilbert space $L^2(\mathcal{D}_\Gamma)$ in this region. Moreover we set $a_m(y, s, n, \Gamma)$ to be the nth Fourier coefficient of $P_m(z, s, \Gamma)$, namely

$$\begin{cases} P_m(z, s, \Gamma)=\displaystyle\sum_{n=-\infty}^{\infty}a_m(y, s, n, \Gamma)e\left(\frac{n}{q}x\right), \\ a_m(y, s, n, \Gamma)=\displaystyle\frac{1}{q}\int_0^q P_m(x+iy, s, \Gamma)e\left(-\frac{n}{q}x\right)dx. \end{cases}$$

For two elements f and g of $L^2(\mathcal{D}_\Gamma)$, the inner product denoted by $\langle f, g\rangle$ implies the following integral:

$$\int_{\mathcal{D}_\Gamma} f(z)\overline{g(z)}d\mu(z),$$

where $\bar{g}$ is the complex conjugate of g. Note that $f(z)\overline{g(z)}$ and $d\mu(z)$ are invariant under the action of Γ. Thus the above integral is well-defined.

If we consider the inner product for P_m and P_n, it is well known to hold that

$$\langle P_m(z, s, \Gamma), P_n(z, \bar{w}, \Gamma)\rangle=q\int_0^\infty a_m(y, s, n, \Gamma)y^{w-2}e^{-2\pi|n|y/q}dy \tag{2.2}$$

where $m, n\in\boldsymbol{Z}_{\neq 0}$ and $s, w\in\boldsymbol{C}$ with $\mathrm{Re}(s)>1$, $\mathrm{Re}(w)>1$, furthermore $\bar{w}$ means the complex conjugate of w. Then calculations of the right hand side of (2.2) after substituting the formula described in [**18**] for a_m give the following

PROPOSITION. *Let m and n be nonzero integers, and for two complex numbers s and w, let us denote by $P_m(z, s, \Gamma)$ and $P_n(z, \bar{w}, \Gamma)$ two non-holomorphic Poincaré series defined by* (2.1). *Then, under the conditions* $\mathrm{Re}(s)>1$ *and* $\mathrm{Re}(w)>1$, *we have*

$$\begin{aligned}\langle P_m(z, s, \Gamma), P_n(z, \bar{w}, \Gamma)\rangle &= \delta_{m,n} q\left(4\pi\frac{|m|}{q}\right)^{1-s-w}\Gamma(s+w-1)\\ &\quad +2f_{m,n}(s, w)\sum_{c>0} S(m, n, c, \Gamma)c^{-(1+s)}\alpha^{w-1}K_{w-s}(\alpha)\\ &\quad -\varepsilon_{m,n} f_{m,n}(s, w)\sum_{c>0} S(m, n, c, \Gamma)c^{-(1+s)}\alpha^{w}R_{m,n}(s, w, c, \Gamma)\end{aligned} \tag{2.3}$$

where $\delta_{m,n}$ *is the Kronecker symbol,* $\varepsilon_{m,n}$ *equals* 1 *or* 0 *according as* $mn>0$ *or* $mn<0$, $\alpha=4\pi|mn|^{1/2}(qc)^{-1}$ *and*

$$f_{m,n}(s, w) = 2^{4-s-3w}\pi^{2-w}q^{w-1}\frac{\Gamma(s+w-1)}{\Gamma(s)\Gamma(w)}|m|^{(1-s)/2}|n|^{(s-2w+1)/2}$$

furthermore

$$\begin{aligned}R_{m,n}(s, w, c, \Gamma) &= \int_0^1 K_{w-s}(\alpha u^{1/2})u^{(s+w)/2}(1-u)^{-1/2}J_1(\alpha(1-u)^{1/2})du\\ &\quad +\int_0^1 K_{w-s}(\alpha u^{1/2})u^{(s+w-2)/2}(1-u)^{1/2}J_1(\alpha(1-u)^{1/2})du\end{aligned}$$

in which K_{w-s} *and* J_1 *denote the modified Bessel function and the Bessel function respectively.*

REMARK 2. In the case of $mn<0$, the inner product formula for P_m and P_n is very simple. This formula for $mn<0$ has already been obtained by Deshouillers-Iwaniec [**1**: Lemma 4.3] with complete coincidence, while they used a usual representation for $a_m(y, s, n, \Gamma)$ different from ours. But, it should be noted that even in the case of $mn<0$, we can not prove the assertion of Theorem without using the formula in the case of $mn>0$ as stated in (2.3). This fact is readily shown from taking Lemma in section 3 into our consideration.

The modified Bessel function K_ν is defined, for example, by

$$K_\nu(y) = \frac{1}{2}\left(\frac{y}{2}\right)^{\nu}\int_0^\infty \exp\left(-t-\frac{y^2}{4t}\right)t^{-\nu-1}dt \tag{2.4}$$

for $y>0$([**16**: P. 183, (15)]), and the Bessel function J_ν by

$$J_\nu(y) = \pi^{-1/2}\Gamma\left(\nu+\frac{1}{2}\right)^{-1}\left(\frac{y}{2}\right)^{\nu}\int_{-1}^1 e^{iyt}(1-t^2)^{\nu-(1/2)}dt \tag{2.5}$$

for $\mathrm{Re}(\nu)>-1/2$ and $y>0$([**2**: Vol. 2, P. 81, (7)]). Then it follows immediately from (2.5) that the Bessel function can be estimated by

$$J_\nu(y) = O\left(\left|\Gamma\left(\nu+\frac{1}{2}\right)\right|^{-1} y^{\mathrm{Re}(\nu)}\frac{1}{\mathrm{Re}(\nu)+(1/2)}\right) \tag{2.6}$$

under the restriction $\mathrm{Re}(\nu)>-1/2$ and $y>0$.

PROOF OF PROPOSITION. We will denote, from now on, $P_m(z, s, \Gamma)$, $a_m(y, s, n, \Gamma)$ and $S(m, n, c, \Gamma)$ respectively by $P_m(z, s)$, $a_m(y, s, n)$ and $S(m, n, c)$

for simplicity. As stated in [**18**: Theorem B], the nth Fourier coefficient $a_m(y, s, n)$ of $P_m(z, s)$ has the following representation subject to the condition $\mathrm{Re}(s)>1$:

$$\begin{aligned} a_m(y, s, n) &= \delta_{m,n} y^s e^{-2\pi|n|y/q} \\ &+ \frac{2}{\Gamma(s)}\left(\frac{2\pi}{q}\right)^{s+1} |m|^{(1-s)/2} |n|^{(3s-1)/2} y^s \sum_{c>0} S(m, n, c) c^{-(1+s)} \tilde{A}_m(y, s, n, c) \end{aligned} \tag{2.7}$$

where $\tilde{A}_m(y, s, n, c)$ denotes

$$\int_0^\infty \exp\{-\beta y(1+2t^2)\} t^{2s-1} (1+t^2)^{(s-1)/2} J_{s-1}(\alpha(1+t^2)^{1/2}) dt$$

for $mn>0$, and

$$\int_0^\infty \exp\{-\beta y(1+2t^2)\} t^s (1+t^2)^{s-1} J_{s-1}(\alpha t) dt$$

for $mn<0$, in which $\alpha=4\pi|mn|^{1/2}(qc)^{-1}$ and $\beta=2\pi|n|/q$.

