J. Math. Soc. Japan
Vol. 44, No. 1, 1992

Non-commutative lens spaces

By Kengo MATSUMOTO and Jun TOMIYAMA

(Received July 18, 1990)
(Revised Nov. 16, 1990)

0. Introduction.

It is based on the classical Gelfand-Naimark theorem for commutative C^*-algebras to think of the theory of C^*-algebras as non-commutative general topology. In the early part of the 1980's, the explosive development of K-theory for C^*-algebras began to make a much stronger case for C^*-algebras as non-commutative topology ([**BDF**], [**Co1**], [**El1**], [**Ka**], [**PV**], [**Ro**], interalia). As part of such a development of K-theory in operator algebras, Connes began to investigate the theory of non-commutative differential geometry in [**Co2**]. Since then, many operator algebraists have investigated non-commutative geometry and topology in the setting of C^*-algebras, for example, higher dimensional non-commutative tori in [**El2**], Yang-Mills on non-commutative 2-tori in [**CR**], the Künneth and universal coefficient theorems of K-theory in [**RS**], deformation quantization by Poisson structures in [**Ri2**], and quantum groups in [**Wo1**], [**Wo2**]. These special non-commutative C^*-algebras, resembling ordinary manifolds in a sense, are sometimes called "non-commutative manifolds". Although we have not found yet the decisive definition of non-commutative manifold, these C^*-algebras are often considered as non-commutative manifolds, and the theory of non-commutative geometry and topology on them has been developing. Among the examples of non-commutative manifolds, the irrational rotation C^*-algebras, called non-commutative tori, have been playing a central rôle, especially in Connes's non-commutative differential geometry. Since many examples support the prosperity of ordinary topology and differential geometry today, the presentation of new examples of "non-commutative manifolds" would perhaps be significant for the development of the theory of non-commutative manifolds in the future. With this in mind, we shall provide some new examples of non-commutative manifolds in the category of C^*-algebras and investigate their structures.

This paper is the second step (following [**Ma1**]) of our plan, in which we aim toward the construction of a theory of non-commutative 3-manifolds.

The first author (in [**Ma1**]) has introduced one method of deforming a 3-sphere into non-commutative C^*-algebras which gives rise to a one-parameter family $\{S^3_\theta\}_{\theta\in\boldsymbol{R}}$ of C^*-algebras. He has characterized them as the universal C^*-algebras determined by the following relations:

$$S^*S=SS^*,\qquad T^*T=TT^*,\qquad TS=e^{2\pi i\theta}ST,$$
$$(1-T^*T)(1-S^*S)=0,\qquad \|S\|=\|T\|=1.$$

To recall their construction, they have been obtained by attaching two non-commutative solid tori, the crossed product C^*-algebras $C(D^2)\times_\theta \boldsymbol{Z}$ of continuous function algebras on the 2-disk in the complex plane by rotation by θ around the origin, on their boundaries. Namely, they are defined as non-commutative versions of genus one Heegaard splitting of S^3.

It is known to many topologists that a 3-manifold which can be obtained by a genus one Heegaard splitting, in the class of compact connected orientable 3-manifolds without boundary, is a lens space. In the present paper, following the idea used in the construction of the non-commutative 3-sphere mentioned above, we shall define non-commutative versions of lens spaces and study their structure. We shall then prove a complete structure theorem for these non-commutative lens spaces as C^*-algebras, and realize them as the universal C^*-algebras determined by certain relations.

Recall that, for coprime integers p, q, the lens space $L(p, q)$ of type (p, q) can be obtained by sewing two solid tori on their boundaries, which are 2-tori, by identifying a meridian on one solid torus with a (q, p)-torus knot on the other solid torus. Here a (q, p)-torus knot means a knot written on the boundary of a solid torus which runs q times along a longitude p times along a meridian (cf. [**He**], [**Ja**]). We will imitate this construction in the category of non-commutative C^*-algebras to get non-commutative versions of $L(p, q)$.

In this connection, note that we already have non-commutative versions of solid torus, $D_\theta=C(D^2)\times_\theta \boldsymbol{Z}$, defined in [**Ma1**], and of the 2-torus, $A_\theta=C(S^1)\times_\theta \boldsymbol{Z}$, defined in [**Ri1**]. By using the toral automorphisms on A_θ given in [**Bre**] and [**Wa**], we shall construct non-commutative lens spaces $L_\theta(p, q)$ with a deformation parameter $\theta\in\boldsymbol{R}$. This construction is carried out in §1. It is almost unnecessary to say that this construction is a generalization of that of the 3-sphere S^3_θ. In fact, for $(p, q)=(1, 0)$, we have $L_\theta(1, 0)=S^3_\theta$. In §2, we show that the algebras $L_\theta(p, q)$ are nuclear and that they are not of type I for irrational θ's. Their K-groups have torsion depending on p. Note that in the commutative case we have, as very interesting examples of these manifolds, the 3-sphere S^3, three demensional real projective space $\boldsymbol{R}P^3$, and a trivial S^2-bundle over S^1. We will discuss non-commutative versions of these manifolds in §3. There is another description of lens spaces. By considering S^3 as the unit sphere of $\boldsymbol{C}^2$, we have a cyclic group action τ on S^3 given by $\tau(z, w)=(e^{2\pi i/p}z, e^{2q\pi i/p}w)$, $(z, w)\in S^3$. It is known that $L(p, q)$ can be realized as the orbit space of S^3 under the action τ. Namely, the universal covering space of $L(p, q)$, whose covering transformation group is generated by the action τ, is S^3. In §4 and §5, we shall present corresponding discussions for our non-commutative versions and prove the following theorem:

THEOREM A (Theorem 5.1). *The non-commutative lens space $L_\theta(p, q)$ can be realized as a fixed point subalgebra of the non-commutative 3-sphere $S^3_{\theta/p}$ under a certain cyclic group action on it.*

Let V and U be unitary generators of A_θ satisfying the relation $VU = e^{2\pi i\theta}UV$. In §6, we will analyze the structure of $L_\theta(p, q)$ to obtain the following structure theorem:

THEOREM B (Theorem 6.8). *The non-commutative lens space $L_\theta(p, q)$ can be realized as the C^*-algebra of all A_θ-valued continuous functions on closed interval $[-1, 1]$ whose values at -1 and 1 belong to $C^*(V)$ and $C^*(V^qU^p)$, the C^*-subalgebras of A_θ generated by V and V^qU^p respectively.*

This result is based on the fact that our non-commutative solid torus becomes the C^*-algebra of all A_θ-valued continuous functions on $[0, 1]$ whose values at 0 belong to the C^*-subalgebra $C^*(V)$ generated by V. By Theorem B we know the complete structure of $L_\theta(p, q)$ as C^*-algebra modulo A_θ. In particular, our non-commutative 3-sphere S^3_θ has its structure as the C^*-algebra of all A_θ-valued continuous functions, whose values at -1 and 1 belong to the C^*-subalgebra $C^*(U)$ and $C^*(V)$ respectively.

Furthermore, in §7, we will realize $L_\theta(p, q)$ as a universal C^*-algebra generated by three elements with some relations. Under the assumption that q must be 1 when $p=0$, the result is the following one:

THEOREM C (Theorem 7.6). *For non-negative coprime integers q, p, $L_\theta(p,q)$ can be realized as the universal C^*-algebra generated by three normal operators B, C and N with the following relations:*

$$CB = e^{2p\pi i\theta}BC \tag{1}$$

$$CN = e^{2\pi i\theta}NC, \qquad BN = e^{2q\pi i\theta}NB \tag{2}$$

$$B^*B + C^*C = 1 \tag{3}$$

$$N^*N = B^*B\cdot C^*C \tag{4}$$

$$C^qN^p = (C^*C)^{(p+q)/2}(B^*B)^{(p-1)/2}\cdot B \qquad (p\neq 0) \tag{5}$$

$$(B^*B)^{1/2}\cdot C = (C^*C)^{1/2}\cdot B \qquad (p=0).$$

Theorem C may be regarded as a generalization of the main theorem in [**Ma1**]. However, the proof of Theorem C will be given by a completely different method from the extended version of the proof of the theorem in [**Ma1**]. We will use Theorem B for our proof this time.

The above operator relations seem to be different from a generalized form appeared in [**Ma1**]. We shall study, in §8, the operator relations of $L_\theta(1, 0)=S_\theta^3$ in detail so that we shall know that they are very similar to those of Woronowicz's quantum group $S_\nu U(2)$. Finally, we will refer to an interesting example of non-commutative lens spaces, that is $L_\theta(2, 1)=\boldsymbol{RP}_\theta^3$ the non-commutative real projective space. We will describe explicitly the operator relations of $\boldsymbol{RP}_\theta^3$ as the universal C^*-algebra.

We remark that a non-commutative version of 2-sphere has recently constructed and studied in [**BEEK**] by using a non-commutative 2-torus. But the method of its construction is completely different from that of our construction of non-commutative lens spaces.

It is possible to generalize our discussions for higher dimensional lens spaces and deformation C^*-algebras of $L(p, q)$ by continuous functions in their centers instead of constant θ. These discussions will be appeared somewhere (cf. [**Ma2**], [**Ma3**]).

We are deeply indebted to S. L. Woronowicz for his helpful discussions and suggestions about the structure of the C^*-algebras $L_\theta(p, q)$. Thanks are also due to Y. Kametani for fruitful discussions in the presentation of this paper.

1. Preliminaries and the construction of non-commutative lens spaces.

Let θ be an arbitrary but fixed real number. We provide first some notations following [**Ma1**].

Let D^2 be the closed unit disk in the complex plane whose boundary is the unit circle S^1. We consider the θ-rotation homeomorphism α_θ on D^2 given by

$$\alpha_\theta(re^{2\pi i\xi}) = re^{2\pi i(\xi+\theta)} \qquad 0 \leqq r, \quad \xi \leqq 1.$$

It then induces an automorphism on the C^*-algebra $C(D^2)$ of all complex valued continuous functions on D^2, which is also denoted by α_θ. Note that the restriction of α_θ to the boundary $\partial D^2=S^1$ also gives a homeomorphism on it. Then the non-commutative solid torus D_θ has been defined, in [**Ma1**], as the crossed product C^*-algebra $C(D^2)\times_\theta \boldsymbol{Z}$.