What we must carry out is to compute the integral on the right side of (2.2) after replacing $a_m(y, s, n)$ by the right in (2.7). To do this, we first need to verify the absolute convergence of such representation. It follows directly from (2.6) that

$$\tilde{A}_m(y, s, n, c) = O\left\{e^{-\beta y} \alpha^{\sigma-1}\left(1+\frac{(\beta y)^{1-2\sigma}}{|\Gamma(s-(1/2))|} \cdot \frac{1}{\sigma-(1/2)}\right)\right\}$$

for $\sigma=\mathrm{Re}(s)>1/2$ in both cases $mn>0$ and $mn<0$. Since $\alpha=4\pi|mn|^{1/2}(qc)^{-1}$, the integrand on the right side of (2.2) can be estimated, after substituting the above into (2.7), by

$$O\left\{\delta_{m,n} y^{\sigma'+\sigma-2} e^{-2\beta y} + \sum_c \frac{|S(m, n, c)|}{c^{2\sigma}} y^{\sigma'+\sigma-2} e^{-2\beta y}\left(1+\frac{(\beta y)^{1-2\sigma}}{|\Gamma(s-(1/2))|} \cdot \frac{1}{\sigma-(1/2)}\right)\right\}$$

where $\sigma=\mathrm{Re}(s)$ with at least $\sigma>1/2$, $\sigma'=\mathrm{Re}(w)$ and $\beta=2\pi|n|/q$. Here, if we temporarily suppose that $\sigma'>\sigma>1$, the absolute convergence of the integral in (2.2) follows. Thus, under this condition, we can change not only the order of summation over c and integration over y, but also that of integrations over y and t. Then after integration with respect to y under the integral sign with respect to t, the integral in (2.2), and hence the inner product $\langle P_m(z, s), P_n(z, \bar{w})\rangle$ can be expressed, for which $\mathrm{Re}(w)>\mathrm{Re}(s)>1$, as

$$\begin{aligned} &\delta_{m,n} q\left(4\pi\frac{|m|}{q}\right)^{1-s-w} \Gamma(s+w-1) \\ &\quad + f_{m,n}(s, w) 2^w \Gamma(w) \sum_{c>0} S(m, n, c) c^{-(1+s)} \tilde{B}_m(s, w, n, c) \end{aligned} \tag{2.8}$$

where $f_{m,n}(s, w)$ is as in (2.3) and $\tilde{B}_m(s, w, n, c)$ presents the following integral according to the condition of mn:

$$\int_0^\infty t^{2s-1}(1+t^2)^{(1-s-2w)/2} J_{s-1}(\alpha(1+t^2)^{1/2})dt \tag{2.9}$$

for $mn>0$ and

$$\int_0^\infty t^s(1+t^2)^{-w} J_{s-1}(\alpha t)dt \tag{2.10}$$

for $mn<0$.

In case of $mn<0$, one knows from [**2**: Vol. 2, P. 95, (51)] that

$$\int_0^\infty t^{\mu+1}(1+t^2)^{-\nu} J_\mu(bt)dt = 2^{1-\nu}b^{\nu-1}\Gamma(\nu)^{-1}K_{\nu-\mu-1}(b)$$

for $2\mathrm{Re}(\nu)-(1/2)>\mathrm{Re}(\mu)>-1$ and $b>0$. By using this, the integral in (2.10) turns out to be equal to $2^{1-w}\alpha^{w-1}\Gamma(w)^{-1}K_{w-s}(\alpha)$ if we recall $K_{w-s}=K_{s-w}$. Thus considering (2.8), we obtain the desired formula still under the assumption that $\mathrm{Re}(w)>\mathrm{Re}(s)>1$. The condition $\mathrm{Re}(w)>\mathrm{Re}(s)$ is not essential. Indeed, we can exclude it later. The formula obtained just now completely coincides with that of [**1**: Lemma 4.3].

Next, let us consider the case $mn>0$. Thus $\tilde{B}_m(s, w, n, c)$ stands for the integral in (2.9), which itself converges absolutely for $\mathrm{Re}(w)>\mathrm{Re}(s)>1/2$. This case requires more calculations than that of $mn<0$. We first recall the recurrence relation for the Bessel function ([**16**: P. 45, (1)]):

$$J_{\nu-1}(y) = 2\nu y^{-1}J_\nu(y) - J_{\nu+1}(y).$$

Utilizing this, we then divide the integrand in (2.9) into two terms and realize

$$\begin{aligned}\tilde{B}_m(s, w, n, c) = {} & 2s\alpha^{-1}\int_0^\infty t^{2s-1}(1+t^2)^{-w-(s/2)} J_s(\alpha(1+t^2)^{1/2})dt \\ & -\int_0^\infty t^{2s-1}(1+t^2)^{-w+1-(s+1)/2} J_{s+1}(\alpha(1+t^2)^{1/2})dt.\end{aligned} \tag{2.11}$$

We note here that the first term in the above converges absolutely for $\mathrm{Re}(w)>\mathrm{Re}(s)>0$ and the second term for $\mathrm{Re}(w)-(1/2)>\mathrm{Re}(s)>0$. Both the first term and the second term on the right hand side of (2.11) have quite similar process of evaluation to each other. Thus, we mainly focus our attention on the first term and will discuss this term in detail.

From now on, we will denote the first term in (2.11) by $\tilde{B}_{m,1}(s, w, n, c)$. In the beginning, from the addition formula for the Bessel function stated in [**16**: P. 366, (13)], we have

$$(1+t^2)^{-s/2} J_s(\alpha(1+t^2)^{1/2}) = 2^s \sum_{l=0}^\infty g(s, l)\frac{J_{s+2l}(\alpha t)}{t^s}\cdot\frac{J_{s+2l}(\alpha)}{\alpha^s} \tag{2.12}$$

where $g(s, l)=(-1)^l(s+2l)\Gamma(s+l)/l\,!$. Then, it is found from [**16**: P. 50, (3) or

P. 369, (8)] that

$$J_{s+2l}(y)=\pi^{-1/2}(-1)^l\frac{\Gamma(2s)(2l)!}{\Gamma(s+(1/2))\Gamma(2s+2l)}\left(\frac{y}{2}\right)^s\int_{-1}^{1}e^{iyx}(1-x^2)^{s-(1/2)}C_{2l}^{s}(x)dx$$

for any non-negative integer l, $\mathrm{Re}(s)>-1/2$ and $y>0$, where $C_{2l}^{s}(x)$ is Gegenbauer's polynomial defined, for example, by [**2**: Vol. 1, P. 176, (10) or Vol. 2, P. 175, (11)]. By using the integral representation [**2**: Vol. 2, P. 177, (31)], $C_{2l}^{s}(x)$ can be estimated by

$$O\left(\frac{\Gamma(2s+2l)}{(2l)!}\cdot\frac{1}{\mathrm{Re}(s)}\right)$$

for $\mathrm{Re}(s)>0$ and $|x|\leqq 1$, where the constant in O-symbol depends on s alone. Thus, it is readily shown from this to hold that

$$J_{s+2l}(y)=O\left(y^{\mathrm{Re}(s)}\frac{1}{\mathrm{Re}(s)}\right) \tag{2.13}$$

for $\mathrm{Re}(s)>0$, where the implied constant depends only on s. Applying (2.13) to $J_{s+2l}(\alpha t)$ and (2.6) to $J_{s+2l}(\alpha)$, we easily obtain

$$\sum_l\left|g(s,l)\frac{J_{s+2l}(\alpha t)}{t^s}\cdot\frac{J_{s+2l}(\alpha)}{\alpha^s}\right|=O\left(\frac{1}{\mathrm{Re}(s)}\right)$$

for $\mathrm{Re}(s)>0$, where the constant in O-symbol depends on s and α. Here, if we impose the condition on s and w that $\mathrm{Re}(w)>\mathrm{Re}(s)>0$, it is derived from the last estimate that the function $\tilde{B}_{m,1}(s,w,n,c)$ after substitution as in (2.12) converges absolutely. Therefore, under such condition for s and w, we can change the order of summation over l and integration over t. After this, we further use the following integral representation:

$$(1+t^2)^{-w}=\frac{1}{\Gamma(w)}\int_0^\infty e^{-(1+t^2)v}v^{w-1}dv$$

for $\mathrm{Re}(w)>0$. Then, again from (2.13), it is easily seen that the two multiple integral over t and v in each summand with respect to l converges absolutely under $\mathrm{Re}(w)>\mathrm{Re}(s)>0$. Thus, we can also interchange the order of integrations with respect to t and v. Consequently, these arguments bring

$$\begin{aligned}\tilde{B}_{m,1}(s,w,n,c)&=2^{s+1}\frac{s\alpha^{-1-s}}{\Gamma(w)}\sum_{l=0}^{\infty}g(s,l)J_{s+2l}(\alpha)\\&\quad\times\int_0^\infty e^{-v}v^{w-1}\int_0^\infty e^{-vt^2}t^{s-1}J_{s+2l}(\alpha t)dtdv\end{aligned} \tag{2.14}$$

for $\mathrm{Re}(w)>\mathrm{Re}(s)>0$, where $g(s,l)=(-1)^l(s+2l)\Gamma(s+l)/l!$

We now proceed further. Noting [**2**: Vol. 2, P. 50, (22)] or [**16**: P. 393, (2)] and using the integral representation for a confluent hypergeometric function, we see

$$\int_0^\infty e^{-vt^2}t^{s-1}J_{s+2l}(\alpha t)dt$$
$$= 2^{-(s+1+2l)}v^{-(s+l)}\frac{\alpha^{s+2l}}{\Gamma(1+l)}\int_0^1 \exp\Big(-\frac{\alpha^2}{4v}(1-u)\Big)u^l(1-u)^{s-1+l}du$$

for $\mathrm{Re}(s)>0$. Then repeating partial integration l-times, the last formula may be rewritten as

$$(2.15)\qquad (-1)^l 2^{-(1+s)}v^{-s}\frac{\alpha^s}{l\,!}\int_0^1 \exp\Big(-\frac{\alpha^2}{4v}(1-u)\Big)\frac{d^l}{du^l}[u^l(1-u)^{s-1+l}]du$$

for $\mathrm{Re}(s)>0$. In order to evaluate the integral in (2.15) more explicitly, we shall introduce Jacobi's polynomial. As stated in [**2**: Vol. 2, P. 169, (10)], it is defined, after slight modification by putting $x=2u-1$, by the equation:

$$(2.16)\qquad \tilde{P}_l(\mu,\ \eta,\ 2u-1)=\frac{(-1)^l}{l\,!}(1-u)^{-\mu}u^{-\eta}\frac{d^l}{du^l}[u^{\eta+l}(1-u)^{\mu+l}]\,.$$

Moreover, the formula described in [**2**: Vol. 2, P. 170, (16)] shows

$$(2.17)\qquad \tilde{P}_l(\mu,\ \eta,\ 2u-1)=(-1)^l\frac{\Gamma(\eta+1+l)}{\Gamma(\eta+1)l\,!}F(-l,\ \mu+\eta+1+l\,;\ \eta+1\,;\ u)$$

where $F={}_2F_1$ is a hypergeometric function. Then the equality (2.16) under the assumption $\mathrm{Re}(\mu)\geqq 0$, $\mathrm{Re}(\eta)\geqq 0$ and $0<u<1$ gives

$$(2.18)\qquad |\tilde{P}_l(\mu,\ \eta,\ 2u-1)|\leqq 2^l l\,!\left|\frac{\Gamma(\mu+1+l)}{\Gamma(\mu+1)l\,!}\right|\cdot\left|\frac{\Gamma(\eta+1+l)}{\Gamma(\eta+1)l\,!}\right|.$$

Keeping (2.14) and (2.15) in mind, and taking $\mu=s-1$, $\eta=0$ in (2.16) or (2.17), we now consider the following series:

$$(2.19)\qquad \begin{aligned}&\sum_{l=0}^\infty g(s,\ l)\tilde{P}_l(s-1,\ 0,\ 2u-1)J_{s+2l}(\alpha)\\ &=\sum_{l=0}^\infty(-1)^l g(s,\ l)F(-l,\ s+l\,;\ 1\,;\ u)J_{s+2l}(\alpha)\end{aligned}$$

where $g(s,\ l)=(-1)^l(s+2l)\Gamma(s+l)/l\,!$. In view of (2.18) and (2.6), it follows immediately that the series in (2.19) converges absolutely for $\mathrm{Re}(s)\geqq 1$. Then we further show that the function $\tilde{B}_{m,1}(s,\ w,\ n,\ c)$ defined by (2.14) after replacing the integral over t by the right in (2.15) also converges absolutely for $\mathrm{Re}(w)>\mathrm{Re}(s)$. After all, in such representation for $\tilde{B}_{m,1}(s,\ w,\ n,\ c)$, it is possible to change the order of summation over l and integrations over v and u as far as $\mathrm{Re}(w)>\mathrm{Re}(s)\geqq 1$. Here, recalling the formula stated in [**16**: P. 140, (3)], we easily find that the series in (2.19) is identical with $2^{-s}\alpha^s J_0(\alpha u^{1/2})$. Collecting these facts, we can obtain the following equality:

$$\tilde{B}_{m,1}(s, w, n, c)=2^{-s}\alpha^{s-1}\frac{s}{\Gamma(w)}\int_0^\infty e^{-v}v^{w-s-1} \times\int_0^1 \exp\left(-\frac{\alpha^2}{4v}(1-u)\right)(1-u)^{s-1}J_0(\alpha u^{1/2})dudv \tag{2.20}$$

for $\mathrm{Re}(w)>\mathrm{Re}(s)\geqq 1$.