In the C^*-algebra D_θ, there exist two canonical generators y and v. The former is the normal operator coming from the generating function of $C(D^2)$ defined by

$$y(re^{2\pi i\xi}) = re^{2\pi i\xi} \qquad 0 \leqq r, \quad \xi \leqq 1.$$

And the latter is the unitary operator coming from the positive generator of the infinite cyclic group $\boldsymbol{Z}$ in $C(D^2)\times_\theta \boldsymbol{Z}$. Hence they satisfy the following covariant relation in D_θ:

$$vyv^* = e^{2\pi i\theta}y. \tag{1-1}$$

Next, consider $(-\theta)$-rotation homeomorphism on D^2. Let x and u be the elements in $D_{-\theta}=C(D^2)\times_{-\theta}\boldsymbol{Z}$ corresponding to the y and v respectively. Thus we have the similar covariant relation to (1-1) in $D_{-\theta}$:

$$uxu^* = e^{-2\pi i\theta}x. \tag{1-2}$$

Now the surjection from $C(D^2)$ to $C(S^1)$, which is induced by restricting functions of $C(D^2)$ to the boundary $\partial D^2=S^1$, is compatible with the θ-rotation homeomorphism on them. Hence it lifts to a surjection π_θ from $C(D^2)\times_\theta\boldsymbol{Z}=D_\theta$ onto $C(S^1)\times_\theta\boldsymbol{Z}$, which is written as A_θ throughout this paper regardless as θ is irrational or not. Similarly we have a surjection $\pi_{-\theta}$ from $D_{-\theta}$ to $A_{-\theta}$. Put

$$U=\pi_\theta(y),\quad V=\pi_\theta(v)\quad\text{and}\quad \hat{U}=\pi_{-\theta}(x),\quad \hat{V}=\pi_{-\theta}(u).$$

Then V and U are the generating pair of unitaries in A_θ as well as the pair $(\hat{V}, \hat{U})$ in $A_{-\theta}$ satisfying familiar relations:

$$VU=e^{2\pi i\theta}UV\quad\text{and}\quad \hat{V}\hat{U}=e^{-2\pi i\theta}\hat{U}\hat{V} \tag{1-3}$$

Let ρ_0 be the isomorphism from $A_{-\theta}$ to A_θ given by

$$\rho_0(\hat{U})=V\quad\text{and}\quad \rho_0(\hat{V})=U.$$

Then the C^*-algebra, our non-commutative 3-sphere, S_θ^3 has been defined as

$$S_\theta^3=\{(a, b)\in D_{-\theta}\oplus D_\theta \mid \rho_0\circ\pi_{-\theta}(a)=\pi_\theta(b)\}.$$

We will consider next to replace the above ρ_0 by another isomorphisms and construct a number of non-commutative 3-manifolds, which we call non-commutative lens spaces.

Now fix coprime integers p, q, and take two integers l and m satisfying $ql-pm=1$ so that the determinant of the matrix $A=\begin{bmatrix} q & m \\ p & l\end{bmatrix}$ is one. Then the matrix A induces the automorphism ρ_A on A_θ as in the following way:

$$\rho_A(V)=e^{-pq\pi i\theta}V^qU^p\quad\text{and}\quad \rho_A(U)=e^{-lm\pi i\theta}V^mU^l. \tag{1-4}$$

It is easy to see that $\rho_{AB}=\rho_A\circ\rho_B$ for $A, B\in SL(2, \boldsymbol{Z})$. Namely, ρ gives an action of $SL(2, \boldsymbol{Z})$ on A_θ (cf. [**Bre**], [**Wa**]).

We define the C^*-algebra $L_\theta(p, q)$, which we call the non-commutative lens space of type (p, q), by the following way;

$$L_\theta(p, q)=\{(a, b)\in D_\theta\oplus D_\theta \mid \rho_A\circ\pi_\theta(a)=\pi_\theta(b)\}.$$

We shall show that the above definition is independent of the choice of integers l and m. The fact is rather obvious in the commutative case.

The following lemma plays crucial rôles in our further discussions.

LEMMA 1.1. *Let ρ_A be the automorphism on A_θ defined from a matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \boldsymbol{Z})$ by*

$$\rho_A(V) = e^{-ac\pi i\theta} V^a U^c \quad \text{and} \quad \rho_A(U) = e^{-bd\pi i\theta} V^b U^d.$$

Then there exists an automorphism $\hat{\rho}_A$ on D_θ satisfying

$$\pi_\theta \circ \hat{\rho}_A = \rho_A \circ \pi_\theta \tag{1-5}$$

if and only if $a=d=\pm 1$ and $c=0$.

PROOF. Assume that there exists an automorphism $\hat{\rho}_A$ on D_θ satisfying (1-5). Then we have the following commutative diagram on K-theory:

$$\begin{tikzcd} K_1(D_\theta) \arrow[r, "(\pi_\theta)_*"] \arrow[d, "(\hat{\rho}_A)_*"] & K_1(A_\theta) \arrow[d, "(\rho_A)_*"] \\ K_1(D_\theta) \arrow[r, "(\pi_\theta)_*"'] & K_1(A_\theta). \end{tikzcd}$$

We note that $K_1(D_\theta)$ is isomorphic to $\boldsymbol{Z}$ whose generator is the class of v so that we see

$$(\hat{\rho}_A)_*([v]) = \pm[v] \quad \text{in} \quad \boldsymbol{Z}[v] = K_1(D_\theta).$$

Therefore we have

$$(\pi_\theta)_* \circ (\hat{\rho}_A)_*([v]) = (\pi_\theta)_*(\pm[v]) = \pm[V] \qquad \text{in} \quad K_1(A_\theta)$$

and

$$(\rho_A)_* \circ (\pi_\theta)_*([v]) = (\rho_A)_*([V]) = a[V] + c[U] \qquad \text{in} \quad K_1(A_\theta).$$

This implies that $c=0$, $a=\pm 1$ and hence $d=\pm 1$.

Conversely, for a given matrix $A=\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, we put

$$\hat{\rho}_A(v) = v \quad \text{and} \quad \hat{\rho}_A(y) = e^{-b\pi i\theta} v^b y.$$

Notice that $\hat{\rho}_A(y)$ and the unit 1 generate a commutative C^*-algebra. As $\hat{\rho}_A(v)$ and $\hat{\rho}_A(y)$ satisfy the covariance relation:

$$\hat{\rho}_A(v)\hat{\rho}_A(y)\hat{\rho}_A(v)^* = e^{2\pi i\theta} \hat{\rho}_A(y),$$

$\hat{\rho}_A$ can yield an automorphism on D_θ because of the universality of the crossed product $C(D^2) \times_\theta \boldsymbol{Z}$. It is then clear that $\hat{\rho}_A$ satisfies the condition (1-5).

When the matrix A is of the form $\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}$, by putting

$$\hat{\rho}_A(v) = v^* \quad \text{and} \quad \hat{\rho}_A(y) = e^{b\pi i\theta} v^b y^*,$$

we obtain a desired automorphism on D_θ by the same reason as above. $\square$

PROPOSITION 1.2. *In the construction of $L_\theta(p, q)$, the isomorphism class of $L_\theta(p, q)$ does not depend on the choice of integers l, m satisfying $ql-pm=1$.*

PROOF. Let j, k be another pair of integers satisfying $qj-pk=1$. Put

$$A=\begin{bmatrix} q & m \\ p & l \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} q & k \\ p & j \end{bmatrix}.$$

For the difference between $L_\theta(p, q)$ constructed through A and $L_\theta(p, q)$ done through B, we write them as $L_\theta^A(p, q)$ and $L_\theta^B(p, q)$ respectively. As the matrix $D=A^{-1}B$ is of the form $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$, there exists an automorphism $\hat{\rho}_D$ on D_θ by Lemma 1.1, which satisfies $\pi_\theta \circ \hat{\rho}_D = \rho_D \circ \pi_\theta$. Define the automorphism Φ on $D_\theta \oplus D_\theta$ by $\Phi(a, b)=(\hat{\rho}_D(a), b)$, $(a, b) \in D_\theta \oplus D_\theta$. One easily sees that Φ induces an isomorphism from $L_\theta^B(p, q)$ to $L_\theta^A(p, q)$. In fact, for $(a, b) \in D_\theta \oplus D_\theta$ satisfying $\rho_B \circ \pi_\theta(a) = \pi_\theta(b)$, we have

$$\rho_A \circ \pi_\theta(\hat{\rho}_D(a)) = \rho_A \circ \rho_D \circ \pi_\theta(a) = \rho_{AD} \circ \pi_\theta(a) = \pi_\theta(b).$$

Thus $\Phi(a, b)$ belongs to $L_\theta^A(p, q)$. Since the construction of the inverse isomorphism from $L_\theta^A(p, q)$ to $L_\theta^B(p, q)$ goes in a similar way, they are mutually isomorphic. $\square$

We have another, but similar, construction of $L_\theta(p, q)$ by using attaching matrix of determinant of minus one.

For given coprime integers p, q, we take integers r, s satisfying $ps-qr=1$. The matrix $E=\begin{bmatrix} q & s \\ p & r \end{bmatrix}$ is of determinant minus one. Let $\hat{V}$ and $\hat{U}$ be canonical unitary generators for $A_{-\theta}$ satisfying $\hat{V}\hat{U}=e^{-2\pi i\theta}\hat{U}\hat{V}$. The matrix E induces an isomorphism ρ_E from $A_{-\theta}$ to A_θ such that

$$\rho_E(\hat{V}) = e^{-pq\pi i\theta} V^q U^p \quad \text{and} \quad \rho_E(\hat{U}) = e^{-rs\pi i\theta} V^s U^r.$$

Here we define a new C^*-algebra $\hat{L}_\theta(p, q)$ as

$$\hat{L}_\theta(p, q) = \{(a, b) \in D_{-\theta} \oplus D_\theta \mid \rho_E \circ \pi_{-\theta}(a) = \pi_\theta(b)\}$$

Similarly to Lemma 1.1, the following lemma assures that it is independent of the choice of the integers r, s as long as $ps-qr=1$. We skip the proof since it goes along with the same lines of the proof of Lemma 1.1.

LEMMA 1.3. *Let ρ_B an isomorphism from $A_{-\theta}$ to A_θ defined from the matrix $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $GL(2, \boldsymbol{Z})$ of determinant minus one by*

$$\rho_B(\hat{V}) = e^{-ac\pi i\theta} V^a U^c \quad \textit{and} \quad \rho_B(\hat{U}) = e^{-bd\pi i\theta} V^b U^d.$$

Then there exists an isomorphism $\hat{\rho}_B$ from $D_{-\theta}$ to D_θ satisfying

$$\pi_\theta \circ \hat{\rho}_B = \rho_B \circ \pi_{-\theta}$$

if and only if $a=-d=\pm 1$ *and* $c=0$. *Thus, by using similar discussions to the previous one, we also have the following:*

PROPOSITION 1.4. *The isomorphism class of* C^**-algebra* $\hat{L}_\theta(p, q)$ *does not depend on the choice of integers* r, s *as long as* $ps-qr=1$.

Furthermore one may see the next proposition.

PROPOSITION 1.5. $L_\theta(p, q)$ *is isomorphic to* $\hat{L}_\theta(p, q)$.