To deduce the desired formula, more computation is necessary. Regarding $s(1-u)^{s-1}$ as $-(d/du)\{(1-u)^s\}$, we first integrate by parts over u under the integral sign with respect to v and at that time note that

$$\frac{d}{du}(u^{-\nu/2}J_\nu(\sqrt{au}))=-\frac{a^{1/2}}{2}u^{-(\nu+1)/2}J_{\nu+1}(\sqrt{au}),$$

which is readily seen from [**2**: Vol. 2, P. 11, (51)]. Thus, by using $J_0(0)=1$, we can reformulate the integral over u completely. Secondly, we replace the inner integral over u on the right in (2.20) by such formula obtained just now and make a change of variable u to $1-u$, then moreover interchange the order of integrations with respect to v and u. After these processes, we now obtain the final formula for $\tilde{B}_{m,1}(s, w, n, c)$:

$$\begin{aligned}\tilde{B}_{m,1}(s, w, n, c)&=\frac{2^{-s}}{\Gamma(w)}\alpha^{s-1}\int_0^\infty \exp\left(-v-\frac{\alpha^2}{4v}\right)v^{w-s-1}dv\\ &+\frac{2^{-2-s}}{\Gamma(w)}\alpha^s\Big\{\alpha\int_0^1\int_0^\infty \exp\left(-v-\frac{\alpha^2}{4v}u\right)v^{w-s-2}dvu^sJ_0(\alpha(1-u)^{1/2})du\\ &-2\int_0^1\int_0^\infty \exp\left(-v-\frac{\alpha^2}{4v}u\right)v^{w-s-1}dvu^s(1-u)^{-1/2}J_1(\alpha(1-u)^{1/2})du\Big\}.\end{aligned} \tag{2.21}$$

The second term on the right in the above converges absolutely at least for $\mathrm{Re}(w)-1>\mathrm{Re}(s)>-1$, the first term for $\mathrm{Re}(w)>\mathrm{Re}(s)$ and the third term for $\mathrm{Re}(w)>\mathrm{Re}(s)>-1$. Thus considering the condition in (2.20), we temporarily realize that the equation in (2.21) is valid for $\mathrm{Re}(w)-1>\mathrm{Re}(s)\geqq 1$, while this restriction is not essential. Anyway we avoid here the arguments of analytic continuation.

As for the second term on the right side of (2.11), completely analogous process of evaluation is possible. We will denote such term by $\tilde{B}_{m,2}(s, w, n, c)$ and describe the results of calculation without detailed proof. Hence we start with the fact that the function $\tilde{B}_{m,2}(s, w, n, c)$ converges absolutely for $\mathrm{Re}(w)-1>\mathrm{Re}(s)>0$. At first, we have the following representation which corresponds to (2.14):

$$\tilde{B}_{m,2}(s, w, n, c) = -2^{s+1}\frac{\alpha^{-1-s}}{\Gamma(w-1)}\sum_{l=0}^{\infty} g(s+1, l)J_{s+1+2l}(\alpha)$$
$$\times\int_0^{\infty} e^{-v}v^{w-2}\int_0^{\infty} e^{-vt^2}t^{s-2}J_{s+1+2l}(\alpha t)dtdv$$

for $\mathrm{Re}(w)-1>\mathrm{Re}(s)>0$, where $g(s+1, l)=(-1)^l(s+1+2l)\Gamma(s+1+l)/l\,!$. Then,

$$\int_0^{\infty} e^{-vt^2}t^{s-2}J_{s+1+2l}(\alpha t)dt$$
$$=(-1)^l 2^{-(2+s)}v^{-s}\frac{\alpha^{s+1}}{\Gamma(2+l)}\int_0^1 \exp\left(-\frac{\alpha^2}{4v}(1-u)\right)\frac{d^l}{du^l}[u^{1+l}(1-u)^{s-1+l}]du$$

for $\mathrm{Re}(s)>0$. If we consider the following series:

$$\frac{1}{\Gamma(2)}\sum_{l=0}^{\infty}\frac{l\,!}{\Gamma(2+l)}g(s+1, l)\tilde{P}_l(s-1, 1, 2u-1)J_{s+1+2l}(\alpha)$$
$$=\frac{1}{\Gamma(2)}\sum_{l=0}^{\infty}(-1)^l g(s+1, l)F(-l, s+1+l\,;\, 2;\, u)J_{s+1+2l}(\alpha),$$

it follows from (2.18) and (2.6) that this converges absolutely for $\mathrm{Re}(s)\geqq 1$. Moreover from [**16**: P. 140, (3)] again, it turns out to be equal to $2^{-s}u^{-1/2}\alpha^s\times J_1(\alpha u^{1/2})$. Therefore we find

$$\tilde{B}_{m,2}(s, w, n, c) = -2^{-1-s}\alpha^s\frac{1}{\Gamma(w-1)}\int_0^{\infty} e^{-v}v^{w-s-2}$$
$$\times\int_0^1 \exp\left(-\frac{\alpha^2}{4v}(1-u)\right)(1-u)^{s-1}u^{1/2}J_1(\alpha u^{1/2})dudv$$

for $\mathrm{Re}(w)-1>\mathrm{Re}(s)\geqq 1$ which corresponds to (2.20). The condition for s and w is not essential, in fact the above formula itself converges absolutely for $\mathrm{Re}(w)-1>\mathrm{Re}(s)>0$. But we adopt the former in order to avoid the arguments of analytic continuation.

Before proceeding further, we first make a change of variable v to $v=\alpha^2(1-u)v'$ after interchanging the order of integrations over u and v. Again changing the order of integrations with respect to u and v, the last formula may be rewritten as

$$-2^{-1-s}\alpha^{2(w-1)-s}\frac{1}{\Gamma(w-1)}\int_0^{\infty} e^{-1/(4v)}v^{w-s-2}$$
$$\times\int_0^1 \exp(-\alpha^2 v(1-u))(1-u)^{w-2}u^{1/2}J_1(\alpha u^{1/2})dudv$$

for $\mathrm{Re}(w)-1>\mathrm{Re}(s)\geqq 1$. Then, regarding $(1-u)^{w-2}$ as $-(1/(w-1))(d/du)\{(1-u)^{w-1}\}$, and noting $J_1(0)=0$ and

$$\frac{d}{du}(u^{\nu/2}J_\nu(\sqrt{au})) = \frac{a^{1/2}}{2}u^{(\nu-1)/2}J_{\nu-1}(\sqrt{au})$$

which follows immediately from [**2**: Vol. 2, P. 11, (50)], we apply partial integration to the inner integral over u. After this, by making a change of variable v to $\{\alpha^2(1-u)\}^{-1}v'$ and u to $1-u$, we obtain the following equality:

$$
\begin{aligned}
&\tilde{B}_{m,2}(s,w,n,c)\\
(2.22)\qquad &=-\frac{2^{-2-s}}{\Gamma(w)}\alpha^s\Big\{\alpha\int_0^1\int_0^\infty\exp\Big(-v-\frac{\alpha^2}{4v}u\Big)v^{w-s-2}dvu^sJ_0(\alpha(1-u)^{1/2})du\\
&\quad+2\int_0^1\int_0^\infty\exp\Big(-v-\frac{\alpha^2}{4v}u\Big)v^{w-s-1}dvu^{s-1}(1-u)^{1/2}J_1(\alpha(1-u)^{1/2})du\Big\}
\end{aligned}
$$

for $\mathrm{Re}(w)-1>\mathrm{Re}(s)\geqq 1$.

We will return to the equation in (2.11). The function $\tilde{B}_m(s,w,n,c)$ has been defined by the integral in (2.9) and it converges absolutely for $\mathrm{Re}(w)>\mathrm{Re}(s)>1/2$. On the other hand, we have just now calculated each term on the right side of (2.11) explicitly which is expressed as (2.21) or (2.22). Substituting (2.21) and (2.22) into the first term and the second term in (2.11) respectively, we can obtain the following formula:

$$
\begin{aligned}
\tilde{B}_m(s,w,n,c)&=\tilde{B}_{m,1}(s,w,n,c)+\tilde{B}_{m,2}(s,w,n,c)\\
&=\frac{2^{-s}}{\Gamma(w)}\alpha^{s-1}\int_0^\infty\exp\Big(-v-\frac{\alpha^2}{4v}\Big)v^{w-s-1}dv\\
(2.23)\qquad &-\frac{2^{-1-s}}{\Gamma(w)}\alpha^s\Big\{\int_0^1\int_0^\infty\exp\Big(-v-\frac{\alpha^2}{4v}u\Big)v^{w-s-1}dvu^s(1-u)^{-1/2}J_1(\alpha(1-u)^{1/2})du\\
&+\int_0^1\int_0^\infty\exp\Big(-v-\frac{\alpha^2}{4v}u\Big)v^{w-s-1}dvu^{s-1}(1-u)^{1/2}J_1(\alpha(1-u)^{1/2})du\Big\}
\end{aligned}
$$

for $\mathrm{Re}(w)-1>\mathrm{Re}(s)\geqq 1$. Here if we replace the integration over v by $K_{w-s}=K_{s-w}$ in view of (2.4) and recalling the equality (2.8) which is valid for $\mathrm{Re}(w)>\mathrm{Re}(s)>1$, the desired formula stated in (2.3) follows, but still under the condition $\mathrm{Re}(w)-1>\mathrm{Re}(s)>1$ in the case of $mn>0$.

The inner product $\langle P_m(z,s), P_n(z,\bar{w})\rangle$ clearly determines a regular function of the other variable in the half plane $\mathrm{Re}(s)>1$ and $\mathrm{Re}(w)>1$ for a fixed value s and w. Hence it remains to verify that, under the conditions $\mathrm{Re}(s)>1$ and $\mathrm{Re}(w)>1$, the series in (2.8) converges absolutely after substitution (2.23) or only the first term in (2.23) for $B_m(s,w,n,c)$, according as $mn>0$ or $mn<0$. Then by the principle of analytic continuation, we can discard our earlier assumption during the computations that $\mathrm{Re}(w)-1>\mathrm{Re}(s)>1$ for $mn>0$ and $\mathrm{Re}(w)>\mathrm{Re}(s)>1$ for $mn<0$. As is easily seen, we have only to consider the case $mn>0$, namely the right hand side of (2.23), because this case includes that of $mn<0$. In general, for $a>0$, we have the following estimate:

$$\int_0^\infty \exp\Big(-v-\frac{a}{v}\Big)v^{\nu-1}dv = \begin{cases} O(1) & \text{for } \mathrm{Re}(\nu)>0, \\ O(1+|\log a|) & \text{for } \mathrm{Re}(\nu)=0, \\ O(a^{\mathrm{Re}(\nu)}) & \text{for } \mathrm{Re}(\nu)<0. \end{cases}$$

Utilizing this estimate, (2.6) and recalling $\alpha=4\pi|mn|^{1/2}(qc)^{-1}$, we see that $c^{-(1+s)}\tilde{B}_m(s, w, n, c)$ can be estimated by

$$O\Big\{c^{-2\min(\sigma,\sigma')}\Big(1+c^{-1}\frac{1}{\min(\sigma, \sigma')}\Big)\Big\}$$

for $\sigma\neq\sigma'$ and

$$O\{c^{-2\sigma}\log c+c^{-1-2\sigma}(\lim_{u\to 0}|\log u\cdot u^{\sigma-1}|+\log c)\}$$

for $\sigma=\sigma'$ where $\sigma=\mathrm{Re}(s)$ and $\sigma'=\mathrm{Re}(w)$. This implies that the series in (2.8) is absolutely convergent under the conditions $\mathrm{Re}(s)>1$ and $\mathrm{Re}(w)>1$, and determines a holomorphic function of s and w in these regions. Consequently, we can remove the earlier restriction, and hence the assertion of Proposition is completely proved.

§3. Proof of Theorem.

In this section, we also denote $P_m(z, s, \Gamma)$, $S(m, n, c, \Gamma)$, etc by $P_m(z, s)$, $S(m, n, c)$ etc, for the sake of simplicity. In the equation (2.3), the most important term is

$$2f_{m,n}(s, w)\sum_{c>0} S(m, n, c)c^{-(1+s)}\alpha^{w-1}K_{w-s}(\alpha). \tag{3.1}$$

Then, one knows from [**9**: P. 15] the following integral representation:

$$K_\nu(y) = 2^{\nu-1}\pi^{-1/2}y^{-\nu}\Gamma\Big(\nu+\frac{1}{2}\Big)\int_{-\infty}^{\infty}\frac{e^{-iyt}}{(1+t^2)^{\nu+(1/2)}}dt \tag{3.2}$$

for $\mathrm{Re}(\nu)>0$ and $y>0$. Regarding $e^{-iyt}=1+(e^{-iyt}-1)$ and noting

$$\int_{-\infty}^{\infty}\frac{1}{(1+t^2)^{\nu+(1/2)}}dt = \pi^{1/2}\frac{\Gamma(\nu)}{\Gamma(\nu+(1/2))},$$

we can decompose the function $K_\nu(y)$ into two terms, that is

$$K_\nu(y) = 2^{\nu-1}y^{-\nu}\Gamma(\nu)+2^{\nu-1}\pi^{-1/2}y^{-\nu}\Gamma\Big(\nu+\frac{1}{2}\Big)Q(\nu, y) \tag{3.3}$$

where

$$Q(\nu, y) = \int_{-\infty}^{\infty}\frac{(e^{-iyt}-1)}{(1+t^2)^{\nu+(1/2)}}dt. \tag{3.4}$$

Then, it follows from $yt\in\boldsymbol{R}$ that $|e^{-iyt}-1|\leqq|yt|$. Thus the function $Q(\nu, y)$ converges absolutely for $\mathrm{Re}(\nu)>1/2$ and satisfies

$$|Q(\nu, y)| \leqq y\Big(2+\frac{1}{\mathrm{Re}(\nu)-(1/2)}\Big). \tag{3.5}$$

Hence, the decomposition (3.3) is valid only for $\mathrm{Re}(\nu)>1/2$. If we apply the decomposition (3.3) to the function $K_{w-s}(\alpha)$ in (3.1), and moreover if we take $w=s+1$, then the equality (2.3) can be rewritten as

$$\begin{aligned} &2^{-2s}q\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}|n|^{-1}Z_{m,n}(s) \\ &=\langle P_m(z, s), P_n(z, \bar{s}+1)\rangle-\delta_{m,n}q\Big(4\pi\frac{|m|}{q}\Big)^{-2s}\Gamma(2s) \\ &\quad -f_{m,n}(s, s+1)\sum_{c>0}S(m, n, c)c^{-(1+s)}\alpha^{s-1}Q(1, \alpha) \\ &\quad +\varepsilon_{m,n}f_{m,n}(s, s+1)\sum_{c>0}S(m, n, c)c^{-(1+s)}\alpha^{s+1}R_{m,n}(s, s+1, c) \end{aligned} \tag{3.6}$$

for $\mathrm{Re}(s)>1$, where $\alpha=4\pi|mn|^{1/2}(qc)^{-1}$ and

$$f_{m,n}(s, s+1)=2^{1-4s}\pi^{1-s}q^{s}\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}|m|^{(1-s)/2}|n|^{-(1+s)/2}.$$

In order to derive the assertion of Theorem, it is necessary to verify that each term on the right hand side of (3.6) can be continued to $\mathrm{Re}(s)>1/2$ and majorized by $|m|^{1/2}|n|^{-1/2}$ at least for m and n. In the following, we will denote $s=\sigma+i\tau$.