PROOF. For the coprime integers p and q, we take two pairs of two integers (l, m) and (r, s) such that $ql-pm=1$ and $qr-ps=-1$. Put

$$A=\begin{bmatrix} q & m \\ p & l \end{bmatrix} \quad \text{and} \quad E=\begin{bmatrix} q & s \\ p & r \end{bmatrix}.$$

Then we have $A^{-1}E=\begin{bmatrix} 1 & ls-mr \\ 0 & -1 \end{bmatrix}$. Hence by Lemma 1.3, the isomorphism $\rho_{A^{-1}E}$ from $A_{-\theta}$ to A_θ can be extended to an isomorphism $\hat{\rho}_{A^{-1}E}$ from $D_{-\theta}$ to D_θ satisfying

$$\pi_\theta \circ \hat{\rho}_{A^{-1}E} = \rho_{A^{-1}E} \circ \pi_{-\theta}.$$

It follows that the isomorphism Ψ from $D_{-\theta} \oplus D_\theta$ to $D_\theta \oplus D_\theta$ defined by

$$\Psi(a, b) = (\hat{\rho}_{A^{-1}E}(a), b), \qquad (a, b) \in D_{-\theta} \oplus D_\theta$$

induces an isomorphism from $\hat{L}_\theta(p, q)$ to $L_\theta(p, q)$ as in the proof of Proposition 1.2. □

Thus we need not make difference between $L_\theta(p, q)$ and $\hat{L}_\theta(p, q)$ henceforth. Hence we unify and write them as $L_\theta(p, q)$.

We remark that in the definition of $L_\theta(p, q)$, we may rearrange the attaching isomorphism ρ_E as the following one:

$$\tilde{\rho}_E(\hat{V}) = V^q U^p \quad \text{and} \quad \tilde{\rho}_E(\hat{U}) = V^s U^r$$

for $E=\begin{bmatrix} q & s \\ p & r \end{bmatrix}$ with minus one determinant. In fact, we can show the following lemma:

LEMMA 1.6. *Put*

$$\tilde{L}_\theta(p, q) = \{(a, b) \in D_{-\theta} \oplus D_\theta \mid \tilde{\rho}_E \circ \pi_\theta(a) = \pi_\theta(b)\}.$$

Then the algebra $\tilde{L}_\theta(p, q)$ *is isomorphic to* $L_\theta(p, q)$.

PROOF. Since

$$e^{-pq\pi i\theta} \tilde{\rho}_E(\hat{V}) = \rho_E(\hat{V}) \quad \text{and} \quad e^{-rs\pi i\theta} \tilde{\rho}_E(\hat{U}) = \rho_E(\hat{U}),$$

we have

$$\rho_E^{-1}\circ\tilde{\rho}_E(\hat{V}) = e^{pq\pi i\theta}\hat{V} \quad\text{and}\quad \rho_E^{-1}\circ\tilde{\rho}_E(\hat{U}) = e^{rs\pi i\theta}\hat{U}.$$

Then the following correspondence ξ in $D_{-\theta}$

$$\xi(u) = e^{pq\pi i\theta}u \quad\text{and}\quad \xi(x) = e^{rs\pi i\theta}x$$

gives rise to an automorphism on $D_{-\theta}$ and the following diagram becomes commutative:

$$\begin{tikzcd} D_{-\theta} \arrow[r, "\xi"] \arrow[d, "\pi_{-\theta}"'] & D_{-\theta} \arrow[d, "\pi_{-\theta}"] \\ A_{-\theta} \arrow[r, "\rho_E^{-1}\circ\tilde{\rho}_E"'] & A_{-\theta} \end{tikzcd}$$

Therefore the restriction of the automorphism $\xi\oplus id$ on $D_{-\theta}\oplus D_\theta$ to $\tilde{L}_\theta(p, q)$ gives an isomorphism from it to $L_\theta(p, q)$. Actually, for a pair $(a, b)\in D_{-\theta}\oplus D_\theta$ satisfying $\tilde{\rho}_E\circ\pi_{-\theta}(a)=\pi_\theta(b)$, we have

$$\rho_E\circ\pi_{-\theta}(\xi(a)) = \rho_E\circ(\rho_E^{-1}\circ\tilde{\rho}_E)\circ\pi_{-\theta}(a) = \tilde{\rho}_E\circ\pi_{-\theta}(a) = \pi_\theta(b). \qquad \square$$

Similar arguments to the above discussions are also applied to a matrix of determinant one. Hence we sometimes adopt $\tilde{\rho}_E$ in place of ρ_E making no difference between them henceforth.

2. Basic properties of $L_\theta(p, q)$.

As we know from the construction, $L_\theta(p, q)$ may be regarded as a C^*-algebra not so badly behaved from the structural point of view. For θ being irrational its structure is however not so close to that of a commutative C^*-algebra. In fact, we have the following:

PROPOSITION 2.1. *For each pair of coprime integers p, q, we have*

(i) *$L_\theta(p, q)$ is nuclear for each $\theta\in\mathbf{R}$,*

(ii) *$L_\theta(p, q)$ is not of type I for each irrational $\theta\in\mathbf{R}$*

(see [**Pe**] *or* [**To1**] *as references for nuclearity and type I).*

PROOF. (i) Take two integers r, s such that $ps-qr=1$ and put $E=\begin{bmatrix} q & s \\ p & r\end{bmatrix}$. We denote by I and J the kernel of the upper horizontal surjection and that of the lower one respectively in the following pull back diagram

$$\begin{tikzcd} L_\theta(p, q) \arrow[r] \arrow[d] & D_\theta \arrow[d, "\pi_\theta"] \\ D_{-\theta} \arrow[r, "\rho_E\circ\pi_{-\theta}"'] & A_\theta \end{tikzcd}$$

where each surjection from $L_\theta(p, q)$ to D_θ and $D_{-\theta}$ is defined as the projection onto each component. Then the next diagram is commutative in the two squares and both horizontal sequences are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & L_\theta(p, q) & \longrightarrow & D_\theta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_\theta} & & \\ 0 & \longrightarrow & J & \longrightarrow & D_{-\theta} & \xrightarrow[\rho_E \circ \pi_{-\theta}]{} & A_\theta & \longrightarrow & 0. \end{array}$$

One can then verify easily that I and J are mutually isomorphic and nuclear. In fact, they are isomorphic to the crossed product C^*-algebra $C_0(D^2)\times_{-\theta}\boldsymbol{Z}$ of the algebra $C_0(D^2)$ of all continuous functions on the disk D^2 with vanishing at the boundary $\partial D(=S^1)$ by the action of rotation by $(-\theta)$ on D^2. Hence it follows that $L_\theta(p, q)$ is nuclear.

(ii) Since there exists a surjective homomorphism from $L_\theta(p, q)$ to A_θ, $L_\theta(p, q)$ is not of type I as long as neither is A_θ. □

Let $L(p, q)$ be the usual lens space. When $\theta=0$, our lens space $L_0(p, q)$ is naturally isomorphic to the C^*-algebra of all complex valued continuous functions on $L(p, q)$. In roughly classifying the spaces $L(p, q)$ concerning about p, q, we observe first their homology groups. They are known as

$$H_1(L(p, q);\ \boldsymbol{Z}) = \boldsymbol{Z}/p\boldsymbol{Z} \qquad (\text{cf. [}\mathbf{He}\text{]}).$$

In comparison with the above homology group, it is worth computing K-groups of our lens spaces.

PROPOSITION 2.2. *For all p, q and θ, we have*

$$K_0(L_\theta(p, q)) = (\boldsymbol{Z}/p\boldsymbol{Z})\oplus\boldsymbol{Z} \quad \text{and} \quad K_1(L_\theta(p, q)) = \begin{cases} \boldsymbol{Z} & (p\neq 0) \\ \boldsymbol{Z}\oplus\boldsymbol{Z} & (p=0). \end{cases}$$

PROOF. Let $E=\begin{bmatrix} q & s \\ p & r \end{bmatrix}$ be a matrix of determinant minus one. We apply Mayer-Vietris sequence of K-groups for the following pull back diagram (cf. [**Bl**]):

$$\begin{array}{ccc} L_\theta(p, q) & \xrightarrow{\ i\ } & D_\theta \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \pi_\theta} \\ D_{-\theta} & \xrightarrow[\rho_E \circ \pi_{-\theta}]{} & A_\theta . \end{array}$$

Hence we have the next exact sequence:

$$\begin{array}{ccccc} K_0(L_\theta(p,q)) & \xrightarrow{(i,j)_*} & K_0(D_{-\theta})\oplus K_0(D_\theta) & \xrightarrow{\rho_E\circ\pi_{-\theta}-\pi_{\theta *}} & K_0(A_\theta) \\ \uparrow & & & & \downarrow \\ K_1(A_\theta) & \xleftarrow[\rho_E\circ\pi_{-\theta}-\pi_{\theta *}]{} & K_1(D_{-\theta})\oplus K_1(D_\theta) & \xleftarrow[(i,j)_*]{} & K_1(L_\theta(p,q)). \end{array}$$

It follows that

$$\begin{array}{ccccc} K_0(L_\theta(p,q)) & \xrightarrow{(i,j)_*} & \boldsymbol{Z}[1]\oplus\boldsymbol{Z}[1] & \xrightarrow{\rho_E\circ\pi_{-\theta}-\pi_{\theta *}} & \boldsymbol{Z}[p]\oplus\boldsymbol{Z}[1] \\ \uparrow & & & & \downarrow \\ \boldsymbol{Z}[V]\oplus\boldsymbol{Z}[U] & \xleftarrow[\rho_E\circ\pi_{-\theta}-\pi_{\theta *}]{} & \boldsymbol{Z}[u]\oplus\boldsymbol{Z}[v] & \xleftarrow[(i,j)_*]{} & K_1(L_\theta(p,q)), \end{array}$$

where a projection p of A_θ at the upper right corner means one of two generators for $K_0(A_\theta)$ which is called a Rieffel projection (cf. [**Ri1**]). Now it is easy to see that the upper right horizontal arrow gives the mapping

$$(m,n)\in\boldsymbol{Z}\oplus\boldsymbol{Z}\longrightarrow(0,m-n)\in\boldsymbol{Z}[p]\oplus\boldsymbol{Z}$$

whereas the lower left one gives the mapping

$$(m,n)\in\boldsymbol{Z}[u]\oplus\boldsymbol{Z}[v]\xrightarrow{\begin{bmatrix} q & -1 \\ p & 0\end{bmatrix}}(mq-n,mp)\in\boldsymbol{Z}[V]\oplus\boldsymbol{Z}[U].$$

Thus it follows that

$$K_0(L_\theta(p,q))=(\boldsymbol{Z}/p\boldsymbol{Z})\oplus\boldsymbol{Z}\quad\text{and}\quad K_1(L_\theta(p,q))=\begin{cases}\boldsymbol{Z} & (p\neq 0)\\ \boldsymbol{Z}\oplus\boldsymbol{Z} & (p=0)\end{cases}$$

by routine calculations. □

3. Some examples.

In this section, we observe some interesting examples of non-commutative lens spaces keeping in mind the usual lens spaces.

First of all, if we take $\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$ as the attaching matrix, the resulting C^*-algebra $L_\theta(1,0)$ is nothing but the non-commutative 3-sphere S_θ^3 introduced in [**Ma1**]. Furthermore, we have, as in the case of the usual lens spaces, the following fact.

PROPOSITION 3.1. *$L_\theta(1,q)$ is isomorphic to S_θ^3 for each $q\in\boldsymbol{Z}$.*

PROOF. Take $\begin{bmatrix} q & 1 \\ 1 & 0\end{bmatrix}$ as an attaching matrix to construct $L_\theta(p,q)$. Put $E=\begin{bmatrix} q & 1 \\ 1 & 0\end{bmatrix}$ and $F=\begin{bmatrix} 1 & -q \\ 0 & 1\end{bmatrix}$. Then we have $E=F^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$. By Lemma 1.1, there exists an automorphism $\hat{\rho}_F$ on D_θ satisfying $\pi_\theta\circ\hat{\rho}_F=\rho_F\circ\pi_\theta$. Consider the automorphism ξ on $D_{-\theta}\oplus D_\theta$ defined by

$$\xi:(a,\, b)\in D_{-\theta}\oplus D_\theta \longrightarrow (a,\, \hat{\rho}_F(b))\in D_{-\theta}\oplus D_\theta\,.$$

Then ξ induces an isomorphism from $L_\theta(1, q)$ to $L_\theta(1, 0)(=S_\theta^3)$. In fact, for a pair $(a, b)\in D_{-\theta}\oplus D_\theta$ such that $\rho_E\circ\pi_{-\theta}(a)=\pi_\theta(b)$, we have

$$\rho_{F^{-1}}\circ\rho_0\circ\pi_{-\theta}(a)=\pi_\theta(b),$$

where ρ_0 means the isomorphism from $A_{-\theta}$ to A_θ induced by the matrix $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. It follows that $\rho_0\circ\pi_{-\theta}(a)=\rho_F\circ\pi_\theta(b)=\pi_\theta(\hat{\rho}_F(b))$ so that $(a, \hat{\rho}_F(b))$ gives an element of $L_\theta(1, 0)$, that is of S_θ^3. □

Next example is a non-commutative version of a product 3-manifold $S^2\times S^1$, which is realized as the manifold attached two solid tori on their boundaries without twisting. Hence we can take the identity matrix as an attaching matrix. Now consider the action on 2-sphere defined by θ-rotation along longitudes round an axis. The action on $C(S^2)$ induced by the θ-rotation is also denoted by θ. We have then,

PROPOSITION 3.2. *$L_\theta(0, 1)$ is isomorphic to $C(S^2)\times_\theta \boldsymbol{Z}$.*

PROOF. By definition of $L_\theta(0, 1)$, we have

$$\begin{aligned} L_\theta(0, 1) &= \{(a, b)\in D_\theta\oplus D_\theta \mid \pi_\theta(a)=\pi_\theta(b)\}\\ &= (C(D^2)\times_\theta \boldsymbol{Z}) \underset{C(S^1)\times_\theta Z}{\cup} (C(D^2)\times_\theta \boldsymbol{Z})\\ &= (C(D^2) \underset{C(S^1)}{\cup} C(D^2))\times_\theta \boldsymbol{Z}\\ &= C(S^2)\times_\theta \boldsymbol{Z}, \end{aligned}$$

where the above notations $A\underset{C}{\cup}B$ for C^*-algebras A, B and C mean the fibre product C^*-algebra as in [**Ma1**]. □

The final example is the algebra $L_\theta(2, 1)$. When $\theta=0$, it is known as three dimensional real projective space. Thus we may call the C^*-algebra $L_\theta(2, 1)$ non-commutative real projective space and write it as $\boldsymbol{RP}_\theta^3$. As $L(2, 1)$ is also homeomorphic to the compact Lie group $SO(3)$, we think $L_\theta(2, 1)=\boldsymbol{RP}_\theta^3$ as a deformation of $SO(3)$. On the other hand $SO(4)$ is topologically regarded as the space $S^3\times SO(3)$, so that the tensor product C^*-algebra $S_\theta^3\otimes\boldsymbol{RP}_\theta^3$ may be thought of a deformation of $SO(4)$ into non-commutative C^*-algebras. These deformations of compact Lie groups are completely different ones from those algebras invented by Rieffel and Woronowicz (cf. [**Ri2**], [**Wo1**]). We will refer to the C^*-algebra $\boldsymbol{RP}_\theta^3$ in the final section again.

4. Non-commutative fibred solid tori of type (p, q).

There is another description of the usual lens space $L(p, q)$ $(p\neq 0)$ in topology. Consider S^3 as the unit sphere of $\boldsymbol{C}^2$. Let $\tau_{p,q}$ be the homeomorphism on S^3 defined by

$$\tau_{p,q}(z, w)=(\omega z, \omega^q w) \qquad (z, w)\in S^3$$

where $\omega=e^{2\pi i/p}$ is the principal p-th root of unity. Hence $\tau_{p,q}$ gives a $\boldsymbol{Z}_p(=\boldsymbol{Z}/p\boldsymbol{Z})$-action on S^3. It is then known that $L(p, q)$ is realized as the orbit space of this action. Therefore S^3 is the universal covering space of $L(p, q)$.

In this section as well as the next one, we shall prove corresponding facts in our non-commutative context. Namely, we realize our non-commutative lens spaces as fixed point subalgebras of S_θ^3 under finite cyclic group actions. Henceforth we fix coprime integers p, q.

We try first to give a $\boldsymbol{Z}_p$-action on S_θ^3. Consider the following correspondence on S_θ^3.

$$S \longrightarrow \omega^q S, \qquad T \longrightarrow \omega T$$

where the operators S and T, satisfying the relations cited in §0, are defined by (u, y) and (x, v) in the subalgebra S_θ^3 of $D_{-\theta}\oplus D_\theta$ respectively (see [**Ma1**]). One then easily verifies that the above correspondence gives rise to an automorphism on S_θ^3 by the universality of the algebra with respect to the operator relations between S and T. We denote it by $g(p, q)$ or simply g. Notice that this action may be lifted up to each C^*-algebra $D_{-\theta}$ and D_θ, and hence to $D_{-\theta}\oplus D_\theta$. Actually the following correspondences give automorphisms on them:

$$x,\ u\in D_{-\theta} \longrightarrow \omega x,\ \omega^q u\in D_{-\theta}, \qquad y,\ v\in D_\theta \longrightarrow \omega^q y,\ \omega v\in D_\theta$$

We write them also as g above.

Before looking at the structure of the fixed point subalgebra $(S_\theta^3)^g$, we study the fixed point subalgebras $(D_{-\theta})^g$ and $(D_\theta)^g$.

LEMMA 4.1. *$(D_\theta)^g$ and $(D_{-\theta})^g$ are the C^*-subalgebras generated by*

$$\{y^i y^{*j} v^k \mid q(i-j)+k\in p\boldsymbol{Z}\} \quad \text{and} \quad \{x^l x^{*m} u^n \mid l-m+qn\in p\boldsymbol{Z}\},$$

respectively.

PROOF. As in [**Ma1**, Lemma 2.1], D_θ is the C^*-algebra generated by elements of polynomials of the forms:

$$\{\sum_{\substack{i,j\geq 0\\ k\in \boldsymbol{Z}}} d_{ijk} y^i y^{*j} v^k \mid d_{ijk}\in \boldsymbol{C}\}$$

whose coefficients $\{d_{ijk}\}$ are uniquely determined by the elements. By using the canonical expectation E from D_θ to $(D_\theta)^g$ given by

$$E(a)=(1/p)\sum_{k=1}^{p} g^k(a)\in(D_\theta)^g \qquad a\in D_\theta\,,$$

we can see that any element of $(D_\theta)^g$ can be approximated by polynomials fixed by g. Thus, by comparing the coefficients between

$$\sum_{\substack{i,j\geq 0\\ k\in \mathbf{Z}}} d_{ijk}y^i y^{*j} v^k \quad \text{and} \quad \sum_{\substack{i,j\geq 0\\ k\in \mathbf{Z}}} d_{ijk}\omega^{q(i-j)+k} y^i y^{*j} v^k\,,$$

we can conclude the assertion for the case of $(D_\theta)^g$. The other C^*-algebra, $(D_{-\theta})^g$, we can deal with in a similar way. □

Next, we attempt to find minimal generators for $(D_\theta)^g$ and $(D_{-\theta})^g$.

LEMMA 4.2. *Any polynomial of $y^i y^{*j} v^k$ such that $q(i-j)+k\in p\mathbf{Z}$ is generated by yv^{p-q} and v^p.*

PROOF. Take an integer m such that $q(i-j)+k=pm$. Then, by the covariance relation (1−1), we see easily that $y^i y^{*j} v^k$ is a scalar multiple of $(yv^{p-q})^i(yv^{p-q})^{*j}(v^p)^{m-(i-j)}$. □

On the other hand, the situation for $x^l x^{*m} u^n$ such that $l-m+qn\in p\mathbf{Z}$ is a little different from the above one. Take two integers r, s such that $ps-qr=1$. We fix such a pair r, s. Corresponding to Lemma 4.2, we have:

LEMMA 4.3. *Any polynomial of $x^l x^{*m} u^n$ such that $l-m+qn\in p\mathbf{Z}$ is generated by xu^r and u^p.*

PROOF. Take an integer k such that $l-m+qn=pk$. As in the proof of the previous lemma, we see that $x^l x^{*m} u^n$ is a scalar multiple of $(xu^r)^l(xu^r)^{*m}(u^p)^{ns-rk}$. □

Furthermore we must provide one another lemma.

LEMMA 4.4.

(i) *The C^*-algebra $C^*(yv^{p-q}, v^p)$ generated by yv^{p-q} and v^p is isomorphic to the C^*-algebra $C^*(y, v^p)$ generated by y and v^p.*

(ii) *Similarly, $C^*(xu^r, u^p)$ is isomorphic to $C^*(x, u^p)$.*

PROOF. It is easy to see that, in D_θ, the following correspondence Φ gives an automorphism:

$$\Phi(y)=yv^{p-q}, \qquad \Phi(v)=v\,,$$

because of the universality of D_θ. By restricting Φ to the C^*-subalgebra $C^*(y, v^p)$, we obtain a desired isomorphism between $C^*(yv^{p-q}, v^p)$ and $C^*(y, v^p)$. We similarly conclude the assertion (ii). □

With these lemmas, we know the structure of the fixed point subalgebras

$(D_\theta)^g$ and $(D_{-\theta})^g$ in the following way.