3.1. *Estimation of the third and fourth terms in* (3.6).

Since $|S(m, n, c)|\leqq qc$, it follows from (3.5) that

$$\Big|\sum_{c}S(m, n, c)c^{-(1+s)}\alpha^{s-1}Q(1, \alpha)\Big| \leqq 2^{1+\sigma}\pi^{\sigma}q^{1-\sigma}|mn|^{\sigma/2}\sum_{c}c^{-2\sigma}.$$

This means that the third term on the right in (3.6) converges absolutely and uniformly for $\sigma>1/2$. Thus it becomes a regular function in $\sigma>1/2$ and can be estimated by

$$O\Big\{\Big|\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}\Big|\cdot\Big|\frac{m}{n}\Big|^{1/2}q\Big(1+\frac{1}{\sigma-(1/2)}\Big)\Big\} \tag{3.7}$$

with an absolute constant in the O-symbol.

We next consider the fourth term on the right side of (3.6), in which the function $R_{m,n}(s, s+1, c)$ stands for

$$\int_0^1 K_1(\alpha u^{1/2}) u^{(2s+1)/2}(1-u)^{-1/2} J_1(\alpha(1-u)^{1/2})du$$
$$+\int_0^1 K_1(\alpha u^{1/2}) u^{(2s-1)/2}(1-u)^{1/2} J_1(\alpha(1-u)^{1/2})du\,.$$

As shown in [**9**: P. 29], if we perform partial integration on the right in (3.2), it is found that

$$K_\nu(y)=2^\nu \pi^{-1/2} i y^{-(\nu+1)} \Gamma\Big(\nu+\frac{3}{2}\Big)\int_{-\infty}^{\infty} \frac{te^{-iyt}}{(1+t^2)^{\nu+(3/2)}}dt\,.$$

This representation gives

$$K_\nu(y)=O\Big\{\Big|\Gamma\Big(\nu+\frac{3}{2}\Big)\Big| y^{-\mathrm{Re}(\nu)-1}\Big(1+\frac{1}{\mathrm{Re}(\nu)+(1/2)}\Big)\Big\} \tag{3.8}$$

for $\mathrm{Re}(\nu)>-1/2$. Therefore we have $K_1(\alpha u^{1/2})=O(\alpha^{-2}u^{-1})$. By using this estimate and recalling (2.6), the function $R_{m,n}(s, s+1, c)$ is now majorized by

$$O\Big(\alpha^{-1}\frac{1}{\sigma-(1/2)}\Big).$$

Consequently, we see that the fourth term on the right hand side of (3.6) can be estimated by

$$O\Big\{\Big|\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}\Big|\cdot\Big|\frac{m}{n}\Big|^{1/2} q\Big(1+\frac{1}{\sigma-(1/2)}\Big)\Big(\frac{1}{\sigma-(1/2)}\Big)\Big\} \tag{3.9}$$

for $\sigma>1/2$, in which the O-symbol is an absolute constant. This estimate also implies that the series in question converges absolutely and uniformly for $\sigma>1/2$ and determines a holomorphic function in this region.

3.2. *Estimation of the inner product* $\langle P_m(z, s), P_n(z, \bar{s}+1)\rangle$.

Suppose that $\mathrm{Re}(s)>1$. Then the non-holomorphic Poincaré series $P_m(z, s)$ belongs to the Hilbert space $L^2(\mathcal{D}_\Gamma)$, and satisfies the following recursion relation:

$$\{s(1-s)+D\}P_m(z, s)=-4\pi\frac{|m|}{q}sP_m(z, s+1).$$

In other words,

$$P_m(z, s)=-4\pi\frac{|m|}{q}s\mathfrak{R}_\lambda P_m(z, s+1) \tag{3.10}$$

for $\mathrm{Re}(s)>1$, where $\mathfrak{R}_\lambda=(\lambda+D)^{-1}$, $\lambda=s(1-s)$ is the resolvent of the Laplace operator D. As is well known, the resolvent $\mathfrak{R}_\lambda$ is meromorphic in $\mathrm{Re}(s)>1/2$ with at most a finite number of simple poles at the points $s=s_j$ for $1/2<s_j<1$

which correspond to exceptional eigenvalues. It then follows from (3.10) that the function $P_m(z, s)$ may be continued holomorphically to $\mathrm{Re}(s)>1/2$ except possibly at the points s_j and becomes an element of $L^2(\mathcal{D}_\Gamma)$ in this region. And moreover, as stated in [**5**: P. 688], we see that the inner product $\langle P_m(z, s), P_n(z, \bar{s}+1)\rangle$ also determines a meromorphic function of s in $\mathrm{Re}(s)>1/2$.

In order to evaluate the order of growth for such inner product, we first use the Cauchy-Schwarz inequality:

$$(3.11)\quad |\langle P_m(z, s), P_n(z, \bar{s}+1)\rangle| \leqq 4\pi\frac{|m|}{q}|s|\,\|\mathcal{R}_\lambda\|\cdot\|P_m(z, s+1)\|\cdot\|P_n(z, \bar{s}+1)\|,$$

where $\|\cdot\|$ is the norm of $L^2(\mathcal{D}_\Gamma)$. As for the norm of $P_m(z, s+1)$, we have the following

LEMMA. *Let M be an arbitrary fixed positive number such that $M>1$. For any complex variable s with $1/2<\mathrm{Re}(s)<M$, we see*

$$(3.12)\qquad \|P_m(z, s+1)\| = O\left\{q^{1+\sigma}|m|^{-1/2}\frac{1}{\sigma-(1/2)}\right\},$$

where $\sigma=\mathrm{Re}(s)$ and the implied constant depends on M alone.