PROPOSITION 4.5.
(i) *$(D_\theta)^g$ is isomorphic to $D_{p\theta}$.*
(ii) *$(D_{-\theta})^g$ is isomorphic to $D_{-p\theta}$.*

PROOF. Once we know that $(D_\theta)^g$ is isomorphic to $C^*(y, v^p)$ in D_θ, it is a standard fact that the algebra is isomorphic to $D_{p\theta}$. We leave the details to the reader. □

REMARK 4.6. In general, for any $\theta \in \boldsymbol{R}$, the following correspondence π from $D_{-\theta}$ to D_θ:

$$\pi(x) = y^*, \qquad \pi(u) = v$$

gives an isomorphism between them. Hence $D_{-p\theta}$ is isomorphic to $D_{p\theta}$.

REMARK 4.7. When we regard $D_{p\theta}$ as the fixed point subalgebra $(D_\theta)^g$, it may be interpreted as a non-commutative version of a fibered solid torus of type (p, q) (cf. [**Ja**]).

5. Realization of $L_\theta(p, q)$ as a fixed point subalgebra of S_θ^3.

Continued from the previous section, this section is devoted to prove Theorem A. Namely, we shall show the following theorem:

THEOREM 5.1. *The fixed point subalgebra of S_θ^3 under the action $g(p, q)$ of the cyclic group $\boldsymbol{Z}_p$ is isomorphic to $L_{p\theta}(p, q)$.*

To prove the theorem, we need several steps. We notice first the following fact:

LEMMA 5.2. *The fixed point subalgebra $(S_\theta^3)^g$ forms the algebra*

$$\{(a, b) \in (D_{-\theta})^g \oplus (D_\theta)^g \mid \rho_0 \circ \pi_{-\theta}(a) = \pi_\theta(b)\}.$$

PROOF. It is obvious. □

Note that the next correspondences on $A_{-\theta}$ and A_θ

$$\begin{array}{ccc} A_{-\theta} \longrightarrow A_{-\theta} & \qquad & A_\theta \longrightarrow A_\theta \\ \hat{V} \longrightarrow \omega^q \hat{V} & & U \longrightarrow \omega^q U \\ \hat{U} \longrightarrow \omega \hat{U} & & V \longrightarrow \omega V \end{array}$$

give rise to automorphisms on them respectively. We write them also as $g(p, q)$ or simply g. Then the above automorphisms are compatible with those corresponding ones on $D_{-\theta}$ and D_θ through $\pi_{-\theta}$ and π_θ respectively. Namely, we have

$$g\circ\pi_{-\theta}=\pi_{-\theta}\circ g \quad \text{and} \quad g\circ\pi_{\theta}=\pi_{\theta}\circ g.$$

Let U_p and V_p (resp. $\hat{U}_p$ and $\hat{V}_p$) be unitary generators for $A_{p\theta}$ (resp. $A_{-p\theta}$) satisfying

$$V_pU_p=e^{2\pi ip\theta}U_pV_p \qquad (\text{resp. } \hat{V}_p\hat{U}_p=e^{-2\pi ip\theta}\hat{U}_p\hat{V}_p).$$

Then, by using similar discussions to Lemma 4.1, 4.2 and 4.3, we can see that the following map Ψ from $(A_\theta)^g$ to $A_{p\theta}$

$$\Psi(UV^{p-q})=U_p, \qquad \Psi(V^p)=V_p$$

induces an isomorphism between them. Similarly the map Φ from $(A_{-\theta})^g$ to $A_{p\theta}$ defined by

$$\Phi(\hat{U}\hat{V}^{p-q})=\hat{U}_p, \qquad \Phi(\hat{V}^p)=\hat{V}_p$$

gives an isomorphism between them. Thus, we have

LEMMA 5.3.
(i) *$(A_\theta)^g$ is isomorphic to $A_{p\theta}$ through Ψ.*
(ii) *$(A_{-\theta})^g$ is isomorphic to $A_{-p\theta}$ through Φ.*

Let Λ be the isomorphism from $(D_\theta)^g$ to $D_{p\theta}$ given by

$$\Lambda(yv^{p-q})=y_p, \qquad \Lambda(v^p)=v_p$$

as in the previous section. Similarly we denote by Ξ the isomorphism from $(D_{-\theta})^g$ to $D_{-p\theta}$ obtained by

$$\Xi(xu^r)=x_p, \qquad \Xi(u^p)=u_p.$$

The following lemma is a direct consequence from the above arguments.

LEMMA 5.4. *Keep the above notations. Then both the following diagrams are commutative:*

$$\text{(i)} \quad \begin{array}{ccc} (D_\theta)^g & \xrightarrow{\Lambda} & D_{p\theta} \\ \pi_\theta\downarrow & \Psi & \downarrow\pi_{p\theta} \\ (A_\theta)^g & \longrightarrow & A_{p\theta} \end{array} \qquad \text{(ii)} \quad \begin{array}{ccc} (D_{-\theta})^g & \xrightarrow{\Xi} & D_{-p\theta} \\ \pi_{-\theta}\downarrow & \Phi & \downarrow\pi_{-p\theta} \\ (A_{-\theta})^g & \longrightarrow & A_{-p\theta}. \end{array}$$

Namely, we have $\pi_{p\theta}\circ\Lambda=\Psi\circ\pi_\theta$ and $\pi_{-p\theta}\circ\Xi=\Phi\circ\pi_{-\theta}$.

LEMMA 5.5. *Put*

$$\mu_1=e^{-p(p+1)(p-q)\pi i\theta} \quad \textit{and} \quad \mu_2=e^{-r(r+1)(p-q)\pi i\theta}.$$

The map Π given by

$$\Pi(\hat{V}_p)=\mu_1V_p^{q-p}U_p \quad \textit{and} \quad \Pi(\hat{U}_p)=\mu_2V_p^{s-r}U_p^r.$$

defines an isomorphism from $A_{-p\theta}$ to $A_{p\theta}$. Then the fixed point subalgebra $(S_\theta^3)^g$ is isomorphic to the C-algebra $L'_{p\theta}(p, q-p)$ defined by*

$$L'_{p\theta}(p, q-p) = \{(a, b) \in D_{-p\theta} \oplus D_{p\theta} \mid \Pi \circ \pi_{-p\theta}(a) = \pi_{p\theta}(b)\}.$$

PROOF. Set $\Pi(\hat{V}_p) = \mu_1 V_p^{q-p} U_p^p$ and $\Pi(\hat{U}_p) = \mu_2 V_p^{s-r} U_p^r$. Then we remark that Π gives an isomorphism from $A_{-p\theta}$ to $A_{p\theta}$. In fact, the determinant of the matrix $\begin{bmatrix} q-p & s-r \\ p & r \end{bmatrix}$ is $(q-p)r - p(s-r) = -1$ so that it follows that

$$\Pi(\hat{V}_p)\Pi(\hat{U}_p) = e^{2\pi i p\theta}\Pi(\hat{U}_p)\Pi(\hat{V}_p)$$

and $\Pi(\hat{V}_p)$ and $\Pi(\hat{U}_p)$ generate $A_{p\theta}$. We notice that the direct sum $\Xi \oplus \Lambda$ yields an isomorphism from $(D_{-\theta})^g \oplus (D_\theta)^g$ to $D_{-p\theta} \oplus D_{p\theta}$. Since, by Lemma 5.2,

$$(S_\theta^3)^g = \{(a, b) \in (D_{-\theta})^g \oplus (D_\theta)^g \mid \rho \circ \pi_{-\theta}(a) = \pi_\theta(b)\},$$

it suffices to show that $\Pi \circ \pi_{-p\theta}(\Xi(a)) = \pi_{p\theta}(\Lambda(b))$ if $\rho_0 \circ \pi_{-\theta}(a) = \pi_\theta(b)$ for $(a, b) \in (D_{-\theta})^g \oplus (D_\theta)^g$. Here, by Lemma 5.4, we have

$$\pi_{p\theta}(\Lambda(b)) = \Psi(\pi_\theta(b)) = \Psi(\rho_0(\pi_{-\theta}(a)). \tag{5-1}$$

Now, by definition of Π, we see that

$$\begin{aligned} \Psi \circ \rho_0 \circ \pi_{-\theta}(xu^r) &= \Psi \circ \rho_0(\hat{U}\hat{V}^r) = \Psi(VU^r) = \mu_2 \Psi((V^p)^{s-r}(UV^{p-q})^r) \\ &= \mu_2(V_p)^{s-r}(U_p)^r = \Pi(\hat{U}_p) \end{aligned}$$

and

$$\begin{aligned} \Psi \circ \rho_0 \circ \pi_{-\theta}(u^p) &= \Psi \circ \rho_0(\hat{V}^p) = \Psi(U^p) = \mu_1 \Psi((V^p)^{q-p}(UV^{p-q})^p) \\ &= \mu_1(V_p)^{q-p}(U_p)^p = \pi(\hat{V}_p). \end{aligned}$$

Furthermore, as we know

$$\pi_{-p\theta} \circ \Xi(xu^r) = \hat{U}_p \quad \text{and} \quad \pi_{-p\theta} \circ \Xi(u^p) = \hat{V}_p$$

we have

$$\Psi \circ \rho_0 \circ \pi_{-\theta}(xu^r) = \Pi \circ \pi_{-p\theta} \circ \Xi(xu^r) \quad \text{and} \quad \Psi \circ \rho_0 \circ \pi_{-\theta}(u^p) = \Pi \circ \pi_{-p\theta} \circ \Xi(u^p).$$

Since xu^r and u^p generate $(D_{-\theta})^g$, we have

$$\Psi \circ \rho_0 \circ \pi_{-\theta} = \Pi \circ \pi_{-p\theta} \circ \Xi. \tag{5-2}$$

By combining (5-1) and (5-2), we see that $\Pi \circ \pi_{-p\theta}(\Xi(a)) = \pi_{p\theta}(\Lambda(b))$. Hence $\Xi(a) \oplus \Lambda(b)$ belongs to $L'_{p\theta}(p, q-p)$. As one can easily construct the inverse isomorphism of $\Xi \oplus \Lambda$ from $L'_{p\theta}(p, q-p)$ to $(S_\theta^3)^g$, $\Xi \oplus \Lambda$ gives an isomorphism between them. This completes the proof. □

LEMMA 5.6. *$L'_{p\theta}(p, q-p)$ is isomorphic to $L_{p\theta}(p, q-p)$.*

PROOF. Set $E=\begin{bmatrix} q-p & s-r \\ p & r \end{bmatrix}$ and $F=E^{-1}=\begin{bmatrix} -r & s-r \\ p & p-q \end{bmatrix}$. We consider the isomorphism ρ_F from $A_{p\theta}$ to $A_{-p\theta}$ defined in §1. Namely it is determined as

$$\rho_F((V_p)^{q-p}(U_p)^p) = e^{-p(q-p)\pi i\theta}\hat{V}_p$$

and

$$\rho_F((V_p)^{s-r}(U_p)^r) = e^{-r(s-r)\pi i\theta}\hat{U}_p .$$

Then we have

$$\rho_F\circ\Pi(\hat{V}_p) = \mu_1 e^{-p(q-p)\pi i\theta}\hat{V}_p$$

and

$$\rho_F\circ\Pi(\hat{U}_p) = \mu_2 e^{-r(s-r)\pi i\theta}\hat{U}_p .$$

Now it is easy to see that the following correspondence ξ in $D_{-p\theta}$

$$\xi(u_p) = \mu_1 e^{-p(q-p)\pi i\theta}u_p \quad\text{and}\quad \xi(x_p)=\mu_2 e^{-r(s-r)\pi i\theta}x_p$$

gives rise to an automorphism on $D_{-p\theta}$, and it makes the following diagram commutative:

$$\begin{array}{ccc} D_{-p\theta} & \xrightarrow{\ \xi\ } & D_{-p\theta} \\ {\scriptstyle \pi_{-p\theta}}\downarrow & & \downarrow{\scriptstyle \pi_{-p\theta}} \\ A_{-p\theta} & \xrightarrow[\rho_F\circ\Pi]{} & A_{-p\theta} . \end{array}$$

Namely we have $\pi_{-p\theta}\circ\xi=\rho_F\circ\Pi\circ\pi_{-p\theta}$. Consider the automorphism $\xi\oplus id$ on $D_{-p\theta}\oplus D_{p\theta}$. Its restriction to the C^*-subalgebra $L'_{p\theta}(p, q-p)$ then defines an isomorphism to $L_{p\theta}(p, q-p)$. In fact, for a pair $(a, b)\in D_{-p\theta}\oplus D_{p\theta}$ satisfying $\Pi\circ\pi_{-p\theta}(a)=\pi_{p\theta}(b)$, we have

$$\rho_E\circ\pi_{-p\theta}(\xi(a)) = \rho_E\circ\rho_F\circ\Pi\circ\pi_{-p\theta}(a) = \Pi\circ\pi_{-p\theta}(a) = \pi_{p\theta}(b).$$

One may also easily construct the inverse isomorphism of $\xi\oplus id$. Therefore we see that $L'_{p\theta}(p, q-p)$ is isomorphic to $L_{p\theta}(p, q-p)$. □

Having finishing these steps, we reach one of our main results.

FINAL PROOF OF THEOREM 5.1. By Lemma 5.5 and Lemma 5.6, we have

$$(S^3_\theta)^{g(p,q)} \cong L_{p\theta}(p, q-p).$$

As $\omega^q=\omega^{p+q}$, where $\omega=e^{2\pi i/p}$, we see that $g(p, q)=g(p, p+q)$. Moreover if a paire (p, q) is coprime, so is a pair $(p, p+q)$. Therefore we obtain the desired conclusion. □

Following the commutative case, we prove the next result.

PROPOSITION 5.7. *If two integers q and q' satisfy $qq' \equiv -1$ (modulo p), then we have $L_\theta(p, q) = L_\theta(p, q')$.*

PROOF. Take an integer k such that $qq'+1=pk$, whence $1/p=k-(qq'/p)$. Let η be the automorphism on S_θ^3 defined by

$$\eta(T) = S^* \quad \text{and} \quad \eta(S) = T,$$

which is well defined because of the universality of S_θ^3 with respect to its relations. Then we have $\eta \circ g(p, q) = g(p, q')^q \circ \eta$, because

$$g(p, q')^q(S^*) = ((\omega^{q'})^q S)^* = \omega S^* \quad \text{and} \quad g(p, q')^q(T) = \omega^q T.$$

Hence η induces an isomorphism between $(S_\theta^3)^{g(p,q)}$ and $(S_\theta^3)^{g(p,q')^q}$. Therefore by Theorem 5.1 together with the next lemma, we end the proof. □

LEMMA 5.8. *For two pairs (p, q) and (p, q') of co-prime integers, the algebra $(S_\theta^3)^{g(p,q')^q}$ coincides with $(S_\theta^3)^{g(p,q')}$.*

PROOF. It is clear that $(S_\theta^3)^{g(p,q')}$ is contained in $(S_\theta^3)^{g(p,q')^q}$. Conversely, take an element $a \in S_\theta^3$ such that $g(p, q')^q(a) = a$. Choose two integers r, s satisfying $ps-qr=1$ as usual, and note that $g(p, q)^p$ and hence $g(p, q)^{ps}$ are identity. Hence we have

$$g(p, q')(a) = g(p, q')^{ps} \circ g(p, q')^{-qr}(a) = a.$$

Thus a is fixed by the action $g(p, q')$. □

6. The structure of $L_\theta(p, q)$ as a C^*-algebra of A_θ-valued functions.

In this section, we study the structure of $L_\theta(p, q)$ as C^*-algebra and prove Theorem B, which appears as Theorem 6.8. Namely, we show that $L_\theta(p, q)$ is realized as a C^*-algebra of A_θ-valued continuous functions on the interval $[-1, 1]$. As $L_\theta(p, q)$ is a C^*-algebra constructed by attaching two non-commutative solid tori D_θ, we observe first the structure of D_θ and prove that it is a C^*-algebra of A_θ-valued continuous functins on $[0, 1]$.

For a real number r such that $0 \leqq r \leqq 1$, let S_r be the circle with radius r round the origin in the complex plane, where S_0 means the one point $\{0\}$. Then the restriction of a function of $C(D^2)$ to S_r yields a surjection ρ_r from the crossed product $C(D^2) \times_\theta \boldsymbol{Z}$ to the crossed product $C(S_r) \times_\theta \boldsymbol{Z}$. For any $r \in (0, 1]$, we denote by y_r the restriction of y to S_r, where the function y is defined as $y(re^{2\pi i\xi})=re^{2\pi i\xi}$, $r, \xi\in[0, 1]$. Notice that the following isomorphism given by

$$y_r \in C(S_r) \longrightarrow ry \in C(S^1)$$

induces an isomorphism from $C(S_r)\times_\theta \boldsymbol{Z}$ to $C(S^1)\times_\theta \boldsymbol{Z}=A_\theta$. We denote it by Ψ_r. When $r=0$, we denote by Ψ_0 the natural imbedding of the group C^*-algebra $C_r^*(\boldsymbol{Z})$ to the C^*-subalgebra $C^*(V)$ generated by the left regular representation of $\boldsymbol{Z}$ in $C(S^1)\times_\theta \boldsymbol{Z}$. Thus each $a\in D_\theta$ gives rise to the A_θ-valued function $\Lambda(a)$ on $[0, 1]$ as in the following way:

$$\Lambda(a)(r) = \Psi_r\circ\rho_r(a)\in A_\theta \qquad r\in[0, 1].$$

Then a straightforward calculation shows that, for any polynomial $a=\sum d_{ijk}y^i y^{*j} v^k$ in D_θ, the above function $\Lambda(a)$ is continuous on $[0, 1]$. Moreover, since the family of such polynomials is dense in D_θ, we can see that the same conclusion holds for any element $a\in D_\theta$.

Hence we obtain the following.

LEMMA 6.1. *For each $a\in D_\theta$, $\Lambda(a)$ is an A_θ-valued continuous function on* $[0, 1]$.

Thus, for each $a\in D_\theta$, $\Lambda(a)$ gives an element of $C([0, 1], A_\theta)$ the C^*-algebra of all continuous functions from $[0, 1]$ to A_θ. We set the C^*-algebra

$$C_V([0, 1], A_\theta) = \{a\in C([0, 1], A_\theta) \mid a(0)\in C^*(V)\}.$$

By construction of Λ, we see that, for each $a\in D_\theta$, $\Lambda(a)$ belongs to $C_V([0, 1], A_\theta)$. Therefore the algebra D_θ is embedded into $C_V([0, 1], A_\theta)$ through the map Λ. In fact, we have:

LEMMA 6.2. *For each $a\in D_\theta$, if $\Lambda(a)=0$, then $a=0$.*

PROOF. For each element a of the crossed product $D_\theta=C(D^2)\times_\theta \boldsymbol{Z}$, we may define the n-th Fourier coefficient $a(n)\in C(D^2)$, $n\in \boldsymbol{Z}$, of a by

$$a(n) = E(av^{-n}), \qquad n\in\boldsymbol{Z},$$

where E means the canonical conditional expectation from D_θ to $C(D^2)$ (cf. [**To1**, Chapter 3]). For $r\in(0, 1]$, let E_r be the canonical conditional expectation from $C(S_r)\times_\theta \boldsymbol{Z}$ to $C(S_r)$. As these expectations are compatible with the homomorphisms ρ_r's, we have

$$a(n)(re^{2\pi i\xi}) = E_r(\rho_r(a)v^{-n})(re^{2\pi i\xi}), \quad r, \xi\in(0, 1], \quad n\in\boldsymbol{Z},$$

Hence, if $\Lambda(a)=0$ and hence $\rho_r(a)=0$ for all $r\in(0, 1]$, then $a(n)=0$ for all $n\in\boldsymbol{Z}$, and $a=0$. $\square$

Henceforth we identify D_θ as a C^*-subalgebra of $C_V([0, 1], A_\theta)$ through Λ. We shall show that it coincides with $C_V([0, 1], A_\theta)$.

Let I_r be the kernel of $\Psi_r\circ\rho_r$, which is an ideal of D_θ. Then we have the following lemma.

LEMMA 6.3. *For a distinct pair t, s, the ideal I_t+I_s contains the unit of D_θ, hence $I_t+I_s=D_\theta$.*

PROOF. Assume that $t<s$ and define the function y_t as

$$y_t(re^{2\pi i\xi})=\begin{cases} 0 & \text{for } 0\leqq r\leqq t \\ (r-t)/(s-t) & \text{for } t\leqq r\leqq s \\ 1 & \text{for } s\leqq r\leqq 1. \end{cases}$$

Write $y_s=1-y_t$, then obviously $y_t\in I_t$, $y_s\in I_s$ and $y_s+y_t=1$. $\square$

PROPOSITION 6.4. *D_θ coincides with $C_V([0, 1], A_\theta)$. Namely the non-commutative solid torus D_θ is realized as the C^*-algebra of all A_θ-valued continuous functions on* $[0, 1]$ *whose values at* 0 *belong to the C^*-subalgebra $C^*(V)$ generated by V.*

PROOF. By the non-commutative Stone-Weierstrass Theorem for continuous cross section C^*-algebra [**To2**, Theorem 2.2], we have only to prove the following lemma: $\square$

LEMMA 6.5. *Put $A(r)=A_\theta$ $r\in(0, 1]$ and $A(0)=C^*(V)$. For any distinct points t, $s\in[0, 1]$, and elements of fibers $c\in A(t)$ and $d\in A(s)$, there exists an A_θ-valued continuous function $h\in D_\theta$ with $h(t)=c$ and $h(s)=d$.*

PROOF. As the homomorphism $\Psi_r\circ\rho_r$ from D_θ to $A(r)$ is surjective for each $r\in[0, 1]$, we may take f and $g\in D_\theta$ such that $f(t)=c$ and $g(s)=d$. By the previous lemma, we may write $f-g$ as a_t-a_s for some $a_t\in I_t$ and $a_s\in I_s$. Then put $h=f-a_t=g-a_s$, which is a desired one. $\square$

The assertion for the algebra $D_{-\theta}$ also holds if we replace θ by $-\theta$. We use this fact in our coming discussions. For an element $a\in D_{-\theta}$ we also denote a the corresponding function in $C_{\bar{V}}([0, 1], A_{-\theta})$.