PROOF. Since $\|P_m(z, s+1)\|^2=\langle P_m(z, s+1), P_m(z, s+1)\rangle$, the formula of Proposition in section 2 is again applicable. Indeed, after taking $w=\bar{s}+1$, $n=m$ and replacing s by $s+1$, we have

$$\begin{aligned}\|P_m(z, s+1)\|^2 &= q\left(4\pi\frac{|m|}{q}\right)^{-(s+\bar{s}+1)}\Gamma(s+\bar{s}+1)\\ &\quad+2^{-(s+3\bar{s})}\pi^{1-\bar{s}}q^{\bar{s}}\frac{\Gamma(s+\bar{s}+1)}{\Gamma(s+1)\Gamma(\bar{s}+1)}|m|^{-\bar{s}}\sum_{c>0}S(m, m, c)c^{-(2+s)}\\ &\quad\times(2\alpha^{\bar{s}}K_{\bar{s}-s}(\alpha)-\alpha^{\bar{s}+1}R_{m,m}(s+1, \bar{s}+1, c))\end{aligned}$$

for $\mathrm{Re}(s+1)>1$, where $\alpha=4\pi|m|(qc)^{-1}$ and

$$\begin{aligned}&R_{m,m}(s+1, \bar{s}+1, c)\\ &\quad=\int_0^1 K_{\bar{s}-s}(\alpha u^{1/2})u^{(s+\bar{s}+2)/2}(1-u)^{-1/2}J_1(\alpha(1-u)^{1/2})du\\ &\quad\quad+\int_0^1 K_{\bar{s}-s}(\alpha u^{1/2})u^{(s+\bar{s})/2}(1-u)^{1/2}J_1(\alpha(1-u)^{1/2})du.\end{aligned}$$

Regarding $(1-u)^{-1/2}J_1(\alpha(1-u)^{1/2})=d/du\{(2/\alpha)J_0(\alpha(1-u)^{1/2})\}$, and noting (3.8) and $J_0(0)=1$, if we integrate by parts each term in $R_{m,m}$, we have

$$R_{m,m}(s+1,\ \bar{s}+1,\ c)=2\alpha^{-1}K_{\bar{s}-s}(\alpha)$$
$$-2\alpha^{-1}\int_0^1\frac{d}{du}\{K_{\bar{s}-s}(\alpha u^{1/2})u^{(s+\bar{s}+2)/2}\}J_0(\alpha(1-u)^{1/2})du$$
$$-2\alpha^{-1}\int_0^1\frac{d}{du}\{K_{\bar{s}-s}(\alpha u^{1/2})u^{(s+\bar{s})/2}(1-u)\}J_0(\alpha(1-u)^{1/2})du\,.$$

Then, it is found from [**16**: P. 79, (2)] or [**2**: Vol. 2, P. 79, (23)] and $K_{-\nu}=K_\nu$ that

$$\frac{d}{du}K_{\bar{s}-s}(\alpha u^{1/2})=-\frac{\alpha}{4}u^{-1/2}(K_{s-\bar{s}+1}(\alpha u^{1/2})+K_{\bar{s}-s+1}(\alpha u^{1/2}))\,.$$

Thus, further calculation yields

$$2\alpha^{\bar{s}}K_{\bar{s}-s}(\alpha)-\alpha^{\bar{s}+1}R_{m,m}(s+1,\ \bar{s}+1,\ c)$$
$$=(s+\bar{s})\alpha^{\bar{s}}\int_0^1 K_{\bar{s}-s}(\alpha u^{1/2})u^{(s+\bar{s}-2)/2}J_0(\alpha(1-u)^{1/2})du$$
$$-\frac{1}{2}\alpha^{\bar{s}+1}\int_0^1(K_{s-\bar{s}+1}(\alpha u^{1/2})+K_{\bar{s}-s+1}(\alpha u^{1/2}))u^{(s+\bar{s}-1)/2}J_0(\alpha(1-u)^{1/2})du\,.$$

From (3.8) and $J_0(y)=O(1)$, the last formula is estimated by

$$O\left\{\alpha^{\sigma-1}\left(2\sigma\left|\Gamma\left(\frac{3}{2}-2i\tau\right)\right|+2\left|\Gamma\left(\frac{5}{2}+2i\tau\right)\right|\right)\frac{1}{\sigma-(1/2)}\right\}$$

for $\sigma>1/2$, where $s=\sigma+i\tau$ and the O-symbol is an absolute constant. Hence, as a conclusion, by using Stirling's formula, we have

$$\|P_m(z,\ s+1)\|^2=O(q^{2(1+\sigma)}|m|^{-(1+2\sigma)})+O\left(q^2|m|^{-1}\sum_c c^{-2\sigma}\left(\frac{1}{\sigma-(1/2)}\right)\right)$$

for $1/2<\sigma<M$, where the constant in the O-symbol depends only on M. This completes the proof.

In the same way as in the preceding case, we also show

$$\|P_n(z,\ \bar{s}+1)\|=O\left\{q^{1+\sigma}|n|^{-1/2}\frac{1}{\sigma-(1/2)}\right\} \tag{3.13}$$

for $1/2<\sigma<M$, where the implied constant depends on M alone.

As for the norm of the resolvent, it is known from the general theory of the Hilbert space that

$$\|\mathfrak{R}_\lambda\|\leqq\sup_{\lambda_j}\frac{1}{|s(1-s)-\lambda_j|},\qquad \lambda_j=s_j(1-s_j)\,.$$

As stated in [**5**: P. 672] for example, it is majorized more explicitly by

$$(3.14)\qquad \|\mathcal{R}_\lambda\| \leqq \begin{cases} \dfrac{1}{|\tau|(2\sigma-1)} & \text{for } \sigma>1/2 \text{ and } |\tau|\geqq 1, \\ \dfrac{1}{(\sigma-(1/2))\sqrt{\tau^2+(\sigma-(1/2))^2}} & \text{for } s\in U_\varepsilon, \end{cases}$$

where U_ε is a set defined by (1.5).

Hence, collecting (3.11) through (3.14), we can obtain that the inner product $\langle P_m(z, s), P_n(z, \bar{s}+1)\rangle$ is estimated by

$$(3.15)\qquad O\left(q^{1+2\sigma}\left|\frac{m}{n}\right|^{1/2}|\tau|^{1/2}\frac{1}{(\sigma-(1/2))^3}\right)$$

for $1/2<\sigma<M$ and $|\tau|\geqq 1$, where the implied constant depends solely on M, and furthermore

$$(3.16)\qquad O\left(q^{1+2\sigma}\left|\frac{m}{n}\right|^{1/2}\frac{1}{(\sigma-(1/2))^3\sqrt{\tau^2+(\sigma-(1/2))^2}}\right)$$

for $s\in U_\varepsilon$ with an absolute constant in O-symbol.

3.3. PROOF OF THEOREM.

Substituting (3.7) and (3.9) into (3.6), we finally obtain

$$\begin{aligned} Z_{m,n}(s) &= \delta_{m,n}O(q^{2\sigma}|m|^{1-2\sigma}|\Gamma(s)\Gamma(s+1)|) \\ &\quad + O\left(|mn|^{1/2}\frac{1}{\sigma-(1/2)}\right)+\varepsilon_{m,n}O\left(|mn|^{1/2}\frac{1}{(\sigma-(1/2))^2}\right) \\ &\quad + \left|\frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2s)}\right| q^{-1}|n|\cdot O(\langle P_m(z, s), P_n(z, \bar{s}+1)\rangle) \end{aligned}$$

for $1/2<\sigma<M$. In view of Stirling's formula and combining this formula with (3.15) or (3.16), we arrive at the assertion of Theorem.

3.4. PROOF OF COROLLARY.

The process to derive the assertion of Corollary is almost analogous to that in Hejhal [**5**: Appendix E]. Thus we only give an outline here.