To realize $L_\theta(p, q)$, as an algebra of A_θ-valued continuous functions on the interval $[-1, 1]$, we need further discussions.

Let $E=\begin{bmatrix} q & s \\ p & r \end{bmatrix}$ be an attaching matrix of determinant minus one in the construction of $L_\theta(p, q)$. Recall that ρ_E gives an isomorphism from $A_{-\theta}$ to A_θ defined by

$$\rho_E(\hat{V}) = e^{-pq\pi i\theta} V^q U^p \quad \text{and} \quad \rho_E(\hat{U}) = e^{-rs\pi i\theta} V^s U^r$$

and $L_\theta(p, q)$ is defined as

$$\{(a, b) \in D_{-\theta} \oplus D_\theta \mid \rho_E \circ \pi_{-\theta}(a) = \pi_\theta(b)\}.$$

LEMMA 6.6. *$D_{-\theta}$ is isomorphic to the following C^*-algebra*

$$C_{(q,p)}([-1, 0], A_\theta) = \{a \in C([-1, 0], A_\theta) \mid a(-1) \in C^*(V^q U^p)\}.$$

PROOF. By the previous discussions, we have

$$D_{-\theta} = \{a \in C([-1, 0], A_{-\theta}) \mid a(-1) \in C^*(\hat{V})\}.$$

Under the identification between $C([-1, 0], A_{\pm\theta})$ and $C([-1, 0]) \otimes A_{\pm\theta}$, one sees that the isomorphism $id \otimes \rho_E$ from $C([-1, 0]) \otimes A_{-\theta}$ to $C([-1, 0]) \otimes A_\theta$ induces the desired isomorphism from $D_{-\theta}$ to $C_{(q,p)}([-1, 0], A_\theta)$. □

We also write the isomorphism from $D_{-\theta}$ to $C_{(q,p)}([-1, 0], A_\theta)$ given in the above proof as $id \otimes \rho_E$. Let $\tilde{\pi}_\theta$ be the surjective homomorphism from $C_{(q,p)}([-1, 0], A_\theta)$ to A_θ defined by $\tilde{\pi}_\theta(a) = a(0)$.

We then have the next lemma.

LEMMA 6.7. *The following diagram is commutative:*

$$\begin{array}{ccc} D_{-\theta} & \xrightarrow{id \otimes \rho_E} & C_{(q,p)}([-1, 0], A_\theta) \\ {\scriptstyle \pi_{-\theta}} \downarrow & & \downarrow {\scriptstyle \tilde{\pi}_\theta} \\ A_{-\theta} & \xrightarrow{\rho_E} & A_\theta . \end{array}$$

PROOF. It suffices to check the generators x, u of $D_{-\theta}$. We first easily see that x, u in $C_{\hat{V}}([-1, 0], A_{-\theta}) = D_{-\theta}$ can be written as

$$x(t) = (1+t)\hat{U}, \qquad u(t) = \hat{V} \qquad t \in [-1, 0].$$

Hence it follows that

$$\tilde{\pi}_\theta \circ (id \otimes \rho_E)(x) = (id \otimes \rho_E)(x)(0) = \rho_E(\hat{U}) = e^{-rs\pi i\theta} V^s U^r$$

and

$$\tilde{\pi}_\theta \circ (id \otimes \rho_E)(u) = (id \otimes \rho_E)(u)(0) = \rho_E(\hat{V}) = e^{-pq\pi i\theta} V^q U^p,$$

which are nothing but the images of x, u through $\rho_E \circ \pi_{-\theta}$. □

THEOREM 6.8. *$L_\theta(p, q)$ is isomorphic to the following C^*-algebra of A_θ-valued continuous functions on $[-1, 1]$:*

$$\{f \in C([-1, 1], A_\theta) \mid f(-1) \in C^*(V^q U^p),\ f(1) \in C^*(V)\}.$$

PROOF. By construction and previous lemmas, we first see that $L_\theta(p, q)$ is isomorphic to the following C^*-algebra:

$$\left\{(a, b) \in C([-1, 0], A_{-\theta}) \oplus C([0, 1], A_\theta) \mid \begin{matrix} a(-1) \in C^*(\hat{V}),\ b(1) \in C^*(V) \\ \rho_E \circ (a(0)) = b(0) \end{matrix}\right\}.$$

Since we have $\rho_E(a(0)) = (id \otimes \rho_E)(a)(0)$ by Lemma 6.7, the above C^*-algebra is isomorphic, through $(id \otimes \rho_E) \oplus id$, to the following one:

$$\left\{(a, b) \in C([-1, 0], A_\theta) \oplus C([0, 1], A_\theta) \mid \begin{matrix} a(-1) \in C^*(V^q U^p),\ b(1) \in C^*(V) \\ a(0) = b(0) \end{matrix}\right\}.$$

The above C^*-algebra is a desired one so that we complete the proof. □

7. Realization of $L_\theta(p, q)$ as a universal C^*-algebra.

The purpose of this section is to prove Theorem C, which is our final goal. In the case of $p=0$, we assume that q must be 1. Throughout this section, we fix arbitrary coprime non-negative integers p, q.

Recall that $L_\theta(p, q)$ is realized as the following C^*-subalgebra of the algebra of all A_θ-valued continuous functions on $[0, \pi/2]$:

$$L_\theta(p, q) = \{f \in C([0, \pi/2], A_\theta) \mid f(0) \in C^*(V^q U^p),\ f(\pi/2) \in C^*(V)\}.$$

Now, we define the three candidate operators in $C([0, \pi/2], A_\theta)$ for Theorem C as in the following way:

$$B(t) = \cos t \cdot V^q U^p, \quad C(t) = \sin t \cdot V, \quad N(t) = \sin t \cdot \cos t \cdot U \qquad t \in [0, \pi/2].$$

The following result is obtained by easy caluculation.

LEMMA 7.1. *Keep the above notations. Then the operators B, C and N belong to $L_\theta(p, q)$ and satisfy the operator identities from* (1) *to* (5) *in Theorem C.*

Next, we prove that the three operators B, C and N generate $L_\theta(p, q)$ as C^*-algebra. We put $\mathcal{S} = C^*(B, C, N)$, the C^*-subalgebra of $L_\theta(p, q)$ generated by them. We regard $L_\theta(p, q)$ as an algebra of A_θ-valued continuous functions on $[0, \pi/2]$ as in the previous section. For each $t \in [0, \pi/2]$, we set

$$L_\theta(t) = \begin{cases} C^*(V^q U^p) & t=0 \\ A_\theta & t\in(0, \pi/2) \\ C^*(V) & t=\pi/2. \end{cases}$$

We denote by λ_t the natural surjection from $L_\theta(p, q)$ to $L_\theta(t)$. Put $J_t=\mathcal{S}\cap\ker(\lambda_t)$: the intersection between $\mathcal{S}$ and the kernel of λ_t.

We then have the following.

LEMMA 7.2. *For any distinct points t, $s\in[0, \pi/2]$, J_t+J_s contains the unit of $L_\theta(p, q)$, hence $J_t+J_s=\mathcal{S}$.*

PROOF. We may assume $t<s$. We notice $B^*B(t)=\cos^2 t$, $C^*C(t)=\sin^2 t$, $t\in[0, \pi/2]$ and hence $B^*B+C^*C=1$. Thus the subalgebra of all scalar valued continuous functions on $[0, \pi/2]$ is generated by C^*C and B^*B so that they belong to $\mathcal{S}$. Define the function y_t as

$$y_t(r) = \begin{cases} 0 & \text{for } 0\leqq r\leqq t \\ (r-t)/(s-t) & \text{for } t\leqq r\leqq s \\ 1 & \text{for } s\leqq r\leqq \pi/2 \end{cases}$$

and put $y_s=1-y_t$. We then easily see that $y_t\in J_t$, $y_s\in J_s$ and $y_t+y_s=1$. $\square$

LEMMA 7.3. *For any two elements $c\in L_\theta(t)$, $d\in L_\theta(s)$ of fibers at two distinct points t, $s\in[0, \pi/2]$, there exists an element $h\in\mathcal{S}$ such that $\lambda_t(h)=c$ and $\lambda_s(h)=d$.*

Proof is similar to that of Lemma 6.5.

Therefore by the Stone-Weierstrass Theorem for continuous cross section C^*-algebras as in [**To2**], we conclude the following:

PROPOSITION 7.4. *$\mathcal{S}$ coincides with $L_\theta(p, q)$. Namely the C^*-algebra $L_\theta(p, q)$ is generated by B, C and N.*

Next, we shall show the universality of $L_\theta(p, q)$ as the C^*-algebra corresponding to the operator relations in Theorem C.

Let $\mathcal{L}_\theta(p, q)$ be the universal C^*-algebra with respect to the relations in Theorem C. The algebra is defined as the quotient C^*-algebra of the universal unital C^*-algebra with three normal generators by the ideal generated by the relations from (1) to (5) in Theorem C. We denote by Q, R and Y the generators of $\mathcal{L}_\theta(p, q)$ corresponding to B, C and N respectively. Then, by the universality of $\mathcal{L}_\theta(p, q)$, there exists a surjective homomorphism Φ from $\mathcal{L}_\theta(p, q)$ to $L_\theta(p, q)$ such that

$$\Phi(Q)=B, \quad \Phi(R)=C, \quad \Phi(Y)=N.$$

We shall show that Φ is isometric and hence it becomes an isomorphism.

LEMMA 7.5. *For any irreducible representation* $(\pi, \mathcal{H})$ *of* $\mathcal{L}_\theta(p, q)$, *there exists an irreducible representation* $(\rho_\pi, \mathcal{H})$ *of* $L_\theta(p, q)$ *such that*

$$\pi(Q)=\rho_\pi(B), \quad \pi(R)=\rho_\pi(C), \quad \pi(Y)=\rho_\pi(N)$$

and hence $\pi(\mathcal{L}_\theta(p, q))=\rho_\pi(L_\theta(p, q)$.

PPOOF. By the operator relations of Q, R and Y corresponding to (1), (2) in Theorem C, and [**Fug**], [**Put**], we see that the three operators Q^*Q, R^*R and Y^*Y belong to the center of $\mathcal{L}_\theta(p, q)$. Hence $\pi(R^*R)$ is a scalar which is written as μ.

CASE 1. $0<\mu<1$. As $\pi(Q^*Q)=1-\mu$ by the corresponding relation to (3), the relation (5) implies the following:

$$\begin{aligned} \pi(R)^q\pi(Y)^p &= \mu^{(p+q)/2}(1-\mu)^{(p-1)/2}\pi(Q) && (p\neq 0)\\ (1-\mu)^{1/2}\pi(R) &= \mu^{1/2}\pi(Q) && (p=0) \end{aligned}$$

so that $\pi(Q)$ is generated by $\pi(R)$ and $\pi(Y)$. Since it follows that

$$\pi(R)\pi(Y)=e^{2\pi i\theta}\pi(Y)\pi(R)$$

we see that $\pi(\mathcal{L}_\theta(p, q))$ is a homomorphic image of A_θ. It induces the irreducible representation ρ_π of $L_\theta(p, q)$ through A_θ with the required properties.

CASE 2. $\mu=0$. In this case, by (4), the C^*-algebra $\pi(\mathcal{L}_\theta(p, q))$ becomes a commutative algebra generated by $\pi(Q)$ and 1. As π is irreducible, it is nothing but $\boldsymbol{C}1$. Hence the quotient map from $L_\theta(p, q)$ to $L_\theta(0)$ induces the desired irreducible representation of $L_\theta(p, q)$.

CASE 3. $\mu=1$. By the same discussions as the case of $\mu=0$, we can conclude the desired assertion.

This completes the proof. □

Now we reach the following final theorem.

THEOREM 7.6. *The natural surjection* Φ *from* $\mathcal{L}_\theta(p, q)$ *to* $L_\theta(p, q)$ *is isometric. Hence it induces an isomorphism between them.*

PROOF. By Lemma 7.5, for each irreducible representation π of $\mathcal{L}_\theta(p, q)$, we can find an irreducible representation ρ_π of $L_\theta(p, q)$ which satisfies $\pi=\rho_\pi\circ\Phi$. Hence we have for each $X\in\mathcal{L}_\theta(p, q)$,

$$\|\pi(X)\|=\|\rho_\pi\circ\Phi(X)\|\leqq\|\Phi(X)\|$$

so that

$$\sup_{\pi\in\Pi}\|\pi(X)\|\leqq\|\Phi(X)\|$$

where Π means the set of all irreducible representations of $\mathcal{L}_\theta(p, q)$. Since the norm of X in $\mathcal{L}_\theta(p, q)$ is given by $\sup_{\pi\in\Pi}\|\pi(X)\|$, we obtain the inequality $\|X\|\leqq\|\Phi(X)\|$. Hence we complete the proof. □

Here we refer to the classification of $L_\theta(p, q)$ concerning about θ under fixed coprime integers p and q. Recall the case of A_θ. As in [**Ri1**], for irrational numbers θ_1, θ_2, if $\theta_1\neq\pm\theta_2$ mod $\boldsymbol{Z}$, A_{θ_1} may not be isomorphic to A_{θ_2}. Therefore we have

PROPOSITION 7.7. *Fix two integers p and q. For irrational numbers θ_1, θ_2, if $L_{\theta_1}(p, q)$ and $L_{\theta_2}(p, q)$ are mutually isomorphic, then $\theta_1\equiv\pm\theta_2$* mod $\boldsymbol{Z}$.

PROOF. It is immediate from the Rieffel's result of A_θ by considering irreducible representations of $L_{\theta_i}(p, q)$. □

8. Non-commutative S^3 and RP^3.

In this section, we treat the two special non-commutative lens spaces, three dimensional sphere and real projective space.

As we stated before, the lens space of type (1, 0) is homeomorphic to the 3-sphere. Thus when we take (1, 0) as (p, q) in Theorem C, we have some operator relations which realize S_θ^3 as a universal C^*-algebra. In fact, one sees the following.

PROPOSITION 8.1. *The non-commutative 3-sphere S_θ^3 becomes the universal C^*-algebra generated by two operators B, C with the following relations:*

$$B^*B = BB^*,\quad C^*C = CC^*,\quad CB = e^{2\pi i\theta}BC,\quad B^*B+C^*C = 1.$$

PROOF. In Theorem C, when $(p, q)=(1, 0)$, one sees that N is generated by B, C by the relation (5). Hence only the relations (1) and (3) survive. □

By the above corollary, we come to know that the operator relations of S_θ^3 in [**Ma1**], which are stated in the section 0, are equivalent to those in Proposition 8.1. In fact, the following correspondence connect the two operator relations:

$$B = S/(S^*S+T^*T)^{1/2},\qquad C = T/(S^*S+T^*T)^{1/2}.$$

We are very interested in the operator relations cited in Proposition 8.1 because they are similar to those of the quantum $SU(2)$ of Woronowicz which is written as $S_\nu U(2)$ in [**Wo1**] and [**Wo2**]. Roughly speaking, the Woronowicz's quantum groups $S_\nu U(2)$ are obtained by deformation of $SU(2)$ along real line reminding its group structure. However the above proposition means that our non-commutative 3-spheres S_θ^3 are the deformation C^*-algebra of $SU(2)$ deformed along the complex numbers with modulus one.

Now we refer to the non-commutative version of real projective space. Since the lens space $L(2, 1)$ of type $(2, 1)$ is homeomorphic to three dimensional real projective space $\boldsymbol{R}P^3$, we may consider $L_\theta(2, 1)$ as a non-commutative version of $\boldsymbol{R}P^3$ which we write as $\boldsymbol{R}P_\theta^3$. Thus as a corollary of Theorem C, we may write down operator relations of $\boldsymbol{R}P_\theta^3$, as well as those of S_θ^3, as a univesal C^*-algebra.

PROPOSITION 8.2. *The non-commutative real projective space $\boldsymbol{R}P_\theta^3$ is the universal C^*-algebra generated by three operators determined by the following relations:*

$$B^*B = BB^*, \quad C^*C = CC^*, \quad N^*N = NN^*,$$

$$CB = e^{4\pi i\theta}BC, \quad CN = e^{2\pi i\theta}NC, \quad BN = e^{2\pi i\theta}NB,$$

$$B^*B + C^*C = 1,$$

$$N^*N = B^*B \cdot C^*C,$$

$$CN^2 = (C^*C)^{3/2}(B^*B)^{1/2}B.$$

It is nuclear for each $\theta \in \boldsymbol{R}$ and is not of type I for irrational θ. The K-groups of the algebra are

$$K_0(\boldsymbol{R}P_\theta^3) = \boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z} \quad \text{and} \quad K_1(\boldsymbol{R}P_\theta^3) = \boldsymbol{Z}.$$

PROOF. The operator relations are seen directly from Theorem C. The rest of the proposition is also clear from our previous discussions. □

REMARK. Since $\boldsymbol{R}P^3$ is homeomorphic to the Lie group $SO(3)$, we may also regard $\boldsymbol{R}P_\theta^3$ as a non-commutative version of $SO(3)$.

References

[Bl] B. Blackadar, K-theory for operator algebras, Springer-Verlag, 1986.

[BEEK] O. Bratteli, G. Elliott, D. Evans and A. Kishimoto, Non-commutative spheres I, II, preprint.

[Bre] B. A. Brenken, Representations and automorphisms of the irrational rotation algebras, Pacific. J. Math., **111** (1984), 257-282.

[BDF] L. Brown, R. Douglas and P. Fillmore, Extensions of C^*-algebras and K-homology, Ann. of Math., **105** (1977), 265-324.

[Co1] A. Connes, An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\boldsymbol{R}$, Advances in Math., **39** (1981), 31-51.

[Co2] A. Connes, Non Commutative Differential Geometry. Chapter I: The Chern Character in K-Homology. Chapter II: De Rham Homology and Non Commutative Algebra, Inst. Hautes Etudes Sci. Publ. Math., **62** (1986), 257-360.

[CR] A. Connes and M. A. Rieffel, Yang-Mills for non-commutative two-tori, Contemp. Math., Amer. Math. Soc., **62** (1987), 237-266.

[El1] G. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J. Algebra, **38** (1976), 29-44.

[El2] G. Elliott, On the K-theory of the C^*-algebra generated by a projective representation of a torsion free discrete abelian group, Operator algebras and group representations, **1**, Pitman, 1983, 157-184.

[Fug] B. Fuglede, A commutative theory for normal operators, Proc. Nat. Acad. Sci., **36** (1950), 35-40.

[He] J. Hempel, 3-Manifolds, Princeton University Press, 1976.

[Ja] W. Jaco, Lectures on Three-Manifold Topology, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, Number 43, Amer. Math. Soc. and National Science Foundation, 1980.

[Ka] G. Kasparov, The operator K-functor and extensions of C^*-algebras, Akad. Nauk SSSR, Ser. Math., **44** (1980), 571-636; Math. USSR Izvestijia, **16** (1981), 513-572.

[Ma1] K. Matsumoto, Non-commutative three dimensional spheres, preprint.

[Ma2] K. Matsumoto, Non-commutative three dimensional spheres, II. Non-commutative Hopf fibering, to appear in Yokohama Math. J..

[Ma3] K. Matsumoto, Non-commutative 3-spheres, to appear in the Proceedings of Current Topics in Operator Algebras, A Satellite conference of ICM-90, World Scientific Press.

[Pe] G.K. Pedersen, C^*-algebras and their automorphism groups, Academic Press, 1979.

[PV] M. Pimsner and D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain cross products of C^*-algebras, J. Operator Theory, **4** (1980), 93-118.

[Put] C.R. Putnam, On normal operators in Hilbert space, Amer. J. Math., **73** (1951), 357-361.

[Ri1] M.A. Rieffel, C^*-algebras associated with irrational rotations, Pacifc J. Math. **93** (1981), 415-429.

[Ri2] M.A. Rieffel, Deformation quantizations and operator algebras, preprint.

[Ro] J. Rosenberg, The role of K-theory in non-commutative algebraic topology, Operator Algebras and K-Theory. Contemporary Mathematics, **10**, Amer. Soc., 1982.

[RS] J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, Duke Math. J., **55** (1987), 431-474.

[To1] J. Tomiyama, Invitation to C^*-algebras and topological dynamics, World Scientific Advanced Series in Dynamical Systems, **3**, World Scientific Press, 1987.

[To2] J. Tomiyama, Topological representation of C^*-algebras, Tôhoku Math. J., **14** (1962), 187-204.

[Wa] Y. Watatani, Toral automorphisms on irrational rotation algebras, Math. Japonica, **26** (1981), 479-484.

[Wo1] S.L. Woronowicz, Twisted SU(2) group. An example of a non-commutative differential calculus, Publ. Res. Inst. Math. Sci., **23** (1987), 117-181.

[Wo2] S.L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys., **111** (1987), 613-665.

Kengo MATSUMOTO
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Hachioji 192-03
Japan

Jun TOMIYAMA
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Hachioji 192-03
Japan