Let R be the positive number defined by

$$R = \limsup_{c\to\infty}\frac{\log|S(m, n, c)|}{\log c},$$

and $A=A(\delta)$ be a positive constant which satisfies

$$|S(m, n, c)| \leqq Ac^{R+\delta}$$

for all c, where δ is an arbitrary positive number. From $|S(m, n, c)| \leqq qc$ we see that $R \leqq 1$. To prove the assertion of Corollary, it is enough to consider the case $x = \rho + (1/2)$, where $\rho \in N$ and sufficiently large. Under these, let T be any fixed positive number which satisfies $1 \leqq T \leqq x^R$, while this restriction is not essential. Moreover we put $\varepsilon = (\log x)^{-1}$. Notice here that the notation ε corresponds to the one in (1.5) or (1.7).

Under the assumption of Corollary, namely Γ being the Hecke congruence group $\Gamma_0(N)$ with $N \leqq 17$, the Kloosterman zeta function turns out to be a regular function in $\mathrm{Re}(s) > 1/2$. Thus, it follows that

$$\int_{\partial E} Z_{m,n}\left(\frac{1+s}{2}\right)\frac{x^s}{s}ds = 0,$$

where $E = [\varepsilon, R+\varepsilon] \times [-T, T]$.

The Phragmén-Lindelöf principle and (1.6) show that

$$Z_{m,n}\left(\frac{1+s}{2}\right) = O\left(\frac{A+|mn|^{1/2}}{\varepsilon^3}|\tau|^{(R+\varepsilon-\sigma)/(2R)}\right)$$

for $|\tau| \geqq 2$ and $\varepsilon \leqq \sigma \leqq R+\varepsilon$, where the O-symbol is an absolute constant because $R \leqq 1$. Therefore

$$\begin{aligned}
&\int_{horiz} Z_{m,n}\left(\frac{1+s}{2}\right)\frac{x^s}{s}ds \\
&= O(1)\frac{A+|mn|^{1/2}}{\varepsilon^3}\cdot\frac{x^\varepsilon}{T^{1/2}}\int_0^R (xT^{-1/(2R)})^u du \\
&= O(1)\frac{A+|mn|^{1/2}}{\varepsilon^3 T^{1/2}}(\log(xT^{-1/(2R)}))^{-1}[xT^{-1/(2R)}]_0^R.
\end{aligned}$$

Since $1 \leqq T \leqq x^R$, the last formula is reduced to

$$O\left\{\frac{A+|mn|^{1/2}}{T}x^R(\log x)^2\right\}$$

with an absolute constant in O-symbol.

Next, it follows from (1.6) and (1.7) that

$$\begin{aligned}
&\int_{\varepsilon-iT}^{\varepsilon+iT} Z_{m,n}\left(\frac{1+s}{2}\right)\frac{x^s}{s}ds \\
&= O(1)\int_2^T \frac{|mn|^{1/2}t^{1/2}}{\varepsilon^3}\cdot\frac{x^\varepsilon}{t}dt + O(1)\int_0^2 \frac{|mn|^{1/2}}{\varepsilon^3\sqrt{\varepsilon^2+t^2}}\cdot\frac{x^\varepsilon}{\sqrt{\varepsilon^2+t^2}}dt \\
&= O\{|mn|^{1/2}(T^{1/2}+(\log x)^3)\log x\},
\end{aligned}$$

where the O-symbols are absolute constants.

Moreover, one knows from analytic number theory the following equality:

$$\int_{R+\varepsilon-iT}^{R+\varepsilon+iT} Z_{m,n}\left(\frac{1+s}{2}\right)\frac{x^s}{s}ds=\sum_{c<x}\frac{S(m,n,c)}{c}+O\left(\frac{Ax^R}{T}\log x\right)$$

with an absolute constant in O-symbol.

Gathering together and taking $T=x^{(2R)/3}$, we obtain

$$\begin{aligned}\sum_{c<x}\frac{S(m,n,c)}{c}&=O\{(A+|mn|^{1/2})x^{R/3}(\log x)^2\}\\&\quad+O\{|mn|^{1/2}(x^{R/3}+(\log x)^3)\log x\}\\&\quad+O(Ax^{R/3}\log x)\end{aligned}$$

where the O-symbols are absolute constants. Here, if we use Weil's estimate for the Kloosterman sum (cf. [**17**] or [**1**: Lemma 2.6]), we see $R\leqq 1/2$ and at least $A=\min\{|m|^{1/2}, |n|^{1/2}\}$. This completes the proof.

References

[1] J.M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, Invent. Math., **70** (1982), 219-288.

[2] A. Erdelyi et al., Higher Transcendental Functions, **1**, **2**, McGraw-Hill, New York, 1954.

[3] A. Erdelyi et al., Tables of Integral Transforms, **1**, McGraw-Hill, New York, 1954.

[4] D. Goldfeld and P. Sarnak, Sums of Kloosterman sums, Invent. Math., **71** (1983), 243-250.

[5] D.A. Hejhal, The Selberg Trace Formula for PSL(2, $\boldsymbol{R}$), **2**, Lecture Notes in Math., no. 1001, Springer, Berlin-New York, 1983.

[6] D.A. Hejhal, The Selberg trace formula and the Riemann zeta function, Duke Math. J., **43** (1976), 441-482.

[7] M.N. Huxley, Introduction to Kloostermania, Banach Center Publ., **17**, PWN-Polish Scientific Publ., 1985, 217-306.

[8] H. Iwaniec, Non-holomorphic modular forms and their applications, in Modular forms (Ed. Rankin), Ellis Horwood Ltd, 1984, 157-196.

[9] T. Kubota, Elementary Theory of Eisenstein Series, Kodansha and John Willey, Tokyo-New York, 1973.

[10] N.V. Kuznetsov, Petersson's conjecture for cusp forms of weight zero and Linnik's conjecture: sums of Kloosterman sums, Math. Sb., **39** (1981), 299-342.

[11] H. Maass, Lectures on Modular Functions of One Complex Variable, Tata Institute, Bombay, 1964.

[12] N.V. Proskurin, Summation formulas for general Kloosterman sums, J. Soviet Math., **17** (1981), 925-950.

[13] W. Roelcke, Über die Wellengleichung bei Grenzkreisgruppen erster Art, Sitz. Ber. Heidel. Akad. der Wiss., 1956, 4. Abh.

[14] P. Sarnak, Additive number theory and Maass forms, Lecture Notes in Math., no. 1052, Springer, Berlin-New York, 1984, 286-309.

[15] A. Selberg, On the estimation of Fourier coefficients of modular forms, Proc. Symp. Pure Math., **8**, AMS 1965, 1-15.
[16] G. N. Watson, Theory of Bessel Functions, 2nd edition, Cambridge Univ. Press, 1944.
[17] A. Weil, On some exponential sums, Proc. Nat. Akad. Sci. USA, **34** (1948), 204-207.
[18] E. Yoshida, On Fourier coefficients of Non-holomorphic Poincaré series, Mem. Fac. Sci. Kyushu Univ., Ser. A, **45** (1991), 1-17.

Eiji YOSHIDA
Department of Mathematics
Faculty of Scienee
Kyushu University
Fukuoka 812
Japan